Toroidal Harmonics Definition for the DARPA project

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1 Defnition of toroidal Harmonics

Several work from plasma physics consider toroidal harmonics. Works by Garapati *et al.* [1, 2] suggest defining the general solution to the Laplace equations as:

$$\Phi = f(\mu, \eta) \sum_{n,m=-\infty}^{\infty} \left[C_{mn} P^{m}_{n-\frac{1}{2}}(\cosh \mu) + D_{mn} Q^{m}_{n-\frac{1}{2}}(\cosh \mu) \right] \exp(in\theta) \exp(im\phi)$$

$$f(\mu, \eta) = \sqrt{\cosh \mu - \cos \eta}$$

1.1 Sign of n, m

From DLMF 14.9.3 [3]:

$$\left\{ \begin{array}{c} P \\ Q \end{array} \right\}_{\nu}^{-m} = (-1)^m \frac{\Gamma(\nu - m + 1)}{\Gamma(\nu + m + 1)} \left\{ \begin{array}{c} P \\ Q \end{array} \right\}_{\nu}^{m}$$

Note that here $\nu \in \mathbb{R}$, so it applies for our case. Mathematica confirms this for our functions. It follows that for m and $-m$ the $P\&Q$ are essentially equal up to standard normalization factor. In our case:

$$\left\{ \begin{array}{c} P \\ Q \end{array} \right\}_{n-\frac{1}{2}}^{-m} = (-1)^m \frac{\Gamma\left(n - \frac{1}{2} - m + 1\right)}{\Gamma\left(n - \frac{1}{2} + m + 1\right)} \left\{ \begin{array}{c} P \\ Q \end{array} \right\}_{n-\frac{1}{2}}^{m} = (-1)^m \frac{\Gamma\left(n - m + \frac{1}{2}\right)}{\Gamma\left(n + m + \frac{1}{2}\right)} \left\{ \begin{array}{c} P \\ Q \end{array} \right\}_{n-\frac{1}{2}}^{m} \tag{1}$$

From DLMF 14.9.5 [3]:

$$P^{\mu}_{-\nu-1} = P^{\mu}_{\nu}$$

It follows that:

$$P^{m}_{-n-\frac{1}{2}} = P^{m}_{-\left(n-\frac{1}{2}\right)-1} = P^{m}_{n-\frac{1}{2}}$$

Mathematica agrees.
For Q we have DLMF 14.9.6:

$$\pi \cos(\nu\pi) \cos(\mu\pi) P^{\mu}_{\nu} = \sin((\nu + \mu)\pi) Q^{\mu}_{\nu} - \sin((\nu - \mu)\pi) Q^{\mu}_{-\nu-1}$$

$$\pi \cos\left(\left(n - \frac{1}{2}\right)\pi\right) \cos(m\pi) P^{m}_{n-\frac{1}{2}} = \sin\left(\left(n + m - \frac{1}{2}\right)\pi\right) Q^{m}_{n-\frac{1}{2}} - \sin\left(\left(n - m - \frac{1}{2}\right)\pi\right) Q^{m}_{-n-\frac{1}{2}}$$

$$0 = (-1)^{n+m-1} Q^{m}_{n-\frac{1}{2}} - (-1)^{n-m-1} Q^{m}_{-n-\frac{1}{2}}$$

$$Q^{m}_{-n-\frac{1}{2}} = (-1)^{2m} Q^{m}_{n-\frac{1}{2}} = Q^{m}_{n-\frac{1}{2}}$$

Again, Mathematica agrees. So:

$$\left\{ \begin{array}{c} P \\ Q \end{array} \right\}_{-n-\frac{1}{2}}^{m} = \left\{ \begin{array}{c} P \\ Q \end{array} \right\}_{n-\frac{1}{2}}^{m} \tag{2}$$

Note also that by definition, $P\&Q$ are solutions of:

$$\left(1 - x^2\right) \frac{d^2 P}{dx^2} - 2x \frac{dP}{dx} + \nu(\nu + 1) P = 0$$

Which in the case of $\nu = n - \frac{1}{2}$ becomes:

$$\left(1 - x^2\right) \frac{d^2 P^m_{n-\frac{1}{2}}}{dx^2} - 2x \frac{dP^m_{n-\frac{1}{2}}}{dx} + \left(n^2 + \frac{1}{4}\right) P^m_{n-\frac{1}{2}} = 0$$

Thus the sign of n is arbitrary, it does not change the equation.

1.2 Definition

It makes sense to define toroidal harmonics as:

$$\Psi^{(1,2)}_{nm}\left(\eta,\, \theta,\, \phi\right) = \left\{ \begin{array}{c} \Psi \\ \Phi \end{array} \right\}_{nm} \left(\eta,\, \theta,\, \phi\right) = \sqrt{\cosh\eta - \cos\theta} \left\{ \begin{array}{c} P \\ Q \end{array} \right\}^m_{n-\frac{1}{2}} \left(\cosh\eta\right) \exp\left(in\theta\right) \exp\left(im\phi\right) \tag{3}$$

Where:

$$\begin{aligned} 0 \le \eta < & \quad \infty \\ -\pi < \theta \le & \quad \pi \\ 0 \le \phi < & \quad 2\pi \\ m, n = 0, \pm 1&, \pm 2, \ldots \end{aligned}$$

Using Eq. (1,2) to change sign.

2 Decomposition of a scalar function

Given an arbitrary scalar function $f = f\left(\boldsymbol{r}\right)$, how do we write it as in terms of toroidal harmonics?

$$f\left(\boldsymbol{r}\right) = \sum_{n'm'} a^{(1)}_{n'm'} \Psi^{(1)}_{n'm'}\left(\boldsymbol{r}\right) + \sum_{n''m''} a^{(2)}_{n''m''} \Psi^{(2)}_{n''m''}\left(\boldsymbol{r}\right)$$

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \exp\left(-im\phi\right) f\left(\boldsymbol{r}\left(\phi\right)\right) = \sqrt{\cosh\eta - \cos\theta} \sum_{n'} \exp\left(in'\theta\right) \left(a^{(1)}_{n'm} P^m_{n'-\frac{1}{2}}\left(\cosh\eta\right) + a^{(2)}_{n'm} Q^m_{n'-\frac{1}{2}}\left(\cosh\eta\right)\right)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta \exp\left(-in\theta\right)}{\sqrt{\cosh\eta - \cos\theta}} \cdot \frac{1}{2\pi} \int_0^{2\pi} d\phi \exp\left(-im\phi\right) f\left(\boldsymbol{r}\left(\phi\right)\right) = a^{(1)}_{nm} P^m_{n-\frac{1}{2}}\left(\cosh\eta\right) + a^{(2)}_{nm} Q^m_{n-\frac{1}{2}}\left(\cosh\eta\right)$$

$$\frac{1}{\left(2\pi\right)^2} \int_{-\pi}^{\pi} d\theta \int_0^{2\pi} d\phi \frac{\exp\left(-i\left(n\theta + m\phi\right)\right)}{\sqrt{\cosh\eta - \cos\theta}} \cdot f\left(\eta,\, \theta,\, \phi\right) = a^{(1)}_{nm} P^m_{n-\frac{1}{2}}\left(\cosh\eta\right) + a^{(2)}_{nm} Q^m_{n-\frac{1}{2}}\left(\cosh\eta\right)$$

Define:

$$\bar{f} = \frac{f\left(\eta,\, \theta,\, \phi\right)}{\sqrt{\cosh\eta - \cos\theta}}$$

Now let $\phi_s = \frac{2\pi}{M} \cdot s$, for $s = 0 \ldots M-1$ and $\theta_r = \frac{2\pi}{N} \cdot r - \pi$, for $r = 0 \ldots N-1$. One question I have is whether nm coefficients can be only non-negative. As one can see from Eq. (1,2) the values of $P\&Q$ for negative parameters are not linearly independent. So it makes no sense to treat them as additional degrees of freedom. I therefore suggest to compute P and Q as follows:

$$P^m_{n-\frac{1}{2}} = \begin{cases} P^m_{n-\frac{1}{2}}, & \text{for } m \le \left\lfloor \frac{M}{2} \right\rfloor,\ n \le \left\lfloor \frac{N}{2} \right\rfloor \\ P^m_{(N-n)-\frac{1}{2}} & \text{for } m \le \left\lfloor \frac{M}{2} \right\rfloor,\ n > \left\lfloor \frac{N}{2} \right\rfloor \\ \left(-1\right)^{(M-m)} \frac{\Gamma\left(n-(M-m)+\frac{1}{2}\right)}{\Gamma\left(n+(M-m)+\frac{1}{2}\right)} P^{M-m}_{n-\frac{1}{2}}, & \text{for } m > \left\lfloor \frac{M}{2} \right\rfloor,\ n \le \left\lfloor \frac{N}{2} \right\rfloor \\ \left(-1\right)^{(M-m)} \frac{\Gamma\left(n-(M-m)+\frac{1}{2}\right)}{\Gamma\left(n+(M-m)+\frac{1}{2}\right)} P^{M-m}_{(N-n)-\frac{1}{2}}, & \text{for } m > \left\lfloor \frac{M}{2} \right\rfloor,\ n > \left\lfloor \frac{N}{2} \right\rfloor \end{cases}$$

With this I can restrict myself to just the positive values of $n, m = 0,\, 1,\, \ldots$. The way it will be handeled in code is by definition of an additional folding function.

Moving on, and converting the integrals to sums:

$$
\begin{aligned}
a_{nm}^{(1)}P_{n-\frac{1}{2}}^{m}\left(\cosh\eta\right)+a_{nm}^{(2)}Q_{n-\frac{1}{2}}^{m}\left(\cosh\eta\right)=&\frac{1}{\left(2\pi\right)^{2}}\int_{-\pi}^{\pi}d\theta\int_{0}^{2\pi}d\phi\,\exp\left(-i\left(n\theta+m\phi\right)\right)\cdot\bar{f}\left(\theta,\,\phi;\,\eta\right)\\
=&\frac{1}{2\pi}\sum_{r}^{N-1}\frac{2\pi}{N}\cdot\frac{1}{2\pi}\sum_{s}^{M-1}\frac{2\pi}{M}\cdot\exp\left(-i2\pi\left(\frac{n\cdot r}{N}+\frac{m\cdot s}{M}\right)\right)\cdot\exp\left(-in\left(-\pi\right)\right)\cdot\bar{f}_{r,s}\left(\eta\right)\\
=&\frac{\left(-1\right)^{n}}{N\cdot M}\cdot\sum_{r=0}^{N-1}\sum_{s=0}^{M-1}\exp\left(-i2\pi\left(\frac{n\cdot r}{N}+\frac{m\cdot s}{M}\right)\right)\cdot\bar{f}_{r,s}\left(\eta\right)
\end{aligned}
$$

This is the 2D DFT of the $\bar{f}_{r,s}\left(\eta\right)=\bar{f}\left(\theta_{r},\,\phi_{s};\,\eta\right)$:

$$
a_{nm}^{(1)}P_{n-\frac{1}{2}}^{m}\left(\cosh\eta\right)+a_{nm}^{(2)}Q_{n-\frac{1}{2}}^{m}\left(\cosh\eta\right)=\frac{\left(-1\right)^{n}}{N\cdot M}\cdot DFT\left(r\to n,N,s\to m,M\right)\left\{\bar{f}_{r,s}\left(\eta\right)\right\}
$$

So need few steps of η with an good grid of θ, ϕ for each step. Then use DFT to get the nm coefficients.

Assume we did this. So now for each nm we have several values of $\eta=\eta_{1}\ldots\eta_{K}$ and we need to get $a^{(1,2)}$. One can write it as:

$$
\left(\begin{array}{cc}
P_{1} & Q_{1}\\
P_{2} & Q_{2}\\
\vdots & \vdots\\
P_{K} & Q_{K}
\end{array}\right)\left(\begin{array}{c}
a^{(1)}\\
a^{(2)}
\end{array}\right)=\left(\begin{array}{c}
V_{1}\\
V_{2}\\
\vdots\\
V_{K}
\end{array}\right)
$$

$$
\boldsymbol{A}\left(\begin{array}{c}
a^{(1)}\\
a^{(2)}
\end{array}\right)=\boldsymbol{V}
$$

One should assume that $\boldsymbol{P}$ and $\boldsymbol{Q}$ are linearly independent. $\left(\boldsymbol{P}\,\boldsymbol{Q}\right)\left(\begin{array}{c}a^{(1)}\\a^{(2)}\end{array}\right)$ is a linear combination of these two vectors. The best thing we can do is to project vector $\boldsymbol{V}$ into the subspace spanned by $\boldsymbol{P}$ and $\boldsymbol{Q}$. Invariably, this will involve multyplying by $\boldsymbol{A}^{\dagger}=\left(\begin{array}{c}\boldsymbol{P}^{\dagger}\\\boldsymbol{Q}^{\dagger}\end{array}\right)$. Let[1]

$$
\boldsymbol{H}=\boldsymbol{A}^{\dagger}\boldsymbol{A}=\left(\begin{array}{cc}
\boldsymbol{P}^{\dagger}.\boldsymbol{P} & \boldsymbol{P}^{\dagger}.\boldsymbol{Q}\\
\boldsymbol{Q}^{\dagger}.\boldsymbol{P} & \boldsymbol{Q}^{\dagger}.\boldsymbol{Q}
\end{array}\right)=\left(\begin{array}{cc}
d_{1} & h\\
h^{\dagger} & d_{2}
\end{array}\right)=\left(\begin{array}{cc}
d_{1} & h\\
h & d_{2}
\end{array}\right),\quad\boldsymbol{W}=\boldsymbol{A}^{\dagger}\boldsymbol{V}=\left(\begin{array}{c}
\boldsymbol{P}^{\dagger}\boldsymbol{V}\\
\boldsymbol{Q}^{\dagger}\boldsymbol{V}
\end{array}\right)=\left(\begin{array}{c}
W_{1}\\
W_{2}
\end{array}\right)
$$

Then:

$$
\boldsymbol{H}\left(\begin{array}{c}
a^{(1)}\\
a^{(2)}
\end{array}\right)=\boldsymbol{W}
$$

Which we can solve for $a^{(1,2)}$.

$$
\begin{aligned}
\left(\begin{array}{c}
a^{(1)}\\
a^{(2)}
\end{array}\right)\approx&\frac{1}{d_{1}d_{2}-h^{2}}\left(\begin{array}{cc}
d_{2} & -h\\
-h & d_{1}
\end{array}\right)\left(\begin{array}{c}
W_{1}\\
W_{2}
\end{array}\right)\\
a^{(1)}\approx&\frac{d_{2}W_{1}-hW_{2}}{d_{1}d_{2}-h^{2}}\\
a^{(2)}\approx&\frac{d_{1}W_{2}-hW_{1}}{d_{1}d_{2}-h^{2}}
\end{aligned}
$$

3 Decomposition of the vector field

3.1 Basic arguments

Any sensible 3d vector field can be decomposed into[2]:

$$
\boldsymbol{F}=\boldsymbol{\nabla}\phi+\boldsymbol{L}\psi+\boldsymbol{\nabla}\times\boldsymbol{L}\zeta
$$

[1] Bear in mind that $\boldsymbol{P}$ and $\boldsymbol{Q}$ are real-valued

[2] See 'HelmholtzDecomposition.pdf' in scanned notes

where ϕ, ψ, ζ are scalar functions and $\boldsymbol{L} = -i\boldsymbol{r} \times \boldsymbol{\nabla}$ is the angular momentum operator. Key to this decomposition is that the first part is that two latter terms have no divergence, so:

$$\boldsymbol{\nabla}.\boldsymbol{F} = \nabla^2 \phi$$

Now, the problem with simply decomposing ϕ in terms of toroidal harmonics is that all toroidal harmonics are solutions to Laplace's Equation, so the devergence would still vanish. A reasonable way to overcome it, IMHO, is to create longitudinal component with a radius:

$$\boldsymbol{F} = \boldsymbol{r}\phi + \boldsymbol{L}\psi + \boldsymbol{\nabla} \times \boldsymbol{L}\zeta$$

This set should provide enough linear independence for the generic decomposition. So the decomposition we are looking for is (with Einstein summation convention):

$$\boldsymbol{F} = a_{nm}^{(\lambda)} \cdot \boldsymbol{r}\Psi_{nm}^{(\lambda)}(\boldsymbol{r}) + b_{nm}^{(\lambda)} \cdot \boldsymbol{L}\Psi_{nm}^{(\lambda)}(\boldsymbol{r}) + c_{nm}^{(\lambda)} \cdot \boldsymbol{\nabla} \times \boldsymbol{L}\Psi_{nm}^{(\lambda)}(\boldsymbol{r})$$

Then[3]:

$$\begin{aligned}
\boldsymbol{\nabla}.\boldsymbol{F} =& a_{nm}^{(\lambda)} \cdot (3 + \boldsymbol{r}.\boldsymbol{\nabla})\, \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) \\
\boldsymbol{r}.\boldsymbol{F} =& a_{nm}^{(\lambda)} \cdot r^2 \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) + c_{nm}^{(\lambda)} \cdot \boldsymbol{r}.\boldsymbol{\nabla} \times \boldsymbol{L}\Psi_{nm}^{(\lambda)}(\boldsymbol{r}) = \\
=& a_{nm}^{(\lambda)} \cdot r^2 \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) + c_{nm}^{(\lambda)} \cdot i\left(2\boldsymbol{r}.\boldsymbol{\nabla} + r^\alpha r^\beta \partial_{\alpha\beta} - r^2 \nabla^2\right) \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) \\
=& a_{nm}^{(\lambda)} \cdot r^2 \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) + c_{nm}^{(\lambda)} \cdot i\left(2\boldsymbol{r}.\boldsymbol{\nabla} + r^\alpha r^\beta \partial_{\alpha\beta}\right) \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) \\
\boldsymbol{L}.\boldsymbol{F} =& b_{nm}^{(\lambda)} \cdot L^2 \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) = b_{nm}^{(\lambda)} \cdot \left(\partial_r \left(r^2 \partial_r \ldots\right) - r^2 \nabla^2\right) \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) \\
=& b_{nm}^{(\lambda)} \cdot \partial_r \left(r^2 \partial_r \Psi_{nm}^{(\lambda)}(\boldsymbol{r})\right) = b_{nm}^{(\lambda)} \cdot \left(2r\partial_r \Psi_{nm}^{(\lambda)} + r^2 \partial_{rr} \Psi_{nm}^{(\lambda)}\right) \\
=& b_{nm}^{(\lambda)} \cdot (\hat{\boldsymbol{r}}.\boldsymbol{\nabla}) \left(r \cdot (\boldsymbol{r}.\boldsymbol{\nabla})\, \Psi_{nm}^{(\lambda)}(\boldsymbol{r})\right)
\end{aligned}$$

This is, in principle enough for the decomposition. So what I need is:

$$(3 + \boldsymbol{r}.\boldsymbol{\nabla})\, \Psi_{nm}^{(\lambda)} = A_{nm,\bar{n}\bar{m}}^{(\lambda,\bar{\lambda})} \Psi_{\bar{n}\bar{m}}^{(\bar{\lambda})} \tag{4}$$

$$r^2 \Psi_{nm}^{(\lambda)} = B_{nm,\bar{n}\bar{m}}^{(\lambda,\bar{\lambda})} \Psi_{\bar{n}\bar{m}}^{(\bar{\lambda})} \tag{5}$$

$$(2\boldsymbol{r}.\boldsymbol{\nabla} + r_\alpha r_\beta \partial_{\alpha\beta})\, \Psi_{nm}^{(\lambda)} = C_{nm,\bar{n}\bar{m}}^{(\lambda,\bar{\lambda})} \Psi_{\bar{n}\bar{m}}^{(\bar{\lambda})} \tag{6}$$

$$\hat{\boldsymbol{r}}.\boldsymbol{\nabla} \left(r\, \boldsymbol{r}.\boldsymbol{\nabla}\Psi_{nm}^{(\lambda)}\right) = D_{nm,\bar{n}\bar{m}}^{(\lambda,\bar{\lambda})} \Psi_{\bar{n}\bar{m}}^{(\bar{\lambda})} \tag{7}$$

3.2 Covariant treatment

The problem with the previous section is that I will need to work exclusively in toroidal coordinates, and I feel slightly worried about some of the expressions above. The stuff on the left-hand side is coordinate independent, so it is ok. The right-handside is coordinate dependent. Something could be missed, so let us re-do it explicitly covariantly. Assuming flat Eucledian space, there exitst the Cartesian coordinates S and some other coordinate system $\bar{S}$.

3.2.1 Position vector

We will need to use the position vector. It can be defined as:

$$\boldsymbol{r} = \hat{\boldsymbol{e}}_\alpha r^\alpha = \hat{\boldsymbol{e}}_\alpha g^{\alpha\beta} \nabla_\beta \left(\frac{r^2}{2}\right)$$

In Cartesian coordinates. This definition is then readily convertable to other coordinate systems:

$$\begin{aligned}
\bar{r}^\alpha =& \frac{\partial \bar{x}^\alpha}{\partial x^\beta} \cdot r^\beta = \frac{\partial \bar{x}^\alpha}{\partial x^\beta} \cdot g^{\alpha\beta} \nabla_\beta \left(\frac{r^2}{2}\right) \\
=& \bar{g}^{\alpha\beta} \bar{\nabla}_\beta \left(\frac{r^2}{2}\right) = \bar{g}^{\alpha\beta} \bar{\partial}_\beta \left(\frac{r^2}{2}\right) \\
=& r \cdot \bar{g}^{\alpha\beta} \bar{\partial}_\beta r
\end{aligned} \tag{8}$$

[3]Bearing in mind that $\nabla^2 \Psi_{nm}^{(\lambda)} = 0$.

Where $r = r(\boldsymbol{r})$ is a position-dependent scalar, and so is $r^2 = r \cdot r$.

3.2.2 Term $\boldsymbol{\nabla}.\boldsymbol{F}$

We have

$$\boldsymbol{\nabla}.\boldsymbol{F} = \nabla_\alpha F^\alpha = \bar{\nabla}_\alpha \bar{F}^\alpha = \bar{\partial}_\alpha \bar{F}^\alpha + \bar{\Gamma}^\alpha_{\alpha\mu} \bar{F}^\mu$$

Using [4]:

$$\bar{\Gamma}^\alpha_{\alpha\mu} = \frac{\bar{\partial}_\mu \sqrt{\bar{g}}}{\sqrt{\bar{g}}}$$

where $\bar{g}$ is the determinant of the metric, we get:

$$\begin{aligned}\boldsymbol{\nabla}.\boldsymbol{F} =& \bar{\partial}_\alpha \bar{F}^\alpha + \bar{\Gamma}^\alpha_{\alpha\mu} \bar{F}^\mu = \bar{\partial}_\alpha \bar{F}^\alpha + \frac{1}{\sqrt{\bar{g}}} \bar{F}^\mu \bar{\partial}_\mu \sqrt{\bar{g}} \\ =& \frac{1}{\sqrt{\bar{g}}} \cdot \bar{\partial}_\alpha \left(\sqrt{\bar{g}} \cdot \bar{F}^\alpha \right)\end{aligned}$$

the curl and angular-momentum components will vanish irrespective of coordinate system, so we get (using Eq. (8)):

$$\begin{aligned}\boldsymbol{\nabla}.\boldsymbol{F} =& a^{(\lambda)}_{nm} \boldsymbol{\nabla}. \left(\boldsymbol{r} \Psi^{(\lambda)}_{nm} (\boldsymbol{r}) \right) = \frac{a^{(\lambda)}_{nm}}{\sqrt{\bar{g}}} \cdot \bar{\partial}_\alpha \left(\sqrt{\bar{g}} \cdot \bar{r}^\alpha \Psi^{(\lambda)}_{nm} \right) \\ =& \frac{a^{(\lambda)}_{nm}}{\sqrt{\bar{g}}} \cdot \bar{\partial}_\alpha \left(\Psi^{(\lambda)}_{nm} \cdot \bar{g}^{\alpha\beta} \sqrt{\bar{g}} \bar{\partial}_\beta \left(\frac{r^2}{2} \right) \right) \\ \boldsymbol{\nabla}.\boldsymbol{F} =& a^{(\lambda)}_{nm} \cdot \bar{g}^{\alpha\beta} \bar{\partial}_\beta \left(\frac{r^2}{2} \right) \cdot \left(\bar{\partial}_\alpha \Psi^{(\lambda)}_{nm} \right) + a^{(\lambda)}_{nm} \cdot \frac{1}{\sqrt{\bar{g}}} \bar{\partial}_\alpha \left(\bar{g}^{\alpha\beta} \sqrt{\bar{g}} \bar{\partial}_\beta \left(\frac{r^2}{2} \right) \right) \cdot \Psi^{(\lambda)}_{nm} \end{aligned} \tag{9}$$

3.3 Toroidal coordinates and vector basis

3.3.1 Basics

To get the matricies from Eq. (4-7), I will probably need to find analytical expressions for the relevant left-hand sides, then numerically decompose them into right-hand side. It makes sense to start with the simplest one: $r^2 \Psi^{(\lambda)}_{nm}$. From Spencer [5] the toroidal coordinates in terms of Cartesian ones are:

$$\begin{aligned}x =& \frac{a \sinh \eta \cos \phi}{\cosh \eta - \cos \theta} \\ y =& \frac{a \sinh \eta \sin \phi}{\cosh \eta - \cos \theta} \\ z =& \frac{a \sin \theta}{\cosh \eta - \cos \theta}\end{aligned}$$

3.3.2 Metric and basis vectors

We have:

$$dx^2 + dy^2 + dz^2 = \frac{a^2}{(\cosh \eta - \cos \theta)^2} \cdot d\eta^2 + \frac{a^2}{(\cosh \eta - \cos \theta)^2} \cdot d\theta^2 + \frac{a^2 \sinh^2 \eta}{(\cosh \eta - \cos \theta)^2} \cdot d\phi^2$$

$$\boldsymbol{e}_\eta . \boldsymbol{e}_\eta = \frac{a^2}{(\cosh \eta - \cos \theta)^2}$$

$$\boldsymbol{e}_\theta . \boldsymbol{e}_\theta = \frac{a^2}{(\cosh \eta - \cos \theta)^2}$$

$$\boldsymbol{e}_\phi . \boldsymbol{e}_\phi = \frac{a^2 \sinh^2 \eta}{(\cosh \eta - \cos \theta)^2}$$

$$\boldsymbol{e}_\eta . \boldsymbol{e}_\theta = \boldsymbol{e}_\eta . \boldsymbol{e}_\phi = \boldsymbol{e}_\theta . \boldsymbol{e}_\phi = 0$$

$$d\boldsymbol{r} = d\eta \, \boldsymbol{e}_\eta + d\theta \, \boldsymbol{e}_\theta + d\phi \, \boldsymbol{e}_\phi$$

It is also convenient to define normalized basis:

$$\begin{aligned}\hat{\boldsymbol{\eta}} =& \frac{\cosh\eta - \cos\theta}{a} \cdot \boldsymbol{e}_\eta \\ \hat{\boldsymbol{\theta}} =& \frac{\cosh\eta - \cos\theta}{a} \cdot \boldsymbol{e}_\theta \\ \hat{\boldsymbol{\phi}} =& \frac{\cosh\eta - \cos\theta}{a\sinh\eta} \cdot \boldsymbol{e}_\phi\end{aligned}$$

So then:

$$d\boldsymbol{r} = \frac{a}{\cosh\eta - \cos\theta} d\eta\,\hat{\boldsymbol{\eta}} + \frac{a}{\cosh\eta - \cos\theta} d\theta\,\hat{\boldsymbol{\theta}} + \frac{a\sinh\eta}{\cosh\eta - \cos\theta} d\phi\,\hat{\boldsymbol{\phi}}$$

The metric is and related things are:

$$\bar{g}_{\alpha\beta} = \frac{a^2}{(\cosh\eta - \cos\theta)^2} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sinh^2\eta \end{pmatrix}_{\alpha\beta} \tag{10}$$

$$\bar{g} = \det(\bar{g}_{\alpha\beta}) = \frac{a^6\sinh^2\eta}{(\cosh\eta - \cos\theta)^6} \tag{11}$$

$$\sqrt{\bar{g}} = \frac{a^3\sinh\eta}{(\cosh\eta - \cos\theta)^3} \tag{12}$$

$$\bar{g}^{\alpha\beta} = \frac{(\cosh\eta - \cos\theta)^2}{a^2} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\sinh^2\eta} \end{pmatrix}^{\alpha\beta} \tag{13}$$

3.3.3 Chrisoffel symbols

It would seem I also need them. Definition:

$$\bar{\Gamma}^{\mu}_{\alpha\beta} = \frac{1}{2} \cdot \bar{g}^{\mu\kappa} \cdot \left(\bar{\partial}_\alpha \bar{g}_{\kappa\beta} + \bar{\partial}_\beta \bar{g}_{\alpha\kappa} - \bar{\partial}_\kappa \bar{g}_{\alpha\beta}\right)$$

I could not bring myself to work it out manually, so here is the Mathematica result:

$$\Gamma^1_{\ldots,\ldots} = \Gamma^\eta_{\ldots,\ldots} = \begin{pmatrix} \frac{\sinh\eta}{\cos\theta - \cosh\eta} & \frac{\sin\theta}{\cos\theta - \cosh\eta} & 0 \\ \frac{\sin\theta}{\cos\theta - \cosh\eta} & \frac{-\sinh\eta}{\cos\theta - \cosh\eta} & 0 \\ 0 & 0 & \frac{(1 - \cos\theta\cosh\eta)\sinh\eta}{\cos\theta - \cosh\eta} \end{pmatrix} \tag{14}$$

$$\Gamma^2_{\ldots,\ldots} = \Gamma^\theta_{\ldots,\ldots} = \begin{pmatrix} \frac{-\sin\theta}{\cos\theta - \cosh\eta} & \frac{\sinh\eta}{\cos\theta - \cosh\eta} & 0 \\ \frac{\sinh\eta}{\cos\theta - \cosh\eta} & \frac{\sin\theta}{\cos\theta - \cosh\eta} & 0 \\ 0 & 0 & \frac{-\sin\theta\sinh^2\eta}{\cos\theta - \cosh\eta} \end{pmatrix} \tag{15}$$

$$\Gamma^3_{\ldots,\ldots} = \Gamma^\phi_{\ldots,\ldots} = \begin{pmatrix} 0 & 0 & \frac{(\cos\theta\cosh\eta - 1)}{(\cos\theta - \cosh\eta)\sinh\eta} \\ 0 & 0 & \frac{\sin\theta}{\cos\theta - \cosh\eta} \\ \frac{(\cos\theta\cosh\eta - 1)}{(\cos\theta - \cosh\eta)\sinh\eta} & \frac{\sin\theta}{\cos\theta - \cosh\eta} & 0 \end{pmatrix} \tag{16}$$

3.3.4 Position vector

It follows that the radius squared is[4]:

$$r^2 = x^2 + y^2 + z^2 = a^2 \cdot \frac{\cosh\eta + \cos\theta}{\cosh\eta - \cos\theta} \tag{17}$$

[4]Note that Mathematica supports toroidal coordinates, and can compute the necessary associated Legendre functions.

I will also need the derivatives of r^2, which I will use primitive functions:

$$\begin{aligned}
r^2_{,\eta} =& \partial_\eta r^2 = -2a^2 \cdot \frac{\sinh\eta \cdot \cos\theta}{(\cosh\eta - \cos\theta)^2} \\
r^2_{,\theta} =& \partial_\theta r^2 = -2a^2 \cdot \frac{\cosh\eta \cdot \sin\theta}{(\cosh\eta - \cos\theta)^2} \\
r^2_{,\eta\eta} =& \partial_{\eta\eta} r^2 = a^2 \cdot \frac{(\cosh 2\eta + 2\cosh\eta\cos\theta - 3)\cdot\cos\theta}{(\cosh\eta - \cos\theta)^3} \\
r^2_{,\theta\theta} =& \partial_{\theta\theta} r^2 = 2a^2 \cdot \frac{\cosh\eta \cdot \left(\sin^2\theta + 1 - \cosh\eta\cos\theta\right)}{(\cosh\eta - \cos\theta)^3} \\
r^2_{,\eta\theta} =& \partial_{\eta\theta} r^2 = 2a^2 \cdot \frac{\sinh\eta \cdot (\cosh\eta + \cos\theta)\cdot\sin\theta}{(\cosh\eta - \cos\theta)^3}
\end{aligned}$$

Alternatively, I may go for actually keeping the radius vector in toroidal basis as a fundamental function. Using Eq. (17) and expressions above, and the metric from Eq. (13) :

$$\begin{aligned}
\bar{r}^\eta =& \frac{1}{2}\bar{g}^{\eta\eta} r^2_{,\eta} = \frac{1}{2} \cdot \frac{(\cosh\eta - \cos\theta)^2}{a^2} \cdot \left(-2a^2\right) \cdot \frac{\sinh\eta \cdot \cos\theta}{(\cosh\eta - \cos\theta)^2} \\
\bar{r}^\eta =& -\sinh\eta \cdot \cos\theta && (18) \\
\bar{r}^\theta =& \frac{1}{2}\bar{g}^{\theta\theta} r^2_{,\theta} = \frac{1}{2} \cdot \frac{(\cosh\eta - \cos\theta)^2}{a^2} \cdot \left(-2a^2\right) \cdot \frac{\cosh\eta \cdot \sin\theta}{(\cosh\eta - \cos\theta)^2} \\
\bar{r}^\theta =& -\cosh\eta \cdot \sin\theta && (19) \\
\bar{r}^\phi =& 0 && (20)
\end{aligned}$$

The position vector can be defined in normalized basis as (no sum implied):

$$\begin{aligned}
\boldsymbol{e}_\alpha . \boldsymbol{r} =& \sqrt{g_{\alpha\alpha}}\, r^\alpha \\
\boldsymbol{r} =& \left(-\frac{a\ \sinh\eta\ \cos\theta}{\cosh\eta - \cos\theta}\right)\hat{\boldsymbol{\eta}} + \left(-\frac{a\ \cosh\eta\ \sin\theta}{\cosh\eta - \cos\theta}\right)\hat{\boldsymbol{\theta}} && (21)
\end{aligned}$$

Also:

$$\bar{r}^\alpha_{,\alpha} = \bar{\partial}_\alpha \bar{r}^\alpha = \partial_\eta\left(-\sinh\eta \cdot \cos\theta\right) + \partial_\theta\left(-\cosh\eta \cdot \sin\theta\right) = -2\cosh\eta \cdot \cos\theta \tag{22}$$

It is also convenient to use the co-variant position vector:

$$\begin{aligned}
\bar{r}_\alpha =& \bar{g}_{\alpha\beta}\bar{r}^\alpha = \bar{\delta}^\eta_\alpha \bar{g}_{\eta\eta}\bar{r}^\eta + \bar{\delta}^\theta_\alpha \bar{g}_{\theta\theta}\bar{r}^\theta \\
\bar{r}_\eta =& \bar{g}_{\eta\eta}\bar{r}^\eta = \frac{a^2}{(\cosh\eta - \cos\theta)^2} \cdot (-\sinh\eta \cdot \cos\theta) = -\frac{a^2 \cdot \sinh\eta \cdot \cos\theta}{(\cosh\eta - \cos\theta)^2} && (23) \\
\bar{r}_\theta =& -\frac{a^2 \cdot \cosh\eta \cdot \sin\theta}{(\cosh\eta - \cos\theta)^2} && (24)
\end{aligned}$$

3.3.5 Term $\boldsymbol{\nabla} . \boldsymbol{F}$

Using[5] Eq. (9) and Eq. (18, 10, 12, 13):

$$\begin{aligned}
\boldsymbol{\nabla} . \boldsymbol{F} =& \boldsymbol{\nabla} . \left(\boldsymbol{r} a^{(\lambda)}_{nm} \Psi^{(\lambda)}_{nm}\right) = \frac{a^{(\lambda)}_{nm}}{\sqrt{\bar{g}}} \cdot \bar{\partial}_\alpha\left(\sqrt{\bar{g}} \cdot \bar{r}^\alpha \Psi^{(\lambda)}_{nm}\right) = \frac{a^{(\lambda)}_{nm}}{\sqrt{\bar{g}}} \cdot \left(\frac{\bar{g}_{,\alpha}}{2\sqrt{\bar{g}}}\,\bar{r}^\alpha \Psi^{(\lambda)}_{nm} + \sqrt{\bar{g}}\bar{r}^\alpha_{,\alpha}\Psi^{(\lambda)}_{nm} + \sqrt{\bar{g}}\bar{r}^\alpha \bar{\partial}_\alpha \Psi^{(\lambda)}_{nm}\right) \\
=& a^{(\lambda)}_{nm} \cdot \left[\left(\frac{\bar{g}_{,\alpha}\bar{r}^\alpha}{2\bar{g}} + \bar{r}^\alpha_{,\alpha}\right) \cdot \Psi^{(\lambda)}_{nm} + \bar{r}^\alpha \bar{\partial}_\alpha \Psi^{(\lambda)}_{nm}\right] \\
=& a^{(\lambda)}_{nm} \cdot \left[\left(\frac{\bar{g}_{,\alpha}\bar{r}^\alpha}{2\bar{g}} + \bar{r}^\alpha_{,\alpha}\right) \cdot \Psi^{(\lambda)}_{nm} - \sinh\eta \cdot \cos\theta \cdot \partial_\eta \Psi^{(\lambda)}_{nm} - \cosh\eta \cdot \sin\theta \cdot \partial_\theta \Psi^{(\lambda)}_{nm}\right]
\end{aligned}$$

[5] As soon as a replace the generic derivatives ($\bar{\partial}_{\dots}$) with actual ones in toroidal coordinates ($\partial_{\eta,\,\theta,\,\phi}$), I will loose the bar to avoid notation overheads.

Using Eq. (22) and Eq. (11) and some simplification we find:

$$\boldsymbol{\nabla}.\boldsymbol{F} = a_{nm}^{(\lambda)} \cdot \left(3 \cdot \Psi_{nm}^{(\lambda)} - \sinh\eta \cdot \cos\theta \cdot \partial_\eta \Psi_{nm}^{(\lambda)} - \cosh\eta \cdot \sin\theta \cdot \partial_\theta \Psi_{nm}^{(\lambda)}\right) \tag{25}$$

which agrees well with Eq. (4).

3.3.6 Angular Momentum Operator

The best way of handling the remaining terms is to let the Mathematica do the dirty work, but we need to define the angular momentum operator. For the case of angular momentum mapping a scalar function into a 3d vector field, we can generalize the standard definition of the angular momentum as:

$$(\boldsymbol{L}f)^\alpha = -i\,\frac{\epsilon^{\alpha\beta\gamma}}{\sqrt{g}}\,r_\beta \partial_\gamma f$$

Note that the above expression is for the natural basis, Mathematica operates with normalized basis vectors, so if you want to use the agular momentum in other calculations (e.g. taking a curl of it), you need (no summation over α):

$$(\boldsymbol{L}f)^\alpha_{norm} = (\boldsymbol{L}f)^\alpha\,\sqrt{g_{\alpha\alpha}} = -i\sqrt{\frac{g_{\alpha\alpha}}{g}}\epsilon^{\alpha\beta\gamma}\,r_\beta\partial_\gamma f \tag{26}$$

The explicit expression for the angular momentum is simpler for the normalized basis, so I will quote this one:

$$(\boldsymbol{L}f)_{norm} = -i\left\{\left(-\frac{\sin\theta}{\tanh\eta}\,\partial_\phi f\right)\,\hat{\boldsymbol{\eta}} + (\cos\theta\,\partial_\phi f)\,\hat{\boldsymbol{\theta}} + (\cosh\eta\,\sin\theta\,\partial_\eta f - \sinh\eta\,\cos\theta\,\partial_\theta f)\,\hat{\boldsymbol{\phi}}\right\} \tag{27}$$

3.3.7 Term $\boldsymbol{r}.\boldsymbol{F}$

Working from:

$$\begin{aligned}\boldsymbol{r}.\boldsymbol{F} =& a_{nm}^{(\lambda)} \cdot r^2 \Psi_{nm}^{(\lambda)}(\boldsymbol{r}) + c_{nm}^{(\lambda)} \cdot \boldsymbol{r}.\boldsymbol{\nabla}\times\boldsymbol{L}\Psi_{nm}^{(\lambda)}(\boldsymbol{r})\\ \left(\boldsymbol{r}.\boldsymbol{F} - a_{nm}^{(\lambda)}\cdot r^2\Psi_{nm}^{(\lambda)}\right) =& c_{nm}^{(\lambda)}\cdot\boldsymbol{r}.\boldsymbol{\nabla}\times\boldsymbol{L}\Psi_{nm}^{(\lambda)}\end{aligned}$$

I can use the definition of the position vector in normalized basis from Eq. (21) and the definition of the normalized angular momentum of a function from Eq. (27), as well as Mathematica simplification to get[6]:

$$\begin{aligned}\boldsymbol{r}.\boldsymbol{\nabla}\times\boldsymbol{L}\Psi_{nm}^{(\lambda)} =& \frac{i\sinh^2\eta}{8}\Bigg\{\\ &+\left(\frac{-4\cosh\eta}{\sinh^3\eta} + \frac{\sinh 4\eta\,\cos 2\theta}{\sinh^4\eta}\right)\partial_\eta\Psi_{nm}^{(\lambda)} + \left(\frac{8\,\sin 2\theta}{\tanh^2\eta}\right)\partial_\theta\Psi_{nm}^{(\lambda)} +\\ &+\left(\frac{8\,\sin 2\theta}{\tanh\eta}\right)\partial_{\eta\theta}\Psi_{nm}^{(\lambda)} + \left(-\frac{8\,\sin^2\theta}{\tanh^2\eta}\right)\partial_{\eta\eta}\Psi_{nm}^{(\lambda)} +\\ &+\left(-8\cos^2\theta\right)\partial_{\theta\theta}\Psi_{nm}^{(\lambda)} + \frac{4}{\sinh^4\eta}\,(\cos 2\theta - \cosh 2\eta)\,(im)^2\,\Psi_{nm}^{(\lambda)}\\ &\Bigg\}\end{aligned} \tag{28}$$

The equation to solve for coefficient $c_{nm}^{(\lambda)}$ is therefore:

[6] Replacing $\partial_{\phi\phi}\Psi_{nm}^{(\lambda)} = (im)^2\,\Psi_{nm}^{(\lambda)}$ straight away.

$$\left(\boldsymbol{r}.\boldsymbol{F} - a_{nm}^{(\lambda)} \cdot r^2 \Psi_{nm}^{(\lambda)}\right) = c_{nm}^{(\lambda)} \cdot \frac{i \sinh^2 \eta}{8} \Bigg\{$$
$$+ \left(\frac{-4\, \cosh\eta}{\sinh^3\eta} + \frac{\sinh 4\eta\, \cos 2\theta}{\sinh^4\eta}\right) \partial_\eta \Psi_{nm}^{(\lambda)} + \left(\frac{8\, \sin 2\theta}{\tanh^2\eta}\right) \partial_\theta \Psi_{nm}^{(\lambda)} +$$
$$+ \left(\frac{8\, \sin 2\theta}{\tanh\eta}\right) \partial_{\eta\theta} \Psi_{nm}^{(\lambda)} + \left(-\frac{8\, \sin^2\theta}{\tanh^2\eta}\right) \partial_{\eta\eta} \Psi_{nm}^{(\lambda)} +$$
$$+ \left(-8\cos^2\theta\right) \partial_{\theta\theta} \Psi_{nm}^{(\lambda)} + \frac{4}{\sinh^4\eta} \left(\cos 2\theta - \cosh 2\eta\right) (im)^2\, \Psi_{nm}^{(\lambda)}$$
$$\Bigg\}$$

3.3.8 Term $\boldsymbol{L}.\boldsymbol{F}$

We have:

$$\boldsymbol{L}.\boldsymbol{F} = b_{nm}^{(\lambda)} \cdot L^2 \Psi_{nm}^{(\lambda)}(\boldsymbol{r})$$

However in Cartersian coordinates we can establish:

$$\begin{aligned} L^2\psi =& \boldsymbol{L}.\boldsymbol{L}\psi = -i\epsilon_{\alpha\beta\gamma} r_\beta \partial_\gamma L_\alpha \psi \\ =& -i\epsilon_{\beta\gamma\alpha} r_\beta \partial_\gamma L_\alpha \psi = -i\boldsymbol{r}.\boldsymbol{\nabla} \times \boldsymbol{L}\psi \end{aligned}$$

It follows that:

$$i\boldsymbol{L}.\boldsymbol{F} = ib_{nm}^{(\lambda)} \cdot L^2 \Psi_{nm}^{(\lambda)} = b_{nm}^{(\lambda)} \cdot \left(\boldsymbol{r}.\boldsymbol{\nabla} \times \boldsymbol{L}\Psi_{nm}^{(\lambda)}\right)$$

Which I have already computed in Eq. (28).

3.4 Derivatives of the harmonics

From previous sections it is clear that we will need to compute $\partial_\eta \Psi_{nm}^{(\lambda)}$ and $\partial_\theta \Psi_{nm}^{(\lambda)}$. Lets work them out[7]:

$$\begin{aligned} \partial_\eta \Psi_{nm}^{(\lambda)} =& \partial_\eta \left(\sqrt{\cosh\eta - \cos\theta} Z_{n-\frac{1}{2}}^{(\lambda)m}(\cosh\eta)\, \exp(in\theta)\, \exp(im\phi)\right) \\ =& \frac{\cosh\eta \cdot ((2n+1)\cos\theta - 2n\cosh\eta) - 1}{2\sinh\eta \cdot (\cosh\eta - \cos\theta)} \cdot \Psi_{nm}^{(\lambda)} + \frac{\left(n + \frac{1}{2} - m\right) \cdot \exp(-i\theta)}{\sinh\eta} \cdot \Psi_{n+1,\, m}^{(\lambda)} \end{aligned} \tag{29}$$

Also:

$$\begin{aligned} \partial_\theta \Psi_{nm}^{(\lambda)} =& \partial_\theta \left(\sqrt{\cosh\eta - \cos\theta} Z_{n-\frac{1}{2}}^{(\lambda)m}(\cosh\eta)\, \exp(in\theta)\, \exp(im\phi)\right) \\ =& \left(\frac{\sin\theta}{2 \cdot (\cosh\eta - \cos\theta)} + in\right) \Psi_{nm}^{(\lambda)} \end{aligned} \tag{30}$$

And:

$$\begin{aligned} \partial_\phi \Psi_{nm}^{(\lambda)} =& \partial_\phi \left(\sqrt{\cosh\eta - \cos\theta} Z_{n-\frac{1}{2}}^{(\lambda)m}(\cosh\eta)\, \exp(in\theta)\, \exp(im\phi)\right) \\ =& (im)\, \Psi_{nm}^{(\lambda)} \end{aligned} \tag{31}$$

[7]Luckily, since Mathematica supports these functions as `LegendreP(n-1/2, m, 3, z)` and `LegendreQ[n-1/2, m, 3, z]`, and can differentiate using recurrence relations.

Let us obtain the diagonal second derivatives:

$$\partial_{\eta\eta}\Psi_{nm}^{(\lambda)} = \partial_\eta\left(\frac{\cosh\eta\cdot((2n+1)\cos\theta-2n\cosh\eta)-1}{2\sinh\eta\cdot(\cosh\eta-\cos\theta)}\cdot\Psi_{nm}^{(\lambda)}+\frac{\left(n+\frac{1}{2}-m\right)\cdot\exp\left(-i\theta\right)}{\sinh\eta}\cdot\Psi_{n+1,\,m}^{(\lambda)}\right)$$

$$\begin{aligned}\partial_{\eta\eta}\Psi_{nm}^{(\lambda)} =& \frac{1}{16\,\sinh\eta\,\left(\cosh\eta-\cos\theta\right)^2}\cdot\Bigg\{\frac{-4\,\cos\theta\,\left(\left(4n^2+2n+1\right)\cosh 2\eta+5+2n\left(7+2n\right)\right)}{\tanh\eta}+\\ &+\frac{6\left(3+4n\left(2+n\right)\right)+\left(1+2n\right)\left(5+2n+\left(1+2n\right)\cosh 2\eta\right)\cos 2\theta}{\sinh\eta}+18\sinh\eta+\\ &+8n\left(5+4n+n\cosh 2\eta\right)\sinh\eta\Bigg\}\cdot\Psi_{nm}^{(\lambda)}+\\ &+\\ &+\frac{\left(2m-2n-1\right)\exp\left(-i\theta\right)}{2\sinh\eta^2\left(\cosh\eta-\cos\theta\right)}\cdot\Big\{2+n-\left(3+2n\right)\cos\theta\,\cosh\eta+\left(1+n\right)\cosh 2\eta\Big\}\Psi_{n+1,m}^{(\lambda)}+\\ &+\\ &+\frac{\exp\left(-i2\theta\right)\left(2m-3-2n\right)\left(2m-2n-1\right)}{4\sinh^2\eta}\cdot\Psi_{n+2,m}^{(\lambda)}\end{aligned}$$

and the other one:

$$\partial_{\theta\theta}\Psi_{nm}^{(\lambda)} = \partial_\theta\left(\left(\frac{\sin\theta}{2\cdot(\cosh\eta-\cos\theta)}+in\right)\Psi_{nm}^{(\lambda)}\right)$$

$$\begin{aligned}\partial_{\theta\theta}\Psi_{nm}^{(\lambda)} =& \frac{1}{4\left(\cosh\eta-\cos\theta\right)^2}\cdot\Big\{2\left(\cosh\eta-\cos\theta\right)\left(\cos\theta-2n^2\left(\cosh\eta-\cos\theta\right)\right)+\\ &+i4n\left(\cosh\eta-\cos\theta\right)\sin\theta-\sin^2\theta\Big\}\cdot\Psi_{nm}^{(\lambda)}\end{aligned}$$

The off-diagonal one:

$$\begin{aligned}\partial_{\eta\theta}\Psi_{nm}^{(\lambda)} =& \partial_\eta\left(\left(\frac{\sin\theta}{2\cdot(\cosh\eta-\cos\theta)}+in\right)\Psi_{nm}^{(\lambda)}\right)\\ =& \left(-\frac{\sin\theta\cdot\sinh\eta}{2\left(\cosh\eta-\cos\theta\right)^2}\right)\cdot\Psi_{nm}^{(\lambda)}+\left(\frac{\sin\theta}{2\cdot(\cosh\eta-\cos\theta)}+in\right)\partial_\eta\Psi_{nm}^{(\lambda)}\end{aligned}$$

3.5 Getting out the constants

Once a suitable analytic expression has been found, for all the terms in Eq. (4-7), we have a generic problem like this:

$$h\left(\boldsymbol{r}\right)=q_{nm}^{(\lambda)}Q_{nm}^{(\lambda)}\left(\boldsymbol{r}\right)$$

Where we need the constants $q_{nm}^{(\lambda)}$. It makes sence to convert the problem into a matrix form:

$$h_{ijk}=Q_{ijk,\lambda nm}q_{\lambda nm}$$

We then contract it with a suitable tensor, e.g.[8]:

$$\left(\sqrt{g}\Psi\right)^*_{\bar\lambda\bar n\bar m,ijk}Q_{ijk,\lambda nm}q_{\lambda nm}=\left(\sqrt{g}\Psi\right)^*_{\bar\lambda\bar n\bar m,ijk}h_{ijk}=\Lambda_{\bar\lambda\bar n\bar m}$$

We can calculate:

$$M_{\bar\lambda\bar n\bar m,\lambda nm}=\left(\sqrt{g}\Psi\right)^*_{\bar\lambda\bar n\bar m,ijk}Q_{ijk,\lambda nm}$$

Finally, all we need to do is solve the tensor equation:

$$M_{\bar\lambda\bar n\bar m,\lambda nm}q_{\lambda nm}=\Lambda_{\bar\lambda\bar n\bar m}$$

[8]Clearly the choice of tensor to contract with, will be important, but lets keep it simple for now. I will stick in density ($\sqrt{g}$) to approximate an integral - this works well if we start on a uniform Cartesian lattice.

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