Supplementary Appendix of *'Decentralization and consumer welfare with substitutes or complements'*

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Computations of the equilibrium under (centralized) foreclosure. From the inverse demand $p_1 = \alpha - q_1 - \gamma q_2$, we find that the VIP obtains inverse demand $p_1 = \alpha - q_1$ when it forecloses the rival (as in that case $q_2 = 0$). This gives the demand function $q_1(p_1) = \alpha - p_1$. By backward induction, the VIP maximizes the monopoly profit $V(p_1) = q_1(p_1).p_1$. The FOC is $-2p_1 + \alpha = 0$ which yields $p_1^{CF} = \frac{\alpha}{2}$. Ultimately, the profit is $V^{CF} = \frac{\alpha^2}{4}$. Ultimately, we find $V^C - V^{CF} = \frac{\alpha^2(1-\gamma)}{(\gamma+1)(\gamma^2+8)} > 0$ for all $\gamma > 0$ but equals 0 for $\gamma = 1$. □

Computations of the equilibrium under (decentralized) foreclosure. From the inverse demand $p_1 = \alpha - q_1 - \gamma q_2$, we find that the VIP obtains inverse demand $p_1 = \alpha - q_1$ when it forecloses the rival (as in that case $q_2 = 0$). This gives the demand function $q_1(p_1) = \alpha - p_1$. By backward induction, the downstream unit maximizes the monopoly profit $\pi_1(p_1) = q_1(p_1).(p_1 - w_1)$. The FOC is $-2p_1 + \alpha + w_1 = 0$ which yields $p_1^{DF}(w_1) = \frac{\alpha + w_1}{2}$. At the contracting stage, the VIP maximizes $V(w_1) = q_1(p_1^{DF}(w_1))p_1^{DF}(w_1)$.[1] The FOC is $-w_1 = 0$ which yields $w_1^{DF} = 0$. Ultimately, the equilibrium profit is $V^{DF} = \frac{\alpha^2}{4}$. Note that foreclosure with centralized or decentralized VIP yields the same profits. □

Proof of Lemma 4. We look for the PBNE in pure strategies of the game with decentralization where the input price w_1 is hidden from D_2 and D_2 knows D_1 observes w_2. We use backward induction to solve the game.

By backward induction, at the competition stage, the downstream unit and the rival maximize their profits $\pi_1(p_1, p_2^{a_1}) = (p_1 - w_1)q_1(p_1, p_2^{a_1})$ and $\pi_2(p_2, p_1^{a_2}) = (p_2 - w_2)q_2(p_2, p_1^{a_2})$ with respect to their prices, where $p_j^{a_i}$ denotes D_i's anticipation on D_j's reaction. The reaction functions write $R_2(w_2, p_1^{a_2})$ and $R_1(w_2, p_2^{a_1})$. We now explain how each firm integrates the other firm's reaction. D_2 observes only w_2 and thus forms beliefs about the downstream unit's input price. Assuming passive beliefs, D_2 thinks w_1 sticks to the equilibrium level w_1^*. Though D_2 observes only w_2, it knows that D_1 observes the two input prices and that the latter knows that D_2 reacts only to w_2 while believing that $w_1 = w_1^*$. Formally, it yields $R_2(w_2, R_1(w_1^*, p_2^{a_1})) = p_2^M(w_1^*, w_2)$. By contrast, D_1 observes the two input prices, and knows that D_2 observes only w_2 but still expects D_1 to observe

[1]and not $q_1(p_1^{DF}(w_1))w_1 as it would be under total separation.$

both prices. Formally, it yields $R_1(w_1, R_2(w_2, p_1^{a_2})) = p_1^M(w_1, w_2)$. Therefore, we end up with the sub-game equilibrium price with observable contracts (Eq. (8)) except that $w_1 = w_1^*$ on D_2's side.

$$p_1^M(w_1, w_2) = \frac{\alpha(2-\gamma-\gamma^2) + 2w_1 + \gamma w_2}{4-\gamma^2} \quad ; \quad p_2^M(w_1^*, w_2) = \frac{\alpha(2-\gamma-\gamma^2) + 2w_2 + \gamma w_1^*}{4-\gamma^2} \qquad (1)$$

At the contracting stage, the VIP maximizes its expected profits $V(p_1^M(w_1, w_2), p_2^M(w_2, w_1^*), w_2)$ with respect to w_1 and w_2. It gives the following FOCs: $\gamma\left(2\gamma w_1 - 2\gamma w_1^* + (\gamma-2)\left(\gamma^2+\gamma-2\right) w_2\right) = 0$ and $\left(\gamma^2+\gamma-2\right)(4\alpha + (\gamma-2)\gamma w_1) - 2\left(\gamma^2+2\gamma-4\right) w_1^* + 2(\gamma((\gamma-2)\gamma-4)+4)w_2 = 0$. Then, considering that D_2's belief about w_1^* is correct at equilibrium $w_1^* = w_1^M$, the FOCs rewrite: $(\gamma-2)\gamma\left(\gamma^2+\gamma-2\right) w_2 = 0$ and $\left(\gamma^2+\gamma-2\right)(4\alpha + (\gamma-2)\gamma w_1) - 2\left(\gamma^2+2\gamma-4\right) w_1 + 2(\gamma((\gamma-2)\gamma-4)+4)w_2 = 0$. Solving the system gives $w_2^M = 0$ and $w_1^M = \frac{4\alpha(1-\gamma)(\gamma+2)}{8-(3-\gamma)\gamma^2(\gamma+2)} \geq 0$.

These values hold as long as $\gamma \leq 0.7$ as the Hessian Matrix is negative semi-definite whenever $\gamma \leq 0.7$. Indeed, we get $\partial^2 V/\partial w_1^2 = -\frac{2\gamma^2}{(1-\gamma^2)(4-\gamma^2)^2} < 0$, $\partial^2 V/\partial w_2^2 = -\frac{2(4-\gamma((2-\gamma)\gamma+4))}{(1-\gamma^2)(4-\gamma^2)^2} < 0$ whenever $-1 < \gamma < 0.8$, and $(\partial^2 V/\partial w_1^2)(\partial^2 V/\partial w_2^2) - (\partial^2 V/\partial w_1 \partial w_2)(\partial^2 V/\partial w_2 \partial w_1) = \frac{\gamma^3(16-\gamma(\gamma+1)(16-(4-\gamma)\gamma(\gamma+1)))}{(1-\gamma^2)^2(\gamma^2+4)^4} > 0$ whenever $-1 < \gamma \leq 0.7$ meaning that the SOC is satisfied (the Hessian matrix is negative semi-definite) only when $-1 < \gamma \leq 0.7$.

We then get the optimal values p_1^M and p_2^M, and in particular the profit

$$V^M = -\frac{\alpha^2(\gamma-1)\gamma\left(\gamma^2+\gamma-4\right)\left(\gamma^3+\gamma^2-4\gamma-8\right)}{(\gamma+1)\left(\gamma^4-\gamma^3-6\gamma^2+8\right)^2}$$

Finally, we find $V^M - V^C = -\frac{\alpha^2\left(\frac{4(1-\gamma)}{\gamma^2+8} - \frac{4\gamma\left(\gamma^3-5\gamma+4\right)\left(8-\gamma\left(\gamma^2+\gamma-4\right)\right)}{((3-\gamma)(\gamma+2)\gamma^2+8)^2} + \gamma + 1\right)}{4(\gamma+1)} < 0$. Besides, we also have $V^M - V^{DF} = -\frac{1}{4}\alpha^2\left(1 - \frac{4\gamma\left(\gamma^3-5\gamma+4\right)\left(8-\gamma\left(\gamma^2+\gamma-4\right)\right)}{(\gamma+1)((\gamma-3)(\gamma+2)\gamma^2+8)^2}\right) < 0$. Therefore, the VIP prefers to foreclose the rival and is worse off (as $V^C \geq V^{CF} = V^{DF}$). □

Computations for Opportunism. Lemma 2 and 3 provide the equilibrium outcomes with public contracts. We here analyse whether the VIP prefers to secretly deviate with the downstream unit. This implicitly means that the downstream unit reacts only to the new input price taken the rival's strategy as given. Formally, the downstream sub-game strategies are $p_1^O(w_1, p_2^D) =$

$\frac{\alpha(1-\gamma)+w_1+\gamma p_2^D}{2}$. The VIP integrates this reaction into its expected profit function $V(p_1^O(w_1), w_1)$ and maximizes it with respect to w_1. We obtain $dV/dw_1 = \frac{\alpha^2(\gamma(\gamma(11-(\gamma-2)\gamma)+16)-24)+16(w_1)^2-16\alpha\gamma w_1}{64(\gamma^2-1)}$. Evaluated at the public contract equilibrium we find $(dV/dw_1)(w_1^D) = \alpha\gamma/(8\gamma+8)$ which takes the sign of γ. Following the same method, we find $(dV/dw_2)(w_2^D) = 0$ on the rival's side. □

Proof of Result with Secret Contracts and passive beliefs. We look for the PBNE in pure strategies of the game with decentralization under secret contracts. We use backward induction to solve the game.

At the downstream competition stage, the independent unit and the rival maximize respectively $\pi_1(p_1, p_2^S) = (p_1 - w_1)q_1(p_1, p_2^S)$ and $\pi_2(p_2, p_1^S) = (p_2 - w_2)q_2(p_2, p_1^S)$ with respect to p_1 and p_2 and given passive beliefs about the other firm's strategy p_2^S and p_1^S. Passive beliefs imply that, upon receiving out-of-equilibrium offer, beliefs are the same as on the equilibrium path (they are passive). This gives the following first order conditions $\text{FOC}_{p_1} : \alpha\gamma - \alpha + 2p_1 - \gamma p_2^S - w_1 = 0$ and $\text{FOC}_{p_2} : \alpha\gamma - \alpha - \gamma p_1^S + 2p_2 - w_2 = 0$. The second order conditions are satisfied ($\text{SOC}_{p_i} : \frac{2}{\gamma^2-1} < 0$, $\forall i$). The FOC system delivers the sub-game pricing strategies $p_1^S(w_1, p_2^S) = \frac{\alpha(1-\gamma)+w_1+\gamma p_2^S}{2}$ and $p_2^S(w_2, p_2^S) = \frac{\alpha(1-\gamma)+w_2+\gamma p_1^S}{2}$.

At the contracting stage, the VIP expects these sub-game strategies so that the VIP's expected profit function is $V(w_1, w_2) = p_1^S(w_1, p_2^S)q_1(p_1^S(w_1, p_2^S), p_2^S(w_2, p_1^S)) + w_2 q_2(p_1^S(w_1, p_2^S), p_2^S(w_2, p_2^S))$. The VIP maximizes this expression with respect to the input prices w_1 and w_2. We obtain the following first order conditions $\text{FOC}_{w_1} : \gamma(\alpha(\gamma - 1) - \gamma p_1^S + 2p_2^S - 3w_2) + 2w_1 = 0$, and $\text{FOC}_{w_2} : \alpha\left(3\gamma^2 - \gamma - 2\right) + 2\gamma p_1^S - 3\gamma(\gamma p_2^S + w_1) + 4w_2 = 0$. We check the second order condition through the Hessian matrix. We get $\partial^2 V/\partial w_1^2 = -\frac{1}{2(1-\gamma^2)} < 0$, $\partial^2 V/\partial w_2^2 = -\frac{1}{1-\gamma^2} < 0$, and $(\partial^2 V/\partial w_1^2)(\partial^2 V/\partial w_2^2) - (\partial^2 V/\partial w_1 \partial w_2)(\partial^2 V/\partial w_2 \partial w_1) = \frac{8-9\gamma^2}{16(1-\gamma^2)^2} > 0$ whenever $|\gamma| < \frac{2\sqrt{2}}{3}$ meaning that the SOC is satisfied (the Hessian matrix is definite semi-negative) only when $|\gamma| < \frac{2\sqrt{2}}{3}$.

The input prices are equilibrium input prices only for low substitutes/complements such that $|\gamma| < \frac{2\sqrt{2}}{3}$. By solving the system composed of the two previous FOC and the fact that equilibrium strategies match equilibrium beliefs $p_i^S(w_i, p_j^S) = p_i^S$, $\forall i \neq j = 1, 2$, we find the equilibrium input prices w_1^S and w_2^S joint with the equilibrium price beliefs p_1^S and p_2^S.

Finally, we substitute these equilibrium price beliefs, which also correspond to the equilibrium prices by construction, into the demand functions, profit functions and consumer surplus to get the last equilibrium outcomes of Lemma 1.

Lemma 1. *Under decentralization and secret contracts, provided* $\gamma^2 < \frac{2\sqrt{2}}{3}$*, the equilibrium input prices are*

$$w_1^S = \gamma \frac{\alpha(\gamma+2)^2}{8(\gamma+1)}\ , \quad w_2^S = \frac{\alpha(\gamma+2)^2}{8(\gamma+1)}$$

and the final prices, profits and consumer surplus, respectively, are

$$p_1^S = \frac{\alpha}{2} + \frac{\alpha\gamma}{8(\gamma+1)} \quad ; \quad p_2^S = \frac{3\alpha}{4} - \frac{\alpha\gamma(\gamma+2)}{8(\gamma+1)}$$
$$V^S = \frac{3\alpha^2}{8} - \frac{\alpha^2\gamma(17\gamma+16)}{64(\gamma+1)^2}\ , \quad \pi_2^S = \frac{\alpha^2(1-\gamma)}{16(\gamma+1)}\ , \textit{and} \quad CS^S = \frac{5\alpha^2}{32} - \frac{\alpha^2\gamma(15\gamma+16)}{128(\gamma+1)^2}$$

For information, by rewriting these values, we find:

$$w_2^S = \frac{\alpha(8+\gamma^3)}{2(8+\gamma^2))} + \frac{3\alpha\gamma^2\left(4-\gamma^2\right)}{8(\gamma+1)\left(\gamma^2+8\right)} \geq \frac{\alpha(8+\gamma^3)}{2(8+\gamma^2))} = w_2^C\ ; \ |\gamma w_2^C| \leq |\gamma w_2^S| = |w_1^S|$$

Last, we compute the differences between the VIP's profits with and without decentralization, and with and without foreclosure. We find:

$$V^S - V^C = -\frac{9\alpha^2\gamma^4}{64(\gamma+1)^2\left(\gamma^2+8\right)} \leq 0 \quad , \quad V^S - V^{DF} = \frac{\alpha^2\left(8-9\gamma^2\right)}{64(\gamma+1)^2} \geq 0 \quad \square$$

Proof for Perfect Bayesian Equilibrium with Symmetric beliefs (or arms-length pricing). Symmetric beliefs or arms-length pricing imply that beliefs about the rival's input price offer is the same as one's firm own offer. Formally, upon receiving any input price w_i, each firm i thinks the other gets $w_j = w_i$. For simplification, we thus suppose $w_j = w_i = w$.

Implementing this new rule into the firms' profits gives $\pi_1(p_1, p_2) = (p_1 - w)q_1(p_1, p_2)$ and symmetrically $\pi_2(p_1, p_2) = (p_2 - w)q_2(p_2, p_1)$. This induces the following first order conditions

$\text{FOC}_{p_1} : \alpha\gamma - \alpha + 2p_1 - \gamma p_2 - w = 0$ and $\text{FOC}_{p_2} : \alpha\gamma - \alpha - \gamma p_1 + 2p_2 - w = 0$. Note that the linear demand specification still enables the second order conditions to be satisfied $\text{SOC}_{p_1} : \frac{2}{\gamma^2-1} < 0$ and $\text{SOC}_{P_2} : \frac{2}{\gamma^2-1} < 0$. The FOC system delivers the sub-game pricing strategies $p_1^A(w) = p_2^A(w) = \frac{\alpha(1-\gamma)+w}{2-\gamma}$.

At the input pricing stage, the VIP accounts for these sub-game strategies. We thus substitute the sub-game strategies into the VIP profit function leading to $V(w) = p_1^A(w)q_1(p_1^A(w), p_2^A(w)) + wq_2(p_2^A(w), p_1^A(w))$ and maximize this expression with respect to the input price w. We obtain the following first order condition $\text{FOC}_w : 2(\alpha + (\gamma - 3)w) = 0$. The second order condition is satisfied $\text{SOC}_w : -\frac{2(3-\gamma)}{(2-\gamma)^2(\gamma+1)} < 0$. By isolating w in the FOC, we obtain the equilibrium input price set by the VIP under centralization:

$$w^A = \frac{\alpha}{3-\gamma}$$

Substituting this price into the subgame pricing strategies, profit functions and consumer surplus leads to the following equilibrium outcome.

$$p_i^A = \frac{\alpha(2-\gamma)}{3-\gamma} \quad ; \; q_i^A = \frac{\alpha}{(3-\gamma)(\gamma+1)} \; , \forall i$$
$$V^A = \frac{\alpha^2}{(3-\gamma)(\gamma+1)} \quad ; \; \pi_2^A = \frac{\alpha^2(1-\gamma)}{(3-\gamma)^2(\gamma+1)} \quad , \text{and } \; CS^A = \frac{\alpha^2}{(3-\gamma)^2(\gamma+1)}$$

By comparing with equilibrium outcomes under centralization displayed by Lemma 3, we find:

$$V^A - V^C = -\frac{\alpha^2\left(4 - \gamma^2((\gamma-2)\gamma+5)\right)}{4(3-\gamma)(\gamma+1)\left(\gamma^2+8\right)} \geq 0 \quad \text{iff } \; \gamma \leq \gamma' \approx -0.752$$

□

Proof for Perfect Bayesian Equilibrium with Wary beliefs. Wary beliefs imply that firm i upon receiving input price offer w_i thinks that the VIP offers the optimal price $W_j(w_i)$ to firm $j \neq i$.

Equilibrium conditions.

Formally, it means that firm i sets $P_i(w_i)$ which maximizes $\pi_i(p_i|P_j(W_j(w_i)), w_i)$ where $W_j(w_i)$

maximizes the VIP's profit $V(w_j|w_i, P(w_i), P(w_j))$. The symmetric reasoning applies to firm j. Note that in our setting, the VIP integrates the downstream unit's profits into its profits function so that there is some asymmetry between the beliefs. Formally, we find the following optimal conditions for prices and beliefs:

$$P_1(w_1) \ : \ 2P_1(w_1) - \gamma P_2(W_2(w_1)) = \alpha(1-\gamma) + w_1 \tag{$\mathcal{P}_1$}$$

$$P_2(w_2) \ : \ 2P_2(w_2) - \gamma P_1(W_1(w_2)) = \alpha(1-\gamma) + w_2 \tag{$\mathcal{P}_j$}$$

$$W_2(w_1) \ : \ (\gamma P_1(w_1) - W_2(w_1))P_2'(W_2(w_1)) + \alpha(1-\gamma) - P_2(W_2(w_1)) + \gamma P_1(w_1) = 0 \tag{$\mathcal{W}_2$}$$

$$W_1(w_2) \ : \ \gamma w_2 - P_1(W_1(w_2)) + \alpha(1-\gamma) - P_1(W_1(w_2)) + \gamma P_2(w_2) = 0 \tag{$\mathcal{W}_1$}$$

We now need to solve the system formed by the equations above. Following the supplementary appendices by Rey and Verge (2004) and Gaudin (2019), we consider polynomial solutions (henceforth polynomial wary beliefs) such that $W_i(w_j) = \sum_{k=0}^{n_i} \mu_{i,k}(w_j)^k$ and $P_i(w_i) = \sum_{k=0}^{m_i} \theta_{i,k}(w_i)^k$, $\forall i$.

Wary beliefs are Affine beliefs.

First, we show that any polynomial solution is affine. Consider equation ($\mathcal{P}_1$), then

$$\underbrace{2P_1(w_1)}_{\text{degree}=m_1} - \underbrace{\gamma P_2(W_2(w_1))}_{\text{degree}=m_2 n_2} = \underbrace{\alpha(1-\gamma) + w_1}_{\text{degree}=1}$$

Three cases can arise:

1. $m_1 < m_2 n_2$. It implies $m_1 = 0$ and $m_2 = n_2 = 1$. Eq. ($\mathcal{P}_1$) implies $-\gamma\theta_{2,1}\mu_{2,1} = 1$ which contradicts $-2\theta_{2,1}\mu_{2,1} = 0$ implied by Eq. ($\mathcal{W}_2$).

2. $m_1 > m_2 n_2$. It implies $m_1 = 1$ and ($m_2 = 0$ or $n_2 = 0$). Suppose $m_2 = 0$, then Eq ($\mathcal{P}_1$) implies $2\theta_{1,1} = 1$ which contradicts $\gamma\theta_{1,1} = 0$ implied by Eq. ($\mathcal{W}_2$). Instead, suppose $n_2 = 0$. Let Eq ($\mathcal{K}_i'$) be the derivative of Eq ($\mathcal{K}_i$) and Eq ($\mathcal{K}_i''$) the derivative of Eq ($\mathcal{K}_i'$), $\forall \mathcal{K} = \mathcal{P}, \mathcal{W}$ and $\forall i = 1, 2$. Eq ($\mathcal{P}_1'$) implies $\theta_{1,1} = 1/2$. Eq ($\mathcal{P}_2''$) joint with the latter result implies $4P_2'' = \gamma W_1''$ which contradicts $W_1'' = \gamma P_2''$ implied by Eq ($\mathcal{W}_i''$) and $\theta_{1,1} = 1/2$ unless n_1 and m_2 are strictly inferior to 2.

3. $m_1 = m_2 n_2 \geq 1$. It implies either all degrees are lower than one, or ($m_2 = m_1 \equiv m \geq 2$ and $n_1 = n_2 = 1$). Suppose the latter situation is true then Eq ($\mathcal{W}_2$) implies

$$\underbrace{(\gamma P_1(w_1) - W_2(w_1))P_2'(W_2(w_1))}_{\text{degree}=2m-1\geq 3} + \underbrace{\alpha(1-\gamma) - P_2(W_2(w_1)) + \gamma P_1(w_1)}_{\text{degree}\leq m} = 0$$

which contradicts $m \geq 2$. This shows that polynomial solutions must be affine.

Equilibrium outcomes.

Substituting the expressions of the affine solutions into the system of equations gives the value of the parameters of these affine solutions. We use Mathematica to find such values (the file is available upon request). Let w_i^b denote the equilibrium input price to firm i then at equilibrium we have $w_i^b = \mu_{j,0} + \mu_{j,1} w_i^b$, $\forall i$ and we find

$$w_1^b = \frac{\alpha(\gamma-2)\gamma(\gamma+2)^2((\gamma-2)\gamma+4)\left(5\gamma^2-8\right)}{\left(\gamma^2+8\right)\left(\gamma(\gamma(\gamma(\gamma(5\gamma+8)+2)-40)-16)+32\right)}$$

$$w_2^b = \frac{\alpha(\gamma-2)(\gamma+2)^2((\gamma-2)\gamma+4)\left(5\gamma^2-8\right)}{\left(\gamma^2+8\right)\left(\gamma(\gamma(\gamma(\gamma(5\gamma+8)+2)-40)-16)+32\right)}$$

Note that these input prices do differ ($w_1^b = \gamma w_2^b$) for any $\gamma \neq 0$, and this happens because the VIP internalizes the downstream unit's profit into its objective function. Using the same method, we find the equilibrium prices and thus the equilibrium profit V^b. By comparing with the benchmark result, we have

$$V^b - V^C = -\frac{\alpha^2((\gamma-2)\gamma+4)^2(\gamma(\gamma(\gamma(\gamma(5\gamma-2)+2)+16)-16)-32)^2}{4(\gamma-2)^2\left(\gamma^2+8\right)\left(\gamma(\gamma(\gamma(\gamma(5\gamma+8)+2)-40)-16)+32\right)^2} \leq 0 \quad \square$$

Proof of Subgame Perfect Nash Equilibrium with Non-linear contracts. We first derive the equilibrium under centralization, then the one under decentralization and finally compare the outcomes.

◇ Centralization

The game changes as follows: the VIP now charges a fixed fee $f_2 \in \Re$ to the rival in addition to the linear input price w_2.

We look for the SPNE in pure strategies. We use backward induction to solve the game. At the downstream competition stage, the VIP and the rival maximize respectively $V(p_1, p_2) = p_1 q_1(p_1, p_2) + w_2 q_2(p_1, p_2) + f_2$ and $\pi_2(p_1, p_2) = (p_2 - w_2) q_2(p_1, p_2) - f_2$ with respect to p_1 and p_2. This gives the same first order conditions as before $\text{FOC}_{p_1} : \alpha(\gamma - 1) + 2p_1 - \gamma(p_2 + w_2) = 0$ and $\text{FOC}_{p_2} : \alpha\gamma - \alpha - \gamma p_1 + 2p_2 - w_2 = 0$. This essentially happens because the fixed fee is sunk at this stage. Obviously, the second order conditions remain satisfied $\text{SOC}_{p_1} : \frac{2}{\gamma^2-1} < 0$ and $\text{SOC}_{P_2} : \frac{2}{\gamma^2-1} < 0$. The FOC system delivers the sub-game pricing strategies $p_1^{TC}(w_2) = p_1^C(w_2)$ and $p_2^{TC}(w_2) = p_1^C(w_2)$ which are displayed in equation 4.

In addition, we find that the rival accepts the contract whenever $\pi_2(w_2, f_2) \geq 0$ leading to $f_2 \leq (p_2^{TC}(w_2) - w_2) q_2(p_1^{TC}(w_2), p_2^{TC}(w_2))$.

At the input pricing stage, the VIP accounts for the sub-game strategies. We thus substitute the sub-game strategies into the VIP's profit leading to $V(w_2, f_2) = p_1^{TC}(w_2) q_1(p_1^{TC}(w_2), p_2^{TC}(w_2)) + w_2 q_2(p_1^{TC}(w_2), p_2^{TC}(w_2)) + f_2$ and maximize this expression with respect to the input price w_2 and the fixed fee f_2. We focus on equilibria where the rival accepts the offer. Since the VIP's profit is increasing with the fixed fee, the VIP sets the fixed fee that extracts all the rival's profit.

It then sets the input price w_2 so as to maximize the following simplified expected profit $V(w_2) = p_1^{TC}(w_2) q_1(p_1^{TC}(w_2), p_2^{TC}(w_2)) + p_2^{TC}(w_2) q_2(p_1^{TC}(w_2), p_2^{TC}(w_2))$. We obtain the following first order condition $\text{FOC}_{w_2} : \alpha\gamma(\gamma + 2)^2 - 2\left(5\gamma^2 + 4\right) w_2 = 0$. Once more, note that the linear demand specification enables the second order condition to be satisfied $\text{SOC}_{w_2} : -\frac{2\left(5\gamma^2+4\right)}{\left(4-\gamma^2\right)^2} < 0$. By isolating w_2 in the FOC, we obtain the equilibrium input price set by the VIP under centralization:

$$w_2^{TC} = \frac{\alpha\gamma(\gamma + 2)^2}{2(5\gamma^2 + 4)}$$

Finally, we substitute w_2 by its equilibrium value into the sub-game pricing strategies to obtain the equilibrium prices $p_1^{TC} = p_1^{TC}(w_2^{TC})$ and $p_2^{TC} = p_2^{TC}(w_2^{TC})$. It remains to substitute these equilibrium prices into the demand functions, profit functions to get the last equilibrium outcomes.

In particular, the VIP's profit is:

$$V^{TC} = \frac{\alpha^2\left(\gamma^3+9\gamma^2+8\right)}{4(\gamma+1)\left(5\gamma^2+4\right)}$$

$\diamond$ Decentralization with observable contracts

The game changes as follows: (i) the VIP now charges a fixed fee $f_1 \in \Re$ and a linear input price w_1 to its downstream unit ; (ii) the downstream unit does not integrate the VIP's upstream profits from the rival's sales.

Once more, we look for the SPNE in pure strategies of the alternative game with decentralization. We again use backward induction to solve the game. At the downstream competition stage, the downstream unit and the rival maximize respectively $\pi_1(p_1,p_2) = (p_1 - w_1)q_1(p_1,p_2) - f_1$ and $\pi_2(p_1,p_2) = (p_2 - w_2)q_2(p_1,p_2) - f_2$ with respect to p_1 and p_2. This gives the same first order conditions as without non-linear contracts as the fixed fees are sunk at this stage. We have FOC_{p_1} : $\alpha\gamma - \alpha + 2p_1 - \gamma p_2 - w_1 = 0$ and $\text{FOC}_{p_2} : \alpha\gamma - \alpha - \gamma p_1 + 2p_2 - w_2 = 0$. Therefore the second order conditions remain satisfied $\text{SOC}_{p_1} : \frac{2}{\gamma^2-1} < 0$ and $\text{SOC}_{P_2} : \frac{2}{\gamma^2-1} < 0$. The FOC system delivers the sub-game pricing strategies $p_1^{TD}(w_1,w_2) = p_1^D(w_1,w_2)$ and $p_2^{TD}(w_1,w_2) = p_2^D(w_1,w_2)$ of equation 6.

In addition, we find that the rival accepts the contract whenever $\pi_2(w_2,f_2) \geq 0$ leading to $f_2 \leq (p_2^{TC}(w_1,w_2) - w_2)q_2(p_2^{TC}(w_1,w_2), p_1^{TC}(w_1,w_2))$. The same applies to the downstream unit and $f_1 \leq (p_1^{TC}(w_1,w_2) - w_1)q_1(p_1^{TC}(w_1,w_2), p_2^{TC}(w_1,w_2))$. However, if the downstream unit rejects the offer then the VIP nevertheless internalizes the downstream unit's profit. This happens because the downstream unit is vertically integrated and is independent solely to set the final price. By internalyzing, the downstream unit's profit, which also includes the fixed fee f_1, the VIP makes f_1 irrelevant in its contracting strategy.

At the input pricing stage, the VIP accounts for sub-game strategies which leads to $V(w_1,w_2,f_2) = p_1^{TD}(w_1,w_2)q_1(p_1^{TD}(w_1,w_2), p_2^{TD}(w_1,w_2)) + w_2q_2(p_1^{TD}(w_1,w_2), p_2^{TD}(w_1,w_2)) + f_2$, for any f_1. The VIP maximizes this expression with respect to the input prices w_1 and w_2, and the fixed fee f_2. We focus on equilibria where the rival accepts the offer. Because the profit is increasing in f_2, the VIP sets the fixed fee to extract all the rival's profit.

The VIP's profit expression simplifies to $V(w_1, w_2) = p_1^{TD}(w_1, w_2)q_1(p_1^{TD}(w_1, w_2), p_2^{TD}(w_1, w_2))+ p_2^{TD}(w_1, w_2)q_2(p_1^{TD}(w_1, w_2), p_2^{TD}(w_1, w_2))$. The VIP then sets w_1 and w_2 to maximize this expression. We obtain the following first order conditions $\text{FOC}_{w_1} : \alpha(\gamma - 1)(\gamma + 2)^2\gamma + \left(8 - 6\gamma^2\right) w_1 - 2\gamma^3 w_2 = 0$, and $\text{FOC}_{w_2} : \alpha(\gamma - 1)(\gamma + 2)^2\gamma - 2\gamma^3 w_1 + \left(8 - 6\gamma^2\right) w_2 = 0$. Note that the Hessian matrix is definite semi-negative (we find $\partial^2 V/\partial w_1^2 = \partial^2 V/\partial w_2^2 = -\frac{8-6\gamma^2}{(1-\gamma^2)(4-\gamma^2)^2} < 0$ and $(\partial^2 V/\partial w_1^2)(\partial^2 V/\partial w_2^2) - (\partial^2 V/\partial w_1 \partial w_2)(\partial^2 V/\partial w_2 \partial w_1) = \frac{4}{(1-\gamma^2)(4-\gamma^2)^2} > 0$). By solving the FOC system, we find the equilibrium input prices set by the VIP under decentralization

$$w_1^{TD} = w_2^{TD} = \frac{\alpha\gamma}{2}$$

Finally, we substitute these input prices by their equilibrium values into the sub-game pricing strategies to obtain the equilibrium prices $p_1^{TD} = p_1^{TD}(w_1^{TD}, w_2^{TD})$ and $p_2^{TD} = p_2^{TD}(w_1^{TD}, w_2^{TD})$. It remains to substitute these equilibrium prices into the demand functions and then profit functions to get the last equilibrium outcomes. In particular, the VIP's profit is:

$$V^{TD} = \frac{\alpha^2}{2(\gamma + 1)}$$

Comparison with centralization.

Finally, we compare the equilibrium profits and find:

$$V^{TD} - V^{TC} = \frac{\alpha^2(1-\gamma)\gamma^2}{4(\gamma + 1)\left(5\gamma^2 + 4\right)} \geq 0$$

⋄ Decentralization with unobservable contracts (passive beliefs)

The game modifies as follows: the VIP makes bilateral offers that are unobservable by the third party. We suppose the firms hold passive beliefs.

We now look for the PBNE in pure strategies. We again use backward induction to solve the game. At the downstream competition stage, the downstream unit and the rival maximize respectively $\pi_1(p_1, p_2) = (p_1 - w_1)q_1(p_1, p_2^{TS}) - f_1$ and $\pi_2(p_1, p_2) = (p_2 - w_2)q_2(p_2, p_1^{TS}) - f_2$ with respect to p_1 and p_2, and given their passive beliefs p_2^{TS} and p_1^{TS}. This gives the same first and second order conditions as without non-linear contracts as the fixed fees are sunk at this stage. We find $p_1^{TS}(w_1) = p_1^S(w_1)$ and $p_2^{TS}(w_2) = p_2^S(w_2)$.

The rival accepts the contract whenever $\pi_2(w_2, f_2) \geq 0$ which leads to $f_2 \leq (p_2^{TC}(w_2) - w_2)q_2(p_1^{TC}, p_2^{TC}(w_2))$. Importantly, note that the demand function uniquely depends on w_2. This happens because when accepting the fixed fee, the rival uses its belief about the downstream unit's strategy. Similarly, the downstream unit accepts the fixed fee whenever $f_1 \leq (p_1^{TC}(w_1) - w_1)q_1(p_1^{TC}(w_1), p_2^{TC})$ which only depends on w_1. But like the observable case, even if the downstream unit rejects the VIP's offer, the VIP still internalizes the whole downstream unit's profit (including f_1). The fixed fee f_1 is thus unsound.

At the input pricing stage, the VIP accounts for sub-game strategies. We thus substitute the sub-game strategies into the VIP profit function leading to $V(w_1, w_2, f_2) = p_1^{TS}(w_1)q_1(p_1^{TS}(w_1), p_2^{TS}(w_2)) + w_2q_2(p_2^{TS}(w_2), p_1^{TS}(w_1)) + f_2$ and maximize this expression with respect to the input prices w_1 and w_2, and the fixed fee f_2. We focus our analysis on equilibria where the rival accepts the offer. Because the profit is increasing in f_2, the VIP sets the fixed fee to extract all the rival's profit.

The profit simplifies to $V(w_1, w_2) = p_1^{TS}(w_1)q_1(p_1^{TS}(w_1), p_2^{TS}(w_2)) + w_2q_2(p_2^{TS}(w_2), p_1^{TS}(w_1)) + (p_2^{TS}(w_2) - w_2)q_2(p_S^{TS}(w_2), p_1^{TS})$. The VIP then sets w_1 and w_2 to maximize this expression. We obtain the following first order conditions FOC_{w_1} : $2w_1 + \gamma(2p_1^{TS} - 3w_2 - \alpha(1-\gamma) - p_1^{TS}\gamma) = 0$, and FOC_{w_2} : $2w_2 + 4\gamma p_1^{TS} - 3\gamma(w_1 + \alpha(1-\gamma) + \gamma p_2^{TS}) = 0$. Once more, we check the second order condition. We get $\partial^2 V/\partial w_1^2 = \partial^2 V/\partial w_2^2 = -\frac{1}{2(1-\gamma^2)} < 0$. Moreover, we find $(\partial^2 V/\partial w_1^2)(\partial^2 V/\partial w_2^2) - (\partial^2 V/\partial w_1 \partial w_2)(\partial^2 V/\partial w_2 \partial w_1) = \frac{4-9\gamma^2}{16(1-\gamma^2)^2} > 0$ whenever $|\gamma| < \frac{2}{3}$ meaning that the SOC is satisfied (the Hessian matrix is definite semi-negative) only when $|\gamma| < \frac{2}{3}$. By solving the system composed of the two previous FOC and the fact that equilibrium strategies match equilibrium beliefs $p_i^{TS}(w_i, p_j^{TS}) = p_i^{TS}$, $\forall i \neq j = 1, 2$, we find the equilibrium input prices set by the VIP under decentralization w_1^{TS} and w_2^{TS} joint with the equilibrium price beliefs p_1^{TS} and p_2^{TS}. In particular, we have

$$w_1^{TS} = \frac{\alpha\gamma^2(\gamma+2)}{4(\gamma+1)} \text{ and } w_2^{TS} = \frac{\alpha\gamma(\gamma+2)}{4(\gamma+1)}$$

Interestingly, note that due to the asymmetry led by the unsoundness of f_1, the input prices differ $w_1^{TS} = \gamma w_2^{TS}$ and are higher than zero. This contrasts the usual outcomes without integration.

It remains to substitute these equilibrium prices into the demand functions, profit functions to get the last equilibrium outcomes. In particular, the VIP's profit is:

$$V^{TS} = \frac{\alpha^2\left(-\gamma^2 + 8\gamma + 8\right)}{16(\gamma+1)^2}$$

Comparison with unobservable contracts.

Finally, we compare the equilibrium profits and find:

$$V^{TS} - V^{TC} = -\frac{9\alpha^2\gamma^4}{16(\gamma+1)^2\left(5\gamma^2+4\right)} \leq 0 \quad \square$$

Proof of Subgame Perfect Equilibrium with Cost asymmetry. We now assume that the downstream bears a per-unit transformation cost denoted c_1 while the rival bears a per-unit transformation cost denoted c_2. We further suppose that $\alpha_i = \alpha - c_i$ denotes firm i's efficiency. In particular, $\alpha_1 \geq \alpha_2$ means the downstream unit is more efficient than the downstream rival whereas $\alpha_1 < \alpha_2$ means the reverse.

⋄ Centralization

The equilibrium with asymmetric transformation costs has been derived by Arya *et al.* (2008). We display their results below. The equilibrium input price is:

$$w_2^C = \frac{\alpha_2}{2} - \frac{(\alpha_2 - \alpha_1\gamma)\gamma^2}{2\left(\gamma^2+8\right)}$$

and the final prices, profits and consumer surplus, respectively, are

$$p_1^C = c_1 + \frac{(8-\gamma^2)\alpha_1 + 2\gamma\alpha_2}{2(8+\gamma^2)}\ , \quad p_2^C = c_2 + \frac{2(6+\gamma^2)\alpha_2 - \gamma(4+\gamma^2)\alpha_1}{2(8+\gamma^2)}$$

$$V^C = \frac{(8-3\gamma^2-\gamma^4)(\alpha_1)^2 + 4(\alpha_2)^2 - 8\gamma\alpha_1\alpha_2}{4(8-7\gamma-\gamma^4)}\ , \ \pi_2^C = \frac{(2+\gamma^2)^2(\alpha_2-\gamma\alpha_1)^2}{(1-\gamma^2)(8+\gamma^2)^2}, \text{ and}$$

$$CS^C = \frac{(64-23\gamma^4-5\gamma^6)(\alpha_1)^2 + 4(4+5\gamma)(\alpha_2)^2 - 4(16+\gamma^2-\gamma^4)\gamma\alpha_1\alpha_2}{8(1-\gamma^2)(8+\gamma^2)^2}$$

Likewise, we find that the VIP does not foreclose the rival under centralization as long as $\gamma < \alpha_2/\alpha_1$.

$\diamond$ Decentralization (observable contracts)

Following the same procedure as with symmetric firms, we find the equilibrium input prices:

$$w_1^D = \gamma\frac{1}{4}(\alpha_2 + \alpha_1\gamma) \text{ and } w_2^D = \frac{\alpha_2}{2}\ ,$$

which leads to the following equilibrium prices, profits and consumer surplus:

$$p_1^D = c_1 + \frac{\alpha_1}{2}\ , \quad p_2^D = c_2 + \frac{3\alpha_2 - \alpha_1\gamma}{4}$$

$$V^D = \frac{(\alpha_1)^2\left(2-\gamma^2\right) - 2\alpha_1\alpha_2\gamma + (\alpha_2)^2}{8\left(1-\gamma^2\right)}\ , \quad \pi_2^D = \frac{(\alpha_2-\alpha_1\gamma)^2}{16\left(1-\gamma^2\right)}\ , \text{ and}$$

$$CS^P = \frac{(\alpha_1)^2\left(4-3\gamma^2\right) - 2\alpha_1\alpha_2\gamma + (\alpha_2)^2}{32\left(1-\gamma^2\right)}.$$

$\diamond$ Comparison between centralization and decentralization (observable contracts)

General result.

By comparing the VIP's profits under the two schemes we find

$$V^D - V^C = \frac{\gamma^2(\alpha_2-\alpha_1\gamma)^2}{8\left(-\gamma^4-7\gamma^2+8\right)}$$

This confirms our results for the asymmetric case: the VIP still benefits from decentralization. However, the gain now depends on the VIP's relative downstream efficiency.

Effect of cost asymmetry on the VIP's gains.

Compare with the symmetric case, we easily see that the main difference is the term $(\alpha_2 - \alpha_1\gamma)^2$ in the expression of gains displayed above. Three cases arise:

1. Case $\alpha_1 = \alpha$. Suppose $\alpha_1 = \alpha$ and $\alpha_2 = \epsilon\alpha$, we then find $(\alpha_2 - \alpha_1\gamma)^2 = (\alpha(\epsilon - \gamma))^2$. If $\epsilon > 1$ then the rival is more efficient than the downstream unit and the reverse holds for $\epsilon < 1$. Importantly, when $\epsilon = 1$, we are back to the symmetric case. We thus have to compare the expression $(\alpha(\epsilon - \gamma))^2$ when $\epsilon \neq 1$ to the expression $(\alpha(1 - \gamma))^2$ to obtain the impact of asymmetric firm on the VIP's gains from decentralization.

$$(\alpha(\epsilon - \gamma))^2 > (\alpha(1 - \gamma))^2$$

   Remind that the VIP foreclose the rival when $\gamma > \frac{\alpha_2}{\alpha_1} = \epsilon$, we thus analyse only the cases where $\epsilon > \gamma$. This implies that the above inequality simplifies to

$$\epsilon > 1$$

   We find the VIP earns more when the downstream unit is more efficient than the rival ($\alpha_2 > \alpha_1 = \alpha$) but earns less when the reverse occurs ($\alpha_2 < \alpha_1 = \alpha$).

2. Case $\alpha_2 = \alpha$. Same result just take $\alpha_1 = \frac{1}{\epsilon}\alpha_2$, and consider again cases where $\gamma < \epsilon$.

3. Case $\alpha_2 \neq \alpha_1 \neq \alpha$. Same result just take $\alpha_2 = \epsilon\alpha_1$ and $\alpha_1 = \epsilon'\alpha$, and consider cases where $\gamma < \epsilon$.

$\square$

Equilibrium when w_1 is hidden from D_2 and D_2 does not know if D_1 observes w_2. In that case, the method is similar to the *mixed regime* scheme in Petrakis and Skartados (2018).

At the last stage, firm 2 only takes into account its own offer when setting its price and does not consider any rival's reaction (set at equilibrium p_1^*). Formally, its sub-game price strategy is the same as under secret contracts $p_2^*(w_2, p_1^*) = \frac{\alpha(1-\gamma)+w_2+\gamma p_1^*}{2}$. Firm 1 instead knows the two input prices and anticipates the reaction of firm 2. Its reaction function is the same as under public contract or total separation $p_1^*(w_1, w_2) = \frac{\alpha(2-\gamma-\gamma^2)+\gamma w_2+2w_1}{4-\gamma^2}$.

At the contracting stage, the VIP maximizes its profits with respect to w_1 and w_2 taking into account the continuation equilibrium prices and the passive beliefs of the firms. Solving the system composed by the first order conditions with respect to w_1 and w_2 and the matching between continuation equilibrium prices and beliefs i.e. $p_1^*(w_1, w_2) = p_1^*$, we obtain the optimal values $w_1^* = \frac{\alpha\gamma(\gamma+2)^2}{8(\gamma+1)}$, $w_2^* = \frac{\alpha(\gamma+2)^2}{8(\gamma+1)}$ and $p_1^* = \frac{\alpha(5\gamma+4)}{8(\gamma+1)}$, which are the same values as under secret contracts. Ultimately, we find the same equilibrium outcome as with our secret contracts setting with passive beliefs (and under the same condition that $-\frac{2\sqrt{2}}{3} < \gamma < \frac{2\sqrt{2}}{3}$). □

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