FOURIER TRANSFORM MODEL OF IMAGE FORMATION

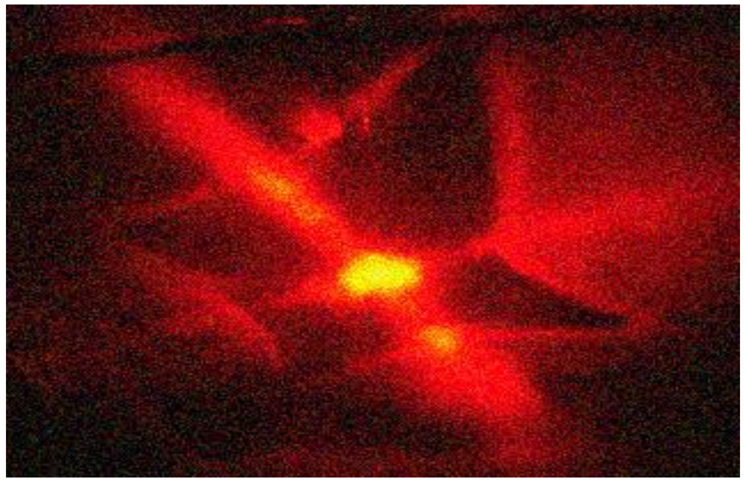
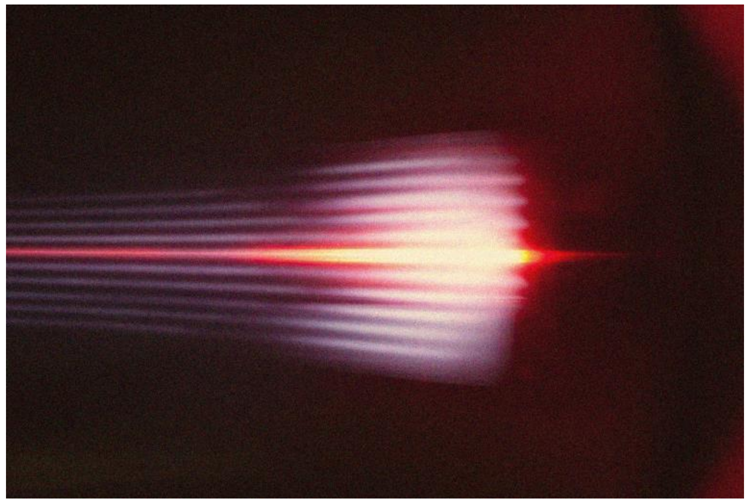


IMAGE AND VIDEO PROCESSING – MODULE 1



NINO PHILIP RAMONES | GITHUB

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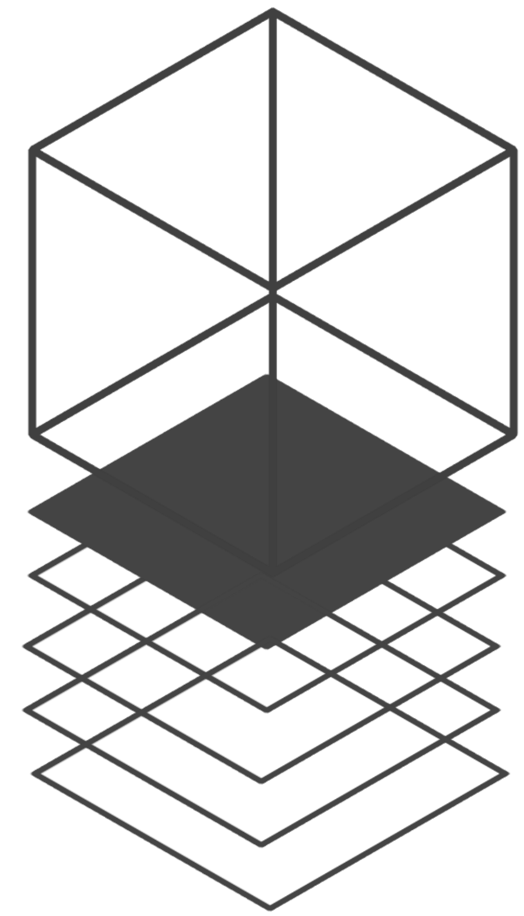
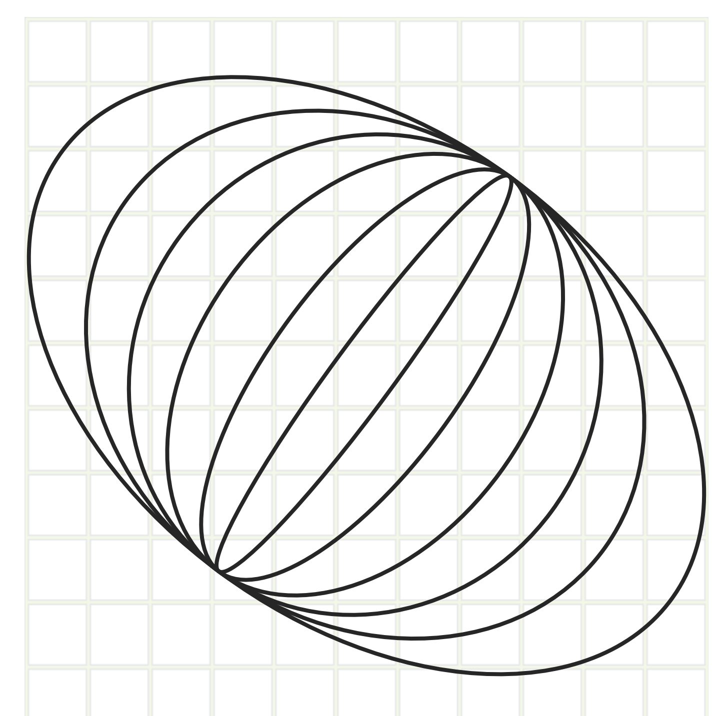
OBJECTIVES

- Familiarize with the rudiments of **Fourier transform** and its various applications in simulating diffraction patterns and various imaging systems
- Observe how Fourier transform comes into play in **image convolution and correlation**



KEY TAKEAWAYS

- Setting your **image resolution** to any powers of two will greatly speed up the calculation process when subjected to FFT algorithms
- Several imaging systems can be simulated using FFT by image convolution
- Different **aperture geometry** yields a unique diffraction pattern
- Some of the **real-life applications** of FFT are template matching and pattern recognition --- through the image correlation algorithm



SOME PITFALLS

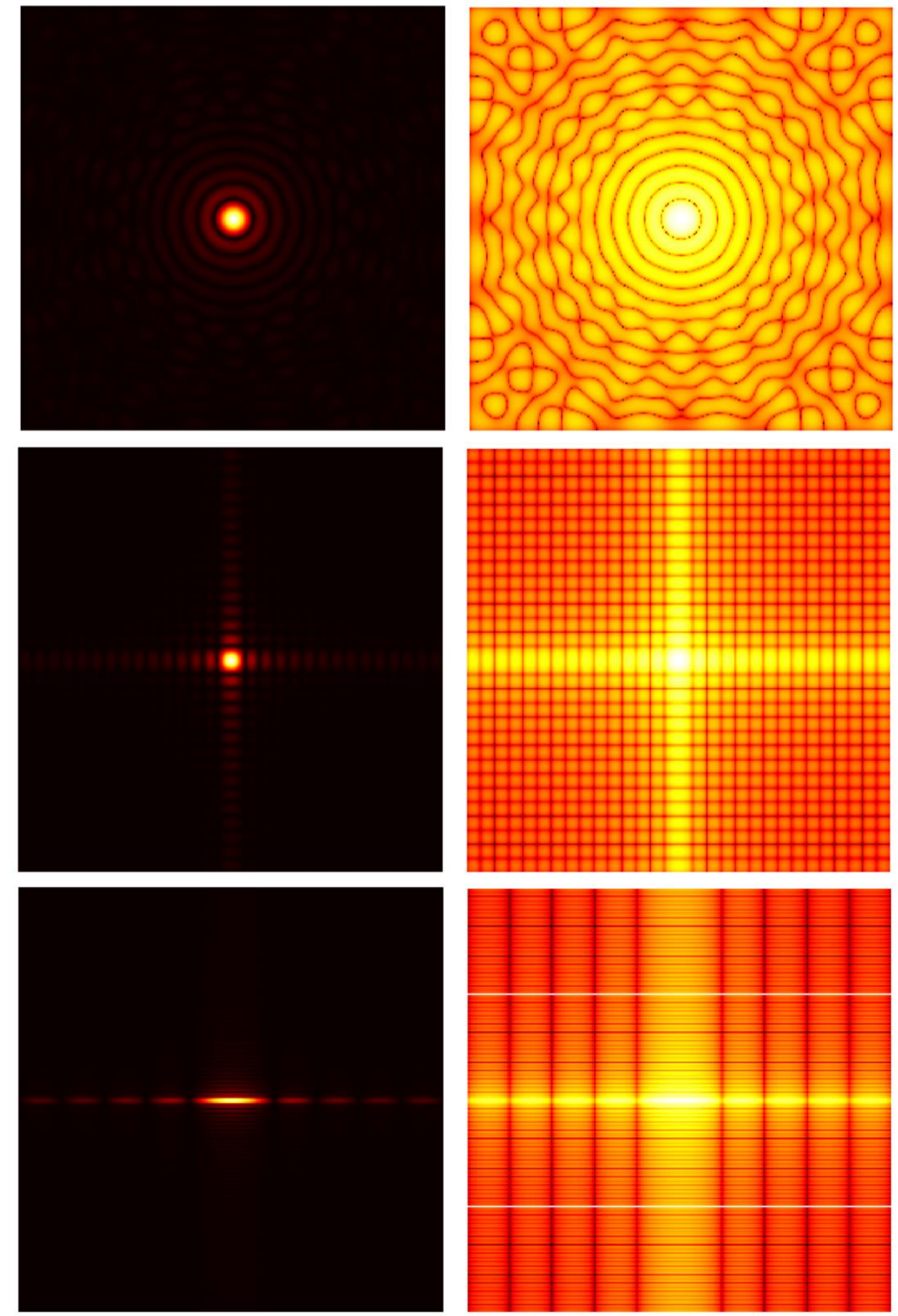
- The two images of interest that must be convolved or correlation must be of the **same size** as it processes both images by checking and comparing it pixel by pixel
- Different image resolutions can somehow produce a **different output** that is far from the expected one; choose the right one through trial and error!

FAR FIELD DIFFRACTION PATTERNS

As described and modelled by Fraunhoffer diffraction

The first part of the activity involves the creation of synthetic images of **apertures** of varying shapes and sizes and taking its corresponding **Fourier transform**. We observe interesting diffraction patterns of such light beam, or monochromatic plane wave in general, as it strikes a given aperture, set in a [256 by 256] image resolution.

For brevity, the projected FFT of the aperture was also set in **logarithmic scale** to see the patterns explicitly. Some of the key results were shown on the right and on the next slides!



WHY DO WE USE 256 OR ANY 2^N?

The Fast Fourier Transform (FFT) is said to be the faster implementation of the Discrete Fourier Transform (DFT), for which the activity intends to simulate. In general, FFT algorithm reduces an n-point FT to about

$$\frac{n}{2}\log_2(n)$$

complex multiplications. Since 256 data points were set for the images in this activity, the DFT calculation would simply go through the resolution by 256 times 256 or around **2^{16} multiplications**. Comparatively, the FFT algorithm would only take, using the equation above, **1024 calculations** only, which is faster by a factor of 64 times! These makes computations much faster for 2^N discrete points as it all boils down to the computer doing all the mathematical rigors with ease.

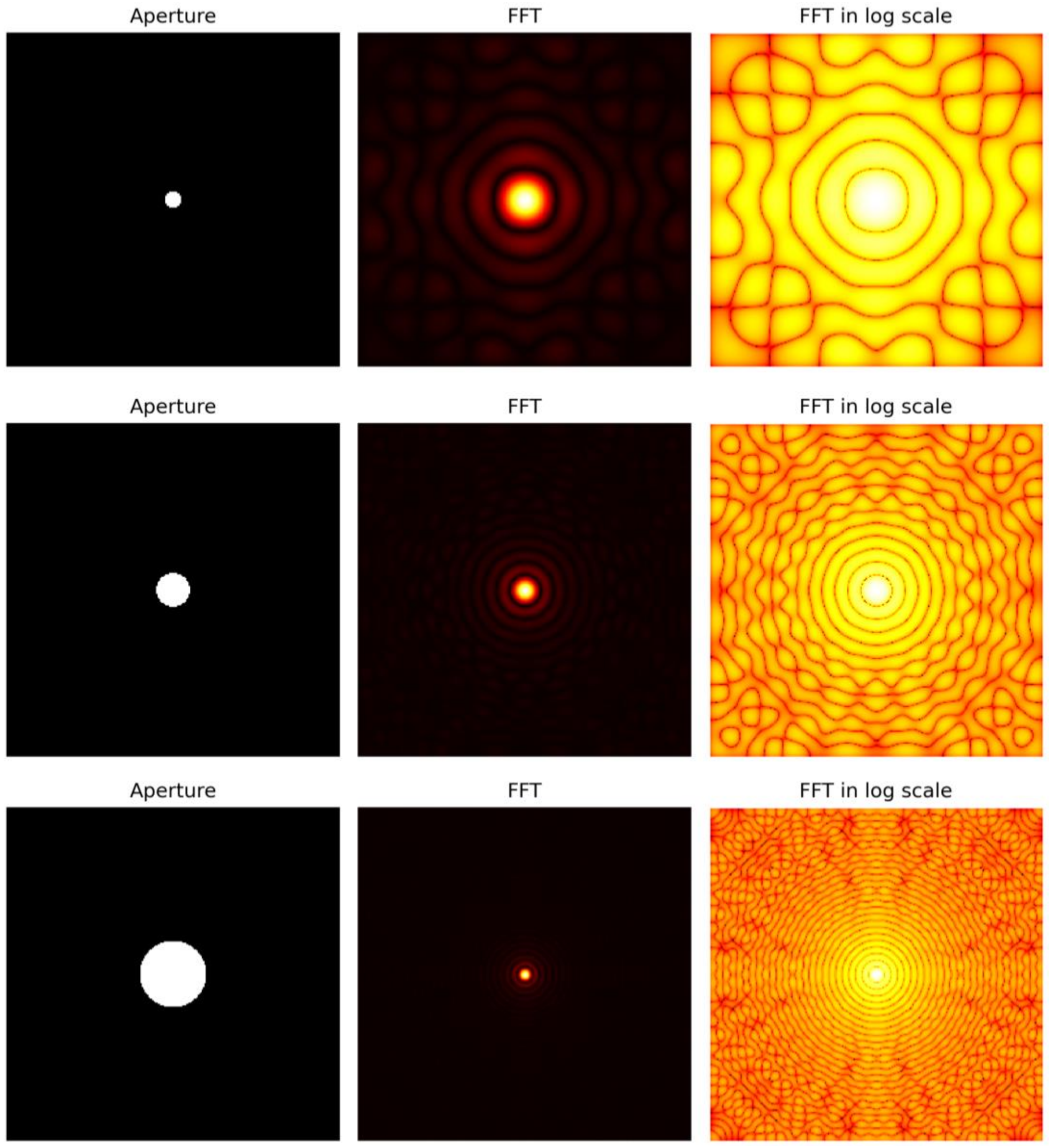


Figure 1. The FFT and FFT in log scale of circular apertures of varying radii.

CIRCULAR APERTURES

Varying radii of circular apertures were considered, and it can be observed that the prominence of **airy pattern** decreases as the aperture radius increases. These airy patterns contribute to the extent of how light rays can get scattered as it passes through the said aperture, i.e. the **radial patterns** that propagate from the center represents the peaks at the real space domain. Interestingly, increasing the radius implies clearer image as the airy pattern diminishes. Hence, **greater recovery** on the details of an object is expected.

$$I(\theta) = I_0 \left[\frac{2J_1\left(ka\sin\theta\right)}{ka\sin\theta} \right]$$

As shown above, the **diffraction pattern intensity** for a circular aperture takes the form of a **Bessel function**, which is evident in the patterns in the logscale FFT. This special function usually arises in systems that invokes radial symmetry like in optical diffraction and similar optical systems.

RECTANGULAR APERTURES

Other aperture geometry were also tried on for this activity. Rectangular apertures of varying widths and heights were prepared to resemble such **single slit**. The following observations were made as follows.

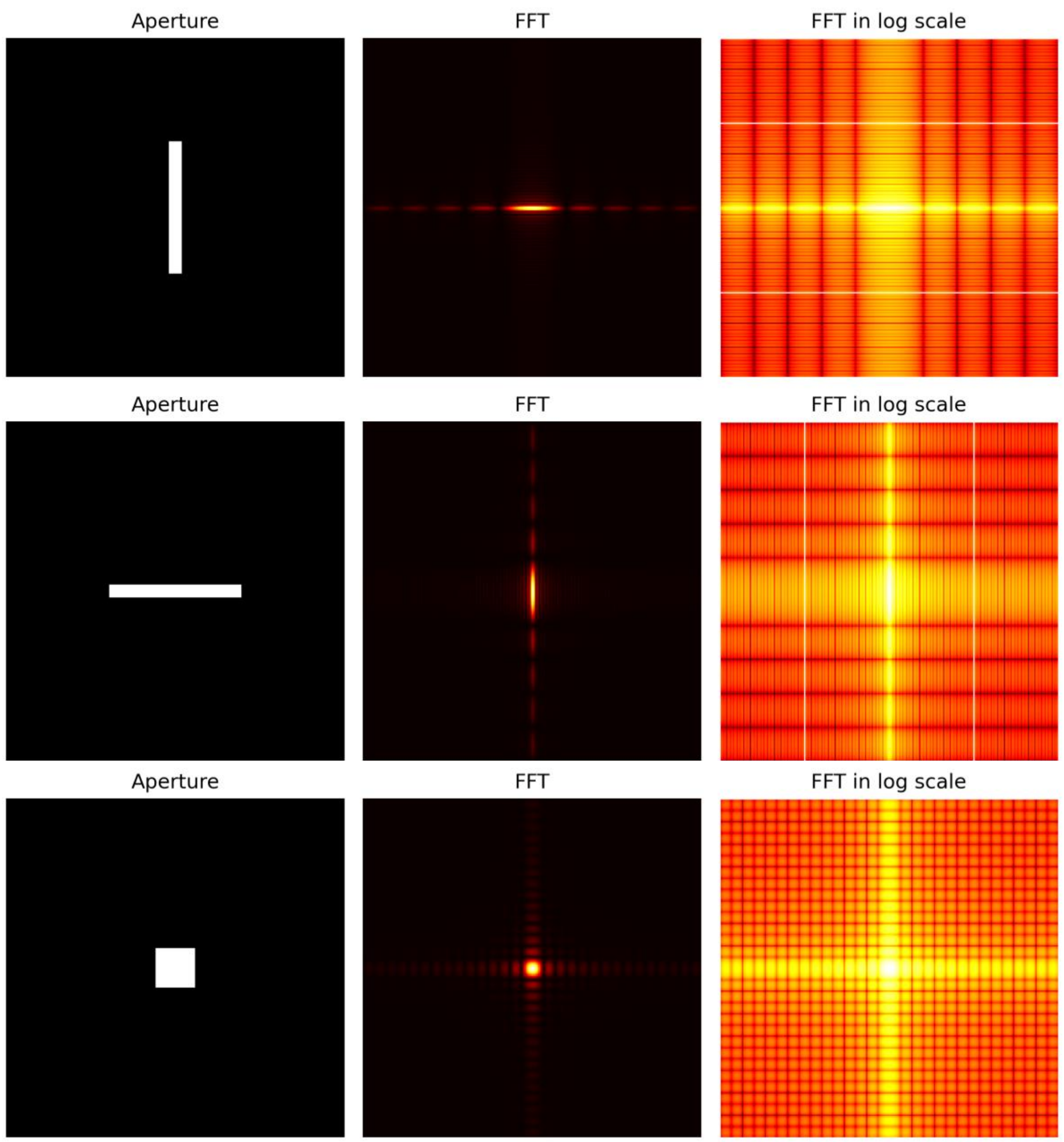


Figure 2. The FFT and FFT in log scale of single slits.

For a single slit oriented vertically, the resulting irradiance (FFT) was along the x-axis of the real plane, while the horizontal slit yielded a diffraction pattern along the y-axis. For both slits, the **peaks in the Fourier space** represent the bright streak of light along the center and faint spots on either side of the band.

For the special case of rectangular aperture --- a **square aperture** resulted into a two-fold symmetry with prominent square in the middle, and smaller less-intense squares coming off away from the center along both axes. These patterns for the three slits can be hypothesized as to why certain animals has **wide range of peripheral vision**, owing to their pupil shapes!

DOUBLE SLIT APERTURES

Given how the double slit interference pioneered the wave-particle duality of light, double slit apertures were also made to mimic this interesting phenomenon. Compared to the single slits, double slits have shown much smaller and several interference fringes on its diffraction pattern. As the slit width increases, the streak band in the middle decreases in size and so are the number of interference fringes.

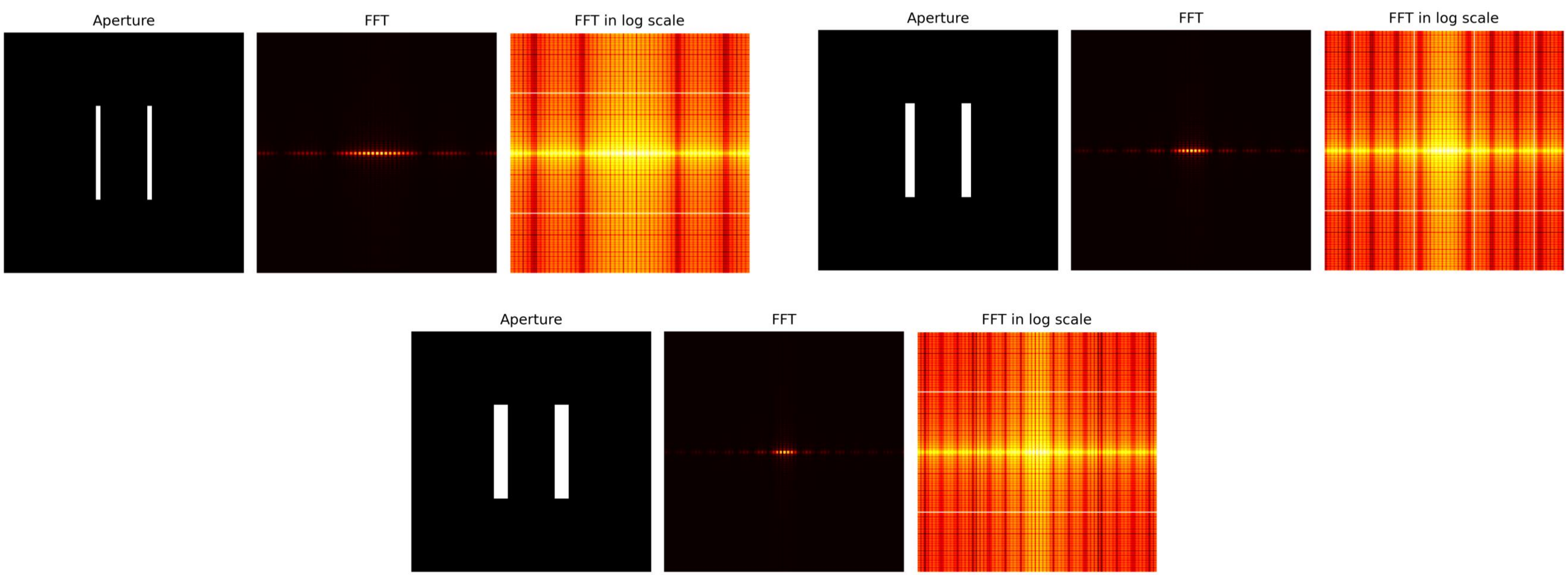
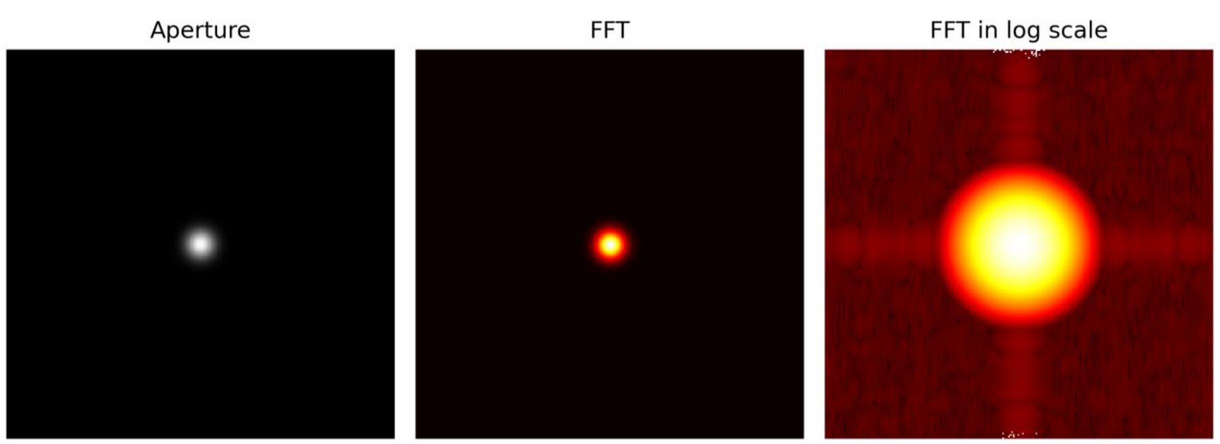
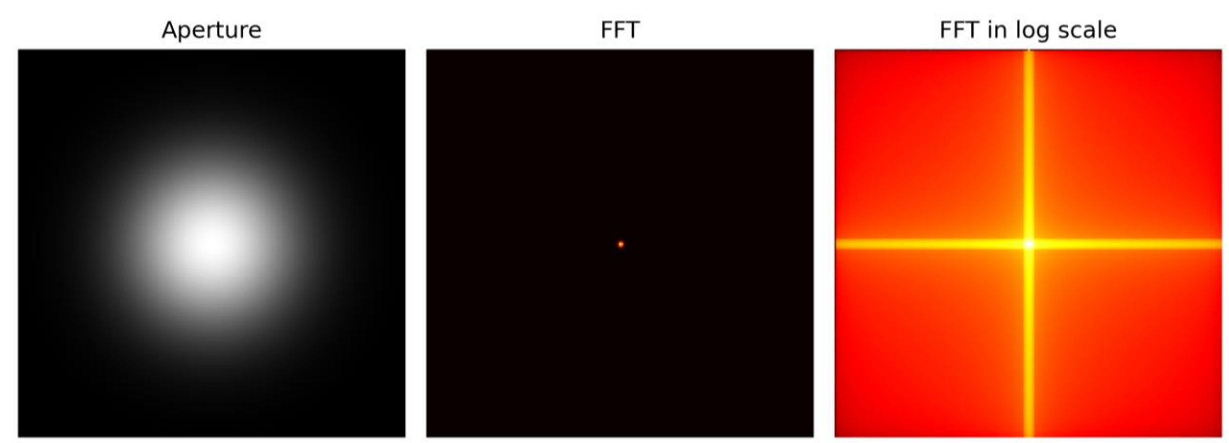


Figure 3. The FFT and FFT in log scale of double slits of varying slit widths.

GAUSSIAN BEAM

A Gaussian beam of varying spread values were also made, and the resulting intensity profiles were also deduced to take the form of a Gaussian as well. This is an interesting property that holds in the field of optics as the linear transformation FT invokes on the aperture preserves its property of being a Gaussian, which can be attributed to its exponential factor.





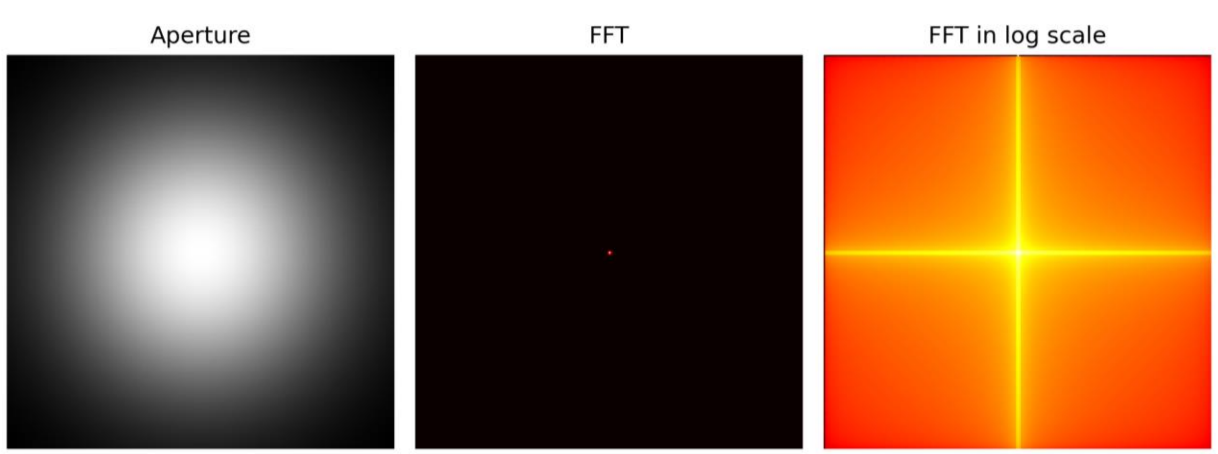


Figure 4. The FFT and FFT in log scale of Gaussian beams.

For reference, the intensity profile of a Gaussian irradiance is given by

$$I(r,z) = I_0 \left[\frac{w_0}{w(z)}\right]^2 \exp\left[-\frac{2r^2}{w(z)^2}\right]$$

FFT AND IFFT ON IMAGES

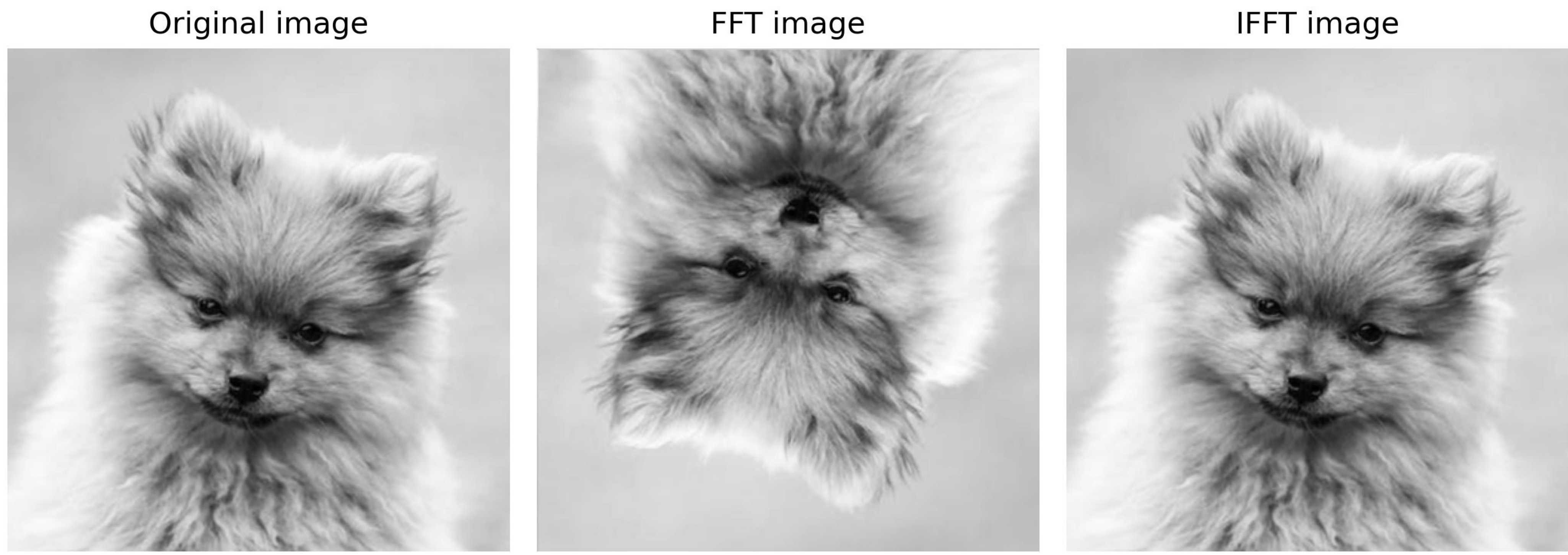


Figure 5. Resulting images after applying FFT and IFFT to the grayscale image.

It was discerned that applying FFT2 twice to the image **inverts** it and applying IFFT2 to the FFT2-image **reverts** it back to its original appearance. On a mathematical standpoint, the grayscale image can be interpreted as a **2D array** or signal in the Fourier space. Hence, applying FFT2 twice would correspond to the **circular inversion** of that array in space and taking the inverse FFT of an FFT2-array would simply revert it back to its original form in space. By these relations, it can be inferred that applying FFT2 to the original image **four times successively** will always result to the original orientation of the image.

IMAGE CONVOLUTION

In theory, the **convolution** between two-dimensional function reads as

$$h(x,y) = f * g = \iint f(x',y')g(x-x',y-y')dx'dy'$$

by which it can be interpreted as a **smearing** of one function against the other such that the resulting function bears a resemblance of both the **input functions**. In imaging, convolution governs systems where an aperture is used to project an object such that the resulting image will be the view when that object is beheld in front of the aperture. A good visualization of this is shown below where the resulting (convolved) image is like how one would perceive it when the text is viewed on the **circular aperture**!

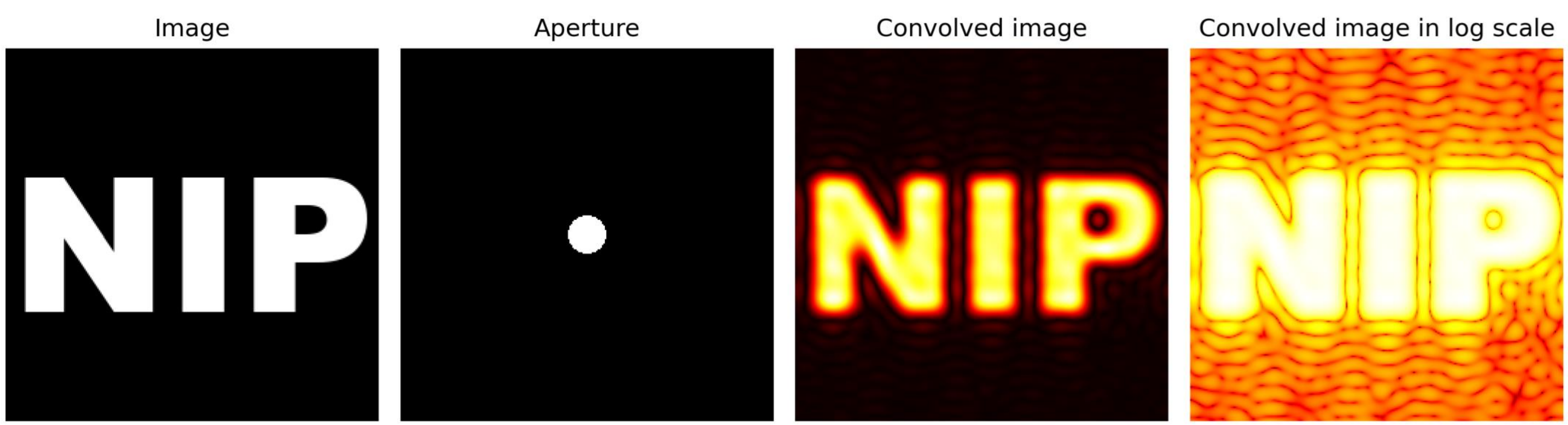


Figure 6. The FFT and FFT in log scale of double slits of varying slit widths.

SIMULATION OF AN IMAGING SYSTEM

The second part of the activity intends to simulate various imaging systems using the underlying concepts introduced in the previous activities. For imaging systems that use circular lens as an aperture, the resulting image solely depends on the **finite size of the camera lens** as shown below.

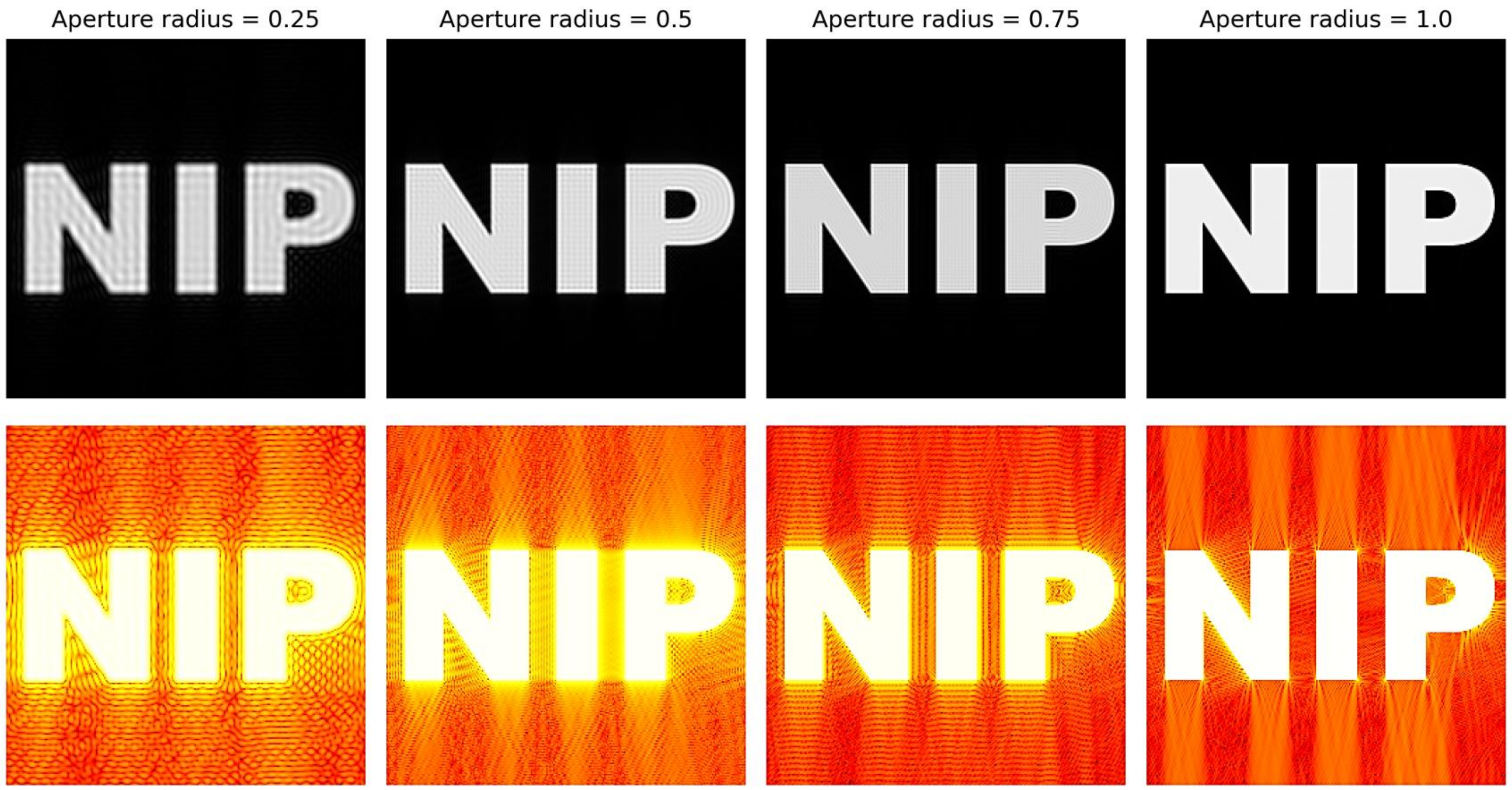


Figure 7. The convolved image of the text for circular apertures of varying radii.

Observe how the increasing **radii of the aperture** increases the image clarity as the **airy patterns** diminish. As expected, **greater recovery** on the object details were acquired for larger radii. On the discussion on circular apertures, these airy patterns are prominent on smaller radii since small bundle of rays that reflect off an object can only be gathered. Hence, the image of an object is not as perfect resemblance compared to larger apertures.

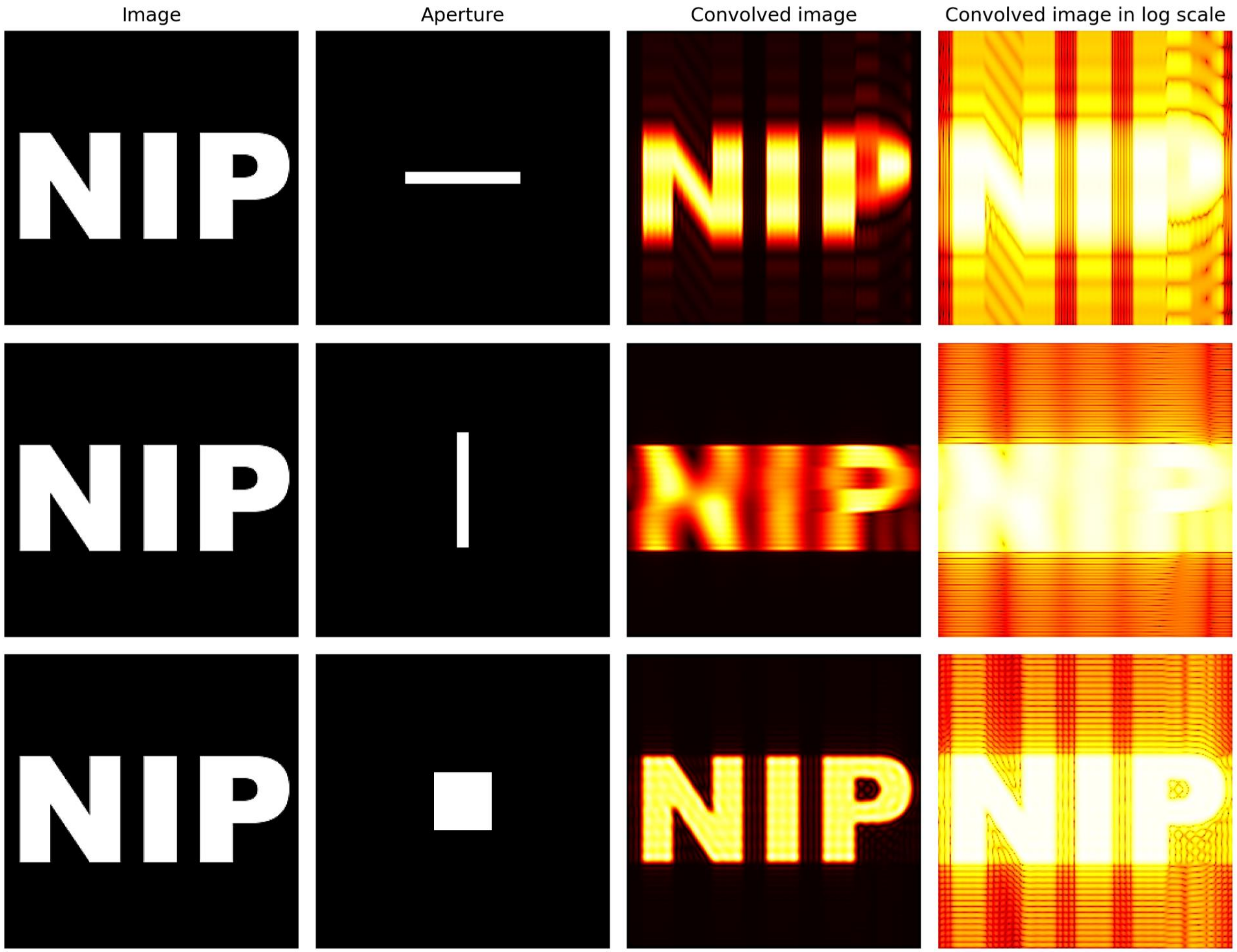


Figure 8. The convolved images using the apertures used in the previous activity.

The resulting images when convolved with the different apertures made in the first activity were parallel to how light is diffracted on the **single slits**. The interference patterns are much more evident in the logarithmic scale for reference.

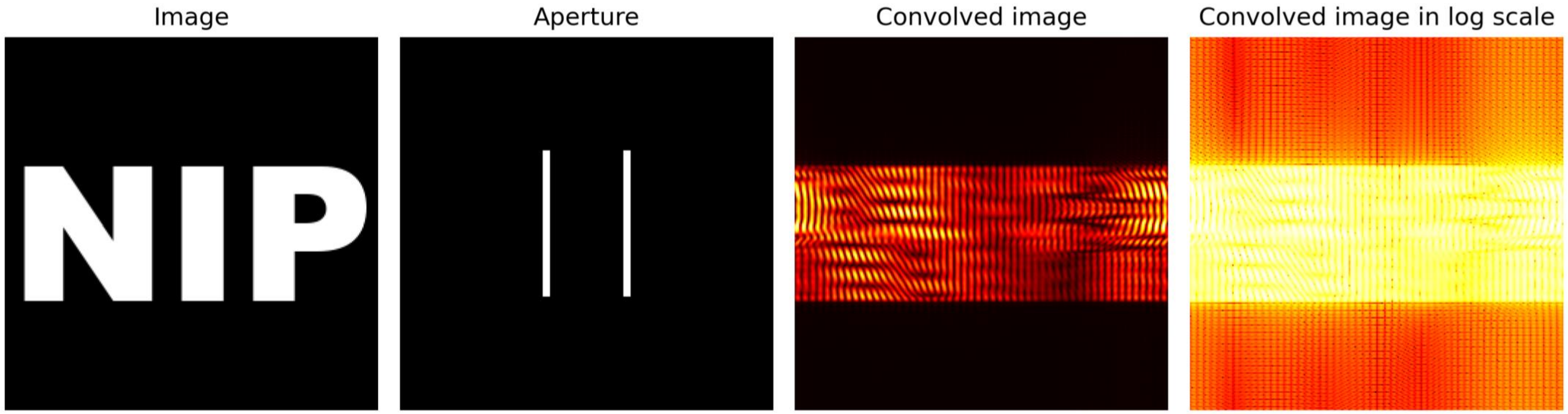


Figure 9. Convolved image with a double slit aperture.

The resulting convolved image for a double slit is dominantly made up of **interference fringes**, making the NIP text indistinguishable. Similarly, a gaussian beam was also tried as shown in Figure 10.

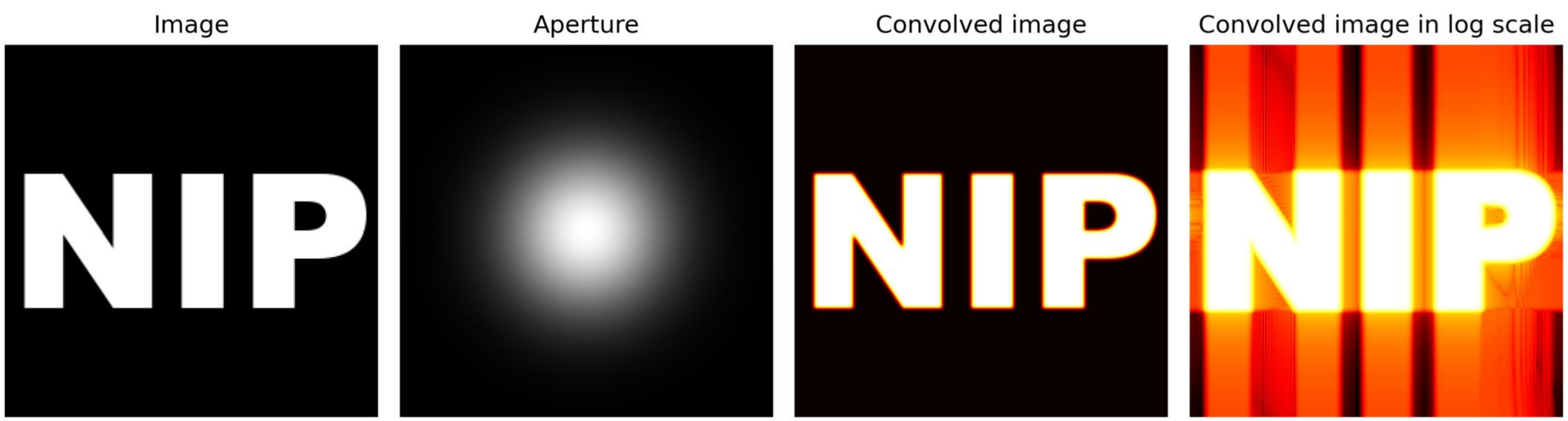
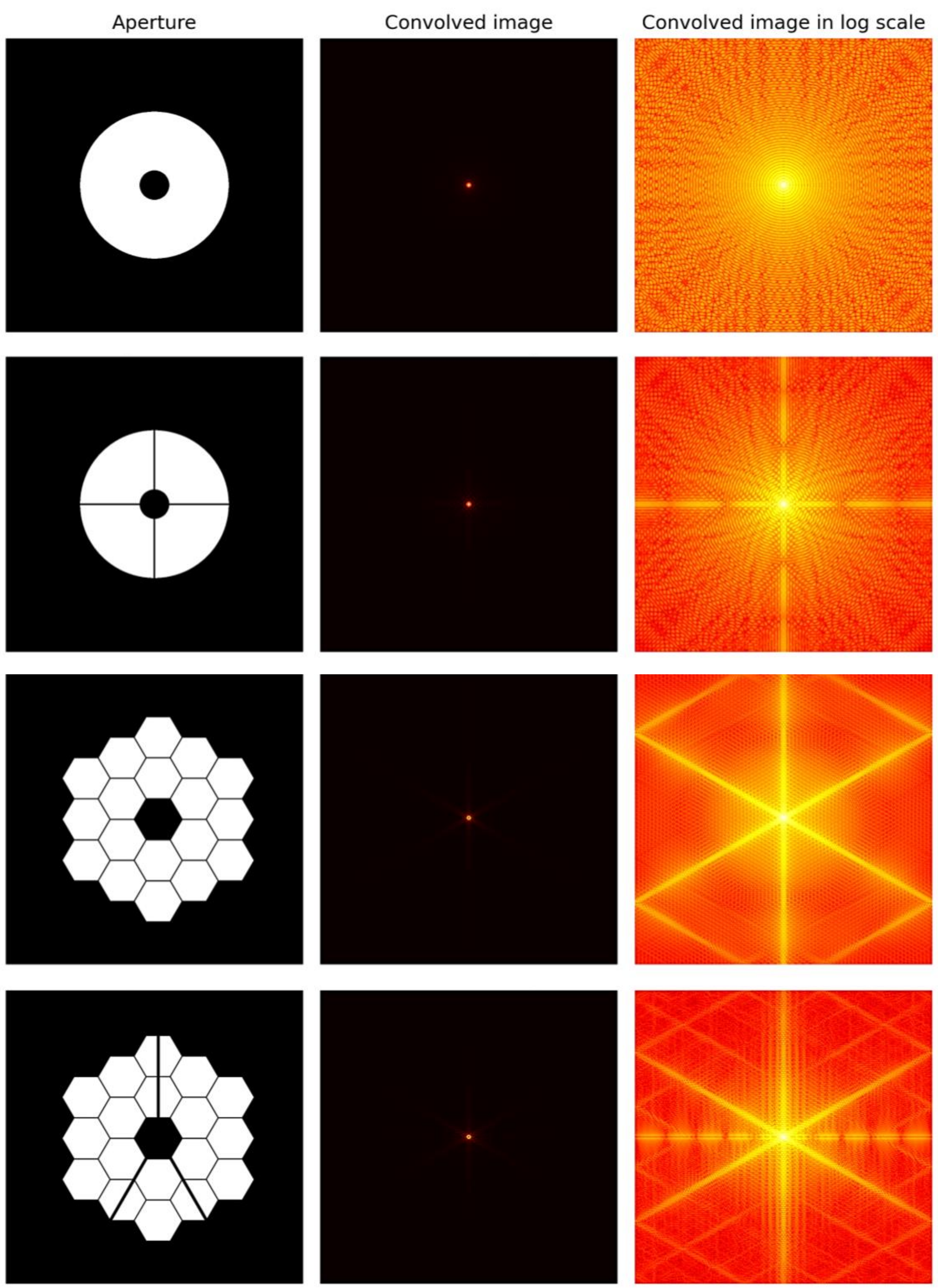


Figure 10. Convolved image with a gaussian beam.

No changes on the convolved image were deduced as the FFT of a Gaussian beam is still **Gaussian**!

IMAGING ON SPACE TELESCOPES

Analysis on the Hubble Primary Mirror (1990) and James Webb Telescope (2021)



One of the interesting **real-life applications** of **image convolution** goes to the simulation of how light from the stars look like when reflected by these space telescopes.

Using the apertures created in the first activity, an **image of a star**, like a point, is convolved with the **telescope assembly**. The resulting images are the convolution of the two and it closely models the appearance of an image captured by these telescopes in general.

For most **reflecting telescopes** like the Hubble and Webb, the diffraction spikes are produced as light from the stars strikes the main mirror and its struts that support the assembly. But how does this exactly happen?

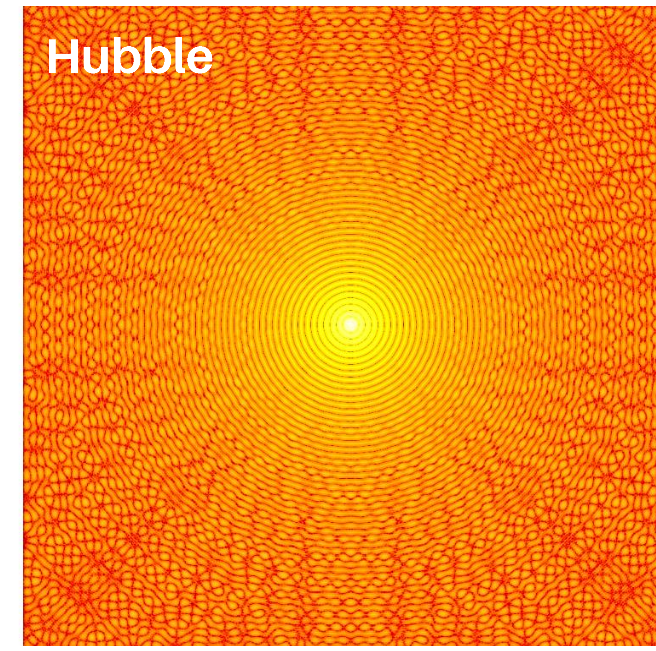
Figure 11. Comparison between the Hubble and James Webb space telescopes with no struts and with the presence of struts in the assembly.

ANALYSIS ON DIFFRACTION PATTERNS

The diffraction patterns seen on the convolved images can be reasoned out to two major factors --- the shape of the primary mirror and the strut influence on the main assembly of the space telescope.

Shape of the primary mirror. Similar to how diffraction patterns varies for different aperture geometries, the primary mirror in space telescopes is already enough to produce such diffraction patterns. In specific, the resulting image of a **point source** like a star totally depends on the **number of sides the reflecting mirrors have**. It has been observed that the diffraction pattern for a circular aperture is a circle, and the resulting pattern for a square aperture are beam spikes radiating into four directions from the center. From these, it can be implied that the diffraction pattern for the Hubble must be a circle, and six-spiked light rays converging at the center are expected for a mirror like James Webb.

Light from the stars interact with the **edges** of an aperture to create **perpendicular diffraction spikes** --- the reason why diffraction patterns are unique to every shape as the number of sides increases the number of visible spikes!



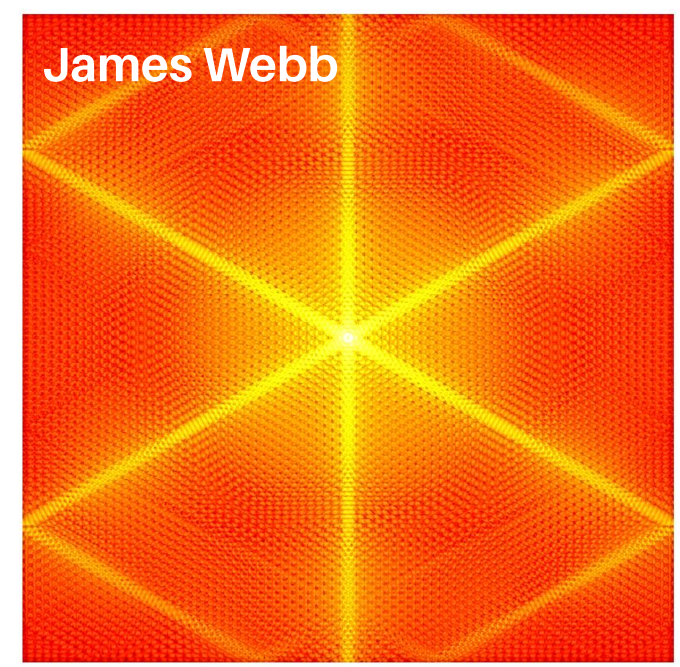


Figure 12. See right image. The primary mirror contribution of the Hubble and James Webb space telescopes to the diffraction pattern

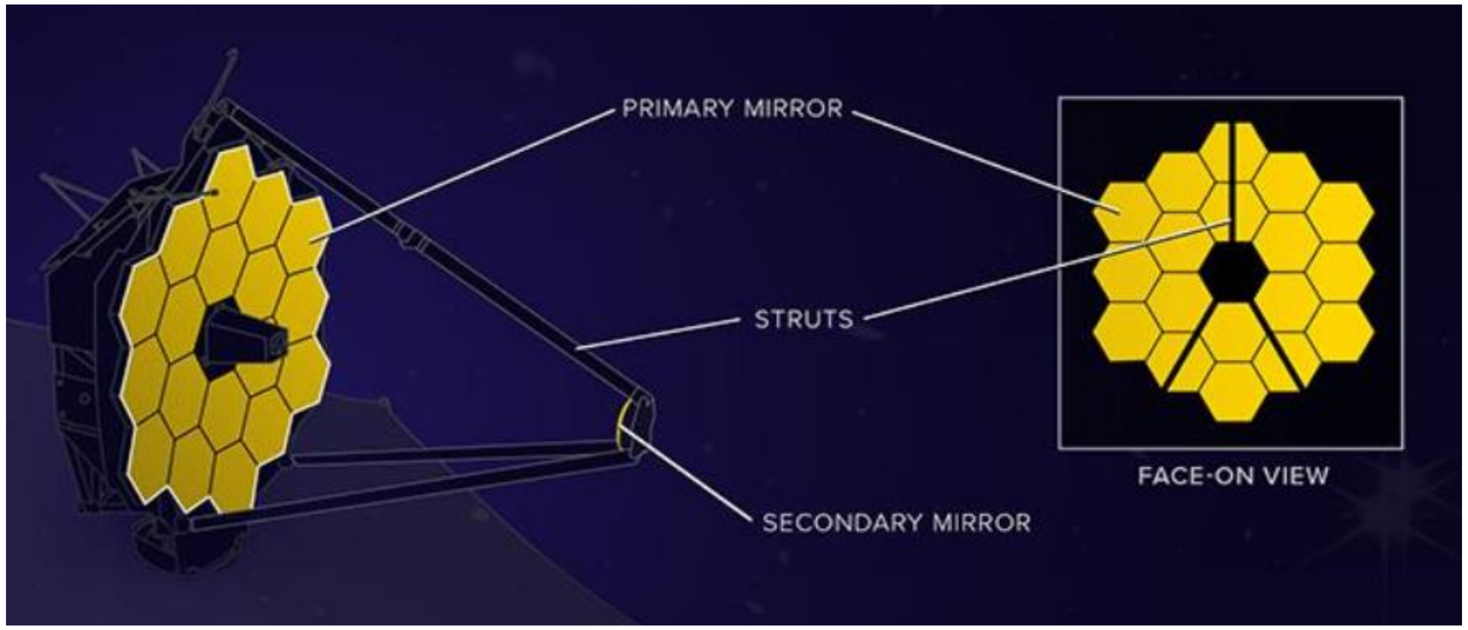
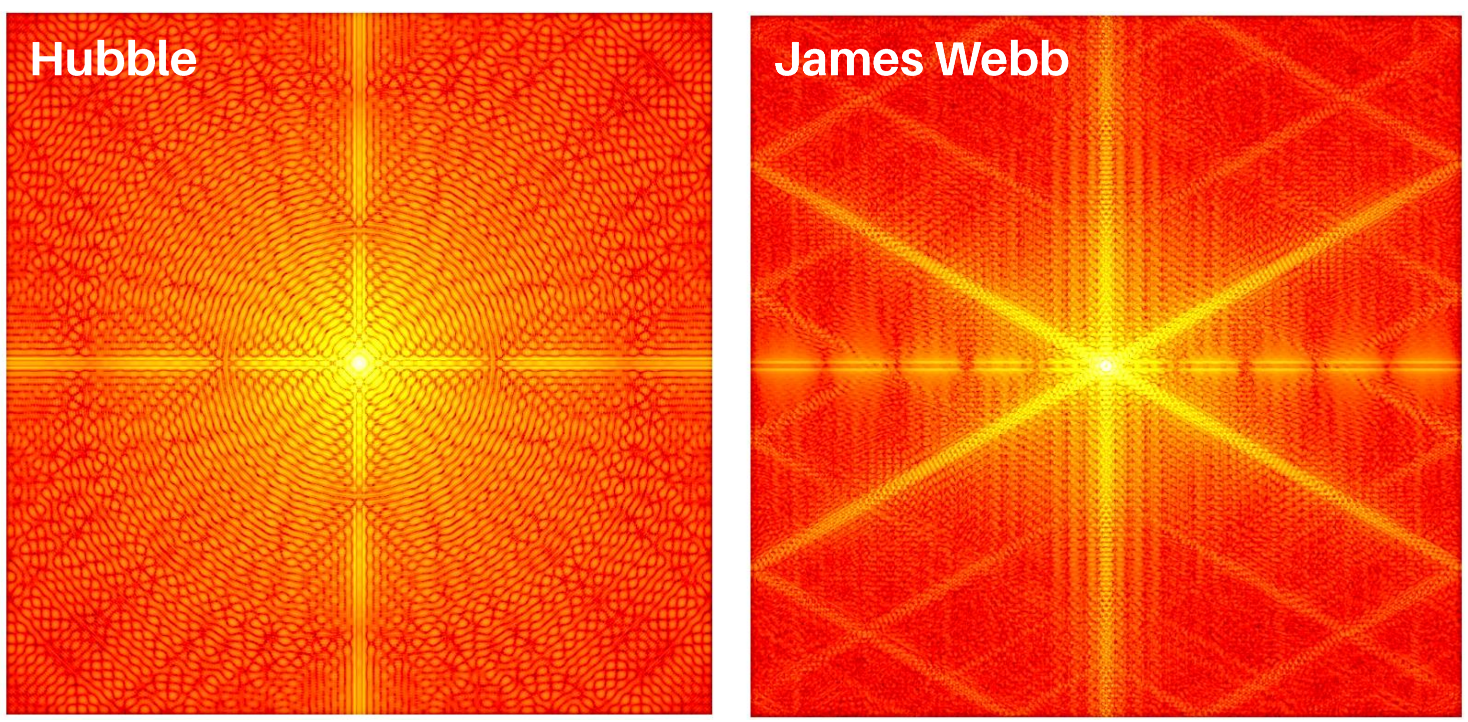


Figure 13. The James Webb space telescope assembly. Image cropped is courtesy of Webb's Diffraction Spikes (webbtelescope.org).



Strut influence. Just like how the edges of an aperture contribute to the appearance of an image point source or a star, the **struts attached to the main mirror** assembly also diffracts the light shined through it. The number and position of struts supporting the secondary mirror determine the diffraction patterns it produces.

Since the diffracted light always bends into a single, perpendicular pattern, the **resulting strut contributions** always converge at the center. These explains why the diffraction pattern for mirrors with strut is different than the the mirror itself, as shown in Figure 12.

In the James Webb, the struts are assembled in a way their diffraction patterns overlap with those created by the primary mirror!

Figure 14. Diffraction pattern of the star accounting for the primary mirror and struts contributions, close to the actual images captured by the telescopes. The James Webb space telescope is known for its complex eight-pointed star pattern as shown.

Figure 15. Comparison between the image of a star captured by the Hubble and James Webb space telescopes.

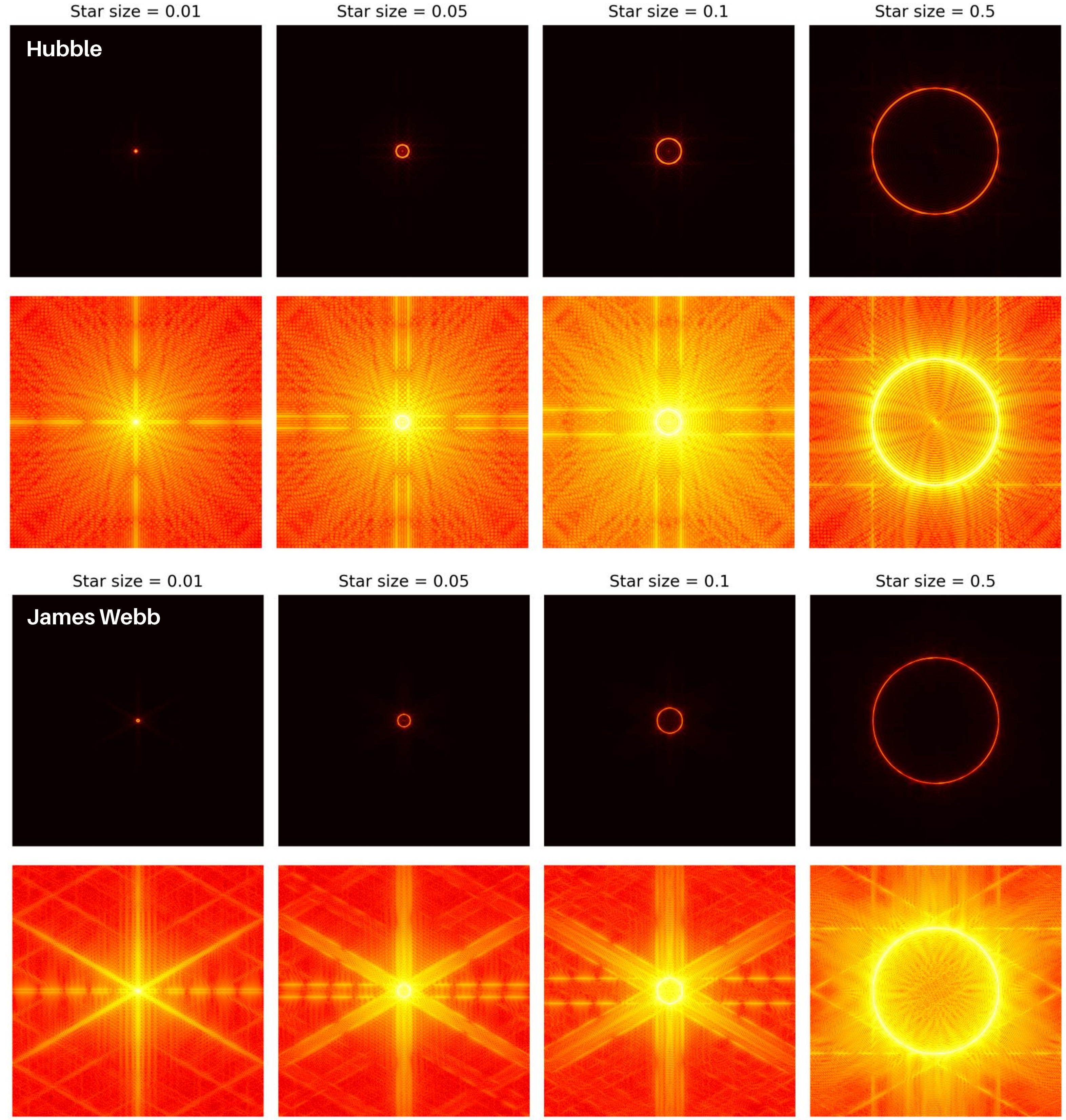


Figure 16. Stars of varying radii when convolved with the Hubble and James Webb space telescopes.

IMAGE CORRELATION

The **correlation** between two-dimensional function is given by the equation

$$p = f \cdot g = \iint f(x', y')g(x + x', y + y')dx'dy'$$

by which it can be interpreted as a **degree of similarity** between two functions, i.e. the more identical they are at a certain position in the image, the higher the correlation value would then be. Image correlation is commonly used as an algorithm in **template matching** or **pattern recognition** in certain scenarios.

As shown on the figure, a phrase was made, and certain letters and certain phrases were used as template to determine how the pixels in template image matches certain parts of the phrase image. The **red peaks** in the correlation map indicates the presence of such text or phrase in the original image!

Figure 17. Image correlation using template matching.

IMAGE CORRELATION

Some limitations due to the template image used

The algorithm used in image correlation has also some of its **limitations**. Since it matches the template image and phrase image **pixel by pixel**, some letter pixel is almost identical to other letters, which somehow limits its ability to template match both images and might yield a high correlation value even though the desired object is not the same as the template. A good concrete example would be shown below for reference.

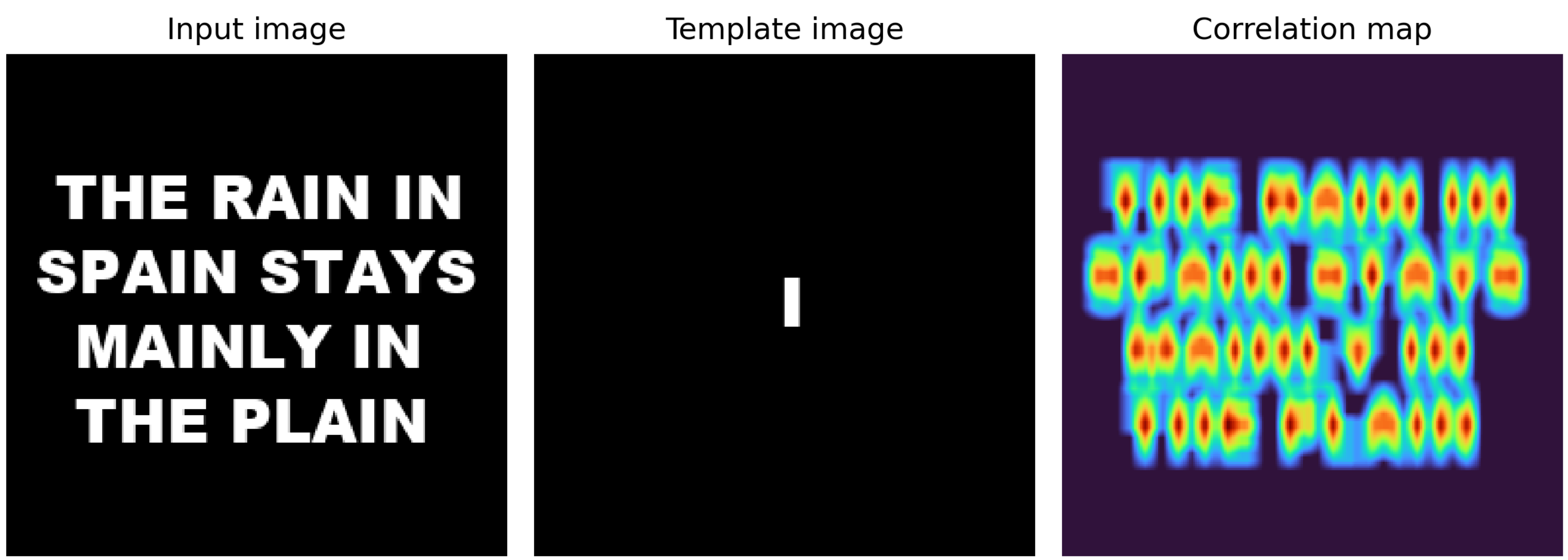


Figure 18. Image correlation using template matching for letter I.

Compared to the letter A used in Figure 17, the correlation map projected several peaks everywhere even though the template image is only in certain parts of the input. This is because "I" or the vertical line it represents were common in almost all of the letters that make up the phrase image. Hence, all strokes of the letter similar to "I" is returned as **high correlation**, equivalent to the red peaks.

IMAGE CORRELATION

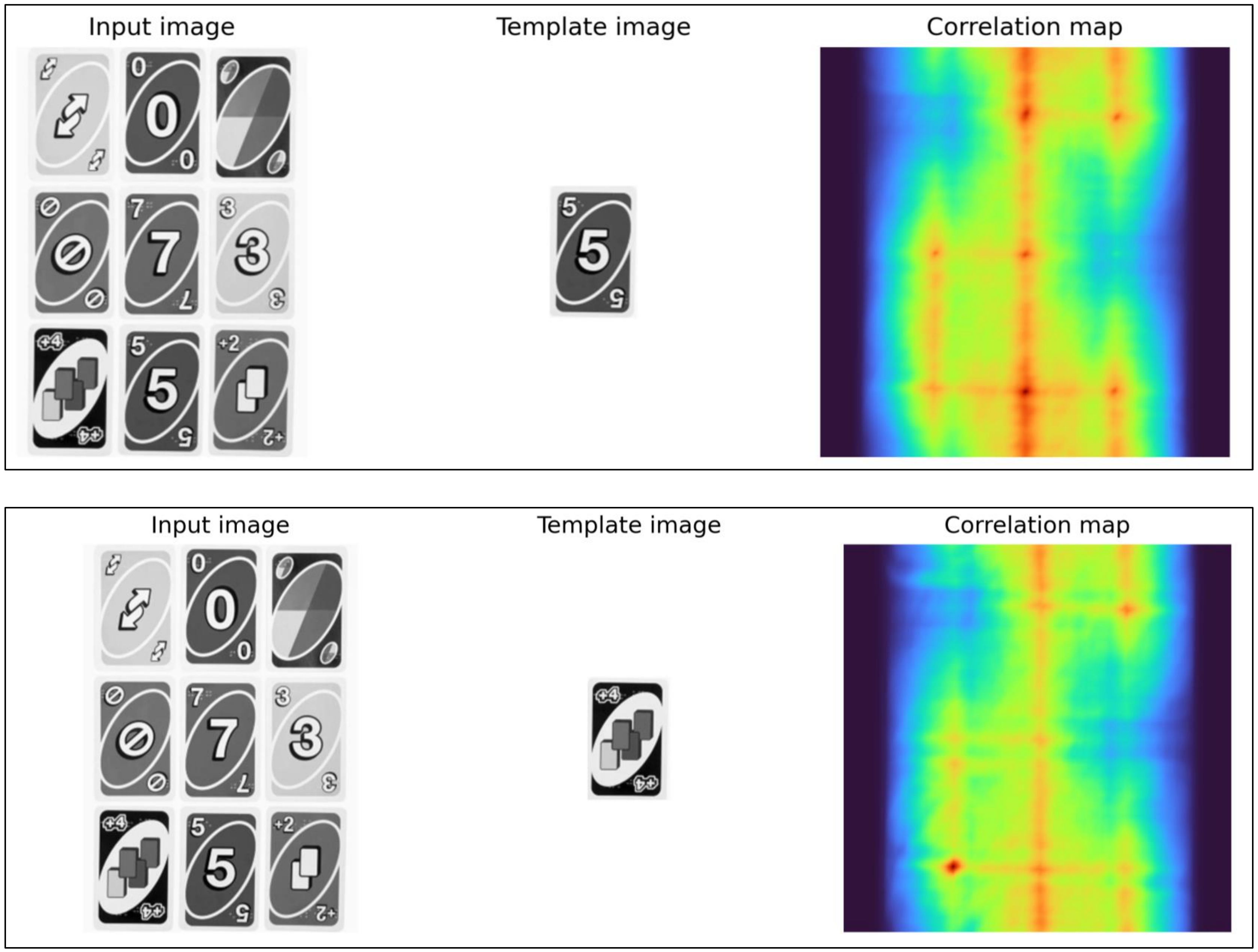


Figure 19. Image correlation using template matching for UNO cards.

As expected, the dark red peaks appeared on the area where the template image matches the input image pixel by pixel. Yay!

REFLECTION

Similar to the previous activity, working on images set in a 2^N resolution is favored by the FFT algorithm as complex calculations are reduced to a reasonable and comparable factor! Overall, I think I was able to execute the **rudimentary of FFT in digital image processing** and even meet what is expected beyond the objectives. I was able to address **some gaps and nuances** in my results as well as the underlying concepts and implications of the resulting images.

I find the activity generally fun since it covers some of the real-world applications of Fourier Transform, which I really look forward to since it transcends beyond the what was taught in the lectures. I was also able to compare the differences of my results to what the theory suggests. Nothing really beats the satisfaction of having your results being close to the theory.

Simulating the imaging systems such as that of **space telescopes** really struck me as I did not know that FFT goes as far as to how stars are captured by the telescopes. Further study into the **mathematics** that describes the intensity pattern that arises from different aperture geometry would be interesting as well! Overall, I would give myself a score of **110/100**.

ACKNOWLEDGEMENT

I would like to acknowledge the **time and knowledge** Sir Rene and Sir Kenneth have shared to us during our laboratory activities and the numerous discussions on the applications of FFT. I also would not have done this without the **help** of Edneil Soriano upon resolving some of the mistakes I made in my result interpretations and working codes.

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