Learning to Consume: Individual versus Social Learning

Nathan Palmer *
Revised September 15, 2012

September 15, 2012

Abstract

Obtaining the exact solution to the consumption-under-uncertainty problem can be an extraordinarily difficult mathematical problem. Allen and Carroll (2001), however, show that a simple linear approximation to the optimal policy rule can yield utility trivially close to that of the exact solution. Furthermore, Allen and Carroll show that consumers can consistently find the near-optimal approximate solution using simple trial-and-error statistical learning, which has an attractive and intuitive interpretation. Their result comes with a negative corollary: the amount of time necessary to consistently find a near-optimal rule is astronomical and unattainable under any plausible scenarios. This paper extends Allen and Carroll's original agent-based model in two ways: first, we incorporate social learning into the process, and second, we introduce an new, intuitively motived estimator of the value of a simple linear consumption rule. Social learning occurs through a simple form of information sharing. This addition retains the original model's results that consumers can consistently find an optimal rule, but lowers the time required to find such a rule arbitrarily close to the lowest possible bound. The time required to find a rule is now a function of the number of agents in the model. Furthermore, the new estimator we identify further decreases the amount of time required to find a near-optimal rule by a full order of magnitude. This estimator also opens the door for the social learning process to incorporate heterogeneous initial endowment values across agents, a welcome extension to the original model.

1 Introduction

Buffer-stock savings is an intuitive and empirically grounded result to emerge from the literature on consumption under uncertainty (Carroll 1992, 2011). Even though buffer-stock savings behavior may be both intuitive and observed, it is difficult to imagine a household coming to this solution via the theory and computation employed by researchers. Allen and Carroll (2001) note this explicitly, and as an alternative, propose that households may learn near-optimal consumption behavior by trial and error. Essentially, they propose that households run a Monte Carlo experiment using their own lifetime experience to estimate the value function associated with a simplified linear consumption rule. They implement this model, and their results are both exciting and discouraging. On the positive side, they show that a simple linear consumption rule can closely approximate the true consumption function. Furthermore, given enough time, consumers using trial and error learning over a set of linear rules can consistently find a near-optimal rule. Their negative result is that "enough time" is anywhere from 200,000 to 4 million years in their model's parameterization. As the authors note, even with a highly efficient search algorithm, the time required to consistently find a near-optimal rule is well outside a human lifetime. The trouble is that the Monte Carlo style estimation relies on random variables which are temporal in nature – a single draw requires agents to literally live through a number of periods.

This paper presents preliminary results for two extensions to Allen and Carroll's framework which seek to address the negative result. First, agents are placed on a network and allowed to share their experiences with one another. This greatly reduces the amount of time required for a population of agents to find a near-optimal rule. With full information sharing among agents, we find it possible to push the number of periods necessary to find a near-optimal consumption rule arbitrarily close to the lowest bound on finding such rules, well within the lifetime of an agent.

*Nathan Palmer, George Mason University. Email: npalmer4@gmu.edu
Many thanks to Rob Axtell and Andrew Crooks for many helpful comments. All mistakes are my own.

The second extension introduces a modified learning rule, such that agents share (and employ) a measure of welfare which is relative to welfare experienced under a "baseline" consumption rule. A natural candidate for the baseline rule is the "consume everything" rule. This extension intends to eventually allow agents with different initial endowments to share information, which is not possible under the original formulation. This is one attempt to make the solution method usable in a larger simulation setting: if endowments can vary, agents can share information freely both across endowments in a single period, and potentially "across generations." Initial results indicate that this modified learning rule performs as well as – and often better than – the original rule in original homogeneous initial endowment setting. In a heterogeneous initial endowment setting, tentative results suggest that the modified rule performs comparably to the original rule in the original setting.

An extensive literature spanning nearly a century exists for the modern formulation of the consumption under uncertainty model. Although Keynes (1936) famously discussed the consumption function in terms of the marginal propensity to consume, it was Friedman (1957) who did much to develop and popularize the idea of the consumption function in its current form, along with the idea that agents acted dynamically and intelligently when making consumption decisions. Due to many technical difficulties, for many years models of the consumption process were bound away from dealing with the issue of income uncertainty – see Hall (1978) for a particularly famous example of this, and Deaton (1992) for an overview of the difficulties this era of models encountered. It wasn't until Zeldes (1989) that the first work on consumption under income uncertainty began to develop. Since then, there has been an explosion of research in this area; Carroll (2001b) offers an excellent overview of this history.

With respect to the particular model extended here, two recent papers have addressed the question of household learning about consumption. Howitt and Ozak (2009) implement an adaptive consumption rule based on the Euler equation, which does not require extensive memory on the part of the agent. Yildizoglu et al. (2012) explicitly seek to address the same negative results this paper addresses, by expanding agent cognition and learning. Their methods included learning through imitation and learning which employs genetic algorithms and artificial neural networks to form outlooks. Our approach differs from both by simply extending the original Allen and Carroll framework to include social learning without introducing much new cognitive architecture.

The rest of the paper is organized as follows: Section 2 reviews the original Allen and Carroll model and explains why agent-based modeling (ABM) is the best approach here. Sections 3 and 4 discusses the first and second extensions to the model, respectively, and algorithmic details of their implementation. Section 5 explores the results, and Section 6 concludes with a discussion of work in progress and directions for future research.

2 Individual Learning

2.1 The Model

Allen and Carroll (2001) construct a simple model of buffer stock savings, as in Carroll (1992)[1]. The consumer[2] solves,

$$\underset{\{C_t\}_0^\infty}{max}\ U = \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(C_t)\right] \tag{1}$$

$$\begin{array}{rcl} & s.t. & \\ X_{t+1} & = & R\left[X_t - C_t\right] + Y_{t+1} \\ C_t & \leq & X_t \\ X_0 & = & S_0 + Y_0 \\ S_0 & & given \end{array}$$

where S_0 is initial savings, X_t is "cash on hand" available to a consumer at time t, C_t is consumption in period t, and R is the interest rate. Here, $R = 1$ for simplicity. Y_t is a simple random income process, (2),

[1]See Carroll (2011) for an extensive discussion of this general class of model.

[2]The terms "consumer" and "agent" are used interchangeably to refer to the agents in this model.

chosen to roughly match the properties of income explored in Carroll (1992):

$$Y_t = \begin{cases} 0.7 & with\ prob\ 0.2 \\ 1.0 & with\ prob\ 0.6 \\ 1.3 & with\ prob\ 0.2. \end{cases} \tag{2}$$

The instantaneous utility function will be the standard Constant Relative Risk Aversion (CRRA) form,

$$u(C) = \frac{C^{(1-\alpha)}}{(1-\alpha)}$$

with $\alpha = 3$ in this model.

The solution to the problem is the sequence of consumption choices $\{C_t\}_0^\infty$, which may be represented as the optimal consumption function is denoted $C^*(X)$.[3]General properties of the function C^* are known. As discussed in Allen and Carroll (1997), when consumers are impatient enough,[4] C^* can be stated in the form:

$$C^*(X_t) = 1 + f(X_t - \bar{X}^*). \tag{3}$$

where $\bar{X}^*$ is a target buffer stock of liquid wealth. When an agent experiences low income, he/she will consume some of the buffer; when income returns to normal, the consumer will attempt to save back to the target level $\bar{X}^*$. The consumption function (3), then, is stated as the average expected income (here, 1) plus a function of the **difference** between current cash on hand, X_t, and the target buffer stock level, $\bar{X}^*$. Thus if the agent's current cash on hand is below the target savings level, f tells the agent how much to consume (and thus how much to save) to move back towards the target savings level.

2.2 Approximating a Solution with Experience-based Learning

The function f, and thus C^*, is non-linear and very difficult to find. As Allen and Carroll (2001) note,

> "Despite its heuristic simplicity, the exact mathematical specification of optimal behavior is given by a thoroughly nonlinear consumption rule [C^*] for which there is no analytical formula. While certain analytical characteristics of the rule can be proven, it is hard to see how a consumer without a supercomputer and a Ph.D. could be expected to determine the exact shape of the nonlinear and non-analytical decision rule."

Anecdotal evidence, however, suggests that many people do behave stylistically similarly to the optimal consumption rule – they save save some fixed level of income for "rainy days," consume it in bad times, and build it back up in better times. Allen and Carroll ask a very natural question: if C^* is so difficult to find, how could we expect the average household to ever discover it? Certainly the fact that it took casts a dim light on the prospects of the average household ever finding such a rule. Allen and Carroll offer a novel hypothesis in the literature: if a simple linear version of the rule could be created, perhaps households could get close to the optimal rule by trial and error. To get a simple linear version of (3) to use in such a process, Allen and Carroll take a first-order Taylor approximation around the target buffer-stock wealth, $\bar{X}$, which yielded a function of the form

$$C^\theta(X_t) = 1 + \gamma[X_t - \bar{X}], \tag{4}$$

and implementing the liquidity constraint,

$$C^\theta(X_t) = \begin{cases} 1 + \gamma(X - \bar{X}) & if\ 1 + \gamma(X - \bar{X}) \leq X \\ X & if\ 1 + \gamma(X - \bar{X}) > X \end{cases}. \tag{5}$$

where now f has been replaced with a simple linear function γx. For convenience, let $\theta := (\gamma, \bar{X})$ be the pair of parameters which define this rule; the superscript θ indicates that C^θ[5] uses rule θ. We can find

[3]We will denote the rule $C^*(X)$ as C^* for brevity.

[4]The condition in this version of the model is $R\beta < 1$. Intuitively, this is a statement is that consumers are impatient; see Carroll (1997, 2011) for a thorough discussion of this problem.

[5]Because the linear consumption rule C^θ is associated with one and only with one parameter-pair $(\gamma, \bar{X})$, we will refer to the linear consumption rule that uses the parameters $\theta = (\gamma, \bar{X})$ interchangeably as both "C^θ" and the shorthand, "θ."

the "optimal approximate" rule $\theta^* = (\gamma^*,\ \bar{X}^*)$ by taking γ^* from the linearizion process applied to (3); the optimal value for $\bar{X}$ is already know from the optimal solution (3). As noted in Allen and Carroll (2001), this optimal rule (to three decimal places) is

$$\theta^* = (\gamma^*,\ \bar{X}) = (0.233, 1.243).$$

Denote the approximate consumption rule that uses these optimal values $C^{\theta*}$.

Both C^{θ} and C^* are possible solutions to the consumption problem (1). To discuss consumer welfare, construct the value function associated with each by plugging them into (1):

$$\begin{aligned} V^{\theta}(\,X_0\,) &= \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(C^{\theta}(X_t))\right] && (6)\\ V^{*}(\,X_0\,) &= \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(C^{*}(X_t))\right], && (7)\end{aligned}$$

with constraints omitted for brevity. By the optimality of C^* for the problem (1), we know that (7) must be greater than all possible (6) values:

$$V^*(X_0) \geq V^{\theta}(X_0)\ \ for\ each\ X_0\ and\ \forall\theta.$$

The question is then which value of V^{θ} is closest to the optimal V^*. One may be tempted to examine simple Euclidean distance to choose the closest rule – that is, for each θ, determine

$$\delta^{\theta}(X_0) = |V^*(X_0) - V^{\theta}(X_0)|$$

and then simply choose the θ for which δ^{θ} is smallest. This approach, however, is dissatisfying from a theoretical perspective. Technically, the left-hand sides of (6) and (7) are in terms of "utility," which is actually a reflection of **ordinal** preferences, and therefore the cardinality of (6) and (7) have no natural value. The cardinality is relevant insofar as it implies an ordinality, but the distances $\delta^{\theta}(X_0)$ have no meaningful value beyond that.

The way around this conundrum is to ask the question "which is better?" in terms of something that is naturally a cardinal value. The approach Allen and Carroll take is to ask the following: "Assume a consumer is using the optimal consumption rule. How much cash-on-hand would he/she be willing to pay to **not** switch from using the optimal rule C^* to some non-optimal rule C^{θ}?"

Call this value the "sacrifice value," denoted ϵ and defined as,

$$\begin{aligned} V^*(X_0-\epsilon) &= V^{\theta}(X_0)\\ \Leftrightarrow \epsilon &= X_0 - V^{*(-1)}(V^{\theta}(X_0))\\ \Leftrightarrow \epsilon^{\theta}(X_0) &= X_0 - V^{*(-1)}(V^{\theta}(X_0)),\end{aligned}$$

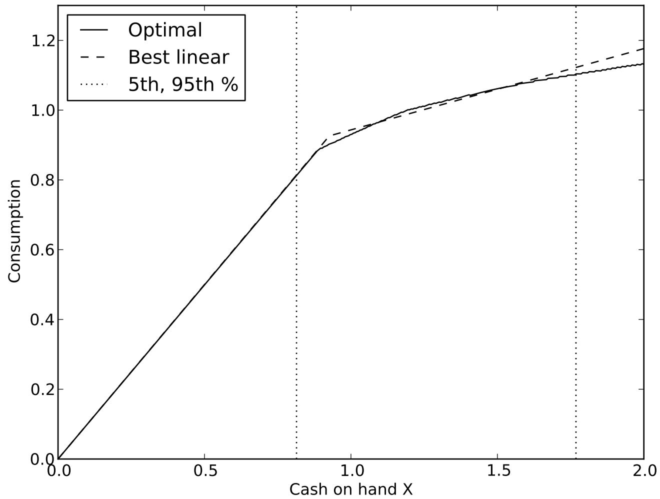
where ϵ^{θ} indicates that this sacrifice value is for rule θ. Note that $V^{*(-1)}$ is the inverse of V^*. Because the sacrifice value clearly depends on the level of wealth that a consumer has at the moment of the decision, X_0, Allen and Carroll opt for one additional step. They construct the **average** of the sacrifice value for each rule θ, which they call $\bar{\epsilon}^{\theta}$:

$$\bar{\epsilon}^{\theta} = \mathbb{E}[\epsilon^{\theta}(X_0)].$$

To find this expectation, Allen and Carroll need a distribution over the X-values. They use the ergodic distribution of cash on hand values constructed via simulation as described in Carroll (2001a). They construct a large number of agents with utility and a budget as in (1) and an income as in (2), and hand them the optimal consumption rule C^*. Then they simply run all these consumers forward through time; after a very short time, the distribution settles down to the ergodic distribution of cash-on-hand wealth.

This brings us to Allen and Carroll's first positive results. With $\bar{\epsilon}^{\theta}$ in hand, a very natural first question is, "what is the sacrifice value of the ruleθ^*? Surprisingly enough, Allen and Carroll find the the answer to be $\bar{\epsilon}^{\theta*} = 0.003$. This is a very small value. Recall that $\bar{\epsilon}^{\theta}$ is in terms of expected cash-on-hand. Recall also the

Figure 1: The Exact Consumption Rule (solid) and the Best Approximation (dashed)



Author's replication of Allen and Carroll's (2001) Figure 2.

calibration of annual income in equation (2) implies that average annual income is normalized to 1. Thus a sacrifice value of 0.003 corresponds to 0.3% of average annual income for an agent in this model – less than one half of one percent. If, for example, average annual income was instead \$50,000, $\bar{\epsilon}^{*}$ would be \$150. To state this differently, if it cost a consumer more than a one-time payment of \$150 to find the "thoroughly nonlinear ... and non-analytical decision rule" C^{*}, the consumer would be better off using the simple linear consumption rule $C^{\theta*}$ for the rest of their lives.

Figure (1) reveals why $\bar{\epsilon}^{\theta*}$ is so low: between the 5^{th} and 95^{th} percentiles of the ergodic cash-on-hand distribution, denoted by the vertical dotted lines, the best linear rule is incredibly close to the exact nonlinear rule. Above the 95^{th} percentile the approximation grows much worse, but the majority of the time consumers will not encounter this region.[6]

Of course, the optimal approximate rule θ^{*} is only one of many possible rules. To get an idea of what the sacrifice value looks like over a wider range of rules, Figure (2) displays a contour map of $\bar{\epsilon}^{\theta}$ over a grid of $(\gamma, \bar{X})$ values. The grid is discussed further below. The grid point closest in utility terms to the exact rule is $(0.25, 1.2)$. Call this rule $\theta^{grid} := (0.25, 1.2)$. The sacrifice value for θ^{grid} is 0.007, slightly more than half a percent of average annual income. That is, for a \$50,000 annual income, this would be \$350. Figure (2) will provide important insight into results below, so it is worthwhile to spend a moment exploring it.

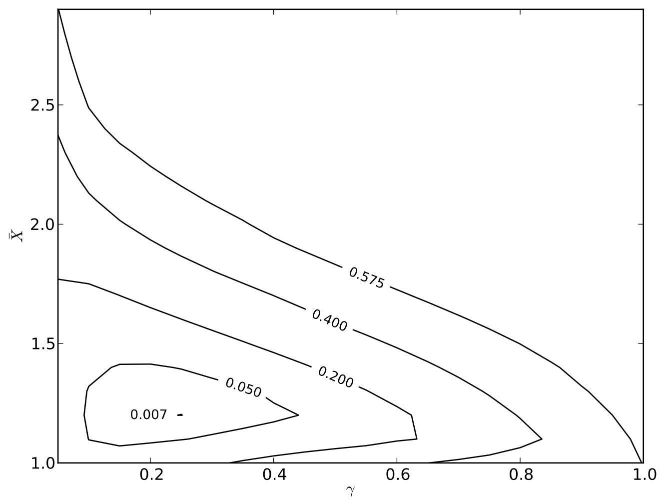
Note that the highest-valued contour line, at 0.575, corresponds to the rule $(1,1)$. Plugging this into (4), we get

$$\begin{aligned} C^{(1,1)}(X_t) &= 1 + (1)[X_t - 1] \\ &= X_t. \end{aligned}$$

That is, we get the "consume everything" rule. This rule, in which agents save nothing and always consume their full income each period, has a sacrifice value of 0.575 – for an average income of \$50,000, about \$29,000 dollars. This is significantly higher than the lowest sacrifice rule, whether we consider θ^{*} or θ^{grid}. The other contour lines are at intermediate values – 5%, 20%, and 40% of average annual income. For an average income of \$50,000, these would correspond to one-time payments of \$2,500, \$10,000, and \$20,000, respectively.

[6]The 45° line is due to the fact that over a certain region of cash-on-hand, the optimal consumption choice is actually higher than the agent's income, and thus the agent consumes everything. This is the "credit constrained" region.

Figure 2: Sacrifice Values For Approximate Linear Consumption Rules



Author's replication of Allen and Carroll's (2001) Figure 1.

2.2.1 Finding the Approximate Rules

To model trial-and-error learning, Allen and Carroll propose a statistically consistent estimator of (6) by defining $\bar{W}^{\theta}$ as follows. Let

$$W_i^{\theta}(X_0) = \sum_{t=0}^{N} \beta^t u(C^{\theta}(X_t)), \text{ and} \tag{8}$$

$$\bar{W}^{\theta} = \frac{1}{M} \sum_{i=1}^{M} W_i^{\theta}(X_0). \tag{9}$$

Let us unpack this statement a little. Each X_t is produced by a different realization of a random income process, starting from the same S_0.[7] As N in (8) goes to ∞, (8) clearly approaches a single-X_t-realization of the summation inside the expectation in equation (6). What we want, of course, is the full expectation in equation (6). Now we employ equation (9). This is the average over M independent "runs" of (8), so as $M \to \infty$, $\bar{W}$ will yield a consistent estimator of (6).

Thus as N and M jointly go to infinity, $\bar{W}$ yields an accurate estimate the value of a particular consumption rule for a consumer. This can now easily be translated into agents' trial-and-error exploration of consumption rules. Assume that the consumer has some number of different consumption rules to try and compare, say, $\{\theta_1, \theta_2, \theta_3, ..., \theta_K\}$. Given some rule θ, the household can try it for N periods, M times, and get an estimate of (6) for that rule. Repeat this for each rule; then at the end of this procedure, simply compare the values attained by each rule. Pick the highest valued rule and go with that. Since the household clearly cannot let N and M go to infinity, the household will have some noisy estimate of the true value of V^{θ} – itself an approximation of V^* – for some finite values of N and M. The question, then, is "what N and M will give consumers a reasonable chance of finding the optimal consumption rule?"

To answer this question, Allen and Carroll create a discrete grid for the consumption rules and instantiate 100 artificial consumers, who explore the space individually with a grid search (discussed further below) for a given (N, M) pair. Specifically, let $\gamma \in (0, 1]$ with steps of 0.05, and $\bar{X} \in [1, 3)$ by steps of 0.1, for a total of 20x20 = 400 possible rules. Formally, the rule space Θ which Allen and Carroll employ is

$$\Theta := \left\{(\gamma, \bar{X}) \ s.t. \ \gamma = \{0.05, ..., 1.0\} \text{ in 0.05 steps; } \bar{X} = \{1.0, ..., 2.9\} \text{ in 0.1 steps}\right\}. \tag{10}$$

[7]Which produced an X_0 value, as described for the problem (1).

Table 1: Allen and Carroll's Individual Search Results

S0 = 1					
		M = 1	M = 10	M = 50	M = 200
N = 10	Mean sacrifice:	0.269	0.122	0.100	0.102
	Success rate:	0.09	0.23	0.29	0.24
	Total Periods:	4000	40000	200000	800000
N = 20	Mean sacrifice:	0.226	0.079	0.053	0.047
	Success rate:	0.18	0.45	0.62	0.68
	Total Periods:	8000	80000	400000	1600000
N = 50	Mean sacrifice:	0.187	0.058	0.036	0.024
	Success rate:	0.26	0.58	0.76	0.91
	Total Periods:	20000	200000	1000000	4000000

Source: Allen and Carroll (2001)

Allen and Carroll sweep over a parameter space of N and M, repeating the above process for each (N, M) pair in the space:

$$\mathbb{L} := \{(N, M) \ s.t. \ N \in \{10, 20, 50\} \ and \ M \in \{1, 10, 50, 200\}\}.$$

Lastly, because the process of (8) and (9) requires the same value of initial cash-on-hand for X_t, Allen and Carroll restrict their consumers to repeated searches from the same initial savings value, S_0 (recalling the budget constraint in equation (1) above); this ensures that (9) is a consistent estimator of (6).[8] To see whether different S_0 values influence how well a consumer can find an optimal rule, Allen and Carroll include S_0 as a parameter over which to sweep.

2.3 Original Model Results

The results of Allen and Carroll's parameter sweeps across N and M are displayed in Table 1, an excerpt of the original results.[9] "Mean Sacrifice" is $\bar{\epsilon}^{\theta}$. "Success Rate" measures the fraction of the 100 consumers who have chosen a rule with a sacrifice value ≤ 0.05 by the end of the simulation – that is, the consumers who fall within the 0.05 contour in Figure 2. "Total periods" is the number of periods it takes a consumer to obtain the solution.

It is immediately clear that for consumers to find a near-optimal linear rule **reliably** – greater than 67% of the time, for example – they must spend hundreds of thousands of years exploring the parameter space. The lowest (N, M) pair which achieves this is $(50, 50)$, which takes 1,000,000 years for each agent to undertake individually. This result turns more negative when we recall that these outcomes apply for only a single value of S_0 – if consumers wanted to look for a best rule across multiple values of S_0, they need to roughly triple their search time. Clearly, part of this time is the inefficient grid search – the fact that each consumer must try out each of the 400 rules immediately implies that the time spent for a given (N, M) combination is $N * M * 400$. Note, however, that a more efficient search will not solve the problem. As Allen and Carroll state,

> "...even if the search could be reduced so that only, say, 4 different rules needed to be evaluated, it would still be necessary to use values of (M, N) large enough to distinguish good rules from bad. Given that the minimum (M, N) combination that appears capable of producing the necessary accuracy is (50,50), even [a] highly efficient hill climbing routine could not reduce the number of periods required to less than $10{,}000 = 50 * 50 * 4$."

The key is really what values of (N, M) provide an acceptable **success rate**; which (N, M) pair allows the consumer to distinguish good rules from bad rules using $\bar{W}^{\theta}$. The top panel of Table 2 shows that, for a

[8] They use S_0 instead of X_0 to make some technical computations easier.

[9] The full results include tables for $S_0 = 0$ and $S_0 = 2$ as well. These results are largely in line with those shown, and are excluded in this paper for the sake of space and clarity.

minimum success rate greater than 67%, we still need an (N, M) pair of at least $(50, 20)$.[10] Even if agents only need to explore 4 rules, this still implies that an agent would need to take 10,000 years to find an adequate rule. The story is broadly the same for $S_0 = 0$ and $S_0 = 2$ (not shown).

Thus Allen and Carroll arrive at two strong positive results and one strong negative result:

1. A linear approximation to the optimal consumption function can get negligibly close to the optimal rule in utility terms;
2. given enough time, consumers can **reliably** find a near-optimal rule with simple trial-and-error learning, and
3. unfortunately, "enough time" is prohibitively high – anywhere from 400,000 to 4 million years.

3 The First Extension: Social Learning

It is clear that the trouble faced by agents in this model is akin to that of a programmer completing a large number of independent tasks in serial on a single machine. If there are 50 independent, identical tasks and 50 available processors for parallel computation, the job could be done in $1/50^{th}$ the time. This relies, of course, on the fact that the 50 tasks are independent of one another. Recall the estimation problem:

$$\begin{aligned} W_i^{\theta}(X_0) &= \sum_{t=0}^{N} \beta^t u(C^{\theta}(X_t)), \text{ and} \\ \bar{W}^{\theta} &= \frac{1}{M} \sum_{i=1}^{M} W_i^{\theta}(X_0). \end{aligned}$$

Estimating W_i^{θ} is an "embarrassingly parallel" problem if each experience of W_i^{θ} is independent. Fortunately, independence of the W_i^{θ} experience is a requirement of the original model, for $\bar{W}^{\theta}$ to be a consistent estimator. Furthermore, the "lifetime experience" of the 100 agents occurs in parallel (in "agent time"). The problem is perfectly suited to be parallelized among the agents.

To do so, we need to specify a method of sharing information between agents. We will need to answer the question of which information is shared, whether any is lost in the process, and with whom agents will share their information – that is, the network of information sharing. As a first pass, we consider and implement the simplest possible case: agents share information perfectly with all other agents. While this may seem unrealistic, it allows us to create an "upper bound" on what may be achieved in this model with information sharing.

The information the agents share is the value W_i^{θ}, which they simply hand off to their neighbors (i.e. all other agents) as soon as they finish estimating it. As it would be a waste of time to explore a rule that has already been explored M times, we need to implement a mechanism for coordination among agents. The next section explains the implementation of the information-sharing system, which is written to be flexible and allow immediate computational exploration of alternate behaviors.

The difference between this and the original model is that now agents are **social agents**, as well as being intelligent agents. Their interactions with one another materially affect the outcomes of the experiment in very positive ways, as we will see.

3.1 Implementation of Full Information Sharing

The general mechanism we use to achieve coordination among agents, on any network topology, is a "bulletin board" and a list of neighbors. The bulletin board is essentially a hash table with an entry for each rule θ;

[10] The replication of the model shown in the middle panel of Table 2 disagrees slightly with the original results in the top panel (and Table 1) regarding which (N, M) pair is needed to achieve a minimum success rate of 67%. Allen and Carroll's model required $(50, 50)$, while the replication only required $(20, 50)$. It is clear in Allen and Carroll's results that $(20, 50)$ is right on the edge of the 67% success rate, and Figure A3 (discussed further in Section 5) makes it clear why these numbers may easily vary: the slope of the "Success Rate" line for $(20, 50)$ is incredibly steep at the cutoff value of 0.05. One can see that even slight differences due to the randomness inherent in the experiment could shift this line around. For the sake of consistency, from here forward we will refer to the replication results of $(N, M) = (20, 50)$ as the parameter pair at which agents can expect to achieve a 67% success rate.

each rule's entry contains an array of W_i^θ values. Each agent owns a bulletin board, which is initially empty, a list of neighbors, and an income process. When a given agent, k, finishes a single N-length estimation of a value W_i^θ, he stores that value under the appropriate rule in his own bulletin board, then pushes the value to all of his neighbors, who store it in their bulletin board. In a not-fully-connected network of agents, there is a question of whether the neighboring agents then pass that information along. In a fully connected network, this is not a concern.

When selecting the next rule, agent k simply chooses a random rule off his board which does not yet have M values in its array. Thus coordination is automatically obtained. If agents are exploring the rule-space in a synchronized fashion, coordination could be costlessly obtained by simply choosing the next rule after all agents update their neighbors. In an asynchronous world, there is a trade-off between waiting to hear from a neighbor (and perhaps saving oneself N periods of rule exploration) and simply starting to explore a random uncompleted rule. This trade off, however, is beyond the scope of this paper to consider. The model terminates when all agents have obtained the number of W_i^θ experiences necessary to estimate all $\bar{W}^\theta$ values.

The above applies in the general case, where the network of agents is neither degenerative ("individual learning") nor fully connected. In the special cases of individual learning and fully connected social learning, agents do not need to use their list of neighbors. This is obvious in the individual learning case; in the fully connected case, it is simpler to assign each agent an empty neighbor list and a reference to a global bulletin board. Coordination and full information sharing then occurs automatically as the algorithm executes.

This model is constructed in Python, employing the NumPy and SciPy scientific libraries when necessary. The object-oriented framework provided by Python is well suited for constructing the bulletin boards each agent uses, and each agent encapsulates their own information and behavior, allowing alternate behaviors to to be quickly introduced. This is aspect of the code is exploited in the second extension, below. Lastly, the language allows the model to be easily run on "cloud" parallel computing platforms, greatly decreasing final execution time.

3.2 Algorithm for Full Information Sharing

To run the model, 100 Consumer agents, a Bulletin Board object, and 100 Income Process objects are created. Each agent is handed an income process and a bulletin board. Each period, the following process occurs:

1. Agents are selected in a random order to try out a rule.
2. Upon selection, each agent chooses a random rule and checks their bulletin board to see if it has already been tried M times.
   (a) If so, another rule is selected until one is found which hasn't been tried M times. If such a rule can't be found, the simulation terminates for that agent.
   (b) If an undepleted rule can be found, the consumer tries the rule for N periods and posts the resulting W_i^θ value to his bulletin board.
3. Repeat the process until all agents terminate their efforts.

3.3 Results for Full Information Sharing

In the full information sharing setup, once every rule has been tried M times, the entire population of agents will choose the same highest-value rule, as they all share the same bulletin board. To learn how well this "full connection" social learning process can be expected to work, and in order to compare these results to those in Table 1, we run the "full connection" experiment 100 times for each M, N pair. Thus we can make a statement about what fraction of the time a population using this process can be expected to fall within $\bar{\epsilon}^\theta \leq 0.05$ of the optimal rule, analogous to the information expressed in Table 1. In fact, besides changing the number of periods it takes an individual consumer to to determine an optimal rule, this should be mathematically identical (accounting for some random variation) to the "individual learning" process outlined in the original study. This is because the "full information sharing" version of the model described above nests the original Allen and Carroll model as a special case. If we instantiate the social learning model with only one agent (instead of a population of agents) and run the model to completion 100 times, the single agent will clearly be forced to explore the entire parameter space alone in each of 100 runs. This replicates the structure of the original Allen and Carroll model entirely.

The results of the full information sharing extension are displayed in the middle panel of Table 2 and as the green line with triangles in Figure 2the y-axis plots the success rate and the x-axis plots the M-values from the table. For ease of reference, the original results from Allen and Carroll (2001) are displayed in the top panel of Table 2 and as the black line in Figure 2. As is expected, the success rate for Full Connection Social Learning model is very close to the success rate of the original Allen and Carroll model – this is illustrated in Figure 2. The main difference arises in the number of periods necessary to obtain these near-identical success rates. These can be observed in the "Total Periods" row in the top and middle panels of Table 2. Consider the pair $(N, M) = (20, 50)$. The original model took 400,000 periods of agent search time to distinguish good rules from bad rules at least 67% of the time. For the full connection model, the equivalent time was 10,000 periods – reduced by a factor of 100. This is entirely due to the fact that 100 agents worked together, in parallel, to explore the parameter space. Note that, with 100 agents and 400 rules, if the agents split up the rules evenly, each would have on average only 4 rules to explore – thus achieving a similar result to the Allen and Carroll quote above, only using social learning instead of a "highly efficient hill climbing routine." In fact, if more agents were added to the full-connection model, this number can be reduced further. If one agent were added to the model for each possible run-trial – $(N, M) = (20, 50) \implies 50 * 400 = 20,000$ agents – then the time required to search the entire space would only be bounded by the time it took to complete a single W_i^θ trial: 20 periods.

This result hinges on the fact that all agents are sharing information fully with all other agents – essentially, that all agents are in a fully connected social network. In practice, of course, it is hard to imagine that 20,000 agents all know each of the other 19,999 agents in a population. The discussions and case studies cited in texts such as Tsvetovat (2011) and Newman (2010) clearly indicate that common human social networks are nowhere near fully connected graphs. The purpose of this full connection model, however, should be viewed as constructing a "ceiling" on how well agents can do if they can share information amongst themselves. The original Allen and Carroll model sets something of a "floor" on how poorly agents can do if they share no information whatsoever. If it was the case that the full connection model was unable to bring agent search time within some reasonable limit, then clearly there would be nothing left to explore of any of the myriad of possible network topologies or information transmission mechanisms. As it stands, however, the full information sharing experiment strongly indicates that social learning can greatly improve individual agent learning. Some combination of network structures and information sharing mechanisms may very well bring agent search efficiency very close to the full connection model. For example, a few highly connected agents with a high transmission rate may produce very similar results; such a network may very well represent a world in which "financial experts" specialize in gathering, analyzing, and disseminating such information.

Before we discuss further possible research along network lines, however, we turn our attention to the our second extension to Allen and Carroll's original framework: considering "relative" rather than "absolute" happiness.

Table 2: Individual Learning versus Full Connection Absolute-Value Learning versus Relative-Value Learning

Original Results with Absolute-Value Estimator

S0=1					
		M = 1	M = 10	M = 50	M = 200
N = 10	Mean Sacrifice:	0.269	0.122	0.100	0.102
	Success Rate:	0.09	0.23	0.29	0.24
	Total Periods:	4,000	40,000	200,000	800,000
N = 20	Mean Sacrifice:	0.226	0.079	0.053	0.047
	Success Rate:	0.18	0.45	0.62	0.68
	Total Periods:	8,000	80,000	400,000	1,600,000
N = 50	Mean Sacrifice:	0.187	0.058	0.036	0.024
	Success Rate:	0.26	0.58	0.76	0.91
	Total Periods:	20,000	200,000	1,000,000	4,000,000

Full Connection Social Learning with Absolute-Value Estimator

S0 = 1					
		M = 1	M = 10	M = 50	M = 200
N = 10	Mean Sacrifice:	0.441	0.238	0.177	0.145
	Success Rate:	0.11	0.31	0.23	0.19
	Total Periods:	40	400	2,000	8,000
N = 20	Mean Sacrifice:	0.349	0.139	0.096	0.08
	Success Rate:	0.15	0.50	0.67	0.74
	Total Periods:	80	800	4,000	16,000
N = 50	Mean Sacrifice:	0.261	0.075	0.039	0.032
	Success Rate:	0.28	0.61	0.85	0.94
	Total Periods:	200	2,000	10,000	40,000

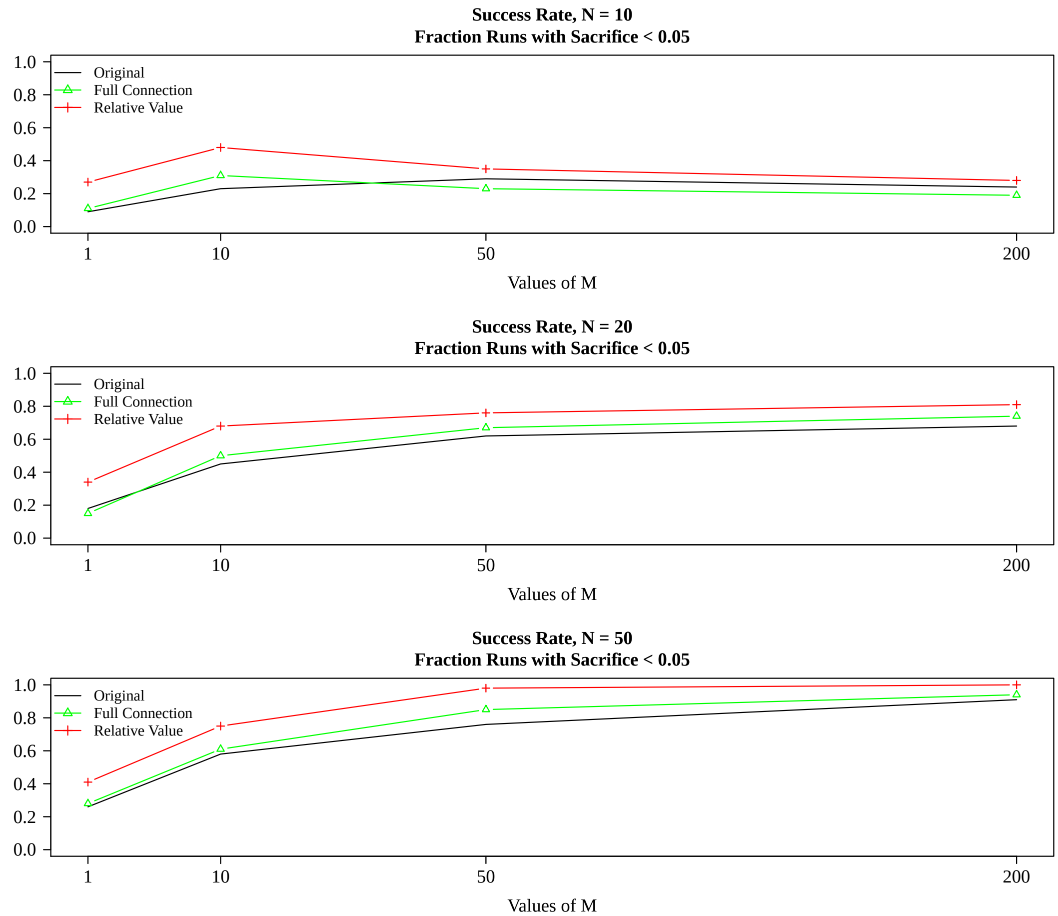
Full Connection Social Learning with Relative-Value Estimator

S0 = 1					
		M = 1	M = 10	M = 50	M = 200
N = 10	Mean Sacrifice:	0.123	0.062	0.079	0.093
	Success Rate:	0.27	0.48	0.35	0.28
	Total Periods:	40	400	2,000	8,000
N = 20	Mean Sacrifice:	0.143	0.052	0.033	0.043
	Success Rate:	0.34	0.68	0.76	0.81
	Total Periods:	80	800	4,000	16,000
N = 50	Mean Sacrifice:	0.119	0.04	0.022	0.013
	Success Rate:	0.41	0.75	0.98	1.00
	Total Periods:	200	2,000	10,000	40,000

4 The Second Extension: A Relative-Value Estimator

Returning to the general model setup, recall that agents are estimating an approximation to the value function V^{θ} in equation (6). This value function is a measurement of the agent's "happiness" with units of this "happiness" are in terms of expected utility. There is, however, another way to think about "how well" a rule works. Instead of asking "how happy does this rule make me," agents might instead ask, "how happy does this rule make me **above and beyond** how happy (or unhappy) I would have been using some other rule?" This entails choosing some baseline rule against which agents compare their current experience. We will refer to this as a "relative-value" measure of well being.

Figure 3: Allen and Carroll Results vs Full Connection and Relative-Value Learning



4.1 Implementation of the Relative-Value Estimator

A natural baseline rule is the "consume everything" rule. Thus instead of estimating the "absolute" value function,

$$V^{\theta}(\,X_0\,) = \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(C^{\theta}(X_t))\right]$$

agents instead employ a relative-value estimator, which simply subtracts off the value an agent would have obtained had he/she employed the "consumer everything" baseline rule:

$$\tilde{V}^{\theta}(\,X\,) = \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(C^{\theta}(X_t)) - \sum_{t=0}^{\infty}\beta^t u(C^{Base}(X_t))\right].$$

As before, the agent estimates this function by experiencing income streams and recording the results. Each consumer will now estimate the following value:

$$
\begin{aligned}
\tilde{W}_i^\theta(X_0) &= \sum_{t=0}^{N} \beta^t u(C^\theta(X_t)) - \sum_{t=0}^{N} \beta^t u(C^{Base}(X_t)), & (11)\\
&= \sum_{t=0}^{N} \beta^t \left[u(C^\theta(X_t)) - u(C^{Base}(X_t)) \right], \text{ and} \\
\bar{W}^\theta &= \frac{1}{M} \sum_{i=1}^{M} \tilde{W}_i^\theta(X_0). & (12)
\end{aligned}
$$

with $\bar{W}^\theta$ calculated as before in equation (9). $C^{Base}(X_t)$ is the baseline "consume everything" consumption rule. Thus $\tilde{W}_i^\theta$ is the relative-value estimator, and the original W_i^θ is the absolute-value estimator.

An additional intended benefit of this relative-value consumption rule is that it may allow for heterogeneous S_0 values at the beginning of each estimation of $\tilde{W}_i^\theta$. Recall from our discussion of equation (8) that each estimation run restarts with the same initial S_0, as in the original model. As it stands, for an agent exploring some rule θ in an $\{N, M, S_0\}$=$\{50, 10, 1\}$ experiment, we must think of the agent as "re-setting" his/her savings to the same S_0 value every 50 periods. If we want agents to carry savings from one N-period stream to the next, as we might imagine an agent doing in practice, we must somehow normalize the initial endowment at the beginning of each estimation of a $W_i^\theta(X)$. If we don't normalize for heterogeneous S_0, a high initial value may skew results in favor of whichever rule just so happened to enjoy the "high S_0." Observe that β^t is a larger number when t is small – that is, early on in the trial of $\tilde{W}_i^\theta$, when S_0 matters the most. If an agent just so happens to have a high value of S_0 for a "bad" rule, but a low value for a "good" rule, the good rule may **appear** to be be bad, simply because of the discounting of β^t over the N periods.[11] If the selection of S_0 is random, this may not be a problem at higher values of M, but if S_0 is not random this may bias results. The relative-value estimator, $\tilde{W}_i^\theta$, seeks to address this by subtracting off the "consume everything" utility and thus normalize agent experience due to initial S_0 values.

4.2 Algorithm for Relative-Value Information Sharing

As a first step towards using the relative-value estimator for experiments with heterogeneous S_0, this procedure is first implemented in the same homogeneous S_0 setting as the absolute-value estimator with full information sharing. If the results of this implementation are much better (or much worse) in the homogeneous-S_0 setting, this can indicate of how well the estimator can be expected to perform when used in a heterogeneous-S_0 setting. Thus algorithmic implementation of the the relative-value estimator is exactly the same as Section 3.2, except that the relative-value measure $\tilde{W}_i^\theta$ is used instead of W_i^θ.

4.3 Results for the Relative-Value Estimator

The results of the relative-value process are displayed in the third panel in Table 2, and in the red line with plus signs in Figure 3. The the "Total Periods" values in Table 2 are the same for both the absolute-value and relative-value estimators; this is to be expected, as both of those experiments have been run in the full-connection social learning setting. What is remarkable, however, is that the relative-value estimator has actually made it easier for consumers to distinguish good rules from bad rule. Previously, in the absolute-value experiment (the second panel in Table 2), consumers needed at least an $(N, M) = (20, 50)$ pair to achieve a minimum 67% success rate. This corresponded to 4,000 periods of search. In the relative-value experiment, however, consumers have improved their rule-finding capabilities by a full order of magnitude: now they can achieve a minimum success rate of 67% at either $(N, M) = (10, 20) \implies 800$ periods. This is less than half that experienced using the absolute-value rule.

The results are more striking when observed in Figure 3. Without fail, the relative-value success rate is above the success rates of the original model and the social learning extension. That is, for every possible (N, M) pair, the relative-value estimator allows more experiments to fall within the $\bar{\epsilon}^\theta \leq 0.05$ cutoff range. To state it differently, this estimator allows the consumers to consistently distinguish "good" rules from "bad" rules, at every (N, M) value.

[11] Note that even if S_0 is held constant across all runs, the fact that X_0 is the sum of both S_0 and a random Y_0 means this dynamic may still play out, albeit to a lesser extent.

5 A Closer Look at Model Output

5.1 Examining Distributional Output

To understand why the relative-value estimator works so well, we need to dive deeper into the functioning of the model. Fortunately our model is a computational simulation, and thus it may be run many times to generate distributions of output. We do this in Figures A1 and A2, in Appendix A. These figures are representations of two-dimensional histograms: each dot represents a number of agents that settled on a given rule $\theta := (\gamma, \bar{X})$ in the Θ space described in equation (10); the area of the dot is proportional to the number of agents at a particular rule on the grid.[12] This is overlaid on the contour plot displayed in Figure 2. Thus we have a non-parametric way to examine the entire space of results, instead of only a slice of the results, as displayed in Table 2. In all of the histograms, the optimal approximate rule, $\theta^* = (0.233, 1.243)$, is represented by a red "+," and the optimal point on the grid, $\theta^{grid} = (0.25, 1.2)$, is represented by a black "x."

In Table 2, the success rate refers to the fraction of agents which fall within the 0.05-contour in Figures A1 and A2. A few stylistic facts emerge as we examine these histograms. In Figure A1, start at the panel corresponding to $(N, M) = (10, 1)$. It is clear that as we move to the right, examining panels $(10, 10)$, $(10, 50)$, and $(10, 200)$, the agents are choosing rules in a tighter groups – the problem is that the rules they are converging on are **just** outside the 0.05-contour. The fact that the agents converge on a location that is not quite the optimal target is stylistically similar to the idea that an estimator can have high bias but low variance. As M increases, the variance drops but the bias does not. If, however, we again start at the upper-left panel $(10, 1)$ in Figure A1 and move down, examining panels $(20, 1)$ and $(50, 1)$, a different story emerges – the cluster of rules the agents arrive at may not tighten as it did along the top row, but it does shift to center around the optimal θ values. This is more apparent when the agents' final choices are less diffuse, for example in the rows corresponding to $M = 50$ or $M = 200$. For the row $M = 50$, it almost appears that the diffusion may increase slightly. The main difference, however, between $(10, 50)$ and $(50, 50)$ is that at $N = 50$, the histogram is centered around the optimal value, instead of a nearby value as at $N = 10$. This makes a significant difference for the success rate. This is stylistically similar to the idea that an estimator can have low variance but high bias. Of course, as both N and M increase, both the bias and the variance decrease. These dynamics are all entirely hidden when we only examine the Fraction Success rows in Table 2.

Given that we can generate distributions from the model, these ideas of bias and variance in the histograms may be examined more rigorously statistically. This is left for future work, however. For current purposes, this initial stylistic understanding is enough to reveal some important first-order questions. It is clear from the discussion above that there is a trade-off occurring in between "bias" and "variance" in the model results. The choice of the $\bar{\epsilon}^{\theta}$ cutoff of 0.05, which we use to determine the success rate in Table 2, appears to be significant when we consider the histograms in Figure A1. We might ask whether the choice of cutoff value makes a significant difference in our results. If so, we need to carefully consider how the choice of cutoff value is made. For panel $(20, 1)$ in Figure A1 a cutoff value of 0.05 instead of 0.075 or 0.1 may not make a significant difference. For panel $(20, 10)$ in the same Figure, however, it may make a large difference.

Before continuing, observe that examining Figure A2 provides an initial indication regarding why the relative-value estimator preforms better than the absolute-value estimator. Broadly speaking, for the absolute-value estimator, the histograms tend to "spread" more along the the horizontal γ axis – this is most apparent in the first two rows of Figure A1, for example. For the relative-value rule, however, the histograms have been shifted – they tend to "spread" along the vertical $\bar{X}$ axis. The practical result of this is that the "teardrop" shape of the $\bar{\epsilon}^{\theta} \leq 0.05$-cutoff region captures more of the relative-value histograms earlier than it captures the absolute-value histograms for a given (N, M) pair. The relative-value histograms in Figure A2 cluster more around the $\bar{\epsilon}^{\theta} \leq 0.05$-cutoff region's wide "base," while the absolute-value histograms in Figure A1 cluster around its narrow "top." As with the bias-variance trade-off above, this complicated relation ship may very well be explored more rigorously, either statistically or analytically, but this is left for future work. The stylistic impressions these histograms and contour plots provide are sufficient to motivate

[12] These Figures are arranges to correspond in layout to the tables of output in Table 2. Although analysis may be run with a high number of agents and model runs, the results shown here are for 100 runs of 100 agents with full information-sharing. Recall that the time spent examining the rule space is only difference between the original Allen and Carroll model and the version employing full information sharing and the absolute-value estimator. Thus the histograms in Figure A1 represent both the original Allen and Carroll model as well as the absolute-value estimator version of the model.

the next step in exploring the model. Furthermore, it is not clear that any simple or informative closed-form analytical expression exists which may describe either the bias-variance trade-off or the relationship between the histograms and the contour plots in Figures A1 and A2.

5.2 Examining the $\bar{\epsilon}^{\theta}$ Cutoff Value

A first-order question that arises from considering the histograms in Figures A1 and A2 is whether the choice of cutoff value 0.05 for $\bar{\epsilon}^{\theta}$ is significant for the success rates displayed in Table 2. The histograms cannot supply any further answers with more precision so the question must be taken to another representation of the data. Figures A3 and A4 in Appendix A display the success rate along the vertical axis as a function of possible cutoff values along the horizontal axis. From Figures A1 and A2, it is clear that very few agents choose rules outside of the cutoff value $\bar{\epsilon}^{\theta} = 0.575$, the value corresponding to the consume-everything rule.[13] Thus the horizontal axes only extends to 0.5.

The dotted gray line at the bottom of the legend in each plot indicates the original cutoff value of 0.05 employed to determine the success rates in Table 2. The red dash-dotted line above the 0.05 line in each plot illustrates what the success rate would be if the cutoff were instead 0.075, and the dashed line indicates what the success rate would be if the cutoff was 0.1. These correspond to 7.5% and 10% of average annual income. In the earlier example of a \$50,000 average income, these values would be \$3,750 and \$5,000, receptively. The solid horizontal line with stars on it indicates the cutoff value that is associated with a success rate of 0.5 across all (N, M) pairs. Thus in Figure A3, in panel $(10, 1)$, agents would need a cutoff value of 0.31, roughly $\frac{1}{3}$ of average annual income, to achieve a success rate of 67%.

It is clear from Figure A3 that the curvature of the "success rate" line is important. Each increase in the cutoff value "buys" additional "success," and the nonlinearity in the success rate function determines how big this marginal increase will be. As noted above, for panel $(20, 1)$ in Figure A3 a cutoff value of 0.075 or 0.1 instead of 0.05 does not significantly shift the success rate greater than 67%. However, for panel $(20, 10)$ in the same Figure, a cutoff value of 0.075 or 0.1 brings the success rate right to 67%. This does change the results of Table 2. Looking at Figure A3, it is clear that if we were to consider the cutoff value of 0.075 for the absolute-value estimator consumers would be able to achieve a 67% success rate at $(N, M) = (20, 10)$ instead of $(20, 50)$ – in 800 periods instead of 4,000 periods, if each consumer only had to explore 4 rules (either through a "highly efficient hill-climbing routine," or through fully connected social learning with 100 agents in parallel). This is a significant improvement in the success rate of the algorithm, simply by considering a slightly increased cutoff value.

What we would like to see, of course, is a 67% success rate for a parameter pair such as $(N, M) = (10, 1)$, $(20, 1)$, or $(50, 1)$. Such values would imply that consumers could reliably find "good" consumption rules within 1 to 3 generations, if a generation lasted roughly 70 periods. This brings us to the results of the relative-value estimator, displayed in Figure A4. A difference which is immediately clear between the absolute-value estimator and the relative-value estimator in Figures A3 and A4, respectively, is the shape of the "success rate" line for the parameter-pairs $(N, M) = (10, 1)$, $(20, 1)$, and $(50, 1)$. In Figure A3, $(10, 1)$ and $(20, 1)$ are almost linear, which is why increasing the cutoff value does not much improve agents' ability to distinguish good rules from bad rules. In Figure A4, however, all three curves in the first column of panels gain a lot of curvature over their their counterparts from the absolute-value estimator. Now increasing the cutoff value has a significant impact on how well the agents distinguish good rules from bad rules. In fact, if we are willing to choose a cutoff of 0.16, we can obtain a success rate of 67% at the lowest possible (N, M) : $(10, 1)$. If we choose a cutoff of 0.12, we can obtain the desired success rate at $(20, 1)$. If an agent only needs to explore 4 rules (either via efficient search or social learning), this translates into agent time of only 40 and 80 years, respectively.

There is a clear trade-off between obtaining an acceptable success rate with less agent effort by raising the cutoff value, and the value lost to the consumers which is directly represented by the subsequently higher cutoff value. An agent who "succeeds" because the cutoff value has been raised to 0.12 is potentially sacrificing 7% more of an average annual income than an agent who simply "explores longer" and "succeeds" with the original cutoff value of 0.05. There may very well be some optimal cutoff value to choose. The simulation code allows many possible experiments to be conducted with this model; this is one possible question that may be addressed in future research. Regardless of whether or not an optimal cutoff value

[13] It is apparently possible to do **worse** than simply consuming everything and saving nothing, but agents rarely select one of these worse rules in practice.

exists the important takeaway is that the choice of cutoff value matters for the results of the model, and needs to be considered a parameter of the model alongside the others. Figures A3 and A4 are an attempt to sweep over this parameter as well; the reader may examine Figures A3 and A4 and see for himself or herself what the influence of the cutoff value is on model results.

6 Conclusions and Future Work

6.1 Summary and Conclusions

Allen and Carroll (2001) uncovered both striking positive and striking negative results. Using a simple linear approximation to the optimal consumption rule,they found that agents can get negligibly close (in utility terms) to the highly non-linear optimal solution. In addition, given enough time, agents could consistently find a near-optimal linear rule via simple trial-and-error search. Unfortunately, "enough time" proved to be prohibitively long – as Table 2 indicates, to reliably find a near-optimal rule – that is, achieve a success rate of at least 67%[14] – agents need to spent 400,000 years searching the parameter space.

This paper addresses the final negative result in two separate steps, each of which greatly reduces the time required to reliably find a near-optimal rule. The first step takes its inspiration from Allen and Carroll themselves. In their 2001 paper, they state that

> "If it takes an individual agent a million periods, ...a population of a million consumers... should collectively obtain essentially the same amount of information in a single period."

In addition they note that,

> "More intriguing [than efficient search algorithms] is the possibility that consumers come by their behavior by a process of social learning, in which rules of thumb that are successful in utility terms are passed along from one consumer to another, or through other mechanisms such as the advice of personal finance experts or advice in personal finance books. ... Elucidating the circumstances under which a process of social learning can be expected to lead the population to reasonably optimal behavior will be an interesting task for future work."

Here and elsewhere, Allen and Carroll suggest two possible extensions to their original model: (1) implementing a more efficient search algorithm over the grid space of possible rules (alluded to above), and (2) some form of social learning. Taking inspiration from the second of these suggestions, we first observed that the agent estimation problem is an embarrassingly parallel computation. We then constructed artificial agents in code and provided them with a bulletin board to store the results of rule-trials, and a network of neighbors with which to share their experiences. When an agent tries a rule, he/she stores it in his own board, and also passes it to all his neighbors. As a first-pass exploration, and in order to have results mathematically comparable to the original model, we instantiated 100 consumers and placed them all on a fully connected social network; the model is then run 100 times and results recorded. This extreme network structure nests the original Allen and Carroll model as a special case (obtained when there is only 1 consumer on the fully connected network) and allows us to explicitly discuss the improvement provided by social information sharing.

The result is that the "Total Periods" in the middle table of Table 2 are reduced by a factor of 100 from those of the Allen and Carroll experiment. Where once it took a minimum of 400,000 years to get a success rate greater than (or very near) 67%, in the full connection experiment it takes only 4,000 periods. The reduction of total periods follows immediately once similar success rates are obtained between the original and the full information sharing procedures. In fact, this result confirms that the "total periods" could by pushed arbitrarily close to the lowest bound for each M, N pair by simply adding more consumers. The lowest bound is simply the N-value, since a rule must be tried at least this many years. This is an encouraging result, and the first step in a multitude of possible research directions. Real networks, of course, do not have anything near a fully connected structure; strategies for addressing this, however, will be discussed further below.

The second step in addressing Allen and Carroll's negative results was to create an estimator of the consumer's value function **relative to a baseline** consumption rule. This estimator takes its inspiration

[14] Or very near 67%; see the footnote at the end of Section 2.3.

from the idea that an agent is likely to think of his happiness not in some absolute terms, but rather in relation to some baseline course of action. Furthermore, in a technical sense, the relative-to-baseline value function is also an attempt to normalize the estimator in the face of heterogeneous initial wealth conditions. If successful, this would allow agents to be instantiated with heterogeneous wealth levels during the experiment. An encouraging result is that this estimator not only works as well as the original absolute-value estimator, but also improves upon its performance. Agents' ability to distinguish "good" rules from "bad" rules improved when they used the relative-value estimator to explored the parameter space.

In addition, the improvement provided by the relative-value estimator is orthogonal to the improvement provided by the social learning mechanism. As discussed in Section 3, the improvement provided by the social learning mechanism was entirely in the "Total Periods" rows of Table 2. The "Success Rate" between the top table and middle table of Table 2 was determined by the same fundamental process, in particular because the experiment that produced the middle table nested the original results as a special case. They only appear different due to random variation. For the relative-value estimator, however, the fundamental estimation process has changed, resulting in a higher success rate for each (N, M) pair. This is reflected in Figure 3, and can be seen more explicitly in Figures A2 and A4. The new estimator has improved properties over the absolute-value estimator.

Lastly, we touched upon the idea that the cutoff value used to determine the success rate of a given (N, M) pair should itself be considered a parameter in the model, and care should be taken to sweep over this as well. We provide one such set of parameter sweeps in Figures A3 and A4.

6.2 Future Work

An extension suggested by Allen and Carroll we did not undertake is exploring more efficient search algorithms for the grid space Θ. In part this is because we already know what a more efficient search can give us. As stated by Allen and Carroll at the end of Section 2 above, if we know which (N, M) pair provides an acceptable success rate, then we can set limits on what a more efficient search algorithm can do. If a more efficient search could allow agents to only need to examine four rules, and we know the minimum (N, M) pair which provides an acceptable success rate is $(10, 10)$, then we know it must take $10*10*4 = 400$ periods to search this space. Of course, if we can **combine** highly efficient search with social learning – which we did not do here, as we do not have a more efficient search algorithm – then those rule-explorations could be spread out over the population of agents and solved in parallel.

This, then, is are some of the immediate next steps in this research program:

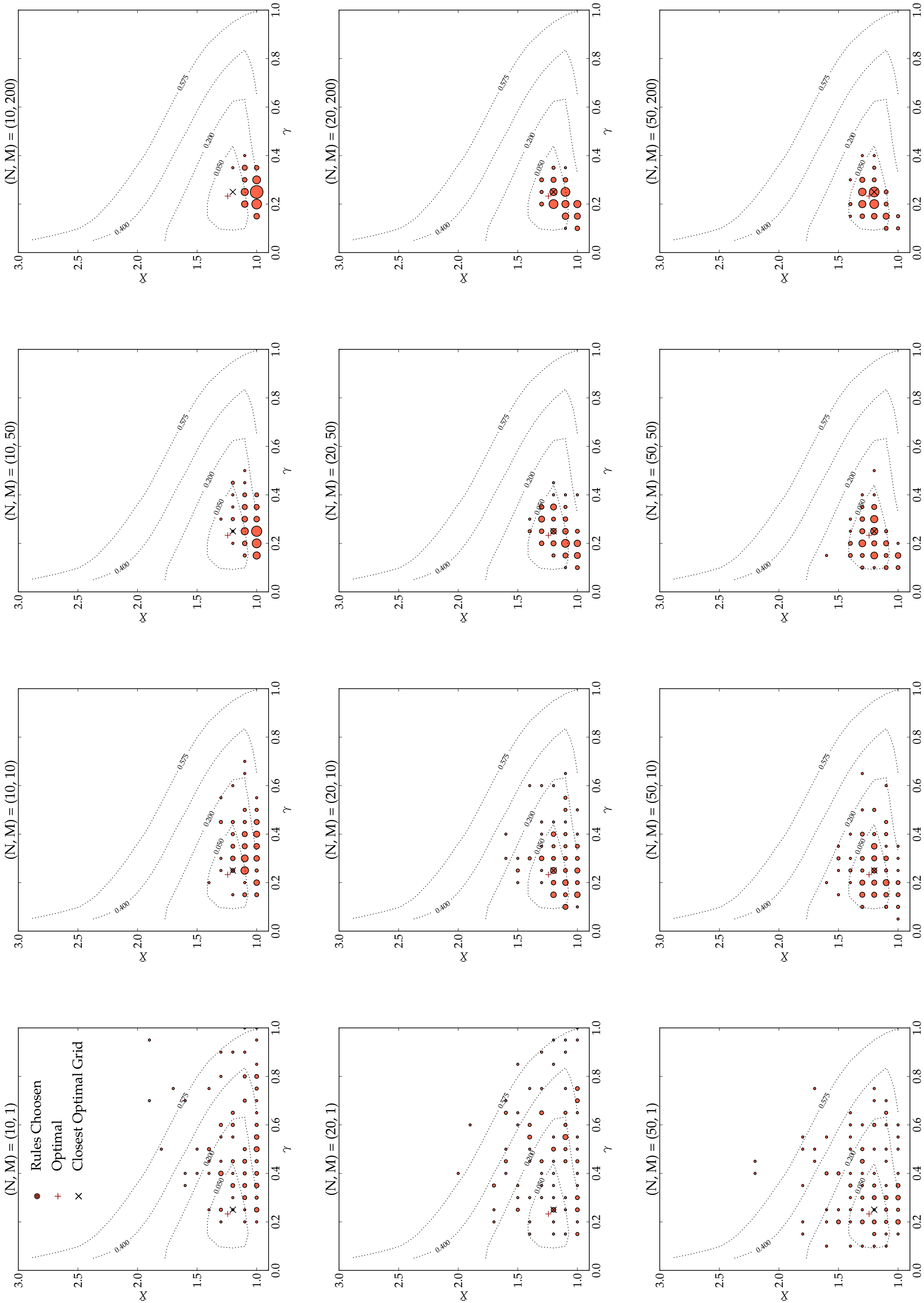
1. Explore a number of more efficient search algorithms,
2. Explore different network topologies beyond the fully connected network, and
3. Work to more fully thoroughly understand the relative-value estimator.

We are optimistic that bringing these three elements together can result in a more realistic social network structure which still allows consumers to estimate a near-optimal linear consumption rule in reasonable time. In addition to standard optimization routines such as hill-climbing, we are interested in using the Particle Swarm Optimization (PSO) as a means of consumer exploration of the Θ space, as we believe it may have an attractive and intuitive interpretation with agents as the particles. With respect to network topologies and information transmission, Carroll (2001c) explores some preliminary models of inflation-expectation transmission on a network. Additional extensions include allowing the S_0 values to be heterogeneous across consumers; allowing agents to only communicate their values in a statistically noisy way as in Chamley (2003), exploring wider ranges of social learning in groups (Young 1997 is an excellent early example of this), easing the credit constraint, and introducing more assets (such as housing) into the consumer's problem.

These are exciting results, and we look forward to exploring them in future research.

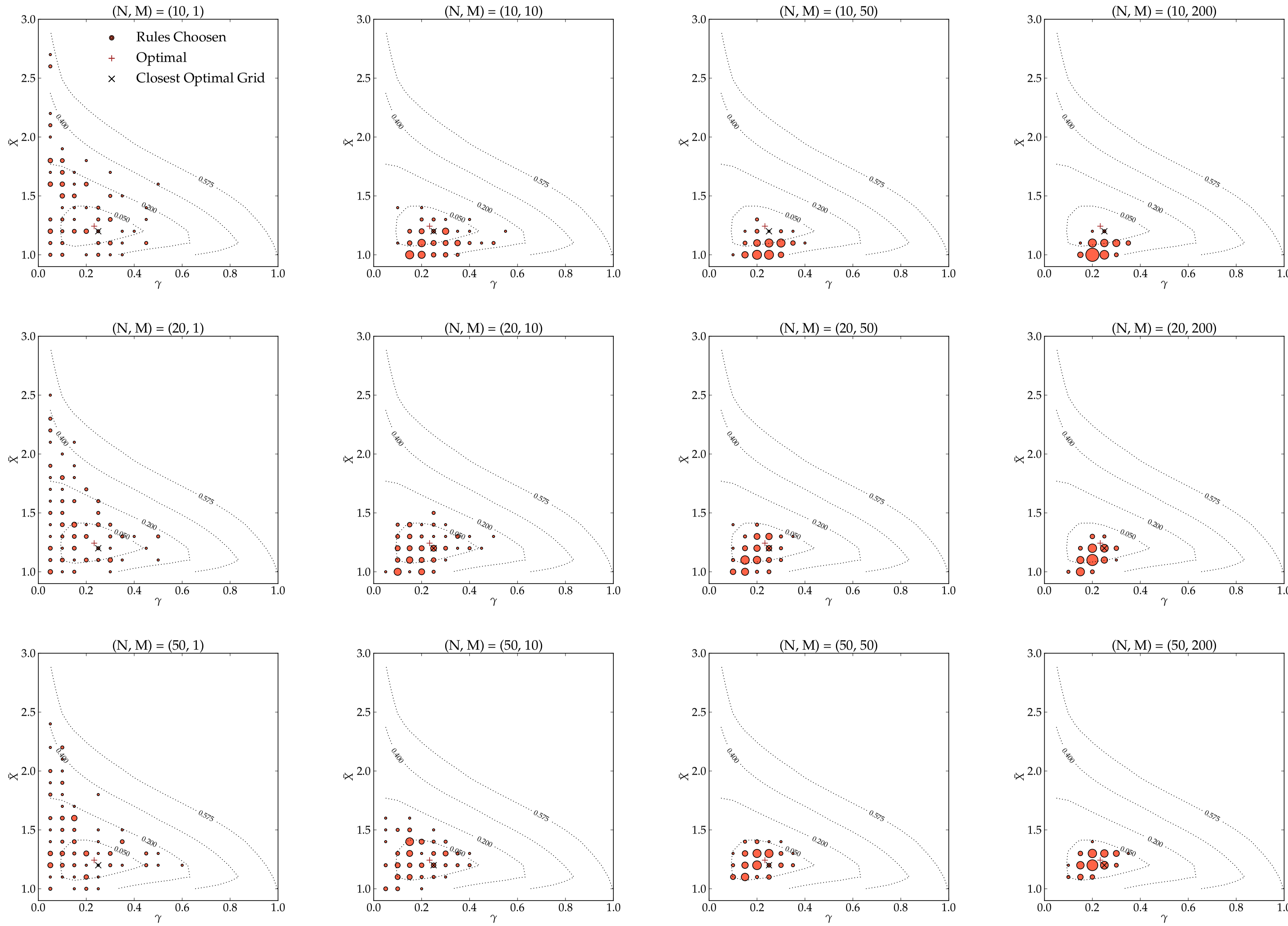
Appendix A

Figure A1: Distribution of Rules Selected by Agents, Original Model



The width of the dots are proportional to the number of agents which choose that $(\gamma, \bar{X})$ rule. Each plot corresponds to results in Table X for the parameter sweeps across N, M. Both the optimal rule, (0.233, 1.243), & the closest grid point, (0.25, 1.2), are shown.

Figure A2: Distribution of Rules Chosen, Relative-Value Model



The width of the dots here are proportional to the number of times the agents collectively arrived at a rule out of 100 runs. In the original model above, the width was the number of agents that selected a rule out of 100 independently acting agents.

Figure A3: Fraction Successful, Original Model

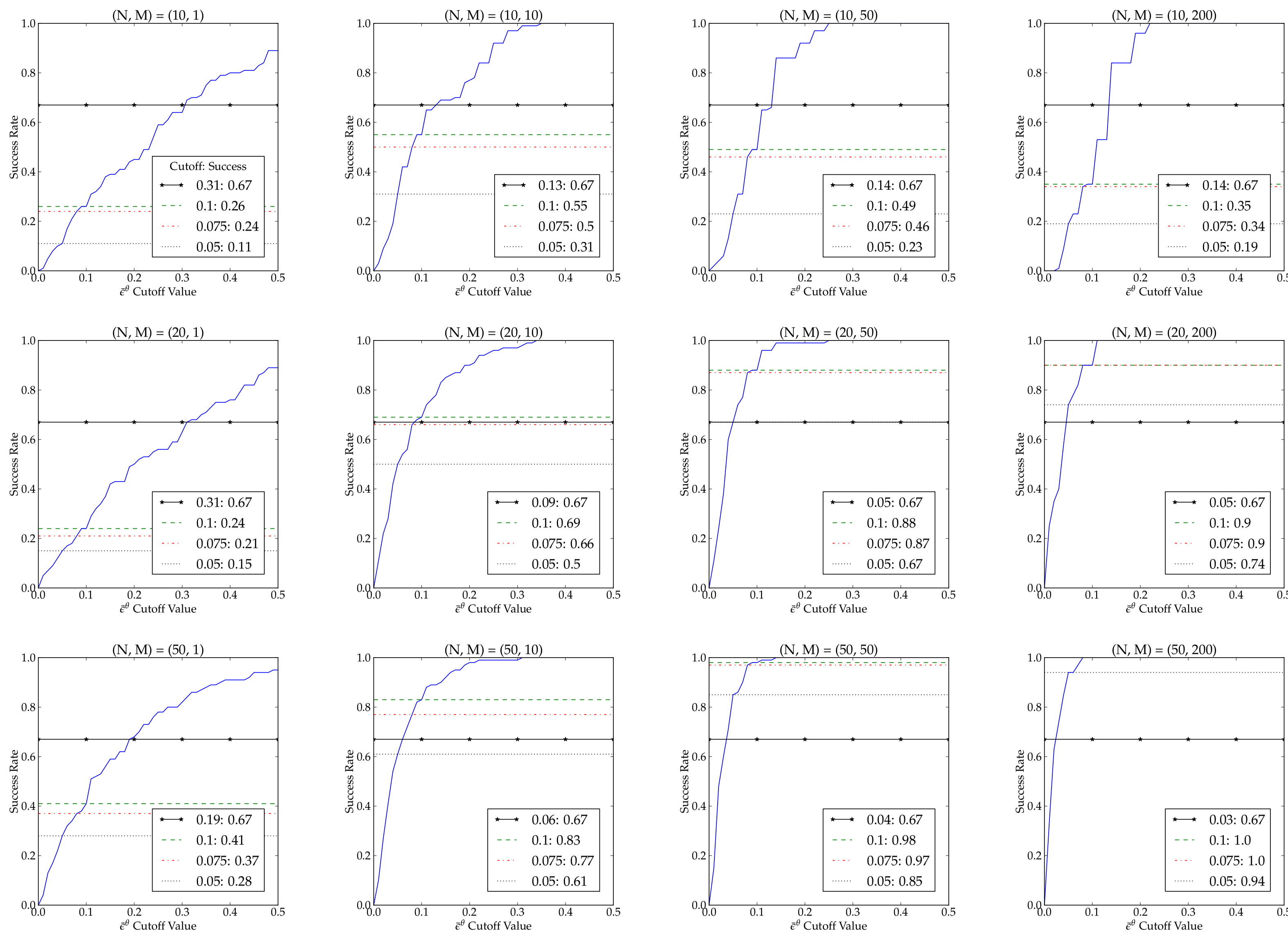
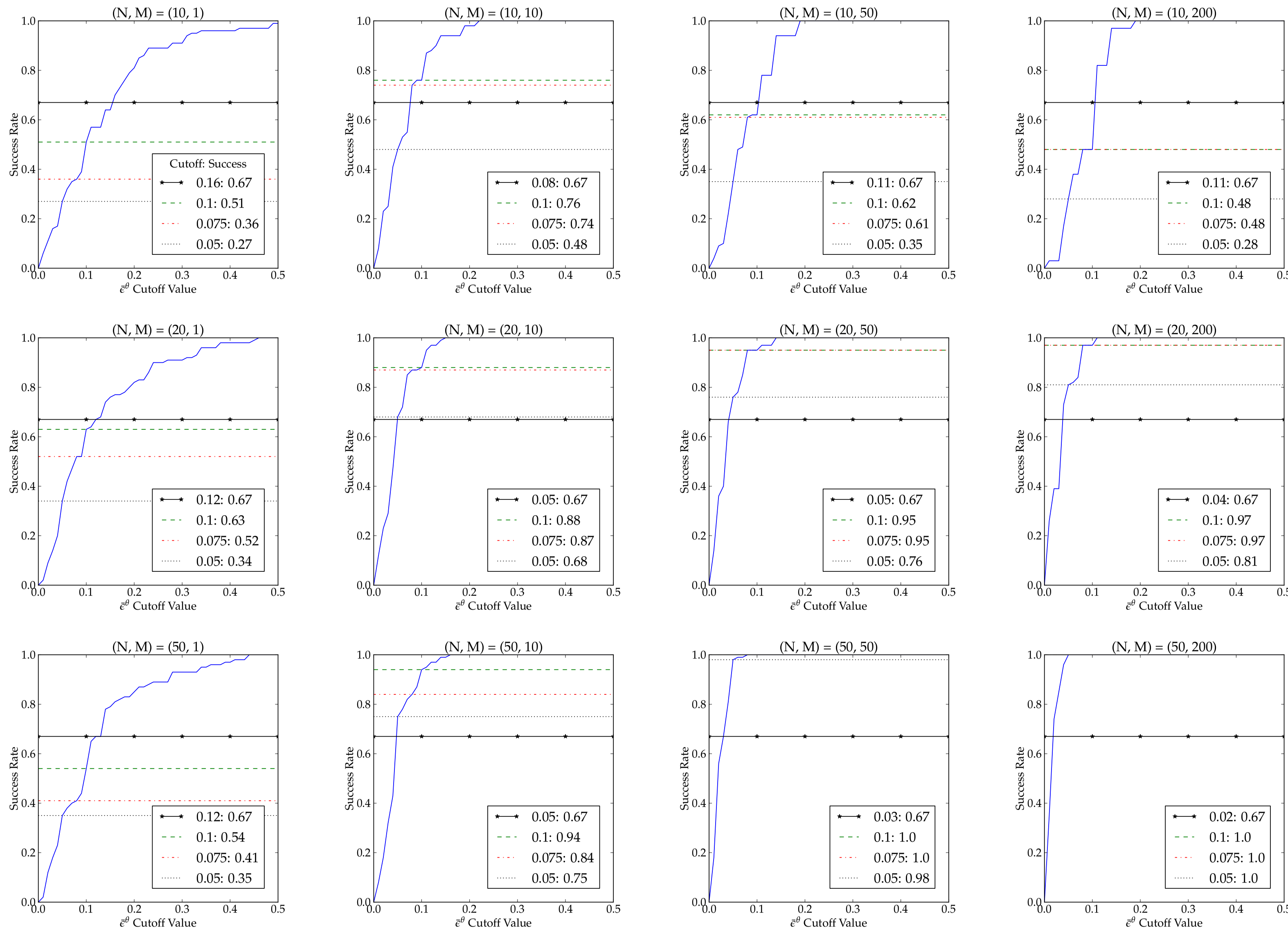


Figure A4: Fraction Successful: Relative-Value Model



References

[1] Allen, T. and Carroll, C. (2001). “Individual Learning About Consumption.” Macroeconomic Dynamics, 5, 255–271.

[2] Backus, D., Routledge, B., and Zin, S. (2004). "Exotic Preferences for Macroeconomists." NBER Working Paper No. 10597.

[3] Carroll, C. (1992). “The Buffer-Stock Theory of Saving: Some Macroeconomic Evidence,” Brookings Papers on Economic Activity, 1992(2), 61– 156.

[4] Carroll, C. (1997). “Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis,” Quarterly Journal of Economics 1997(1), 1-56 .

[5] Carroll, C. (2001a). “Death to the Log-Linearized Consumption Euler Equation! (And Very Poor Health to the Second Order Approximation),” Advances in Macroeconomics 2001.

[6] Carroll, C. (2001b). “A Theory of the Consumption Function, With and Without Liquidity Constraints.” NBER Working Paper No. 8387.

[7] Carroll, C. (2001c). “The Epidemiology of Macroeconomic Expectations.” NBER Working Paper No. 8695.

[8] Carroll, C. (2011). “Theoretical Foundations of Buffer Stock Saving,” Working Paper.

[9] Carroll, C. and M. Kimball (2008). "Precautionary Saving and Precautionary Wealth." The New Palgrave Dictionary of Economics. Second Edition. Eds. Steven N. Durlauf and Lawrence E. Blume. Palgrave Macmillan.

[10] Chamley, C. (2003). Rational Herds: Economic Models of Social Learning. Cambridge University Press.

[11] Freidman, M. (1957). A Theory of the Consumption Function. Princeton University Press.

[12] Fuster, A., Laibson, D., and Mendel, B. (2010). "Natural Expectations and Macroeconomic Fluctuations." Journal of Economic Perspectives, 24(4), 67–84.

[13] Hall, R. E. (1978). “Stochastic Implications of the Life-Cycle/Permanent In-come Hypothesis: Theory and Evidence.” Journal of Political Economy. 96, 971–87

[14] Hubbard, G., Skinner, G., and Zeldes, S. (1994): “The Importance of Precautionary Motives for Explaining Individual and Aggregate Saving,” in The Carnegie-Rochester Conference Series on Public Policy, ed. by Allan H. Meltzer, and Charles I. Plosser, vol. 40, pp. 59–126.

[15] Hubbard, G., Skinner, G., and Zeldes, S. (1995). “Precautionary Saving and Social Insurance,” Journal of Political Economy, 103, 330–399.

[16] Keynes, J.M. (1936). The General Theory of Employment, Interest and Money. Palgrave Macmillan Press.

[17] Laibson, D. (1997). “Golden Eggs and Hyperbolic Discounting.” Quarterly Journal of Economics, 112: 443–77.

[18] Lettau, M., and H. Uhlig (1999). “Rules of Thumb and Dynamic Programming,” American Economic Review, 89, 148-174.

[19] Mas-Colell, A., Whinston, M. and Green, J. (1995). Microeconomic Theory. Oxford University Press.

[20] Newman, M. (2010). Networks: An Introduction. Oxford University Press.

[21] Stachurski, J. (2009). Economic Dynamics: Theory and Computation. The MIT Press.

[22] Tsvetovat, M. and Kouznetsov, A. (2011). Social Network Analysis for Startups. O’Reilly Press.

[23] Yildizoglu, M., Senegas, M., Salle, I., and M. Zumpe, (2012). "Learning the optimal buffer-stock consumption rule of Carroll." Working paper.

[24] Young, H. P. (1997). Learning and Evolution in Games. Princeton University Press.

[25] Zeldes, S. P. (1989). “Optimal Consumption with Stochastic Income: Deviations from Certainty Equivalence.” Quarterly Journal of Economics. 104, 275–298.