c) 1D shallow water eqns:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x} \quad , \quad \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uh) = 0 \quad , \; h = \eta - \eta_b ,$$

At $(u, \eta) = (0,0)$:

$$\boxed{\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}} \quad , \quad \frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x}\cdot h + \cancelto{0}{\frac{\partial h}{\partial x}}\cdot u = 0$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} h = 0$$

Then since η_b is constant and $\eta = 0$, we can approximate h as H where H is the water column height at rest. This gives:

1) $$\boxed{\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}} \qquad \boxed{\frac{\partial \eta}{\partial t} = -H\frac{\partial u}{\partial x}}$$

Putting these into the form of equations 9.29 in the text:

Define the spacing: $- 2\Delta x$

$V = \sqrt{gH}$

$h = \Delta t$

we get:

2) $$\eta(x, t+\Delta t) = \eta(x,t) + \Delta t\, u(x,t)$$
$$u(x, t+\Delta t) = u(x,t) + \Delta t \left(\frac{\sqrt{gH}}{\Delta x}\right)^2 \left[\eta(x+2\Delta x, t) + \eta(x-2\Delta x, t) - 2\eta(x,t)\right]$$

Then considering the two variables to be elements of a 2D vector (η, u), we write the Fourier series solution as:

$$\begin{pmatrix} \eta(x,t) \\ u(x,t) \end{pmatrix} = \begin{pmatrix} C_\eta(t) \\ C_u(t) \end{pmatrix} e^{ikx}$$

Substituting this into equations 2), we get that the coefficients are:

$$C_\eta(t+\Delta t) = C_\eta(t) + \Delta t \, C_u(t) \quad , \quad C_u(t+\Delta t) = C_u(t) - \Delta t \, C_\eta(t) \frac{gH}{\Delta x^2} \sin^2(\Delta x k)$$

In vector form: $\vec{c}(t+\Delta t) = \vec{A}\vec{c}(t)$

where $\vec{A}$ is:

$$\vec{A} = \begin{pmatrix} 1 & \Delta t \\ -\Delta t r^2 & 1 \end{pmatrix} \quad , \quad r = \frac{\sqrt{gH}}{\Delta x} \sin(\Delta x k)$$

The eigenvalues of this matrix can be found using $\det(A - \lambda I) = 0$, where λ is the eigenvalue.

$$\det(A - \lambda I) = (1-\lambda)^2 + \Delta t^2 r^2 = 0$$

Solving for λ:

$$(1-\lambda)^2 = -\Delta t^2 r^2$$

$$\lambda = \pm\sqrt{-\Delta t^2 r^2} + 1$$

$$\lambda = 1 \pm i \Delta t r$$

The magnitude of λ can be written: $|\lambda| = \sqrt{1 + \Delta t^2 r^2}$.

Then substituting $r = \frac{\sqrt{gH}}{\Delta x}\sin(\Delta x k)$, the magnitude of the eigenvalues λ is:

$$\boxed{|\lambda| = \sqrt{1 + \left(\frac{\Delta t}{\Delta x}\right)^2 gH \sin^2(\Delta x k)}}$$

as wanted.

So we have the two linear equations we want.
Applying Euler's method?

$$\left.\begin{aligned} \underset{\frac{du}{dt}\,t+h}{u(x,t+h)} &= \underset{\frac{du}{dt}}{u(x,t+h)} - gh\frac{\partial \eta}{\partial x} \\ \eta(x,t+h) &= \eta(x,t) - hH\,\partial u \end{aligned}\right\} \quad 2)$$

$$\frac{\partial n}{\partial t}\Big|_{t+h} \qquad \frac{\partial n}{\partial t}$$

Consider the two variables to be a vector (u, n) and write as Fourier series solution:

$$\begin{pmatrix} u(x,t) \\ n(x,t) \end{pmatrix} = \begin{pmatrix} c_u(t) \\ c_\eta(t) \end{pmatrix} e^{ikx} \qquad \begin{pmatrix} u(x,t+h) \\ \eta(x,t+h) \end{pmatrix} = \begin{pmatrix} c_u(t+h) \\ c_n(t+h) \end{pmatrix} e^{ikx}$$

Substituting into 2), we get:

$$u(x,t+h) = c_u(t)\, e^{ikx} - gh \frac{\partial n}{\partial x}$$

$$\eta(x,t+h) = c_\eta(t) e^{ikx} - hH \frac{\partial u}{\partial x}$$