Exercise 1

Consider a (univariate) lognormal random variable: $X \sim LogN\left(\mu, \sigma^2\right)$.

a) Write MATLAB function that determines the value of the parameters μ and σ^2 from the expectation $\mathbb{E}\{X\}$ and the variance $\mathbb{V}\{X\}$.

b) Write a MATLAB script in which you:

- Use the function created in a) to determine the parameters μ and σ^2 such that $\mathbb{E}\{X\} = 3$ and $\mathbb{V}\{X\} = 5$.
- Generate a large sample from the distribution of X with such parameters;
- Plot the sample (do not join the observations, show them as dots);
- Plot the histogram (suitably normalized) and superimpose the exact pdf for comparison;
- Plot the empirical cdf and superimpose the exact cdf for comparison.

Note: display title, labels and legend in each plot.

Exercise 2

a) Given a sample $\{\boldsymbol{\epsilon}_t\}_{t=1,\ldots,\bar{t}}$ and a vector of probabilities $\{p_t\}_{t=1,\ldots,\bar{t}}$ expressing the relative weights of the observations, the recursive routine *MaxLikelihoodFPLocDispT* outputs the Maximum Likelihood estimates of the location and dispersion parameters under a multivariate Student t distribution with ν degrees of freedom.

Write a MATLAB function that implements the routine, here written in pseudo-code

$$(\hat{\boldsymbol{\mu}}_{\boldsymbol{\varepsilon}}^{MLFP}, \hat{\boldsymbol{\sigma}}_{\boldsymbol{\varepsilon}}^{2MLFP}) = MaxLikelihoodFPLocDispT(\{\boldsymbol{\epsilon}_t, p_t\}_{t=1,\ldots,\bar{t}}, \nu)$$

$$\begin{array}{ll}
\text{Step 0. Initialize} & \left\{\begin{array}{ll} \boldsymbol{\mu} \leftarrow \hat{\boldsymbol{\mu}}_{\boldsymbol{\varepsilon}}^{HFP} & \text{use (2)} \\ \boldsymbol{\sigma}^2 \leftarrow \hat{\boldsymbol{\sigma}}_{\boldsymbol{\varepsilon}}^{2HFP} & \text{use (3)} \\ w_t \leftarrow 1 & t = 1, \ldots, \bar{t} \end{array}\right. \\
\text{Step 1. Update weights} & w_t \leftarrow \frac{\nu + \bar{\imath}}{\nu + (\boldsymbol{\epsilon}_t - \boldsymbol{\mu})'(\boldsymbol{\sigma}^2)^{-1}(\boldsymbol{\epsilon}_t - \boldsymbol{\mu})} \quad t = 1, \ldots, \bar{t} \\
\text{Step 2. Update output} & \left\{\begin{array}{l} \boldsymbol{\mu} \leftarrow (\sum_{t=1}^{\bar{t}} p_t w_t \boldsymbol{\epsilon}_t)/(\sum_{s=1}^{\bar{t}} p_s w_s) \\ \boldsymbol{\sigma}^2 \leftarrow \sum_{t=1}^{\bar{t}} p_t w_t \left(\boldsymbol{\epsilon}_t - \boldsymbol{\mu}\right)\left(\boldsymbol{\epsilon}_t - \boldsymbol{\mu}\right)' \end{array}\right. \\
\multicolumn{2}{l}{\text{Step 3. If convergence, output } (\boldsymbol{\mu}, \boldsymbol{\sigma}^2)\text{; else go to Step 1}}
\end{array} \tag{1}$$

where

- each $\boldsymbol{\epsilon}_t$ is a $\bar{\imath} \times 1$ vector, each p_t is a scalar, ν is a positive scalar;
- the mean and covariance for the initialization (Step 0) are defined as

$$\hat{\boldsymbol{\mu}}_{\boldsymbol{\varepsilon}}^{HFP} \equiv \sum_{t=1}^{\bar{t}} p_t \boldsymbol{\epsilon}_t \tag{2}$$

$$\hat{\boldsymbol{\sigma}}_{\boldsymbol{\varepsilon}}^{2HFP} \equiv \sum_{t=1}^{\bar{t}} p_t (\boldsymbol{\epsilon}_t - \hat{\boldsymbol{\mu}}_{\boldsymbol{\varepsilon}}^{HFP})(\boldsymbol{\epsilon}_t - \hat{\boldsymbol{\mu}}_{\boldsymbol{\varepsilon}}^{HFP})'; \tag{3}$$

- convergence in the above routine occurs when the relative Euclidean norm $\|\tilde{\boldsymbol{\mu}} - \boldsymbol{\mu}\| / \|\boldsymbol{\mu}\|$ and the relative Frobenius norm $\|\tilde{\boldsymbol{\sigma}}^2 - \boldsymbol{\sigma}^2\|_F / \|\boldsymbol{\sigma}^2\|_F$ between two subsequent updates are smaller than a given threshold (required as an additional input).

b) To test the function, write a script that calls it with the following inputs:

- $\{\boldsymbol{\epsilon}_t\}_{t=1,\ldots,\bar{t}}$ is a sample of $\bar{t}$ =1000 simulations from a standard bivariate normal distribution;
- $p_t = 1/\bar{t}$ for every t (observations are equally weighted);
- $\nu = 100$;
- convergence threshold $= 10^{-9}$.