ECE 5510 Fall 2006: Homework 4 Solutions

1. Y&G 2.9.5. See Page 4.
2. Y&G 2.8.8. See Page 4.
3. Y&G 3.7.3. See Page 4.
4. **Uniform Wait Time**:

   (a) If you arrive at time M between H:00 and H:30, the time to bus arrival $X = 30 - M$. Since M is uniformly distributed between 0 and 30, X is also uniformly distributed between 0 and 30. (You can prove this either by Method of Moments, by the Jacobian method, or by observation). If M had been between H:30 and $H + 1$:00, the same distribution for X would result. Thus, $f_X(x) = 1/30$, for $0 \le X < 30$, and zero otherwise.

   (b) The given event $A = \{X > 10\}$. Note that $P[A] = \int_{10}^{30} 1/30 dx = 2/3$. Plugging into the formula for conditional pdf,

   $$f_{X|A}(x) = \begin{cases} \frac{1/30}{2/3}, & 10 < x < 30 \\ 0, & o.w. \end{cases} = \begin{cases} 1/20, & 10 < x < 30 \\ 0, & o.w. \end{cases}$$

   The additional time that you will wait (beyond the 10 minutes), *i.e.*, $X - 10$, is uniform on a 20 minute interval, no longer a 30 minute interval, so yes the distribution has changed.

5. **Exponential Wait Time**: A different bus system (the EXP line) does not have a regular schedule. Instead, your measurements have shown that X, the time between when you arrive at the bus stop and the bus arrival, is exponentially distributed with parameter $\lambda = 1/15$. Note that $f_X(x) = \frac{1}{15}e^{-\frac{1}{15}x}$, for $x \ge 0$, and zero otherwise.

   (a) What is the pdf of X given that you have already waited for 10 minutes without seeing a bus? Answer: Let event $A = \{X > 10\}$. We have

   $$P[A] = \int_{10}^{\infty} \frac{1}{15} e^{-\frac{1}{15}x} dx = -e^{-\frac{1}{15}x}\Big|_{10}^{\infty} = e^{-\frac{10}{15}}$$

   Then, plugging into the formula for conditional pdf,

   $$f_{X|A}(x) = \begin{cases} \frac{\frac{1}{15}e^{-\frac{1}{15}x}}{e^{-\frac{10}{15}}}, & x > 10 \\ 0, & o.w. \end{cases} = \begin{cases} \frac{1}{15}e^{-\frac{1}{15}(x-10)}, & x > 10 \\ 0, & o.w. \end{cases}$$

   (b) The additional time you'll wait for the bus is $Y = X - 10$. You can see from observation, using the Jacobian method, or the method of moments, that the distribution of $Y|A$ is the same as the distribution of X – exponential with parameter $\lambda = 1/15$.

6. Using simulation to verify analysis:

   (a) Analysis: Let X be uniform on the interval $(0, 1)$. Let $Y = g(X) = -\ln X$. Derive $f_y(Y)$. Answer: Note that $f_X(x) = 1$ for $0 < x < 1$, and zero otherwise. A plot shows that $g(X) = -\ln X$ is a purely decreasing function.

- By the method of moments,

  $$F_Y(y) = P[Y \le y] = P[-\ln X \le y] = P[\ln X \ge -y] = P\left[X \ge e^{-y}\right] = 1 - P\left[X < e^{-y}\right].$$

  Note that the CDF $F_X(x) = x$ for $0 < x < 1$, zero for $x \le 0$ and 1 for $x \le 1$. Then, $F_Y(y) = 1 - e^{-y}$ as long as $0 < e^{-y} < 1$, which is true for $y > 0$. So,

  $$F_Y(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-y}, & y \ge 0 \end{cases}$$

  Then, taking the derivative w.r.t. y,

  $$f_Y(y) = \frac{\partial}{\partial y} F_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & o.w. \end{cases}$$

- By the Jacobian method: Note $X = g^{-1}(Y) = e^{-Y}$. Taking the derivative w.r.t. y, $\frac{\partial}{\partial y} g^{-1}(y) = \frac{\partial}{\partial y} e^{-y} = -e^{-y}$. Then

  $$f_Y(y) = f_X(e^{-y})e^{-y} = \begin{cases} e^{-y}, & y > 0 \\ 0, & o.w. \end{cases}$$

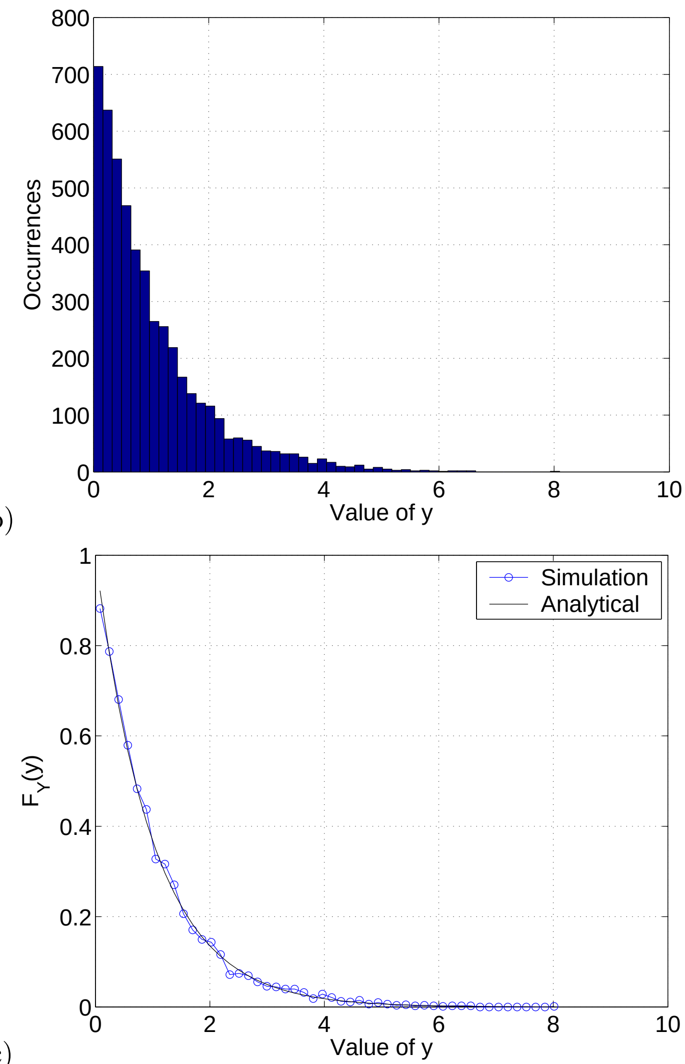
(b) Simulation: Figure 1(b) shows the output of this Matlab script:

```
trials  = 5000;
bins    = 50;
X       = rand(1, trials);          % X is uniform between 0 and 1
Y       = -log(X);
[N,BC]  = hist(Y, bins);
delta_y = BC(2)-BC(1);              % width of bin
est_fY  = (N./trials)./delta_y;
analytical_fY = exp(-BC);
plot(BC, est_fY, 'b-o', BC, analytical_fY, 'k-');
legend('Simulation','Analytical')
set(gca,'FontSize',18);
xlabel('Value of y')
ylabel('F_Y(y)')
grid
```

(c) Verification: Show that the integral of $\tilde{f}_Y(y)$ is one.

$$\begin{aligned} \int_{-\infty}^{\infty} \tilde{f}_Y(y) dy &= \sum_{i=1\dots 50} \frac{N(i)}{(5000)\Delta_y} \left[BC(i) + \frac{\Delta_y}{2} - (BC(i) - \frac{\Delta_y}{2})\right] \\ &= \frac{\sum_{i=1\dots 50} N(i)}{5000\Delta_y} \Delta_y = 1. \end{aligned}$$

Plot is shown in Figure 1(c).



(b)

(c)

Figure 1: Figures from Problem 6(b) and 6(c).

Problem 2.9.5 Solution

The probability of the event B is

$$P[B] = P[X \geq \mu_X] = P[X \geq 3] = P_X(3) + P_X(4) + P_X(5) \tag{1}$$

$$= \frac{\binom{5}{3} + \binom{5}{4} + \binom{5}{5}}{32} = \cancel{21}\,16/32 = 1/2 \tag{2}$$

The conditional PMF of X given B is

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \binom{5}{x}\frac{1}{\cancel{21}\,16} & x = 3,4,5 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The conditional first and second moments of X are

$$E[X|B] = \sum_{x=3}^{5} x P_{X|B}(x) = 3\binom{5}{3}\frac{1}{\cancel{21}\,16} + 4\binom{5}{4}\frac{1}{\cancel{21}\,16} + 5\binom{5}{5}\frac{1}{\cancel{21}\,16} \tag{4}$$

$$= [30 + 20 + 5]/\cancel{21}\,16 = 55/\cancel{21}\,16 \tag{5}$$

$$E[X^2|B] = \sum_{x=3}^{5} x^2 P_{X|B}(x) = 3^2\binom{5}{3}\frac{1}{\cancel{21}\,16} + 4^2\binom{5}{4}\frac{1}{\cancel{21}\,16} + 5^2\binom{5}{5}\frac{1}{\cancel{21}\,16} \tag{6}$$

$$= [90 + 80 + 25]/\cancel{21}\,16 = 195/\cancel{21}\,16 \tag{7}$$

The conditional variance of X is

$$195/16 - (55/16)^2 = \frac{195}{16} - \frac{3025}{256} = \frac{95}{256}$$

$$\text{Var}[X|B] = E[X^2|B] - (E[X|B])^2 = \cancel{65/7 - (55/21)^2 = 1070/441 = 2.43} \tag{8} \simeq 0.37$$

Problem 2.8.8 Solution

Given the following description of the random variable Y,

$$Y = \frac{1}{\sigma_x}(X - \mu_X) \tag{1}$$

we can use the linearity property of the expectation operator to find the mean value

$$E[Y] = \frac{E[X - \mu_X]}{\sigma_X} = \frac{E[X] - E[X]}{\sigma_X} = 0 \tag{2}$$

Using the fact that $\text{Var}[aX + b] = a^2 \text{Var}[X]$, the variance of Y is found to be

$$\text{Var}[Y] = \frac{1}{\sigma_X^2} \text{Var}[X] = 1 \tag{3}$$

Problem 3.7.3 Solution

Since X is non-negative, $W = X^2$ is also non-negative. Hence for $w < 0$, $f_W(w) = 0$. For $w \geq 0$,

$$F_W(w) = P[W \leq w] = P[X^2 \leq w] \tag{1}$$

$$= P[X \leq w] \tag{2}$$

$$= 1 - e^{-\lambda\sqrt{w}} \tag{3}$$

Taking the derivative with respect to w yields $f_W(w) = \lambda e^{-\lambda\sqrt{w}}/(2\sqrt{w})$. The complete expression for the PDF is

$$f_W(w) = \begin{cases} \frac{\lambda e^{-\lambda\sqrt{w}}}{2\sqrt{w}} & w \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$