Portfolio Optimization

An Application of Convex Optimization

A project as part of the course:
"Advanced Topics in Convex Optimization"

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Outline

Contents

Portfolio

Definition

A **portfolio** is a collection of financial investments like stocks, bonds, commodities, cash, and cash equivalents, including closed-end funds and exchange traded funds (ETFs).

A portfolio may also contain:

- real estate
- art
- private investments

Assets

Definition

An **asset** is a resource with economic value that an individual, corporation, or country owns or controls with the expectation that it will provide a future benefit, i.e. generate cash flow, reduce expenses, or improve sales, regardless of whether it is manufacturing equipment or a patent.

Assets are reported on a company's balance sheet. They are classified as:

- current
- fixed
- financial
- intangible

Diversification

Definition

Diversification is a risk management strategy that mixes a wide variety of investments within a portfolio. Diversification can be achieved by buying investments in different asset classes, that are in different countries, industries and sizes of companies.

Diversification's achievments:

- yielding **higher long-term returns**
- **decreasing the risk** of any individual holding or security

Portfolio Investment

Definition

A **portfolio investment** is ownership of a stock, bond, or other financial asset with the expectation that it will earn a return or grow in value over time, or both.

Portfolio investment categories:

- **Strategic investment**, i.e. buying financial assets for long-term growth potential or income yield, or both.
- **Tactical approach**, i.e. active buying and selling activity for short-term gains.

Portfolio Management

Definition

Portfolio management is the art and science of selecting and overseeing a group of investments that meet the long-term financial objectives and risk tolerance of a client, a company, or an institution.

- **Active** portfolio management requires strategic investment.
- **Passive** portfolio management requires tactical approach.

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Return

Definition

A **return**, also known as a financial return, in its simplest terms, is the money made or lost on an investment over some period of time.

A return can be expressed as

- the change in dollar value of an investment over time or
- a percentage derived from the ratio of profit to investment.

Returns can also be presented as

- net results (after fees, taxes, and inflation) or
- gross returns that do not account for anything but the price change.

Rate of Return

Definition

A **Rate of Return (RoR)** is the net gain or loss of an investment over a specified time period, expressed as a percentage of the investment's initial cost. When calculating the rate of return, we determine the percentage change from the beginning of the period until the end.

The formula is:

$$RoR = \frac{\text{Current value} - \text{Initial value}}{\text{Initial value}} \cdot 100\%$$

Return on Investment

Definition

Return on Investment (ROI) is a performance measure used to evaluate the efficiency or profitability of an investment or compare the efficiency of a number of different investments. ROI tries to directly measure the amount of return on a particular investment, relative to the investments cost.

The formula is:

$$ROI = \frac{\text{Current Value of Investment} - \text{Cost of Investment}}{\text{Cost of Investment}}$$

Risk-Adjusted Return

Definition

A **risk-adjusted return** is a calculation of the profit or potential profit from an investment that takes into account the degree of risk that must be accepted in order to achieve it. The risk is measured in comparison to that of a virtually risk-free investment.

Risk-adjusted returns are applied to:

- individual stocks
- investment funds
- entire portfolios

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Risk

Definition

Risk is defined in financial terms as the chance that an outcome or investment's actual gains will differ from an expected outcome or return.

- It includes the possibility of losing some or all of an original investment.
- It is assessed by considering historical behaviors and outcomes.

Volatility

Definition

Volatility is a statistical measure of the dispersion of returns for a given security or market index. The higher the volatility, the riskier the security. It is often measured from either the standard deviation or variance between returns from that same security or market index.

- A **higher volatility** means that a security's value can potentially be spread out over a large range of values in a short time period.
- A **lower volatility** means that a security's value does not fluctuate dramatically, and tends to be more steady.

Risk-free terms

Definition

A **risk-free asset** is one that has a certain future return and virtually no possibility they will drop in value or become worthless altogether. Risk-free assets tend to have low rates of return, since their safety means investors don't need to be compensated for taking a chance.

Definition

Risk-free return is the theoretical return attributed to an investment that provides a guaranteed return with zero risks.

Definition

The **risk-free rate of return** is the theoretical rate of return of an investment with zero risk. The risk-free rate of return represents the interest on an investor's money that would be expected from an absolutely risk-free investment over a specified period of time.

Risk Management

Definition

Risk Management is a crucial process used to make investment decisions. Risk management involves identifying and analyzing risk in an investment and deciding whether or not to accept that risk given the expected returns for the investment. Some common measurements of risk include standard deviation, Sharpe ratio, beta, value at risk (VaR), conditional value at risk (CVaR), and R-squared.

Sharpe ratio

Definition

The **Sharpe ratio** is the average return earned in excess of the risk-free rate per unit of volatility or total risk. It serves as an indicator of whether an investment's return is worth the associated risk. A high Sharpe ratio is good when compared to similar portfolios or funds with lower returns.

The formula is:

$$\text{Sharpe ratio} = \frac{R_p - R_f}{\sigma_p}$$

where

R_p : Rate of return of the portfolio
R_f : Risk-free rate of return
σ_p : Standard deviation of the portfolio's excess return

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Introduction

Modern Portfolio Theory (MPT) or mean variance portfolio optimization assumes that all investors are risk averse. Hence they would prefer:

- a high return portfolio over a low return one for a given level of risk
- a low risk portfolio over a high risk one for a given level of return

This theory states that:

- adding assets to a diversified portfolio that have low correlations can decrease portfolio risk without sacrificing return
- adding diversification should increase the Sharpe ratio, compared to similar portfolios with a lower level of diversification

Investors can achieve their best results by choosing an **optimal mix** of return and risk based on an assessment of their individual tolerance to risk.

To choose the best portfolio from a number of possible portfolios two separate decisions need to be made:

- Determination of a set of efficient portfolios
- Selection of the best portfolio out of the efficient set

Determining the efficient set

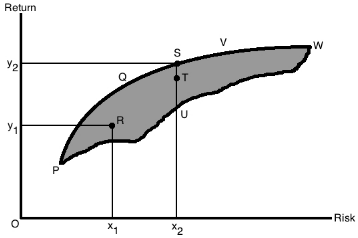

Figure 1: Risk-return figure of possible portfolios

- A portfolio that gives maximum return for a given risk, or minimum risk for given return is an **efficient portfolio**.
- In Figure 1, the shaded area $PVWP$ includes **all the possible securities** an investor can invest in.
- The efficient portfolios for a given risk level are the ones that lie **on** the boundary of $PQVW$, which is called the **Efficient Frontier**.

Portfolios that lie:

- **below** the Efficient Frontier are not good enough because the return would be lower for the given risk.
- to the **right** of the Efficient Frontier would not be good enough, as there is higher risk for a given rate of return.

The Efficient Frontier graphically depicts the benefit of **diversification** and its curvature shows the relation between portfolio's risk and reward profile.

Choosing the best portfolio

For selection of the optimal portfolio or the best portfolio, the risk-return preferences are analyzed.

- An investor who is **highly** risk averse will hold a portfolio on the **lower left** hand of the frontier.
- An investor who is **not** too risk averse will choose a portfolio on the **upper** portion of the frontier.

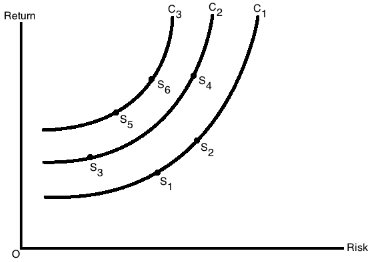

Figure 2: Risk-return indifference curves

- Figure 2 shows the **risk-return indifference curves** for the investors.
- Each of the different points on a particular indifference curve shows a different combination of risk and return, which provide the **same satisfaction** to the investors.
- Each curve to the **left** represents **higher** utility or satisfaction and the goal is to maximize the satisfaction.

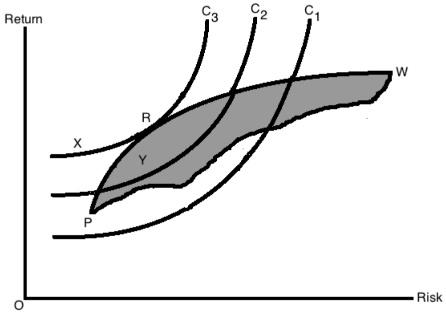

Figure 3: The Efficient Portfolio

- The investor's **optimal portfolio** is found at the point of **tangency** of the efficient frontier with the indifference curve.
- This point marks the **highest** level of satisfaction the investor can obtain.

Capital Market Line (CML)

Definition

The **Capital Market Line (CML)** represents portfolios that optimally combine risk and return. It is a theoretical concept that represents all the portfolios that optimally combine the risk-free rate of return and the market portfolio of risky assets.

All investors will choose a position on the Capital Market Line, in equilibrium, by borrowing or lending at the risk-free rate, since this maximizes return for a given level of risk.

The **slope** of the CML is the **Sharpe ratio** of the market portfolio.

The intercept point of CML and Efficient Frontier would result in the most efficient portfolio called the tangency (market) portfolio.

As a generalization:

- **buy** assets if Sharpe ratio is **above** CML
- **sell** assets if Sharpe ratio is **below** CML

The formula is:

$$R_p = R_f + \frac{R_t - R_f}{\sigma_t}\sigma_p$$

where

R_p : Rate of return of the portfolio
R_t : Rate of return of the market portfolio
R_f : Risk-free rate of return
σ_t : Standard deviation of the market portfolio's excess return
σ_p : Standard deviation of the portfolio's excess return

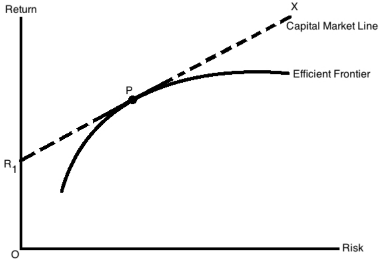

Figure 4: The Combination of Risk-Free Securities with the Efficient Frontier and Capital Market Line (CML)

- Portfolios may also include risk-free securities.
- A portfolio with risk-free securities will enable an investor to achieve a **higher** level of satisfaction.

The Efficient Frontier represents combinations of risky assets.

If we draw a line from the risk-free rate of return, which is **tangential** to the Efficient Frontier, we get the Capital Market Line. The point of tangency is the **most efficient** portfolio.

Moving:

- **up** the CML will increase portfolio's risk and consequently increase return expectation
- **down** the CML will decrease portfolio's risk and consequently decrease return expectation

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Formulation

Let a long only portfolio with n assets. We define the following:

- Invest fraction w_i in asset i with $i = 1, 2, \ldots, n$.
- Vector $\mathbf{w} \in \mathbb{R}^n$ is the portfolio allocation vector.
- Long only portfolio means that $w_i > 0$ for $i = 1, 2, \ldots, n$.
  - A **long position** refers to the purchase of an asset with the expectation it will increase in value.
  - A **short position** is created when a trader sells a security first with the intention of repurchasing it or covering it later at a lower price.
- It must $\mathbf{1}^T\mathbf{w} = 1$.
- The number of the trading days in a year is considered to be 252.

- Vector $\boldsymbol{\mu} \in \mathbb{R}^n$ contains the annual mean returns of each asset. It is being calculated as follows:
  - First, we compute the percentage change on the daily stocks' close prices (the percentage change between each row and its immediately previous row).
  - Then, we get $\boldsymbol{\mu}$ as the annualized geometric mean of the percentage changes calculated before.
  - Let n stocks, data of m days and each percentage change of i-th stock's daily price as $r_{i,k}$, $k = 1, \ldots, m-1$. The formula for stock i is: $\boldsymbol{\mu}_i = ((1 + r_{i,1}) \cdot (1 + r_{i,2}) \cdot \ldots \cdot (1 + r_{i,m}))^{\left(\frac{252}{m-1}\right)} - 1.$

- Matrix $\mathbf{\Sigma} \in \mathbb{R}^{n \times n}$ is the annual covariance matrix of portfolio's assets. It is being calculated as follows:
  - First, we compute the percentage change on the daily stocks' close prices (the percentage change between each row and its immediately previous row).
  - Let n stocks, data of m days and each percentage change of i-th stock's daily price as $r_{i,k}$, $k = 1, \ldots, m-1$. The formula for the covariance between stock i and stock j is:
    $Cov(s_i, s_j) = \frac{\sum_{k=1}^{m-1} (r_{i,k} - \bar{r_i})(r_{j,k} - \bar{r_j})}{m-1}$

- Portfolio's annual variance is given by $\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}$.
- Given that portfolio's volatility risk is defined as the square root of portfolio's variance, then minimization on the portfolio's volatility is equivalent to minimization on the portfolio's variance.
- Portfolio's annual return is given by $\boldsymbol{\mu}^T \mathbf{w}$.
- Quadratic utility is given by $\boldsymbol{\mu}^T \mathbf{w} - \gamma \frac{1}{2} \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}$ where γ is the risk aversion parameter.
- Risk-free rate of return is equal to zero.

Minimum volatility

Let a portfolio as initially described above. We want to obtain the optimal vector **w** that minimizes portfolio's volatility (risk). To do so, we have to solve the following convex optimization problem:

$$
\begin{aligned}
\min_{\mathbf{w}} \quad & \frac{1}{2}\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} \\
\text{s.t.} \quad & \mathbf{1}^T \mathbf{w} = 1 \\
& 0 \leq w_i \leq 1 \ , \ i = 1, 2, \ldots n
\end{aligned}
\tag{1}
$$

Minimum volatility for a given target return

Let a portfolio as initially described above. We want to obtain the optimal vector $\mathbf{w}$ that minimizes portfolio's volatility (risk) and gives annual portfolio's return r. To do so, we have to solve the following convex optimization problem:

$$
\begin{aligned}
\min_{\mathbf{w}} \quad & \frac{1}{2}\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} \\
\text{s.t.} \quad & \boldsymbol{\mu}^T\mathbf{w} = r \\
& \mathbf{1}^T\mathbf{w} = 1 \\
& 0 \leq w_i \leq 1 \ , \ i = 1, 2, \dots n
\end{aligned}
\tag{2}
$$

By problem's nature, we can easily see that constraint $\boldsymbol{\mu}^T\mathbf{w} = r$ is equivalent to constraint $\boldsymbol{\mu}^T\mathbf{w} \geq r$.

Maximum return for a given target volatility

Let a portfolio as initially described above. We want to obtain the optimal vector $\mathbf{w}$ that maximizes portfolio's return and gives annual portfolio's volatility (risk) v. Note that maximizing $\boldsymbol{\mu}^T\mathbf{w}$ is equivalent to minimizing $-\boldsymbol{\mu}^T\mathbf{w}$. To do so, we have to solve the following convex optimization problem:

$$\begin{aligned} \min_{\mathbf{w}} \quad & -\boldsymbol{\mu}^T\mathbf{w} \\ \text{s.t.} \quad & \frac{1}{2}\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} = v \\ & \mathbf{1}^T\mathbf{w} = 1 \\ & 0 \leq w_i \leq 1 \ , \ i = 1, 2, \ldots n \end{aligned} \tag{3}$$

By problem's nature, we can easily see that constraint $\frac{1}{2}\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} = v$ is equivalent to constraint $\frac{1}{2}\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} \leq v$.

Maximum Sharpe ratio

Let a portfolio as initially described above. We want to obtain the optimal vector $\mathbf{w}$ that maximizes portfolio's Sharpe ratio SR. Note that maximizing SR is equivalent to minimizing $-SR$. To do so, we have to solve the following optimization problem:

$$
\begin{aligned}
\min_{\mathbf{w}} \quad & -SR \equiv -\frac{\boldsymbol{\mu}^T\mathbf{w} - R_f}{\sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}}} \\
\text{s.t.} \quad & \mathbf{1}^T\mathbf{w} = 1 \\
& 0 \leq w_i \leq 1 \ , \ i = 1, 2, \ldots n
\end{aligned}
\tag{4}
$$

Next, we will see how we can convert this problem to an equivalent quadratic convex problem.

Maximum quadratic utility given some risk aversion

Let a portfolio as initially described above. We want to obtain the optimal vector $\mathbf{w}$ that maximizes quadratic utility QU given some risk aversion parameter γ. Note that maximizing QU is equivalent to minimizing $-QU$. To do so, we have to solve the following convex optimization problem:

$$\begin{aligned} \min_{\mathbf{w}} \quad & -QU \equiv -\boldsymbol{\mu}^T\mathbf{w} + \gamma\frac{1}{2}\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} \\ \text{s.t.} \quad & \mathbf{1}^T\mathbf{w} = 1 \\ & 0 \le w_i \le 1 \ , \ i = 1, 2, \ldots n \end{aligned} \tag{5}$$

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Python PyPortfolioOpt and Efficient Frontier

PyPortfolioOpt is a Python's library that implements portfolio optimization methods, including classical efficient frontier techniques.

The efficient_frontier module houses the EfficientFrontier class and its descendants, which generate optimal portfolios for various possible objective functions and parameters.

An EfficientFrontier object (inheriting from BaseConvexOptimizer) contains multiple optimization methods that can be called (corresponding to different objective functions) with various parameters.

Mean-variance optimization requires two things: the expected returns of the assets, and the covariance matrix (or more generally, a risk model quantifying asset risk). PyPortfolioOpt provides methods for estimating both (located in expected_returns and risk_models respectively).

Using the methods shown below we solve efficiently problems (1) - (5):

- min_volatility() optimizes for minimum volatility
- max_sharpe() optimizes for maximal Sharpe ratio (a.k.a the tangency portfolio)
- max_quadratic_utility() maximizes the quadratic utility, given some risk aversion.
- efficient_risk() maximizes return for a given target risk
- efficient_return() minimizes risk for a given target return
- portfolio_performance() calculates the expected return, volatility and Sharpe ratio for the optimized portfolio.
- clean_weights() rounds the weights and clips near-zeros.

Python SciPy Optimization (Minimization)

SciPy optimize provides Python functions for minimizing (or maximizing) objective functions, possibly subject to constraints. It includes solvers for nonlinear problems (with support for both local and global optimization algorithms), linear programing, constrained and nonlinear least-squares, root finding, and curve fitting.

In particular, scipy.optimize.minimize is capable of doing constrained minimization of scalar function of one or more variables in an efficent way. The user have to define the objective function, give the constraints in the appropriate form and specify the type of the solver.

In that way, we can solve easily portfolio optimization problems (1) - (5).

Combinatorics using the analytic solutions

The essence of using KKT conditions to solve an optimization problem with inequality constraints is to make it combinatorial.

Based on the analytic solutions cited in the supplementary document, we have developed python code that uses the matrix form (system of equations) solutions and combinatorics to solve problems (1), (2) and (5). In particular, the idea is that we need to test all combinations of activeness of the inequality constraints.

It is obvious that this approach is computationally demanding and absolutely inefficient, as the combinations of activeness of the inequality constraints are of order $\mathcal{O}(2^n)$, where n is the number of portfolio assets.

CVX

CVX is a Matlab-based modeling system for convex optimization. CVX turns Matlab into a modeling language, allowing constraints and objectives to be specified using standard Matlab expression syntax.

We use CVX to solve easily problems (1) - (5). In particular, we solve via CVX problems (1), (2), (4) and (5) exactly in the form stated before. For problem (3), we solve via CVX an equivalent convex problem, which will be explained below.

Iterative Optimization Algorithms

We are using the **projected gradient method** to solve problems (1), (2) and (5).

We are using the **interior point method** and **Newton's algorithm** to solve problem (3) and problem (8), which is the convex equivalent problem of problem (4).

We are using the **FDPG algorithm** to compute the projections needed for the algorithms mentioned before.

Below, we explain the process to convert problem (4) to an equivalent convex problem.

Maximum Sharpe ratio

The original problem we have to solve is problem (4). That problem is not convex and there is not any analytic algorithm to solve it. Next, we will see how we can transform it to another equivalent problem that is convex with linear inequality constraints and affine equality constraints [Daniel Bienstock et al.].

Problem (4) after some unimportant changes in the notations takes the following form:

$$\begin{aligned} \max_{\mathbf{w}} \quad & SR \equiv \frac{\boldsymbol{\mu}^T \mathbf{w} - R_f}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{w} = \mathbf{b} \text{ , where } \mathbf{A} \equiv \mathbf{1}^T \text{ and } \mathbf{b} \equiv 1 \\ & \mathbf{w} \geq \mathbf{0} \\ & \mathbf{C}\mathbf{w} \geq \mathbf{d} \text{ , where } \mathbf{C} \equiv -\mathbf{I}_n \text{ and } \mathbf{d} \equiv -\mathbf{1} \end{aligned} \tag{6}$$

Problem (6) is difficult because of the nature of its objective.

Under the following assumption, it can be converted to a standard convex quadratic program:

- There exists a vector **w** satisfying all the constraints of problem (6) such that $\boldsymbol{\mu}^T \mathbf{w} - R_f > 0$.
- This assumption is reasonable as it says that our set of assets is able to beat the risk-free rate of return.

We define:

$$f_0(\mathbf{w}) = \frac{\boldsymbol{\mu}^T \mathbf{w} - R_f}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}}$$

Since $\mathbf{1}^T \mathbf{w} = 1$, we have:

$$f_0(\mathbf{w}) = \frac{\boldsymbol{\mu}^T \mathbf{w} - R_f}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}} = \frac{\boldsymbol{\mu}^T \mathbf{w} - R_f \mathbf{1}^T \mathbf{w}}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}} = \frac{\left(\boldsymbol{\mu}^T - R_f \mathbf{1}^T\right) \mathbf{w}}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}} = \frac{\hat{\boldsymbol{\mu}}^T \mathbf{w}}{\sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}}$$

where $\hat{\mu}_i = \mu_i - R_f, i = 1, 2, \ldots, n$.

For any scalar $\lambda > 0$, it holds that $f_0(\lambda \mathbf{w}) = f_0(\mathbf{w})$.

Now we can state our optimization problem.

Let $\hat{\mathbf{C}}$ be the matrix whose i, j-entry is $C_{i,j} - d_i$.

The problem we consider is:

$$\begin{aligned} \max_{\mathbf{y}} \quad & \frac{1}{\sqrt{\mathbf{y}^T \boldsymbol{\Sigma} \mathbf{y}}} \\ \text{s.t.} \quad & \hat{\mathbf{A}}\mathbf{y} = \mathbf{b}, \ , \text{ where } \hat{\mathbf{A}} \equiv \hat{\boldsymbol{\mu}}^T \text{ and } \mathbf{b} \equiv 1 \\ & \mathbf{y} \geq \mathbf{0} \\ & \hat{\mathbf{C}}\mathbf{y} \geq \mathbf{0} \end{aligned} \tag{7}$$

To see that problems (6) and (7) are indeed equivalent, suppose that $\mathbf{y}_*$ is an optimal solution to (7).

Notice that because of the equality constraint $\hat{\mathbf{A}}\mathbf{y} = \mathbf{b}$, $\mathbf{y}_*$ is not identically zero, and so by $\mathbf{y} \geq \mathbf{0}$, it is $\sum_{i=1}^{n} y_{*,i} > 0$. Define the vector:

$$\mathbf{w}_* = \frac{\mathbf{y}_*}{\sum_{i=1}^{n} y_{*,i}}$$

Then, by construction it holds that:

$$\sum_{i=1}^{n} w_{*,i} = 1$$

Since $\mathbf{y}$ satisfies $\hat{\mathbf{C}}\mathbf{y} \geq \mathbf{0}$, then for any row i we have that:

$$\begin{aligned}
&\sum_{j=1}^{n} (c_{i,j} - d_i)\, y_{*,j} \geq 0 \qquad \Rightarrow \\
\Rightarrow \quad &\sum_{j=1}^{n} c_{i,j} y_{*,j} \geq \left(\sum_{j=1}^{n} y_{*,j} \right) d_i \quad \Rightarrow \\
\Rightarrow \quad &\sum_{j=1}^{n} c_{i,j} w_{*,j} \geq d_i
\end{aligned}$$

Therefore, $\mathbf{w}_*$ is feasible for problem (6). It holds that:

$$f_0(\mathbf{w}_*) = f_0(\mathbf{y}_*) = \frac{1}{\sqrt{\mathbf{y}_*^T \mathbf{\Sigma} \mathbf{y}_*}}$$

since $\hat{\boldsymbol{\mu}}^T \mathbf{y} = 1$.

In summary, the value of problem (6) is at least as large as the value of problem (7). The converse is proved in a similar way. So, indeed, problems (6) and (7) are equivalent.

We just have to solve problem (7), which is equivalent to problem:

$$\begin{aligned} \min_{\mathbf{y}} \quad & \mathbf{y}^T \mathbf{\Sigma} \mathbf{y} \\ \text{s.t.} \quad & \hat{\mathbf{A}}\mathbf{y} = \mathbf{b}, \ \text{, where } \hat{\mathbf{A}} \equiv \hat{\boldsymbol{\mu}}^T \text{ and } \mathbf{b} \equiv 1 \\ & \mathbf{y} \geq \mathbf{0} \\ & \hat{\mathbf{C}}\mathbf{y} \geq \mathbf{0} \end{aligned} \tag{8}$$

This is just a standard quadratic program.

In order to solve problem (8), and consequently problem (4), using the interior point barrier method, we write it in the following form:

$$
\begin{aligned}
\min_{\mathbf{y}} \quad & f_0(\mathbf{y}) \equiv \mathbf{y}^T \boldsymbol{\Sigma} \mathbf{y} \\
\text{s.t.} \quad & \mathbf{A}\mathbf{y} = \mathbf{b} \text{ , where } \mathbf{A} \equiv \hat{\boldsymbol{\mu}}^T \text{ and } \mathbf{b} \equiv 1 \\
& f_{1,i}(\mathbf{y}) \equiv -y_i \leq 0 \text{ , } i = 1, 2, \ldots n \\
& f_{2,i}(\mathbf{y}) \equiv -\hat{\mathbf{c}}_i^T \mathbf{y} \leq 0 \text{ , } i = 1, 2, \ldots n
\end{aligned}
\tag{9}
$$

Then, the solution to the original problem is given by:

$$
\mathbf{w} = \frac{\mathbf{y}}{\sum_{i=1}^{n} y_i}
$$

Code

Link 1 :

Portfolio_Optimization_1.ipynb

It includes Python code for solving problems (1) - (5) using the PyPortfolioOpt package.

Link 2 :

Portfolio_Optimization_2.ipynb

It includes Python code for solving problems (1) - (5) using the SciPy Optimize Minimize package.

Link 3 :

Portfolio_Optimization_3.ipynb

It includes Python code for solving problems (1), (2) and (5) using the analytic solutions and combinatorics.

Link 4 :

MATLAB_Code

It includes all Matlab files needed for solving problems (1) - (5) using the iterative algorithms explained before.

Bibliography

- Daniel Bienstock et al., Course IEOR4500 Applications Programming for Financial Engineering (Maximizing the Sharpe ratio), Columbia University

Supplementary Documents

- Statistical Terms
- Analytic Solutions
- Algorithmic Solutions - Programming

Thank you!