Toroidal Harmonics Definition for the DARPA project

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1 Defnition of toroidal Harmonics

Several work from plasma physics consider toroidal harmonics. Works by Garapati *et al.* [1, 2] suggest defining the general solution to the Laplace equations as:

$$\Phi = f(\mu,\, \eta) \sum_{n,m=-\infty}^{\infty} \left[C_{mn} P^{m}_{n-\frac{1}{2}}(\cosh\mu) + D_{mn} Q^{m}_{n-\frac{1}{2}}(\cosh\mu) \right] \exp(in\eta) \exp(im\phi)$$

$$f(\mu,\, \eta) = \sqrt{\cosh\mu - \cos\eta}$$

I will therefore define toroidal harmonics as:

$$\Psi^{(1,2)}_{nm}(\eta,\, \theta,\, \phi) = \left\{ \begin{array}{c} \Psi \\ \Phi \end{array} \right\}_{nm} (\eta,\, \theta,\, \phi) = \sqrt{\cosh\eta - \cos\theta} \left\{ \begin{array}{c} P \\ Q \end{array} \right\}^{m}_{n-\frac{1}{2}} (\cosh\eta)\, \exp(in\theta)\, \exp(im\phi) \tag{1}$$

Where:

$$\begin{aligned} 0 \le \eta < &\quad \infty \\ -\pi < \theta \le &\quad \pi \\ 0 \le \phi < &\quad 2\pi \\ m, n = 0, 1, \ldots & \end{aligned}$$

2 Decomposition of a scalar function

Given an arbitrary scalar function $f = f(\boldsymbol{r})$, how do we write it as in terms of toroidal harmonics?

$$f(\boldsymbol{r}) = \sum_{n'm'} a^{(1)}_{n'm'} \Psi^{(1)}_{n'm'}(\boldsymbol{r}) + \sum_{n''m''} a^{(2)}_{n''m''} \Psi^{(2)}_{n''m''}(\boldsymbol{r})$$

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \exp(-im\phi)\, f(\boldsymbol{r}(\phi)) = \sqrt{\cosh\eta - \cos\theta} \sum_{n'} \exp(in'\theta) \left(a^{(1)}_{n'm} P^{m}_{n'-\frac{1}{2}}(\cosh\eta) + a^{(2)}_{n'm} Q^{m}_{n'-\frac{1}{2}}(\cosh\eta) \right)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta \exp(-in\theta)}{\sqrt{\cosh\eta - \cos\theta}} \cdot \frac{1}{2\pi} \int_0^{2\pi} d\phi \exp(-im\phi)\, f(\boldsymbol{r}(\phi)) = a^{(1)}_{nm} P^{m}_{n-\frac{1}{2}}(\cosh\eta) + a^{(2)}_{nm} Q^{m}_{n-\frac{1}{2}}(\cosh\eta)$$

$$\frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} d\theta \int_0^{2\pi} d\phi\, \frac{\exp(-i(n\theta + m\phi))}{\sqrt{\cosh\eta - \cos\theta}} \cdot f(\eta,\, \theta,\, \phi) = a^{(1)}_{nm} P^{m}_{n-\frac{1}{2}}(\cosh\eta) + a^{(2)}_{nm} Q^{m}_{n-\frac{1}{2}}(\cosh\eta)$$

Define:

$$\bar{f} = \frac{f(\eta,\, \theta,\, \phi)}{\sqrt{\cosh\eta - \cos\theta}}$$

Now let $\phi_s = \frac{2\pi}{M} \cdot s$, for $s = 0 \ldots M-1$ and $\theta_r = \frac{2\pi}{N} \cdot r - \pi$, for $r = 0 \ldots N-1$. One question I have is whether

nm coefficients can be only non-negative. Consider:

$$\begin{aligned}
\cos n\theta_r =& \frac{1}{2}\left(\exp\left(in\theta_r\right)+\exp\left(-in\theta_r\right)\right)\\
=&\frac{1}{2}\left(\exp\left(in\left[\frac{2\pi}{N}\cdot r-\pi\right]\right)+\exp\left(-in\left[\frac{2\pi}{N}\cdot r-\pi\right]\right)\right)\\
=&\frac{(-1)^n}{2}\left(\exp\left(i\frac{2\pi}{N}\cdot r\cdot n\right)+\exp\left(-i\frac{2\pi}{N}\cdot r\cdot n\right)\right)\\
=&\frac{(-1)^n}{2}\left(\exp\left(i\frac{2\pi}{N}\cdot r\cdot n\right)+\exp\left(i\frac{2\pi}{N}\cdot (N-r)\cdot n\right)\right)
\end{aligned}$$

and similarly for ϕ, thus on a finite domain it is sufficent to consider just positive indices: $n, m = 0, 1, \ldots$. Moving on, and converting the integrals to sums:

$$\begin{aligned}
a^{(1)}_{nm}P^m_{n-\frac{1}{2}}\left(\cosh\eta\right)+a^{(2)}_{nm}Q^m_{n-\frac{1}{2}}\left(\cosh\eta\right)=&\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}d\theta\int_0^{2\pi}d\phi\,\exp\left(-i\left(n\theta+m\phi\right)\right)\cdot\bar{f}\left(\theta,\,\phi;\,\eta\right)\\
=&\frac{1}{2\pi}\sum_r^{N-1}\frac{2\pi}{N}\cdot\frac{1}{2\pi}\sum_s^{M-1}\frac{2\pi}{M}\cdot\exp\left(-i2\pi\left(\frac{n\cdot r}{N}+\frac{m\cdot s}{M}\right)\right)\cdot\exp\left(-in\left(-\pi\right)\right)\cdot\bar{f}_{r,s}\left(\eta\right)\\
=&\frac{(-1)^n}{N\cdot M}\cdot\sum_{r=0}^{N-1}\sum_{s=0}^{M-1}\exp\left(-i2\pi\left(\frac{n\cdot r}{N}+\frac{m\cdot s}{M}\right)\right)\cdot\bar{f}_{r,s}\left(\eta\right)
\end{aligned}$$

This is the 2D DFT of the $\bar{f}_{r,s}\left(\eta\right)=\bar{f}\left(\theta_r,\,\phi_s;\,\eta\right)$:

$$a^{(1)}_{nm}P^m_{n-\frac{1}{2}}\left(\cosh\eta\right)+a^{(2)}_{nm}Q^m_{n-\frac{1}{2}}\left(\cosh\eta\right)=\frac{(-1)^n}{N\cdot M}\cdot DFT\left(r\to n, N, s\to m, M\right)\left\{\bar{f}_{r,s}\left(\eta\right)\right\}$$

So need few steps of η with an good grid of θ, ϕ for each step. Then use DFT to get the nm coefficients.

Assume we did this. So now for each nm we have several values of $\eta=\eta_1\ldots\eta_K$ and we need to get $a^{(1,2)}$. One can write it as:

$$\begin{pmatrix} P_1 & Q_1\\ P_2 & Q_2\\ \vdots & \vdots\\ P_K & Q_K\end{pmatrix}\begin{pmatrix} a^{(1)}\\ a^{(2)}\end{pmatrix}=\begin{pmatrix} V_1\\ V_2\\ \vdots\\ V_K\end{pmatrix}$$
$$\boldsymbol{A}\begin{pmatrix} a^{(1)}\\ a^{(2)}\end{pmatrix}=\boldsymbol{V}$$

One should assume that $\boldsymbol{P}$ and $\boldsymbol{Q}$ are linearly independent. $(\boldsymbol{P}\,\boldsymbol{Q})\begin{pmatrix} a^{(1)}\\ a^{(2)}\end{pmatrix}$ is a linear combination of these two vectors. The best thing we can do is to project vector $\boldsymbol{V}$ into the subspace spanned by $\boldsymbol{P}$ and $\boldsymbol{Q}$. Invariably, this will involve multyplying by $\boldsymbol{A}^\dagger=\begin{pmatrix}\boldsymbol{P}^\dagger\\ \boldsymbol{Q}^\dagger\end{pmatrix}$. Let[1]

$$\boldsymbol{H}=\boldsymbol{A}^\dagger\boldsymbol{A}=\begin{pmatrix}\boldsymbol{P}^\dagger.\boldsymbol{P} & \boldsymbol{P}^\dagger.\boldsymbol{Q}\\ \boldsymbol{Q}^\dagger.\boldsymbol{P} & \boldsymbol{Q}^\dagger.\boldsymbol{Q}\end{pmatrix}=\begin{pmatrix} d_1 & h\\ h^\dagger & d_2\end{pmatrix}=\begin{pmatrix} d_1 & h\\ h & d_2\end{pmatrix},\quad \boldsymbol{W}=\boldsymbol{A}^\dagger\boldsymbol{V}=\begin{pmatrix}\boldsymbol{P}^\dagger\boldsymbol{V}\\ \boldsymbol{Q}^\dagger\boldsymbol{V}\end{pmatrix}=\begin{pmatrix} W_1\\ W_2\end{pmatrix}$$

Then:

$$\boldsymbol{H}\begin{pmatrix} a^{(1)}\\ a^{(2)}\end{pmatrix}=\boldsymbol{W}$$

[1]Bear in mind that $\boldsymbol{P}$ and $\boldsymbol{Q}$ are real-valued

Which we can solve for $a^{(1,2)}$.

$$\begin{pmatrix} a^{(1)} \\ a^{(2)} \end{pmatrix} \approx \frac{1}{d_1 d_2 - h^2} \begin{pmatrix} d_2 & -h \\ -h & d_1 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$$
$$a^{(1)} \approx \frac{d_2 W_1 - h W_2}{d_1 d_2 - h^2}$$
$$a^{(2)} \approx \frac{d_1 W_2 - h W_1}{d_1 d_2 - h^2}$$

3 Decomposition of the vector field

3.1 Basic arguments

Any sensible 3d vector field can be decomposed into[2]:

$$\boldsymbol{F} = \boldsymbol{\nabla}\phi + \boldsymbol{L}\psi + \boldsymbol{\nabla} \times \boldsymbol{L}\zeta$$

where ϕ, ψ, ζ are scalar functions and $\boldsymbol{L} = -i\boldsymbol{r} \times \boldsymbol{\nabla}$ is the angular momentum operator. Key to this decomposition is that the first part is that two latter terms have no divergence, so:

$$\boldsymbol{\nabla}.\boldsymbol{F} = \nabla^2\phi$$

Now, the problem with simply decomposing ϕ in terms of toroidal harmonics is that all toroidal harmonics are solutions to Laplace's Equation, so the devergence would still vanish. A reasonable way to overcome it, IMHO, is to create longitudinal component with a radius:

$$\boldsymbol{F} = \boldsymbol{r}\phi + \boldsymbol{L}\psi + \boldsymbol{\nabla} \times \boldsymbol{L}\zeta$$

This set should provide enough linear independence for the generic decomposition. So the decomposition we are looking for is (with Einstein summation convention):

$$\boldsymbol{F} = a_{nm}^{(\lambda)} \cdot \boldsymbol{r}\Psi_{nm}^{(\lambda)}(\boldsymbol{r}) + b_{nm}^{(\lambda)} \cdot \boldsymbol{L}\Psi_{nm}^{(\lambda)}(\boldsymbol{r}) + c_{nm}^{(\lambda)} \cdot \boldsymbol{\nabla} \times \boldsymbol{L}\Psi_{nm}^{(\lambda)}(\boldsymbol{r})$$

Then[3]:

$$\begin{aligned}
\boldsymbol{\nabla}.\boldsymbol{F} =& a_{nm}^{(\lambda)} \cdot (3 + \boldsymbol{r}.\boldsymbol{\nabla})\, \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) \\
\boldsymbol{r}.\boldsymbol{F} =& a_{nm}^{(\lambda)} \cdot r^2 \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) + c_{nm}^{(\lambda)} \cdot \boldsymbol{r}.\boldsymbol{\nabla} \times \boldsymbol{L}\Psi_{nm}^{(\lambda)}(\boldsymbol{r}) = \\
=& a_{nm}^{(\lambda)} \cdot r^2 \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) + c_{nm}^{(\lambda)} \cdot i\left(2\boldsymbol{r}.\boldsymbol{\nabla} + r^\alpha r^\beta \partial_{\alpha\beta} - r^2\nabla^2\right) \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) \\
=& a_{nm}^{(\lambda)} \cdot r^2 \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) + c_{nm}^{(\lambda)} \cdot i\left(2\boldsymbol{r}.\boldsymbol{\nabla} + r^\alpha r^\beta \partial_{\alpha\beta}\right) \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) \\
\boldsymbol{L}.\boldsymbol{F} =& b_{nm}^{(\lambda)} \cdot L^2 \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) = b_{nm}^{(\lambda)} \cdot \left(\partial_r\left(r^2\partial_r \ldots\right) - r^2\nabla^2\right) \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) \\
=& b_{nm}^{(\lambda)} \cdot \partial_r\left(r^2 \partial_r \Psi_{nm}^{(\lambda)}(\boldsymbol{r})\right) = b_{nm}^{(\lambda)} \cdot \left(2r\partial_r \Psi_{nm}^{(\lambda)} + r^2 \partial_{rr} \Psi_{nm}^{(\lambda)}\right) \\
=& b_{nm}^{(\lambda)} \cdot (\hat{\boldsymbol{r}}.\boldsymbol{\nabla}) \left(r \cdot (\boldsymbol{r}.\boldsymbol{\nabla})\, \Psi_{nm}^{(\lambda)}(\boldsymbol{r})\right)
\end{aligned}$$

This is, in principle enough for the decomposition. So what I need is:

$$(3 + \boldsymbol{r}.\boldsymbol{\nabla})\, \Psi_{nm}^{(\lambda)} = A_{nm,\bar{n}\bar{m}}^{(\lambda,\bar{\lambda})} \Psi_{\bar{n}\bar{m}}^{(\bar{\lambda})} \tag{2}$$
$$r^2 \Psi_{nm}^{(\lambda)} = B_{nm,\bar{n}\bar{m}}^{(\lambda,\bar{\lambda})} \Psi_{\bar{n}\bar{m}}^{(\bar{\lambda})} \tag{3}$$
$$(2\boldsymbol{r}.\boldsymbol{\nabla} + r_\alpha r_\beta \partial_{\alpha\beta})\, \Psi_{nm}^{(\lambda)} = C_{nm,\bar{n}\bar{m}}^{(\lambda,\bar{\lambda})} \Psi_{\bar{n}\bar{m}}^{(\bar{\lambda})} \tag{4}$$
$$\hat{\boldsymbol{r}}.\boldsymbol{\nabla}\left(r\, \boldsymbol{r}.\boldsymbol{\nabla}\Psi_{nm}^{(\lambda)}\right) = D_{nm,\bar{n}\bar{m}}^{(\lambda,\bar{\lambda})} \Psi_{\bar{n}\bar{m}}^{(\bar{\lambda})} \tag{5}$$

[2] See 'HelmholtzDecomposition.pdf' in scanned notes

[3] Bearing in mind that $\nabla^2 \Psi_{nm}^{(\lambda)} = 0$.

3.2 Covariant treatment

The problem with the previous section is that I will need to work exclusively in toroidal coordinates, and I feel slightly worried about some of the expressions above. The stuff on the left-hand side is coordinate independent, so it is ok. The right-handside is coordinate dependent. Something could be missed, so let us re-do it explicitly covariantly. Assuming flat Eucledian space, there exitst the Cartesian coordinates S and some other coordinate system $\bar{S}$.

3.2.1 Position vector

We will need to use the position vector. It can be defined as:

$$\boldsymbol{r} = \hat{\boldsymbol{e}}_\alpha r^\alpha = \hat{\boldsymbol{e}}_\alpha g^{\alpha\beta} \nabla_\beta \left(\frac{r^2}{2}\right)$$

In Cartesian coordinates. This definition is then readily convertable to other coordinate systems:

$$\begin{aligned}
\bar{r}^\alpha =& \frac{\partial \bar{x}^\alpha}{\partial x^\beta} \cdot r^\beta = \frac{\partial \bar{x}^\alpha}{\partial x^\beta} \cdot g^{\alpha\beta} \nabla_\beta \left(\frac{r^2}{2}\right) \\
=& \bar{g}^{\alpha\beta} \bar{\nabla}_\beta \left(\frac{r^2}{2}\right) = \bar{g}^{\alpha\beta} \bar{\partial}_\beta \left(\frac{r^2}{2}\right) \\
=& r \cdot \bar{g}^{\alpha\beta} \bar{\partial}_\beta r
\end{aligned} \tag{6}$$

Where $r = r(\boldsymbol{r})$ is a position-dependent scalar, and so is $r^2 = r \cdot r$.

3.2.2 Term $\boldsymbol{\nabla}.\boldsymbol{F}$

We have

$$\boldsymbol{\nabla}.\boldsymbol{F} = \nabla_\alpha F^\alpha = \bar{\nabla}_\alpha \bar{F}^\alpha = \bar{\partial}_\alpha \bar{F}^\alpha + \bar{\Gamma}^\alpha_{\alpha\mu} \bar{F}^\mu$$

Using [3]:

$$\bar{\Gamma}^\alpha_{\alpha\mu} = \frac{\bar{\partial}_\mu \sqrt{\bar{g}}}{\sqrt{\bar{g}}}$$

where $\bar{g}$ is the determinant of the metric, we get:

$$\begin{aligned}
\boldsymbol{\nabla}.\boldsymbol{F} =& \bar{\partial}_\alpha \bar{F}^\alpha + \bar{\Gamma}^\alpha_{\alpha\mu} \bar{F}^\mu = \bar{\partial}_\alpha \bar{F}^\alpha + \frac{1}{\sqrt{\bar{g}}} \bar{F}^\mu \bar{\partial}_\mu \sqrt{\bar{g}} \\
=& \frac{1}{\sqrt{\bar{g}}} \cdot \bar{\partial}_\alpha \left(\sqrt{\bar{g}} \cdot \bar{F}^\alpha\right)
\end{aligned}$$

the curl and angular-momentum components will vanish irrespective of coordinate system, so we get (using Eq. (6)):

$$\begin{aligned}
\boldsymbol{\nabla}.\boldsymbol{F} =& a^{(\lambda)}_{nm} \boldsymbol{\nabla}. \left(\boldsymbol{r} \Psi^{(\lambda)}_{nm}(\boldsymbol{r})\right) = \frac{a^{(\lambda)}_{nm}}{\sqrt{\bar{g}}} \cdot \bar{\partial}_\alpha \left(\sqrt{\bar{g}} \cdot \bar{r}^\alpha \Psi^{(\lambda)}_{nm}\right) \\
=& \frac{a^{(\lambda)}_{nm}}{\sqrt{\bar{g}}} \cdot \bar{\partial}_\alpha \left(\Psi^{(\lambda)}_{nm} \cdot \bar{g}^{\alpha\beta} \sqrt{\bar{g}} \bar{\partial}_\beta \left(\frac{r^2}{2}\right)\right) \\
\boldsymbol{\nabla}.\boldsymbol{F} =& a^{(\lambda)}_{nm} \cdot \bar{g}^{\alpha\beta} \bar{\partial}_\beta \left(\frac{r^2}{2}\right) \cdot \left(\bar{\partial}_\alpha \Psi^{(\lambda)}_{nm}\right) + a^{(\lambda)}_{nm} \cdot \frac{1}{\sqrt{\bar{g}}} \bar{\partial}_\alpha \left(\bar{g}^{\alpha\beta} \sqrt{\bar{g}} \bar{\partial}_\beta \left(\frac{r^2}{2}\right)\right) \cdot \Psi^{(\lambda)}_{nm}
\end{aligned} \tag{7}$$

3.2.3 Term $\boldsymbol{r}.\boldsymbol{F}$

Assuming that $a_{nm}^{(\lambda)}$ is already known, we have[4]:

$$
\begin{aligned}
\boldsymbol{r}.\boldsymbol{F} =& a_{nm}^{(\lambda)} \cdot r^2 \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) + c_{nm}^{(\lambda)} \cdot \boldsymbol{r}.\boldsymbol{\nabla} \times \boldsymbol{L} \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) \\
\boldsymbol{r}.\boldsymbol{F} - a_{nm}^{(\lambda)} \cdot r^2 \Psi_{nm}^{(\lambda)} =& c_{nm}^{(\lambda)} \cdot \boldsymbol{r}.\boldsymbol{\nabla} \times \boldsymbol{L} \Psi_{nm}^{(\lambda)} \\
=& c_{nm}^{(\lambda)} \cdot r^\alpha g_{\alpha\beta} \frac{\epsilon^{\beta\gamma\mu}}{\sqrt{g}} \nabla_\gamma \left(\boldsymbol{L} \Psi_{nm}^{(\lambda)} \right)_\mu \\
=& c_{nm}^{(\lambda)} \cdot r^\alpha g_{\alpha\beta} \frac{\epsilon^{\beta\gamma\mu}}{\sqrt{g}} \nabla_\gamma \left(-i \boldsymbol{r} \times \boldsymbol{\nabla} \Psi_{nm}^{(\lambda)} \right)_\mu \\
=& - i c_{nm}^{(\lambda)} \cdot r^\alpha g_{\alpha\beta} \frac{\epsilon^{\beta\gamma\mu}}{\sqrt{g}} \nabla_\gamma \left(g_{\mu\kappa} \frac{\epsilon^{\kappa\phi\zeta}}{\sqrt{g}} r^\sigma g_{\phi\sigma} \nabla_\zeta \Psi_{nm}^{(\lambda)} \right)
\end{aligned}
$$

This last expression is written in covariant terms, so will be valid in any coordinate system. It makes sense to adopt the co-radius, then[5]:

$$
\begin{aligned}
i\left(\boldsymbol{r}.\boldsymbol{F} - a_{nm}^{(\lambda)} \cdot r^2 \Psi_{nm}^{(\lambda)} \right) =& c_{nm}^{(\lambda)} \cdot \bar{r}_\alpha \frac{\bar{\epsilon}^{\alpha\beta\mu}}{\sqrt{\bar{g}}} \bar{\nabla}_\beta \left(\bar{g}_{\mu\kappa} \frac{\bar{\epsilon}^{\kappa\phi\zeta}}{\sqrt{\bar{g}}} \bar{r}_\phi \bar{\partial}_\zeta \Psi_{nm}^{(\lambda)} \right) \\
=& c_{nm}^{(\lambda)} \cdot \bar{r}_\alpha \frac{\bar{\epsilon}^{\alpha\beta\mu}}{\sqrt{\bar{g}}} \cdot \left[\bar{\partial}_\beta \left(\bar{g}_{\mu\kappa} \frac{\bar{\epsilon}^{\kappa\phi\zeta}}{\sqrt{\bar{g}}} \bar{r}_\phi \bar{\partial}_\zeta \Psi_{nm}^{(\lambda)} \right) - \right. \\
& \left. - \bar{\Gamma}^\gamma_{\beta\mu} \bar{g}_{\gamma\kappa} \frac{\bar{\epsilon}^{\kappa\phi\zeta}}{\sqrt{\bar{g}}} \bar{r}_\phi \bar{\partial}_\zeta \Psi_{nm}^{(\lambda)} \right] \\
i\left(\boldsymbol{r}.\boldsymbol{F} - a_{nm}^{(\lambda)} \cdot r^2 \Psi_{nm}^{(\lambda)} \right) =& c_{nm}^{(\lambda)} \cdot \bar{r}_\alpha \frac{\bar{\epsilon}^{\alpha\beta\mu}}{\sqrt{\bar{g}}} \cdot \bar{\partial}_\beta \left(\bar{g}_{\mu\kappa} \frac{\bar{\epsilon}^{\kappa\phi\zeta}}{\sqrt{\bar{g}}} \bar{r}_\phi \bar{\partial}_\zeta \Psi_{nm}^{(\lambda)} \right) \qquad (8)
\end{aligned}
$$

3.2.4 Term $\boldsymbol{L}.\boldsymbol{F}$

The easiest way is:

$$
\boldsymbol{L}.\boldsymbol{F} = b_{nm}^{(\lambda)} \cdot L^2 \Psi_{nm}^{(\lambda)}
$$

So what is the anglar momenutum squared in toroidal coordinates?

$$
\begin{aligned}
L^2 f =& \left(-i \epsilon_{\alpha\beta\gamma} \sqrt{g} r^\beta g^{\gamma\kappa} \nabla_\kappa \right) \left(-i \frac{\epsilon^{\alpha\mu\nu}}{\sqrt{g}} r_\mu \nabla_\nu \right) f \\
=& - \left(\bar{\epsilon}_{\alpha\beta\gamma} \sqrt{\bar{g}} \bar{r}^\beta \bar{g}^{\gamma\kappa} \bar{\nabla}_\kappa \right) \left(\frac{\bar{\epsilon}^{\alpha\mu\nu}}{\sqrt{\bar{g}}} \bar{r}_\mu \bar{\nabla}_\nu \right) f \\
=& - \bar{\epsilon}_{\alpha\beta\gamma} \sqrt{\bar{g}} \bar{r}^\beta \bar{g}^{\gamma\kappa} \bar{\nabla}_\kappa \left(\frac{\bar{\epsilon}^{\alpha\mu\nu}}{\sqrt{\bar{g}}} \bar{r}_\mu \bar{\partial}_\nu f \right) \\
=& - \bar{\epsilon}_{\alpha\beta\gamma} \sqrt{\bar{g}} \bar{r}^\beta \bar{g}^{\gamma\kappa} \bar{\partial}_\kappa \left(\frac{\bar{\epsilon}^{\alpha\mu\nu}}{\sqrt{\bar{g}}} \bar{r}_\mu \bar{\partial}_\nu f \right) - \bar{\epsilon}_{\alpha\beta\gamma} \sqrt{\bar{g}} \bar{r}^\beta \bar{g}^{\gamma\kappa} \bar{\Gamma}^\alpha_{\kappa\zeta} \cdot \left(\frac{\bar{\epsilon}^{\alpha\mu\nu}}{\sqrt{\bar{g}}} \bar{r}_\mu \bar{\partial}_\nu f \right)
\end{aligned}
$$

So:

$$
\boldsymbol{L}.\boldsymbol{F} = b_{nm}^{(\lambda)} \cdot \left[- \bar{\epsilon}_{\alpha\beta\gamma} \sqrt{\bar{g}} \bar{r}^\beta \bar{g}^{\gamma\kappa} \bar{\partial}_\kappa \left(\frac{\bar{\epsilon}^{\alpha\mu\nu}}{\sqrt{\bar{g}}} \bar{r}_\mu \bar{\partial}_\nu \Psi_{nm}^{(\lambda)} \right) - \bar{\epsilon}_{\alpha\beta\gamma} \sqrt{\bar{g}} \bar{r}^\beta \bar{g}^{\gamma\kappa} \bar{\Gamma}^\alpha_{\kappa\zeta} \cdot \left(\frac{\bar{\epsilon}^{\alpha\mu\nu}}{\sqrt{\bar{g}}} \bar{r}_\mu \bar{\partial}_\nu \Psi_{nm}^{(\lambda)} \right) \right] \qquad (9)
$$

3.3 Toroidal coordinates and vector basis

3.3.1 Basics

To get the matricies from Eq. (2-5), I will probably need to find analytical expressions for the relevant left-hand sides, then numerically decompose them into right-hand side. It makes sense to start with the simplest one: $r^2 \Psi_{nm}^{(\lambda)}$.

[4]Bear in mind that for the Levi-Civita, it is $\epsilon^{\alpha\beta\gamma}/\sqrt{g}$ and $\epsilon_{\alpha\beta\gamma} \cdot \sqrt{g}$ that transform like tensors.

[5]Bearing in mind that connection is symmetric and Levi-Civita is anti-symmetric

From Spencer [4] the toroidal coordinates in terms of Cartesian ones are:

$$\begin{aligned}
x =& \frac{a\ \sinh\eta\ \cos\phi}{\cosh\eta-\cos\theta} \\
y =& \frac{a\ \sinh\eta\ \sin\phi}{\cosh\eta-\cos\theta} \\
z =& \frac{a\ \sin\theta}{\cosh\eta-\cos\theta}
\end{aligned}$$

3.3.2 Metric and basis vectors

We have:

$$\begin{gathered}
dx^2+dy^2+dz^2 = \frac{a^2}{(\cosh\eta-\cos\theta)^2}\cdot d\eta^2+\frac{a^2}{(\cosh\eta-\cos\theta)^2}\cdot d\theta^2+\frac{a^2\sinh^2\eta}{(\cosh\eta-\cos\theta)^2}\cdot d\phi^2 \\
\boldsymbol{e}_\eta.\boldsymbol{e}_\eta = \frac{a^2}{(\cosh\eta-\cos\theta)^2} \\
\boldsymbol{e}_\theta.\boldsymbol{e}_\theta = \frac{a^2}{(\cosh\eta-\cos\theta)^2} \\
\boldsymbol{e}_\phi.\boldsymbol{e}_\phi = \frac{a^2\sinh^2\eta}{(\cosh\eta-\cos\theta)^2} \\
\boldsymbol{e}_\eta.\boldsymbol{e}_\theta=\boldsymbol{e}_\eta.\boldsymbol{e}_\phi=\boldsymbol{e}_\theta.\boldsymbol{e}_\phi=0 \\
d\boldsymbol{r}=d\eta\,\boldsymbol{e}_\eta+d\theta\,\boldsymbol{e}_\theta+d\phi\,\boldsymbol{e}_\phi
\end{gathered}$$

It is also convenient to define normalized basis:

$$\begin{aligned}
\hat{\boldsymbol{\eta}} =& \frac{\cosh\eta-\cos\theta}{a}\cdot\boldsymbol{e}_\eta \\
\hat{\boldsymbol{\theta}} =& \frac{\cosh\eta-\cos\theta}{a}\cdot\boldsymbol{e}_\theta \\
\hat{\boldsymbol{\phi}} =& \frac{\cosh\eta-\cos\theta}{a\sinh\eta}\cdot\boldsymbol{e}_\phi
\end{aligned}$$

So then:

$$d\boldsymbol{r}=\frac{a}{\cosh\eta-\cos\theta}d\eta\,\hat{\boldsymbol{\eta}}+\frac{a}{\cosh\eta-\cos\theta}d\theta\,\hat{\boldsymbol{\theta}}+\frac{a\sinh\eta}{\cosh\eta-\cos\theta}d\phi\,\hat{\boldsymbol{\phi}}$$

The metric is and related things are:

$$\bar{g}_{\alpha\beta} =\frac{a^2}{(\cosh\eta-\cos\theta)^2}\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&\sinh^2\eta\end{pmatrix}_{\alpha\beta} \tag{10}$$

$$\bar{g} =\det\left(\bar{g}_{\alpha\beta}\right)=\frac{a^6\sinh^2\eta}{(\cosh\eta-\cos\theta)^6} \tag{11}$$

$$\sqrt{\bar{g}} =\frac{a^3\sinh\eta}{(\cosh\eta-\cos\theta)^3} \tag{12}$$

$$\bar{g}^{\alpha\beta} =\frac{(\cosh\eta-\cos\theta)^2}{a^2}\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{\sinh^2\eta}\end{pmatrix}^{\alpha\beta} \tag{13}$$

3.3.3 Chrisoffel symbols

It would seem I also need them. Definition:

$$\bar{\Gamma}^\mu_{\alpha\beta}=\frac{1}{2}\cdot\bar{g}^{\mu\kappa}\cdot\left(\bar{\partial}_\alpha\bar{g}_{\kappa\beta}+\bar{\partial}_\beta\bar{g}_{\alpha\kappa}-\bar{\partial}_\kappa\bar{g}_{\alpha\beta}\right)$$

I could not bring myself to work it out manually, so here is the Mathematica result:

$$\Gamma^1_{\dots,\dots} = \Gamma^\eta_{\dots,\dots} = \begin{pmatrix} \frac{\sinh\eta}{\cos\theta-\cosh\eta} & \frac{\sin\theta}{\cos\theta-\cosh\eta} & 0 \\ \frac{\sin\theta}{\cos\theta-\cosh\eta} & \frac{-\sinh\eta}{\cos\theta-\cosh\eta} & 0 \\ 0 & 0 & \frac{(1-\cos\theta\cosh\eta)\sinh\eta}{\cos\theta-\cosh\eta} \end{pmatrix} \tag{14}$$

$$\Gamma^2_{\dots,\dots} = \Gamma^\theta_{\dots,\dots} = \begin{pmatrix} \frac{-\sin\theta}{\cos\theta-\cosh\eta} & \frac{\sinh\eta}{\cos\theta-\cosh\eta} & 0 \\ \frac{\sinh\eta}{\cos\theta-\cosh\eta} & \frac{\sin\theta}{\cos\theta-\cosh\eta} & 0 \\ 0 & 0 & \frac{-\sin\theta\sinh^2\eta}{\cos\theta-\cosh\eta} \end{pmatrix} \tag{15}$$

$$\Gamma^3_{\dots,\dots} = \Gamma^\phi_{\dots,\dots} = \begin{pmatrix} 0 & 0 & \frac{(\cos\theta\cosh\eta-1)}{(\cos\theta-\cosh\eta)\sinh\eta} \\ 0 & 0 & \frac{\sin\theta}{\cos\theta-\cosh\eta} \\ \frac{(\cos\theta\cosh\eta-1)}{(\cos\theta-\cosh\eta)\sinh\eta} & \frac{\sin\theta}{\cos\theta-\cosh\eta} & 0 \end{pmatrix} \tag{16}$$

3.3.4 Position vector

It follows that the radius squared is[6]:

$$r^2 = x^2 + y^2 + z^2 = a^2 \cdot \frac{\cosh\eta + \cos\theta}{\cosh\eta - \cos\theta} \tag{17}$$

I will also need the derivatives of r^2, which I will use primitive functions:

$$\begin{aligned} r^2_{,\eta} =& \partial_\eta r^2 = -2a^2 \cdot \frac{\sinh\eta \cdot \cos\theta}{(\cosh\eta - \cos\theta)^2} \\ r^2_{,\theta} =& \partial_\theta r^2 = -2a^2 \cdot \frac{\cosh\eta \cdot \sin\theta}{(\cosh\eta - \cos\theta)^2} \\ r^2_{,\eta\eta} =& \partial_{\eta\eta} r^2 = a^2 \cdot \frac{(\cosh 2\eta + 2\cosh\eta\cos\theta - 3)\cdot\cos\theta}{(\cosh\eta - \cos\theta)^3} \\ r^2_{,\theta\theta} =& \partial_{\theta\theta} r^2 = 2a^2 \cdot \frac{\cosh\eta \cdot \left(\sin^2\theta + 1 - \cosh\eta\cos\theta\right)}{(\cosh\eta - \cos\theta)^3} \\ r^2_{,\eta\theta} =& \partial_{\eta\theta} r^2 = 2a^2 \cdot \frac{\sinh\eta \cdot (\cosh\eta + \cos\theta)\cdot\sin\theta}{(\cosh\eta - \cos\theta)^3} \end{aligned}$$

Alternatively, I may go for actually keeping the radius vector in toroidal basis as a fundamental function. Using Eq. (17) and expressions above, and the metric from Eq. (13) :

$$\bar{r}^\eta = \frac{1}{2}\bar{g}^{\eta\eta} r^2_{,\eta} = \frac{1}{2} \cdot \frac{(\cosh\eta - \cos\theta)^2}{a^2} \cdot \left(-2a^2\right) \cdot \frac{\sinh\eta \cdot \cos\theta}{(\cosh\eta - \cos\theta)^2}$$

$$\bar{r}^\eta = -\sinh\eta \cdot \cos\theta \tag{18}$$

$$\bar{r}^\theta = \frac{1}{2}\bar{g}^{\theta\theta} r^2_{,\theta} = \frac{1}{2} \cdot \frac{(\cosh\eta - \cos\theta)^2}{a^2} \cdot \left(-2a^2\right) \cdot \frac{\cosh\eta \cdot \sin\theta}{(\cosh\eta - \cos\theta)^2}$$

$$\bar{r}^\theta = -\cosh\eta \cdot \sin\theta \tag{19}$$

$$\bar{r}^\phi = 0 \tag{20}$$

Also:

$$\bar{r}^\alpha_{,\alpha} = \bar{\partial}_\alpha \bar{r}^\alpha = \partial_\eta\left(-\sinh\eta \cdot \cos\theta\right) + \partial_\theta\left(-\cosh\eta \cdot \sin\theta\right) = -2\cosh\eta \cdot \cos\theta \tag{21}$$

It is also convenient to use the co-variant position vector:

$$\bar{r}_\alpha = \bar{g}_{\alpha\beta}\bar{r}^\alpha = \bar{\delta}^\eta_\alpha \bar{g}_{\eta\eta}\bar{r}^\eta + \bar{\delta}^\theta_\alpha \bar{g}_{\theta\theta}\bar{r}^\theta$$

$$\bar{r}_\eta = \bar{g}_{\eta\eta}\bar{r}^\eta = \frac{a^2}{(\cosh\eta - \cos\theta)^2} \cdot (-\sinh\eta \cdot \cos\theta) = -\frac{a^2 \cdot \sinh\eta \cdot \cos\theta}{(\cosh\eta - \cos\theta)^2} \tag{22}$$

$$\bar{r}_\theta = -\frac{a^2 \cdot \cosh\eta \cdot \sin\theta}{(\cosh\eta - \cos\theta)^2} \tag{23}$$

[6]Note that Mathematica supports toroidal coordinates, and can compute the necessary associated Legendre functions.

3.3.5 Term $\boldsymbol{\nabla}.\boldsymbol{F}$

Using[7] Eq. (7) and Eq. (18, 10, 12, 13):

$$
\begin{aligned}
\boldsymbol{\nabla}.\boldsymbol{F} =&\boldsymbol{\nabla}.\left(\boldsymbol{r}a_{nm}^{(\lambda)}\Psi_{nm}^{(\lambda)}\right) = \frac{a_{nm}^{(\lambda)}}{\sqrt{\bar{g}}}\cdot\bar{\partial}_\alpha\left(\sqrt{\bar{g}}\cdot\bar{r}^\alpha\Psi_{nm}^{(\lambda)}\right) = \frac{a_{nm}^{(\lambda)}}{\sqrt{\bar{g}}}\cdot\left(\frac{\bar{g}_{,\alpha}}{2\sqrt{\bar{g}}}\,\bar{r}^\alpha\Psi_{nm}^{(\lambda)} + \sqrt{\bar{g}}\bar{r}^\alpha_{,\alpha}\Psi_{nm}^{(\lambda)} + \sqrt{\bar{g}}\bar{r}^\alpha\bar{\partial}_\alpha\Psi_{nm}^{(\lambda)}\right)\\
=&a_{nm}^{(\lambda)}\cdot\left[\left(\frac{\bar{g}_{,\alpha}\bar{r}^\alpha}{2\bar{g}} + \bar{r}^\alpha_{,\alpha}\right)\cdot\Psi_{nm}^{(\lambda)} + \bar{r}^\alpha\bar{\partial}_\alpha\Psi_{nm}^{(\lambda)}\right]\\
=&a_{nm}^{(\lambda)}\cdot\left[\left(\frac{\bar{g}_{,\alpha}\bar{r}^\alpha}{2\bar{g}} + \bar{r}^\alpha_{,\alpha}\right)\cdot\Psi_{nm}^{(\lambda)} - \sinh\eta\cdot\cos\theta\cdot\partial_\eta\Psi_{nm}^{(\lambda)} - \cosh\eta\cdot\sin\theta\cdot\partial_\theta\Psi_{nm}^{(\lambda)}\right]
\end{aligned}
$$

Using Eq. (21) and Eq. (11) and some simplification we find:

$$
\boldsymbol{\nabla}.\boldsymbol{F} =a_{nm}^{(\lambda)}\cdot\left(3\cdot\Psi_{nm}^{(\lambda)} - \sinh\eta\cdot\cos\theta\cdot\partial_\eta\Psi_{nm}^{(\lambda)} - \cosh\eta\cdot\sin\theta\cdot\partial_\theta\Psi_{nm}^{(\lambda)}\right) \tag{24}
$$

which agrees well with Eq. (2).

3.3.6 Term $\boldsymbol{r}.\boldsymbol{F}$

Using[8] Eq. (8) and Eq. (22, 10, 12, 13):

$$
i\left(\boldsymbol{r}.\boldsymbol{F} - a_{nm}^{(\lambda)}\cdot r^2\Psi_{nm}^{(\lambda)}\right) =c_{nm}^{(\lambda)}\cdot\bar{r}_\alpha\frac{\bar{\epsilon}^{\alpha\beta\mu}}{\sqrt{\bar{g}}}\cdot\bar{\partial}_\beta\left(\bar{g}_{\mu\kappa}\frac{\bar{\epsilon}^{\kappa\rho\zeta}}{\sqrt{\bar{g}}}\bar{r}_\rho\bar{\partial}_\zeta\Psi_{nm}^{(\lambda)}\right)
$$

Let us focus on the inner expression first:

$$
\begin{aligned}
\left(\bar{g}_{\mu\kappa}\frac{\bar{\epsilon}^{\kappa\rho\zeta}}{\sqrt{\bar{g}}}\bar{r}_\rho\bar{\partial}_\zeta\Psi_{nm}^{(\lambda)}\right)_{\mu\to\eta} =&\bar{g}_{\eta\eta}\frac{\bar{\epsilon}^{\eta\theta\phi}}{\sqrt{\bar{g}}}\bar{r}_\theta\partial_\phi\Psi_{nm}^{(\lambda)} = \bar{g}_{\eta\eta}\frac{1}{\sqrt{\bar{g}}}\bar{r}_\theta\left(im\right)\Psi_{nm}^{(\lambda)}\\
=&\frac{a^2}{\left(\cosh\eta-\cos\theta\right)^2}\cdot\frac{\left(\cosh\eta-\cos\theta\right)^3}{a^3\sinh\eta}\cdot\left(-\frac{a^2\cdot\sinh\eta\cdot\cos\theta}{\left(\cosh\eta-\cos\theta\right)^2}\right)\cdot\left(im\right)\Psi_{nm}^{(\lambda)}\\
=&-im\cdot\frac{a\cdot\cos\theta}{\left(\cosh\eta-\cos\theta\right)}\cdot\Psi_{nm}^{(\lambda)}
\end{aligned}
$$

Second component:

$$
\begin{aligned}
\left(\bar{g}_{\mu\kappa}\frac{\bar{\epsilon}^{\kappa\rho\zeta}}{\sqrt{\bar{g}}}\bar{r}_\rho\bar{\partial}_\zeta\Psi_{nm}^{(\lambda)}\right)_{\mu\to\theta} =&\bar{g}_{\theta\theta}\frac{\bar{\epsilon}^{\theta\eta\phi}}{\sqrt{\bar{g}}}\bar{r}_\eta\partial_\phi\Psi_{nm}^{(\lambda)} = -\bar{g}_{\theta\theta}\frac{1}{\sqrt{\bar{g}}}\bar{r}_\eta\left(im\right)\Psi_{nm}^{(\lambda)}\\
=&-im\cdot\frac{a^2}{\left(\cosh\eta-\cos\theta\right)^2}\cdot\frac{\left(\cosh\eta-\cos\theta\right)^3}{a^3\sinh\eta}\cdot\left(-\frac{a^2\cdot\cosh\eta\cdot\sin\theta}{\left(\cosh\eta-\cos\theta\right)^2}\right)\Psi_{nm}^{(\lambda)}\\
=&im\cdot\frac{a\cdot\coth\eta\cdot\sin\theta}{\left(\cosh\eta-\cos\theta\right)}\cdot\Psi_{nm}^{(\lambda)}
\end{aligned}
$$

[7] As soon as a replace the generic derivatives ($\bar{\partial}_{\ldots}$) with actual ones in toroidal coordinates ($\partial_{\eta,\,\theta,\,\phi}$), I will loose the bar to avoid notation overheads.

[8] As soon as a replace the generic derivatives ($\bar{\partial}_{\ldots}$) with actual ones in toroidal coordinates ($\partial_{\eta,\,\theta,\,\phi}$), I will loose the bar to avoid notation overheads.

Third component:

$$\left(\bar{g}_{\mu\kappa}\frac{\bar{\epsilon}^{\kappa\rho\zeta}}{\sqrt{\bar{g}}}\bar{r}_\rho\bar{\partial}_\zeta\Psi_{nm}^{(\lambda)}\right)_{\mu\to\phi} = \bar{g}_{\phi\phi}\frac{\bar{\epsilon}^{\phi\rho\zeta}}{\sqrt{\bar{g}}}\bar{r}_\rho\bar{\partial}_\zeta\Psi_{nm}^{(\lambda)}$$
$$= \bar{g}_{\phi\phi}\frac{1}{\sqrt{\bar{g}}}\left(\bar{\epsilon}^{\phi\eta\theta}\bar{r}_\eta\bar{\partial}_\theta\Psi_{nm}^{(\lambda)} + \bar{\epsilon}^{\phi\theta\eta}\bar{r}_\theta\bar{\partial}_\eta\Psi_{nm}^{(\lambda)}\right)$$
$$= \bar{g}_{\phi\phi}\frac{1}{\sqrt{\bar{g}}}\left(\bar{r}_\eta\bar{\partial}_\theta\Psi_{nm}^{(\lambda)} - \bar{r}_\theta\bar{\partial}_\eta\Psi_{nm}^{(\lambda)}\right)$$
$$= \frac{a^2\sinh^2\eta}{(\cosh\eta-\cos\theta)^2}\cdot\frac{(\cosh\eta-\cos\theta)^3}{a^3\sinh\eta}\cdot\left(-\frac{a^2}{(\cosh\eta-\cos\theta)^2}\right)\cdot$$
$$\cdot\left(\sinh\eta\cdot\cos\theta\cdot\partial_\theta\Psi_{nm}^{(\lambda)} - \cosh\eta\cdot\sin\theta\cdot\partial_\eta\Psi_{nm}^{(\lambda)}\right)$$
$$= \frac{a\cdot\sinh\eta}{(\cosh\eta-\cos\theta)}\cdot\left(\cosh\eta\cdot\sin\theta\cdot\partial_\eta\Psi_{nm}^{(\lambda)} - \sinh\eta\cdot\cos\theta\cdot\partial_\theta\Psi_{nm}^{(\lambda)}\right)$$

Now the main bit:

$$i\left(\boldsymbol{r}.\boldsymbol{F} - a_{nm}^{(\lambda)}\cdot r^2\Psi_{nm}^{(\lambda)}\right) = c_{nm}^{(\lambda)}\cdot\bar{r}_\alpha\frac{\bar{\epsilon}^{\alpha\beta\mu}}{\sqrt{\bar{g}}}\cdot\bar{\partial}_\beta\left(\dots\right)_\mu$$
$$= c_{nm}^{(\lambda)}\cdot\left[\bar{r}_\eta\frac{\bar{\epsilon}^{\eta\theta\phi}}{\sqrt{\bar{g}}}\cdot\partial_\theta\left(\dots\right)_\phi + \bar{r}_\eta\frac{\bar{\epsilon}^{\eta\phi\theta}}{\sqrt{\bar{g}}}\cdot\partial_\phi\left(\dots\right)_\theta\right.$$
$$\left. + \bar{r}_\theta\frac{\bar{\epsilon}^{\theta\eta\phi}}{\sqrt{\bar{g}}}\cdot\partial_\eta\left(\dots\right)_\phi + \bar{r}_\theta\frac{\bar{\epsilon}^{\theta\phi\eta}}{\sqrt{\bar{g}}}\cdot\partial_\phi\left(\dots\right)_\eta\right]$$
$$= c_{nm}^{(\lambda)}\cdot\frac{1}{\sqrt{\bar{g}}}\left[\bar{r}_\eta\cdot\partial_\theta\left(\dots\right)_\phi - \bar{r}_\eta\cdot\partial_\phi\left(\dots\right)_\theta - \right.$$
$$\left. - \bar{r}_\theta\cdot\partial_\eta\left(\dots\right)_\phi + \bar{r}_\theta\cdot\partial_\phi\left(\dots\right)_\eta\right]$$
$$= c_{nm}^{(\lambda)}\cdot\frac{1}{\sqrt{\bar{g}}}\left[\bar{r}_\eta\cdot\left(\left(\dots\right)_{\phi,\theta} - im\left(\dots\right)_\theta\right) + \bar{r}_\theta\left(im\left(\dots\right)_\eta - \left(\dots\right)_{\phi,\eta}\right)\right]$$

The expression ends up looking horrible, but it is independent of a, which is encouraging:

$$i\left(\boldsymbol{r}.\boldsymbol{F} - a_{nm}^{(\lambda)}\cdot r^2\Psi_{nm}^{(\lambda)}\right) = c_{nm}^{(\lambda)}\cdot$$
$$\cdot\left\{\left[\frac{\sinh 2\eta\,\cos\theta\,\sin^2\theta}{\cosh\eta-\cos\theta} - \left(\cosh^2\eta - \cos^2\theta\right)\coth\eta\right]\cdot\partial_\eta\Psi_{nm}^{(\lambda)} + \right.$$
$$+ \left[\frac{\cosh\eta\,\left(\sin 2\theta - 2\cosh\eta\,\cos^2\theta\,\sin\theta\right)}{\cosh\eta-\cos\theta}\right]\cdot\partial_\theta\Psi_{nm}^{(\lambda)} +$$
$$\left. + \left[-\cosh^2\eta\,\sin^2\theta\right]\cdot\partial_{\eta\eta}\Psi_{nm}^{(\lambda)} + \left[\sinh^2\eta\,\cos^2\theta\right]\cdot\partial_{\theta\theta}\Psi_{nm}^{(\lambda)}\right\}$$

The second derivatives of the toroidal harmonics are also left as primitive functions.

3.3.7 Term $\boldsymbol{L}.\boldsymbol{F}$

From Eq. (9) and others we have:

$$\boldsymbol{L}.\boldsymbol{F} = b_{nm}^{(\lambda)}\cdot\left[-\bar{\epsilon}_{\alpha\beta\gamma}\sqrt{\bar{g}}\bar{r}^\beta\bar{g}^{\gamma\kappa}\bar{\partial}_\kappa\left(\frac{\bar{\epsilon}^{\alpha\mu\nu}}{\sqrt{\bar{g}}}\bar{r}_\mu\bar{\partial}_\nu\Psi_{nm}^{(\lambda)}\right) - \bar{\epsilon}_{\alpha\beta\gamma}\sqrt{\bar{g}}\bar{r}^\beta\bar{g}^{\gamma\kappa}\bar{\Gamma}^\alpha_{\kappa\zeta}\cdot\left(\frac{\bar{\epsilon}^{\zeta\mu\nu}}{\sqrt{\bar{g}}}\bar{r}_\mu\bar{\partial}_\nu\Psi_{nm}^{(\lambda)}\right)\right]$$

First, it makes sence to define a vector, and it will be a vector:

$$\bar{V}^\alpha = \frac{\bar{\epsilon}^{\alpha\mu\nu}}{\sqrt{\bar{g}}}\bar{r}_\mu\bar{\partial}_\nu\Psi_{nm}^{(\lambda)} = \frac{(\cosh\eta-\cos\theta)^3}{a^3\sinh\eta}\cdot\bar{\epsilon}^{\alpha\mu\nu}\bar{r}_\mu\bar{\partial}_\nu\Psi_{nm}^{(\lambda)}$$

and see what it evaluates to. First component:

$$\begin{aligned}\bar{V}^{\eta} =&\frac{(\cosh\eta-\cos\theta)^3}{a^3\sinh\eta}\cdot\left(\bar{\epsilon}^{\eta\theta\phi}\bar{r}_\theta\partial_\phi\Psi^{(\lambda)}_{nm}\right)=\frac{im\,(\cosh\eta-\cos\theta)^3}{a^3\sinh\eta}\cdot\bar{r}_\theta\Psi^{(\lambda)}_{nm}\\ =&\frac{im\,(\cosh\eta-\cos\theta)^3}{a^3\sinh\eta}\cdot\left(-\frac{a^2\cdot\cosh\eta\cdot\sin\theta}{(\cosh\eta-\cos\theta)^2}\right)\Psi^{(\lambda)}_{nm}=-\frac{im}{a}\cdot\frac{(\cosh\eta-\cos\theta)\cdot\sin\theta}{\tanh\eta}\cdot\Psi^{(\lambda)}_{nm}\end{aligned}$$

Second component:

$$\begin{aligned}\bar{V}^{\theta} =&\frac{(\cosh\eta-\cos\theta)^3}{a^3\sinh\eta}\cdot\bar{\epsilon}^{\theta\eta\phi}\bar{r}_\eta\partial_\phi\Psi^{(\lambda)}_{nm}=-\frac{im\,(\cosh\eta-\cos\theta)^3}{a^3\sinh\eta}\cdot\bar{r}_\eta\Psi^{(\lambda)}_{nm}\\ =&-\frac{im\,(\cosh\eta-\cos\theta)^3}{a^3\sinh\eta}\cdot\left(-\frac{a^2\cdot\sinh\eta\cdot\cos\theta}{(\cosh\eta-\cos\theta)^2}\right)\Psi^{(\lambda)}_{nm}\\ =&\frac{im}{a}\cdot(\cosh\eta-\cos\theta)\cdot\cos\theta\Psi^{(\lambda)}_{nm}\end{aligned}$$

Third component:

$$\begin{aligned}\bar{V}^{\phi} =&\frac{(\cosh\eta-\cos\theta)^3}{a^3\sinh\eta}\cdot\left(\bar{\epsilon}^{\phi\eta\theta}\bar{r}_\eta\partial_\theta\Psi^{(\lambda)}_{nm}+\bar{\epsilon}^{\phi\theta\eta}\bar{r}_\theta\partial_\eta\Psi^{(\lambda)}_{nm}\right)=\frac{(\cosh\eta-\cos\theta)^3}{a^3\sinh\eta}\cdot\left(\bar{\epsilon}^{\phi\eta\theta}\bar{r}_\eta\partial_\theta\Psi^{(\lambda)}_{nm}+\bar{\epsilon}^{\phi\theta\eta}\bar{r}_\theta\partial_\eta\Psi^{(\lambda)}_{nm}\right)\\ =&\frac{(\cosh\eta-\cos\theta)^3}{a^3\sinh\eta}\cdot\left(\bar{r}_\eta\partial_\theta\Psi^{(\lambda)}_{nm}-\bar{r}_\theta\partial_\eta\Psi^{(\lambda)}_{nm}\right)\\ =&\frac{(\cosh\eta-\cos\theta)^3}{a^3\sinh\eta}\cdot\left(\frac{a^2\cdot\cosh\eta\cdot\sin\theta}{(\cosh\eta-\cos\theta)^2}\cdot\partial_\eta\Psi^{(\lambda)}_{nm}-\frac{a^2\cdot\sinh\eta\cdot\cos\theta}{(\cosh\eta-\cos\theta)^2}\cdot\partial_\theta\Psi^{(\lambda)}_{nm}\right)\\ =&\frac{(\cosh\eta-\cos\theta)}{a\sinh\eta}\cdot\left(\cosh\eta\cdot\sin\theta\cdot\partial_\eta\Psi^{(\lambda)}_{nm}-\sinh\eta\cdot\cos\theta\cdot\partial_\theta\Psi^{(\lambda)}_{nm}\right)\end{aligned}$$

Next we can consider the full expression:

$$\begin{aligned}\boldsymbol{L}.\boldsymbol{F} =&b^{(\lambda)}_{nm}\cdot\left[-\bar{\epsilon}_{\alpha\beta\gamma}\sqrt{\bar{g}}\bar{r}^\beta\bar{g}^{\gamma\kappa}\bar{\partial}_\kappa\left(\frac{\bar{\epsilon}^{\alpha\mu\nu}}{\sqrt{\bar{g}}}\bar{r}_\mu\bar{\partial}_\nu\Psi^{(\lambda)}_{nm}\right)-\bar{\epsilon}_{\alpha\beta\gamma}\sqrt{\bar{g}}\bar{r}^\beta\bar{g}^{\gamma\kappa}\bar{\Gamma}^\alpha_{\kappa\zeta}\cdot\left(\frac{\bar{\epsilon}^{\zeta\mu\nu}}{\sqrt{\bar{g}}}\bar{r}_\mu\bar{\partial}_\nu\Psi^{(\lambda)}_{nm}\right)\right]\\ =&b^{(\lambda)}_{nm}\cdot\frac{a^3\sinh\eta}{(\cosh\eta-\cos\theta)^3}\cdot\bar{\epsilon}_{\alpha\beta\gamma}\bar{r}^\beta\bar{g}^{\gamma\kappa}\left[-\bar{\partial}_\kappa\bar{V}^\alpha-\bar{\Gamma}^\alpha_{\kappa\zeta}\cdot\bar{V}^\zeta\right]\\ =&b^{(\lambda)}_{nm}\cdot\frac{a^3\sinh\eta}{(\cosh\eta-\cos\theta)^3}\cdot\bar{\epsilon}_{\alpha\beta\gamma}\bar{r}^\beta\bar{g}^{\gamma\kappa}\left[\ldots\right]^\alpha_\kappa\\ =&b^{(\lambda)}_{nm}\cdot\frac{a^3\sinh\eta}{(\cosh\eta-\cos\theta)^3}\cdot\left(\bar{\epsilon}_{\alpha\eta\gamma}\bar{r}^\eta\bar{g}^{\gamma\kappa}\left[\ldots\right]^\alpha_\kappa+\bar{\epsilon}_{\alpha\theta\gamma}\bar{r}^\theta\bar{g}^{\gamma\kappa}\left[\ldots\right]^\alpha_\kappa\right)\\ =&b^{(\lambda)}_{nm}\cdot\frac{-a^3\sinh\eta}{(\cosh\eta-\cos\theta)^3}\cdot\left(\bar{\epsilon}_{\alpha\eta\gamma}\sinh\eta\cdot\cos\theta\bar{g}^{\gamma\kappa}\left[\ldots\right]^\alpha_\kappa+\bar{\epsilon}_{\alpha\theta\gamma}\cosh\eta\cdot\sin\theta\bar{g}^{\gamma\kappa}\left[\ldots\right]^\alpha_\kappa\right)\\ =&b^{(\lambda)}_{nm}\cdot\frac{-a^3\sinh\eta}{(\cosh\eta-\cos\theta)^3}\cdot\Big\{\bar{\epsilon}_{\theta\eta\phi}\sinh\eta\cdot\cos\theta\bar{g}^{\phi\phi}\left[\ldots\right]^\theta_\phi+\bar{\epsilon}_{\phi\eta\theta}\sinh\eta\cdot\cos\theta\bar{g}^{\theta\theta}\left[\ldots\right]^\phi_\theta\\ &+\bar{\epsilon}_{\eta\theta\phi}\cosh\eta\cdot\sin\theta\bar{g}^{\phi\phi}\left[\ldots\right]^\eta_\phi+\bar{\epsilon}_{\phi\theta\eta}\cosh\eta\cdot\sin\theta\bar{g}^{\eta\eta}\left[\ldots\right]^\phi_\eta\Big\}\\ =&b^{(\lambda)}_{nm}\cdot\frac{-a^3\sinh\eta}{(\cosh\eta-\cos\theta)^3}\cdot\Big\{-\sinh\eta\cdot\cos\theta\cdot\frac{(\cosh\eta-\cos\theta)^2}{a^2\sinh^2\eta}\left[\ldots\right]^\theta_\phi+\sinh\eta\cdot\cos\theta\cdot\frac{(\cosh\eta-\cos\theta)^2}{a^2}\left[\ldots\right]^\phi_\theta\\ &+\cosh\eta\cdot\sin\theta\cdot\frac{(\cosh\eta-\cos\theta)^2}{a^2\sinh^2\eta}\left[\ldots\right]^\eta_\phi-\cosh\eta\cdot\sin\theta\cdot\frac{(\cosh\eta-\cos\theta)^2}{a^2}\left[\ldots\right]^\phi_\eta\Big\}\\ \boldsymbol{L}.\boldsymbol{F} =&b^{(\lambda)}_{nm}\cdot\frac{-a\sinh\eta}{(\cosh\eta-\cos\theta)}\cdot\left\{-\frac{\cos\theta}{\sinh\eta}\left[\ldots\right]^\theta_\phi+\sinh\eta\cdot\cos\theta\cdot\left[\ldots\right]^\phi_\theta+\frac{\cosh\eta\cdot\sin\theta}{\sinh^2\eta}\left[\ldots\right]^\eta_\phi-\cosh\eta\cdot\sin\theta\cdot\left[\ldots\right]^\phi_\eta\right\}\end{aligned}$$

The full expression was implemented on Mathematica and evaluated there. The end result is:

$$
\begin{aligned}
\boldsymbol{L}.\boldsymbol{F} =& b_{nm}^{(\lambda)} \cdot \Bigg\{ \\
&+ \frac{m^2 \cdot \left(\cosh^2\eta - \cos^2\theta\right)}{\sinh^2\eta} \cdot \Psi_{nm}^{(\lambda)} + \\
&+ \frac{\cosh\eta \cdot \left(\cosh 2\eta\, \cos 2\theta - 1\right)}{2\sinh\eta} \cdot \partial_\eta \Psi_{nm}^{(\lambda)} + \\
&+ \cosh^2\eta\, \sin 2\theta\, \cdot \partial_\theta \Psi_{nm}^{(\lambda)} + \\
&+ \frac{1}{2}\sinh 2\eta\, \sin 2\theta \cdot \partial_{\eta\theta} \Psi_{nm}^{(\lambda)} + \\
&+ \left(-\cosh^2\eta\, \sin^2\theta\right) \cdot \partial_{\eta\eta} \Psi_{nm}^{(\lambda)} \\
&+ \left(-\sinh^2\eta\, \cos^2\theta\right) \cdot \partial_{\theta\theta} \Psi_{nm}^{(\lambda)} \Bigg\}
\end{aligned}
$$

3.4 Derivatives of the harmonics

From previous sections it is clear that we will need to compute $\partial_\eta \Psi_{nm}^{(\lambda)}$ and $\partial_\theta \Psi_{nm}^{(\lambda)}$. Lets work them out[9]:

$$
\begin{aligned}
\partial_\eta \Psi_{nm}^{(\lambda)} =& \partial_\eta \left(\sqrt{\cosh\eta - \cos\theta} Z_{n-\frac{1}{2}}^{(\lambda)m}\left(\cosh\eta\right)\, \exp\left(in\theta\right)\, \exp\left(im\phi\right)\right) \\
=& \frac{\cosh\eta \cdot \left(\left(2n+1\right)\cos\theta - 2n\cosh\eta\right) - 1}{2\sinh\eta \cdot \left(\cosh\eta - \cos\theta\right)} \cdot \Psi_{nm}^{(\lambda)} + \frac{\left(n + \frac{1}{2} - m\right) \cdot \exp\left(-i\theta\right)}{\sinh\eta} \cdot \Psi_{n+1,\, m}^{(\lambda)}
\end{aligned} \tag{25}
$$

Also:

$$
\begin{aligned}
\partial_\theta \Psi_{nm}^{(\lambda)} =& \partial_\theta \left(\sqrt{\cosh\eta - \cos\theta} Z_{n-\frac{1}{2}}^{(\lambda)m}\left(\cosh\eta\right)\, \exp\left(in\theta\right)\, \exp\left(im\phi\right)\right) \\
=& \left(\frac{\sin\theta}{2 \cdot \left(\cosh\eta - \cos\theta\right)} + in\right) \Psi_{nm}^{(\lambda)}
\end{aligned} \tag{26}
$$

And:

$$
\begin{aligned}
\partial_\phi \Psi_{nm}^{(\lambda)} =& \partial_\phi \left(\sqrt{\cosh\eta - \cos\theta} Z_{n-\frac{1}{2}}^{(\lambda)m}\left(\cosh\eta\right)\, \exp\left(in\theta\right)\, \exp\left(im\phi\right)\right) \\
=& \left(im\right) \Psi_{nm}^{(\lambda)}
\end{aligned} \tag{27}
$$

Let us obtain the diagonal second derivatives:

$$
\begin{aligned}
\partial_{\eta\eta} \Psi_{nm}^{(\lambda)} =& \partial_\eta \left(\frac{\cosh\eta \cdot \left(\left(2n+1\right)\cos\theta - 2n\cosh\eta\right) - 1}{2\sinh\eta \cdot \left(\cosh\eta - \cos\theta\right)} \cdot \Psi_{nm}^{(\lambda)} + \frac{\left(n + \frac{1}{2} - m\right) \cdot \exp\left(-i\theta\right)}{\sinh\eta} \cdot \Psi_{n+1,\, m}^{(\lambda)}\right) \\
\partial_{\eta\eta} \Psi_{nm}^{(\lambda)} =& \frac{1}{16\, \sinh\eta\, \left(\cosh\eta - \cos\theta\right)^2} \cdot \Bigg\{ \frac{-4\, \cos\theta\, \left(\left(4n^2 + 2n + 1\right)\cosh 2\eta + 5 + 2n\left(7 + 2n\right)\right)}{\tanh\eta} + \\
&+ \frac{6\left(3 + 4n\left(2 + n\right)\right) + \left(1 + 2n\right)\left(5 + 2n + \left(1 + 2n\right)\cosh 2\eta\right)\cos 2\theta}{\sinh\eta} + 18\sinh\eta + \\
&+ 8n\left(5 + 4n + n\cosh 2\eta\right)\sinh\eta \Bigg\} \cdot \Psi_{nm}^{(\lambda)} + \\
&+ \\
&+ \frac{\left(2m - 2n - 1\right)\exp\left(-i\theta\right)}{2\left(\cosh\eta - \cos\theta\right)} \cdot \left\{\frac{2\left(n+1\right)}{\tanh^2\eta} - \frac{\left(3 + 2n\right)\cos\theta}{\tanh\eta\, \sinh\eta} + \frac{1}{\sinh^2\eta}\right\} \Psi_{n+1,m}^{(\lambda)} + \\
&+ \\
&+ \frac{\exp\left(-i2\theta\right)\left(2m - 3 - 2n\right)\left(2m - 2n - 1\right)}{4\sinh^2\eta} \cdot \Psi_{n+2,m}^{(\lambda)}
\end{aligned}
$$

[9] Luckily, since Mathematica supports these functions as `LegendreP(n-1/2, m, 3, z)` and `LegendreQ[n-1/2, m, 3, z]`, and can differentiate using recurrence relations.

and the other one:

$$\partial_{\theta\theta}\Psi_{nm}^{(\lambda)} =\partial_{\theta}\left(\left(\frac{\sin\theta}{2\cdot(\cosh\eta-\cos\theta)}+in\right)\Psi_{nm}^{(\lambda)}\right)$$
$$\partial_{\theta\theta}\Psi_{nm}^{(\lambda)} =\left(\frac{4\cosh\eta\,\cos\theta-\cos 2\theta-3}{8\left(\cosh\eta-\cos\theta\right)^{2}}\right)\cdot\Psi_{nm}^{(\lambda)}$$

The off-diagonal one:

$$\begin{aligned}\partial_{\eta\theta}\Psi_{nm}^{(\lambda)} &=\partial_{\eta}\left(\left(\frac{\sin\theta}{2\cdot(\cosh\eta-\cos\theta)}+in\right)\Psi_{nm}^{(\lambda)}\right)\\ &=\left(-\frac{\sin\theta\cdot\sinh\eta}{2\left(\cosh\eta-\cos\theta\right)^{2}}\right)\cdot\Psi_{nm}^{(\lambda)}+\left(\frac{\sin\theta}{2\cdot(\cosh\eta-\cos\theta)}+in\right)\partial_{\eta}\Psi_{nm}^{(\lambda)}\end{aligned}$$

3.5 Getting out the constants

Once a suitable analytic expression has been found, for all the terms in Eq. (2-5), we have a generic problem like this:

$$h\left(\boldsymbol{r}\right)=q_{nm}^{(\lambda)}Q_{nm}^{(\lambda)}\left(\boldsymbol{r}\right)$$

Where we need the constants $q_{nm}^{(\lambda)}$. It makes sence to convert the problem into a matrix form:

$$h_{ijk}=Q_{ijk,\lambda nm}q_{\lambda nm}$$

We then contract it with a suitable tensor, e.g.[10]:

$$\left(\sqrt{g}\Psi\right)^{*}_{\bar{\lambda}\bar{n}\bar{m},ijk}Q_{ijk,\lambda nm}q_{\lambda nm}=\left(\sqrt{g}\Psi\right)^{*}_{\bar{\lambda}\bar{n}\bar{m},ijk}h_{ijk}=\Lambda_{\bar{\lambda}\bar{n}\bar{m}}$$

We can calculate:

$$M_{\bar{\lambda}\bar{n}\bar{m},\lambda nm}=\left(\sqrt{g}\Psi\right)^{*}_{\bar{\lambda}\bar{n}\bar{m},ijk}Q_{ijk,\lambda nm}$$

Finally, all we need to do is solve the tensor equation:

$$M_{\bar{\lambda}\bar{n}\bar{m},\lambda nm}q_{\lambda nm}=\Lambda_{\bar{\lambda}\bar{n}\bar{m}}$$

References

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[10] Clearly the choice of tensor to contract with, will be important, but lets keep it simple for now. I will stick in density ($\sqrt{g}$) to approximate an integral - this works well if we start on a uniform Cartesian lattice.