Toroidal Harmonics Definition for the DARPA project

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1 Defnition of toroidal Harmonics

Several work from plasma physics consider toroidal harmonics. Works by Garapati *et al.* [1, 2] suggest defining the general solution to the Laplace equations as:

$$\Phi = f(\mu,\, \eta) \sum_{n,m=-\infty}^{\infty} \left[C_{mn} P_{n-\frac{1}{2}}^{m} (\cosh \mu) + D_{mn} Q_{n-\frac{1}{2}}^{m} (\cosh \mu) \right] \exp(in\eta) \exp(im\phi)$$

$$f(\mu,\, \eta) = \sqrt{\cosh \mu - \cos \eta}$$

I will therefore define toroidal harmonics as:

$$\Psi_{nm}^{(1,2)}(\eta,\, \theta,\, \phi) = \left\{ \begin{array}{c} \Psi \\ \Phi \end{array} \right\}_{nm} (\eta,\, \theta,\, \phi) = \sqrt{\cosh \eta - \cos \theta} \left\{ \begin{array}{c} P \\ Q \end{array} \right\}_{n-\frac{1}{2}}^{m} (\cosh \eta) \exp(in\theta) \exp(im\phi) \tag{1}$$

Where:

$$\begin{aligned} 0 \le \eta < &\quad \infty \\ -\pi < \theta \le &\quad \pi \\ 0 \le \phi < &\quad 2\pi \\ m, n = 0, 1, \ldots& \end{aligned}$$

2 Decomposition

Given an arbitrary scalar function $f = f(\boldsymbol{r})$, how do we write it as in terms of toroidal harmonics?

$$f(\boldsymbol{r}) = \sum_{n'm'} a_{n'm'}^{(1)} \Psi_{n'm'}^{(1)}(\boldsymbol{r}) + \sum_{n''m''} a_{n''m''}^{(2)} \Psi_{n''m''}^{(2)}(\boldsymbol{r})$$

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \exp(-im\phi) f(\boldsymbol{r}(\phi)) = \sqrt{\cosh \eta - \cos \theta} \sum_{n'} \exp(in'\theta) \left(a_{n'm}^{(1)} P_{n'-\frac{1}{2}}^{m} (\cosh \eta) + a_{n'm}^{(2)} Q_{n'-\frac{1}{2}}^{m} (\cosh \eta) \right)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta \exp(-in\theta)}{\sqrt{\cosh \eta - \cos \theta}} \cdot \frac{1}{2\pi} \int_0^{2\pi} d\phi \exp(-im\phi) f(\boldsymbol{r}(\phi)) = a_{nm}^{(1)} P_{n-\frac{1}{2}}^{m} (\cosh \eta) + a_{nm}^{(2)} Q_{n-\frac{1}{2}}^{m} (\cosh \eta)$$

$$\frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} d\theta \int_0^{2\pi} d\phi \, \frac{\exp(-i(n\theta + m\phi))}{\sqrt{\cosh \eta - \cos \theta}} \cdot f(\eta,\, \theta,\, \phi) = a_{nm}^{(1)} P_{n-\frac{1}{2}}^{m} (\cosh \eta) + a_{nm}^{(2)} Q_{n-\frac{1}{2}}^{m} (\cosh \eta)$$

Define:

$$\bar{f} = \frac{f(\eta,\, \theta,\, \phi)}{\sqrt{\cosh \eta - \cos \theta}}$$

Now let $\phi_s = \frac{2\pi}{M} \cdot s$, for $s = 0 \ldots M-1$ and $\theta_r = \frac{2\pi}{N} \cdot r - \pi$, for $r = 0 \ldots N-1$. One question I have is whether

nm coefficients can be only non-negative. Consider:

$$\begin{aligned}\cos n\theta_r =& \frac{1}{2}\left(\exp\left(in\theta_r\right)+\exp\left(-in\theta_r\right)\right)\\ =& \frac{1}{2}\left(\exp\left(in\left[\frac{2\pi}{N}\cdot r-\pi\right]\right)+\exp\left(-in\left[\frac{2\pi}{N}\cdot r-\pi\right]\right)\right)\\ =& \frac{(-1)^n}{2}\left(\exp\left(i\frac{2\pi}{N}\cdot r\cdot n\right)+\exp\left(-i\frac{2\pi}{N}\cdot r\cdot n\right)\right)\\ =& \frac{(-1)^n}{2}\left(\exp\left(i\frac{2\pi}{N}\cdot r\cdot n\right)+\exp\left(i\frac{2\pi}{N}\cdot (N-r)\cdot n\right)\right)\end{aligned}$$

and similarly for ϕ, thus on a finite domain it is sufficent to consider just positive indices: $n, m = 0, 1, \ldots$. Moving on, and converting the integrals to sums:

$$\begin{aligned}a^{(1)}_{nm}P^m_{n-\frac{1}{2}}\left(\cosh\eta\right)+a^{(2)}_{nm}Q^m_{n-\frac{1}{2}}\left(\cosh\eta\right) =& \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}d\theta\int_0^{2\pi}d\phi\,\exp\left(-i\left(n\theta+m\phi\right)\right)\cdot\bar{f}\left(\theta,\,\phi;\,\eta\right)\\ =& \frac{1}{2\pi}\sum_r^{N-1}\frac{2\pi}{N}\cdot\frac{1}{2\pi}\sum_s^{M-1}\frac{2\pi}{M}\cdot\exp\left(-i2\pi\left(\frac{n\cdot r}{N}+\frac{m\cdot s}{M}\right)\right)\cdot\exp\left(-in\left(-\pi\right)\right)\cdot\bar{f}_{r,s}\left(\eta\right)\\ =& \frac{(-1)^n}{N\cdot M}\cdot\sum_{r=0}^{N-1}\sum_{s=0}^{M-1}\exp\left(-i2\pi\left(\frac{n\cdot r}{N}+\frac{m\cdot s}{M}\right)\right)\cdot\bar{f}_{r,s}\left(\eta\right)\end{aligned}$$

This is the 2D DFT of the $\bar{f}_{r,s}\left(\eta\right)=\bar{f}\left(\theta_r,\,\phi_s;\,\eta\right)$:

$$a^{(1)}_{nm}P^m_{n-\frac{1}{2}}\left(\cosh\eta\right)+a^{(2)}_{nm}Q^m_{n-\frac{1}{2}}\left(\cosh\eta\right)=\frac{(-1)^n}{N\cdot M}\cdot DFT\left(r\to n, N, s\to m, M\right)\left\{\bar{f}_{r,s}\left(\eta\right)\right\}$$

So need few steps of η with an good grid of θ, ϕ for each step. Then use DFT to get the nm coefficients.

Assume we did this. So now for each nm we have several values of $\eta=\eta_1\ldots\eta_K$ and we need to get $a^{(1,2)}$. One can write it as:

$$\left(\begin{array}{cc}P_1 & Q_1\\ P_2 & Q_2\\ \vdots & \vdots\\ P_K & Q_K\end{array}\right)\left(\begin{array}{c}a^{(1)}\\ a^{(2)}\end{array}\right)=\left(\begin{array}{c}V_1\\ V_2\\ \vdots\\ V_K\end{array}\right)$$

$$\boldsymbol{A}\left(\begin{array}{c}a^{(1)}\\ a^{(2)}\end{array}\right)=\boldsymbol{V}$$

One should assume that $\boldsymbol{P}$ and $\boldsymbol{Q}$ are linearly independent. $\left(\boldsymbol{P}\,\boldsymbol{Q}\right)\left(\begin{array}{c}a^{(1)}\\ a^{(2)}\end{array}\right)$ is a linear combination of these two vectors. The best thing we can do is to project vector $\boldsymbol{V}$ into the subspace spanned by $\boldsymbol{P}$ and $\boldsymbol{Q}$. Invariably, this will involve multyplying by $\boldsymbol{A}^\dagger=\left(\begin{array}{c}\boldsymbol{P}^\dagger\\ \boldsymbol{Q}^\dagger\end{array}\right)$. Let[1]

$$\boldsymbol{H}=\boldsymbol{A}^\dagger\boldsymbol{A}=\left(\begin{array}{cc}\boldsymbol{P}^\dagger.\boldsymbol{P} & \boldsymbol{P}^\dagger.\boldsymbol{Q}\\ \boldsymbol{Q}^\dagger.\boldsymbol{P} & \boldsymbol{Q}^\dagger.\boldsymbol{Q}\end{array}\right)=\left(\begin{array}{cc}d_1 & h\\ h^\dagger & d_2\end{array}\right)=\left(\begin{array}{cc}d_1 & h\\ h & d_2\end{array}\right),\quad \boldsymbol{W}=\boldsymbol{A}^\dagger\boldsymbol{V}=\left(\begin{array}{c}\boldsymbol{P}^\dagger\boldsymbol{V}\\ \boldsymbol{Q}^\dagger\boldsymbol{V}\end{array}\right)=\left(\begin{array}{c}W_1\\ W_2\end{array}\right)$$

Then:

$$\boldsymbol{H}\left(\begin{array}{c}a^{(1)}\\ a^{(2)}\end{array}\right)=\boldsymbol{W}$$

[1] Bear in mind that $\boldsymbol{P}$ and $\boldsymbol{Q}$ are real-valued

Which we can solve for $a^{(1,2)}$.

$$\begin{aligned}
\begin{pmatrix} a^{(1)} \\ a^{(2)} \end{pmatrix} \approx& \frac{1}{d_1 d_2 - h^2} \begin{pmatrix} d_2 & -h \\ -h & d_1 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \\
a^{(1)} \approx& \frac{d_2 W_1 - h W_2}{d_1 d_2 - h^2} \\
a^{(2)} \approx& \frac{d_1 W_2 - h W_1}{d_1 d_2 - h^2}
\end{aligned}$$

References

[1] K. V. Garapati, M. Salhi, S. Kouchekian, G. Siopsis, and A. Passian, "Poloidal and toroidal plasmons and fields of multilayer nanorings," *Phys. Rev. B*, vol. 95, p. 165422, 2017.

[2] K. V. Garapati,M. Bagherian,A. Passian, and S. Kouchekian, "Plasmon dispersion in a multilayer solid torus in terms of three-term vector recurrence relations and matrix continued fractions," *J. Phys. Commun.*, vol. 2, p. 015031, 2018.