Analyzing mathematical concepts behind the Rubik's Cube

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Abstract

This paper introduces the Rubik's Cube and explores its mathematical properties from basic calculus to undergraduate-level abstract algebra. It breaks down the cube's structure and examines the concept of permutations, as well as the idea of groups, with a focus on the Symmetric and Rubik's cube groups. Notation for the Rubik's Cube is explained, and the Rubik's Cube Group is analyzed. The paper also discusses God's Number, which is the minimum number of moves required to solve the Rubik's Cube, and the concept of Cayley Graphs is introduced as a means of understanding the Rubik's Cube Group. The paper concludes by suggesting possible future research directions. Overall, it provides a comprehensive overview of the Rubik's Cube and its mathematical properties in a concise manner.

Contents

1 Introduction to the Rubik's Cube

In 1974 Ernő Rubik, a Hungarian architect, wanted to find a way to model three-dimensional movement. After much trial and error, he created what he called the "magic cube". It was later renamed the famously known "Rubik's cube". The Rubik's cube was born as a puzzle and it fascinated the world. By 1982 it was a household term, and it became a part of the Oxford English Dictionary. The exact definition from Oxford English Dictionary (n.d.) (2023) is: "A three-dimensional puzzle consisting of a multicoloured plastic cube which must be manipulated to adjust the pattern of colours.". That is a very simple definition because it turns out that the Rubik's cube is much more sophisticated than that. It has become an iconic toy with more than 350 million units sold worldwide making it the world's bestselling puzzle game.

Once someone learns the algorithms to find a solution it becomes almost trivial to solve but it turns out that by using different branches of advanced mathematics we can make sense of it all.

With the success of the original Rubik's cube, it didn't take the industry very long to come up with similar puzzles that used the same color concept, creating a large family of combination and mechanical puzzles. Later came the $n \times n$ puzzles, 2×2, 3×3 (the "original"), 4×4, and so on and so forth up until 33×33 was built in 2017. There are cubes with other shapes like the pyraminx, which is a regular tetrahedron puzzle, or the skewb which has a cubical shape but its axes of rotation pass through the corners rather than the centers of the faces. You can observe different kinds of cubes represented in Figure 1.

There are many mathematical topics that can be explored through the lenses of these puzzles that form permutation groups. We will be doing some probability to break down where certain numbers come from, we will also be discussing functions and permutation functions and we will finally dive into groups. Within the realm of groups, one of the things that will be analyzed is its properties and we will prove how the main 3×3 cube forms a group.

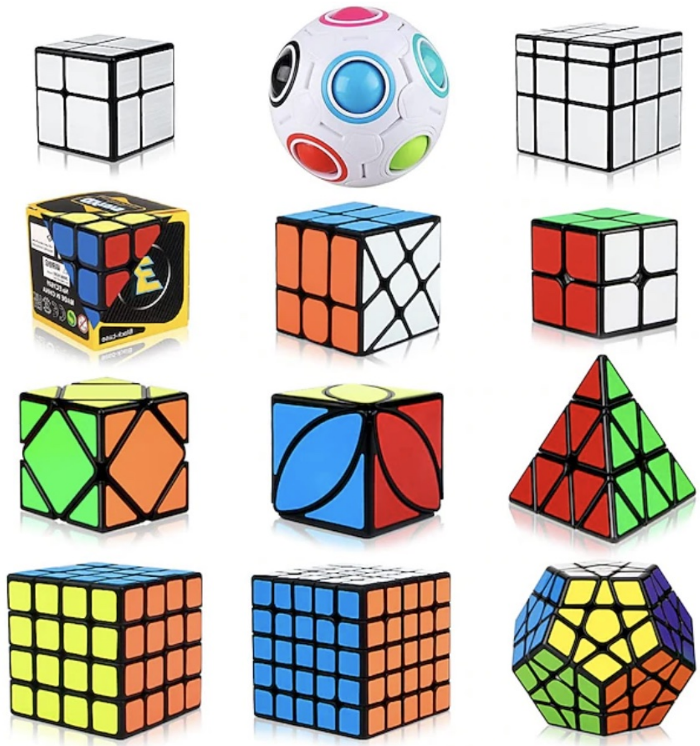


Figure 1

From lightinthebox.com (2023)

Another topic of interest that will be discussed is the cube's "God's number", commonly known among the cubing community. When you mix up the 3×3 Rubik's cube, there is a total number of approximately 43 quintillion combinations you can get. God's number is the maximum minimum amount of movements that are needed in order to return the cube to a solved position. As of 2023, God's number has been proven to be 20. We will also briefly use some graph theory and a Cayley graph to understand this concept.

1.1. Breaking down the structure

As was just mentioned, there are exactly $43,252,003,274,489,856,000$ possible scrambles for a Rubik's cube. But where did the 43 quintillion combinations come from? When we write it in a mathematical way, this is equivalent to:

$$\frac{(3^8 \times 8!) \times (2^{12} \times 12!)}{12} = 43,252,003,274,489,856,000 \tag{1}$$

Before we break this number down it is important to understand what pieces the Rubik's cube contains. We have 6 different faces, each with a different color, each face has a total of 9 facets. Each face contains a centerpiece that doesn't move other than rotating in place. When we disassemble a cube (see Figure 2), we find three types of building blocks, the centers, that we just mentioned; the corners, which each contain 3 different colors; and the edges, each with 2 colors. Each cube has, one core, with 6 centers, 8 corner cubies, and 12 edge cubies.

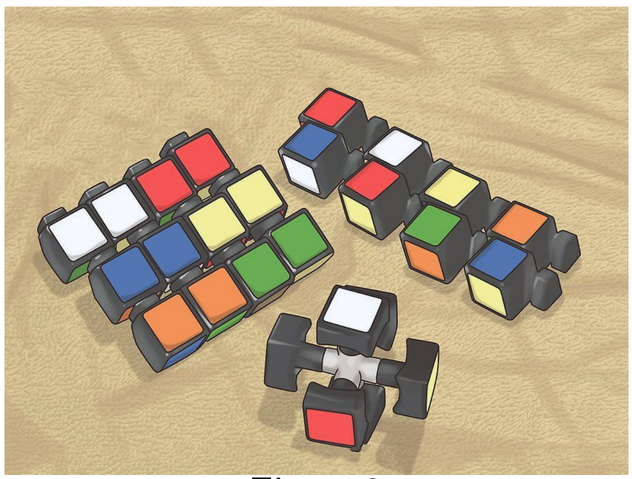


Figure 2
from Linkletter (2020)

As far as our equation goes, the first term 3^8, counts each of the ways we can rotate the corners, each corner cube can have a total of 3 orientations, raising 3 to the power of 8. Next, we have to take into consideration where each of the corners goes. There are a total of 8 slots, so we can consider it a factorial distribution. After we have selected a spot for the 1st corner cubie, now we have 7 spots left, and so on and so forth. Meaning that the first parenthesis is each of the corners' locations and its orientation ($3^8 \times 8!$).

The second part of the numerator follows the same concept but for the edge cubies. Each edge has two possible orientations and there is a total of 12 spots where it can be located, so we multiply ($2^{12} \times 12!$).

So all that is left from the formula is the denominator, so that is division by 12.

This concept is complicated to understand and less straightforward. Before I make an attempt to reasonably describe what is happening take into consideration Figure 2 (from Linkletter (2020)) with all the cubies that form a Rubik's cube. If you were to put it back together to a cube shape, it would look like a regular scramble but in reality, there is only a 1 in 12 probability that the cube is solvable.

Where does that 12 come from? The following explanation is an adaptation from "The Amazing Math inside the Rubik's Cube" by Linkletter (2020). Take a look at Figure 3, the first Row has normal corners, and the 2nd and 3rd rows have one corner rotated in place. Column 1 has normal edges, whereas column 2 has one edge turned in place, column 3 has two edges swapped, and column 4 has two edges swapped and one rotated in place.

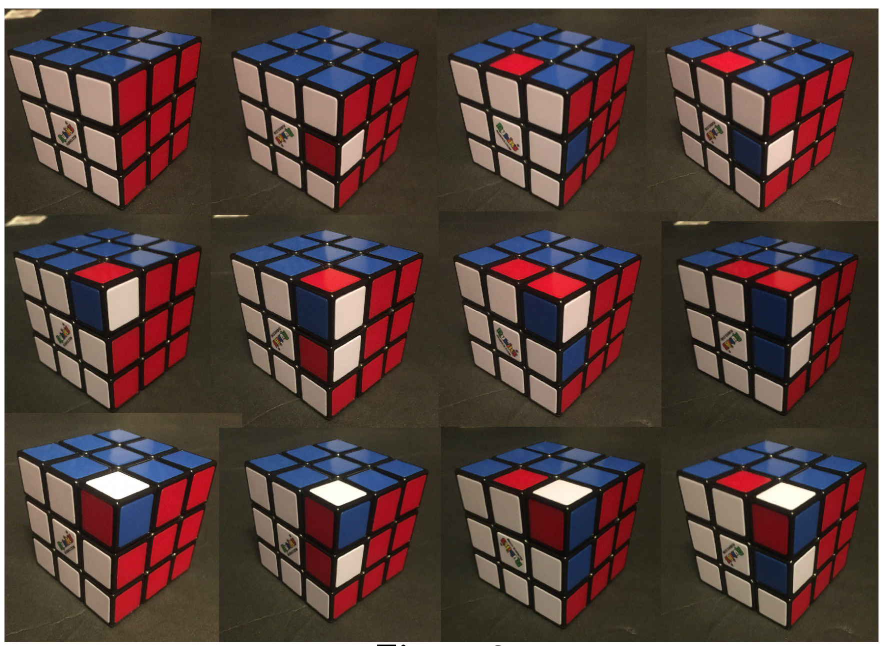


Figure 3
from Linkletter (2020)

In short, none of the following cases can be changed from one to another with the natural movement and rotation of the Rubik's cube. This means, that in reality there is only one possibility in 12 that when you reassemble the cube it can be solved without breaking it apart.

Let's break down that factor of 12 into one factor of 3 and two factors of 2. When we look at what "cubers" understand as algorithms, there is no algorithm that can twist a single corner in place. There exists an algorithm that can twist two separate corners, but no algorithm can twist only one. If we were to manually move another corner, it could be possible that once we performed the algorithm that twists two separate corners the cube returned to only one twisted corner position, which breaks down the 12 to the first factor of 3. Similarly, there is an algorithm that flips in place two separate edges, but no algorithm can flip only one edge in place.

The last factor of 2 involves both edges and corners. Like we mentioned, there is an algorithm that can swap two edges AND two corners, but there is no algorithm that can only swap two edges or two corners, which means that once you perform the algorithm you will either go from the situations in column 1 to the situations in column 3 and from column 2 to column 4.

Although this didn't sound very mathematical, there is somewhat of a mathematical explanation for this, and Linkletter (2020), describes it as the "popular mechanics of the Rubik's cube proof" (p.112), and he provides an example for the first factor of 2 from the factor of 12.

Theorem 1. *Given a* 3×3 *Rubik's cube, there is no algorithm that flips one edge cubie in place, without flipping any other cubies.*

Proof. (Adapted from (Linkletter (2020))

When a face of the cube is turned, four edge cubies get moved. Consider, for instance, an algorithm of 10 moves. For each cubie, follow it through the algorithm, and count how many times it gets moved, and call that its cubie-moves count. Add up those numbers for every edge cubie, and the total must come to 40 cubie-moves. Since each of the 10 moves adds four to the total.

In general, any algorithm's total number of cubie-moves for the edge cubies must be a multiple of 4. Now for a critical pair of facts: If an edge cubie is moved an even number of times and returned back to the same slot, it will have the same orientation. Conversely, if an edge cubie is moved an odd number of times and put back in the same slot, it will be flipped.

Finally, consider a hypothetical algorithm that accomplishes the goal here, flipping one edge cubie in place without changing any other cubie. The one flipped edge was therefore moved an odd number of times by the algorithm, while each of the other 11 edges was moved an even number of times. The sum of 11 even numbers and one odd number is always odd, but we established earlier that this sum must be a multiple of 4, and an odd number can't ever be a multiple of 4. Therefore no such algorithm exists. ■

This breaks down where the entire formula comes from and why when we are following the mechanics of the Rubik's cube without breaking it apart there are exactly $43,252,003,274,489,856,000$ possible scrambles.

2 Functions

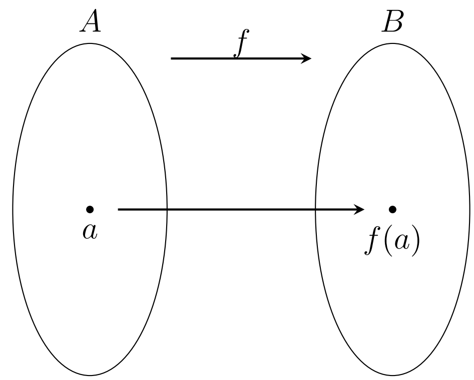
Functions are the main building structure for most branches of mathematics, and group theory is no different. Functions are a way of associating elements from one set to another, and in the context of group theory and in this paper we will focus mostly on 'permutation' functions. Roughly speaking, these mappings mix up and swap different elements from a specific finite set using those rules. Before, let's refresh some concepts and define function terms.

Definition 1. ***Function*** *(From (Gallian (2017)).*

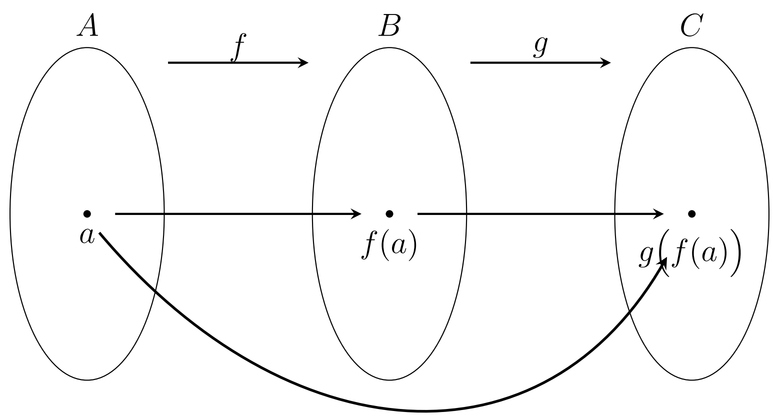
*A **function** (or mapping) f from a set A to a set B is a rule that assigns to each element a of A exactly one element b of B. The set A is called the domain of f, and B is called the range of f. If f assigns b to a, then b is called the image of a under f. The subset of B comprising all the images of elements of A is called the image of A under f.*

Definition 2. ***Composition*** *(From (Joyner (2002)).*

*Let $f : A \to B$ and $g : B \to C$ be two functions. We can compose them to get another function, the **composition**, denoted $fg : A \to C$.*



Function



fg

Composition

Definition 3. ***Surjective*** *(From (Joyner (2002)).*

If the image of the function $f : A \to B$ is all of B, i.e., if $f(A) = B$, then we call f ***surjective*** *(or 'onto', or 'is surjection'). Equivalently, a function f from A to B is surjective if every $b \in B$ is the image of some $a \in A$ under f.*

For example, if you take a $3x3$ Rubik's cube, let S be the set of all 54 facets of the cube and let $f : S \to S$ be the map that sends one facet to the facet that is diametrically opposite. For instance, using Figure 4 for guidance, using our function f, facet 19 would be mapped to facet 52, and f would be considered surjective.

Definition 4. ***Injective*** *(From (Joyner (2002)).*

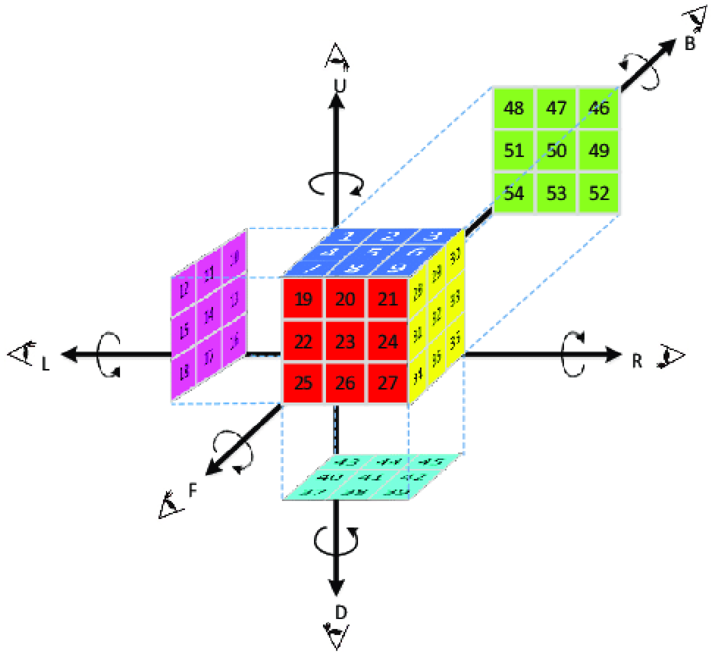
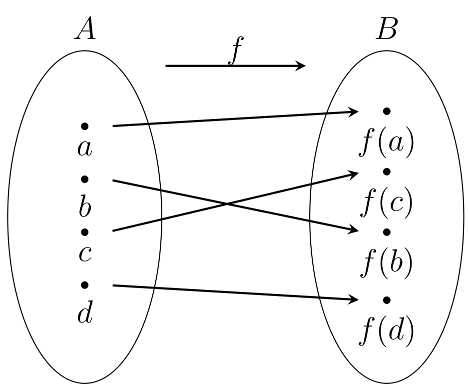


Figure 4

A function $f : A \to B$ is called ***injective*** *(or 'one-to-one', or 'an injection') if each element b belonging to the image $f(A)$ is the image of exactly one a in A.*

The Rubik's cube function to itself is also an example of a one-to-one function since each move can only map each facet to a single specific facet. Take a look at the following illustration showing a function f that is both one-to-one and onto:



As it turns out, the Rubik's cubes functions to itself are bijective, which means both injective and surjective, and therefore they are considered a permutation function, which we mentioned earlier but are about to define.

3 Permutations

When we hear the word permutation and we are thinking about mathematics our mind automatically goes to thinking about probability. Permutation functions and permutations in probability are not completely unrelated.

In probability theory, a permutation refers to a way of selecting a subset of elements from a larger set in a specific order. For example, suppose you have a set of 10 items, and you want to select 3 of them in a specific order. The number of permutations of 3 elements from a set of 10 is $10 \times 9 \times 8 = 720$, because there are 10 choices for the first item, 9 choices for the second item (since one has already been chosen), and 8 choices for the third item (since two have already been chosen). Each of these 720 permutations represents a different way of selecting 3 items from the set.

Here's how we define a permutation function:

Definition 5. ***Permutation*** *(Adapted from Gallian (2017)) A* ***permutation*** *of a set A is a function from A to A that is both one-to-one and onto. A bijection from A to itself.*

In other words, it is a function that maps each element of a set to a unique element in a different order. For example, a permutation of the set 1, 2, 3 might be (2, 1, 3), which rearranges the elements of the set in a different order.

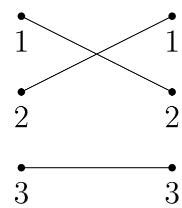
So, the main difference between a permutation function and a permutation in probability is that a permutation function rearranges the elements of a set, while a permutation in probability refers to a way of selecting elements from a set in a specific order.

We have different ways to visually represent permutation functions, but two of the most common ones are array notation and arrow diagram. In array notation, $f : \mathbb{Z}_n \to \mathbb{Z}_n$ is represented by a 2 $\times n$ array, where each integer x, maps to some integer f_x:

$$f \leftrightarrow \begin{pmatrix} 1 & 2 & ... & n \\ f_1 & f_2 & ... & f_n \end{pmatrix}$$

Here is an example of the permutation $f : \mathbb{Z}_3 \to \mathbb{Z}_3$ for which $f(1) = 1$, $f(2) = 1$, $f(3) = 3$ represented in array notation and arrow diagram.

$$f \leftrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$



There are some properties of permutations that are also important to note. Like in many other mathematical branches, we have the **identity** permutation, which is denoted by ε, and it is the permutation that doesn't change the position of any of its elements.

$$\varepsilon = \begin{pmatrix} 1 & 2 & ... & n \\ 1 & 2 & ... & n \end{pmatrix}$$

Let's look at an example that will help us understand the next topic. Take a look at the following two permutations:

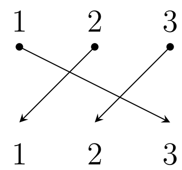
$$\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

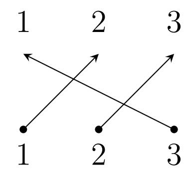
We can compose the two yielding:

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \varepsilon$$

This is what it would look like visually β is on the left and α is on the right, but for α we have switched the top and bottom rows to prove our point. Meaning that the permutation function α is sending the bottom points to the top points:

Unknowingly so, with the above example we have also shown a special property of



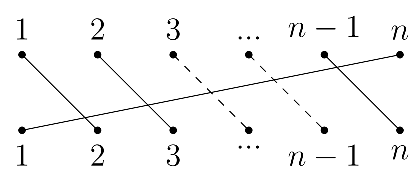


the permutation functions, which is that for any permutation function γ, there will always exist another permutation function δ, such that its composition is equal to the identity element, $\gamma\delta = \delta\gamma = \varepsilon$. The such permutation is defined as an **inverse**.

There is also an **n-cycle**, a special kind of permutation that cyclically permutes the values of the set. A very easy example to understand a cycle permutation is an analog clock, when each number increases by one unit simultaneously. Another example,

$$\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

and visually n-cycle permutations can also be represented:



Just as we had defined for functions, we can also use composition for permutations. By now you may have made this connection already since we mentioned that the Rubik's cube is on itself a permutation function. Therefore, the Rubik's cube belongs to a very specific group of puzzles that we call the permutation group.

The signum function is very commonly used when talking about permutations. It will not be discussed or used throughout the remainder of the paper but it can be useful to note. It can help us prove Theorem 1 using a more mathematical approach, that will be left to the reader.

Definition 6. ***Swapping Number*** *(From (Joyner (2002)).*

Let $f : \mathbb{Z}_n \to \mathbb{Z}_n$ *be a permutation and let*

$$e_f(i) = \#\{j > i | f(i) > f(j)\}, \qquad 1 \leq i \leq n-1.$$

Let

$$swap(f) = e_f(1) + ... + e_f(n-1).$$

We call this the ***swapping number*** *(or* ***length****) of the permutation f since it counts the number of times f swaps the inequality in i<j to f(i) > f(j). If we plot a bar-graph of the unction f then swap(f) counts the number of times the bar at i is higher than the bar at j. We call f* ***even*** *if swap(f) is even and we call f* ***odd*** *otherwise.*

The number

$$sign(f) = (-1)^{swap(f)}$$

is called the ***sign*** *(or* ***signum*** *function) of the permutation f.*

4 Groups

Let us begin by establishing what constitutes a group before delving into its intricacies. We will employ the Rubik's cube to illustrate this concept, and as it turns out while studying permutations, we have unknowingly employed all the fundamental properties of a group. Therefore, we can define a group as follows:

Definition 7. ***Group*** *(From (Gallian (2017)).*

Let G be a set together with a binary operator (usually called multiplication) that assigns to each ordered pair ***(a,b)*** *of elements of G an element in G denoted by* ***ab****. We say G is a* ***group*** *under this operation if the following three properties are satisfied:*

Associativity*. The operation is associative, that is (ab)c=a(bc) for all a,b,c in G.*

Identity*. There is an element* ***e*** *(called identity) in G such that* $ae = ea$ *for all* $a \in G$*.*

Inverses. *For each element $a \in G$, there is an element $b \in G$ (called an inverse of a) such that $ab = ba = e$.*

An important concept to understand is the **cardinality** of a group, which refers to the number of elements in the group. The cardinality of a group G is denoted by $|G|$, and it is a measure of the size or order of the group. The order of a group is an important property as it determines the structure and behavior of the group. For example, groups with a prime order have certain properties that distinguish them from groups with composite order. The cardinality of a group is also used in many theorems and proofs in group theory, and it will take special importance when we discuss God's number for the Rubik's cube.

4.1. The Symmetric Group

When we are talking about permutation groups, which are the groups that consist of a set of permutations of a set that form a group under function composition, there exists a special kind of permutation group denoted Symmetric Group S_n. If you let $A = 1, 2, ..., n$, the set of all permutations of A is called the *symmetric group of degree n,* S_n. The elements of S_n can be expressed in the form:

$$\begin{pmatrix} 1 & 2 & 3 & ... & n \\ \alpha(1) & \alpha(2) & \alpha(3) & ... & \alpha(n) \end{pmatrix}$$

Similarly to what we could use in permutation theory, we can compute the cardinality of the set S_n. There are n choices for $\alpha(a)$. Once $\alpha(1)$ has been chosen, there are $n-1$ possibilities for $\alpha(2)$ and since we are one-to-one, $\alpha(1) \neq \alpha(2)$. Now there are $n-2$ choices for $\alpha(3)$. If we continue this trend, we see that there are $n \times (n-1) \times (n-1) \times ... \times 2 \times 1 = n!$ possible choices for $\alpha(1)$ to $\alpha(n)$. Therefore $|S_n| = n!$.

5 Rubik's cube

You might be wondering how the Rubik's cube falls under the group category. Before we begin here is some cube notation that will be used from this point forward. Later in the section, we will write a formal proof of how the Rubik's cube forms a group.

5.1. Notation

The following image represents all the possible basic moves that we can perform on a Rubik's cube.

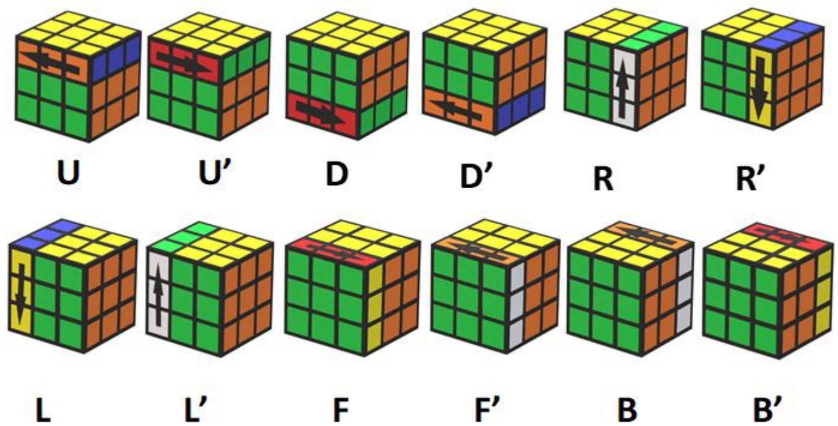


Figure 5

The position of the centers represents each individual move by its first letter. So,

$$R = \text{right},\ L = \text{left},\ U = \text{up},\ D = \text{down},\ F = \text{front},\ B = \text{back}.$$

Each individual letter X denotes a 90-degree rotation on that specific center clockwise. We use prime (X') to determine the inverse of such a move, which can also be simplified as a 90-degree rotation counterclockwise. If we want to represent a 180-degree rotation, for simplification we denote it as X2, for any X that represents a move on that center.

Labeling the cube flat will help better understand the permutations from a group theory perspective. We will only label 48 facets of the cube. We have 6 sides, each of which has $3 \times 3 = 9$ facets. However, since the centerpieces are fixed and will never move regardless of the permutation we perform, $9 \times 6 - 6 = 48$, which leaves 48 facets that are movable. Those are the ones we have represented in the image.

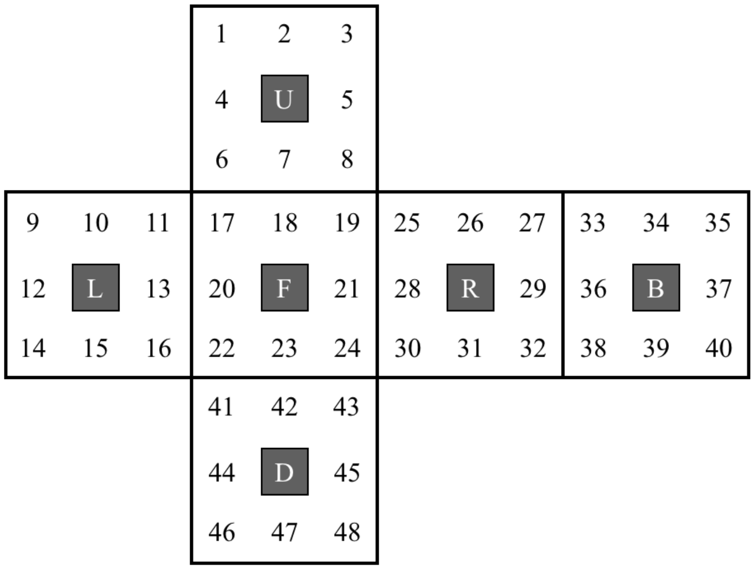


Figure 6
from Mulholland (2021)

5.2. Rubik's Cube Group

The permutations corresponding to each of the basic moves of the cube we just described are:

$$R = (25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24)$$

$$L = (9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35)$$

$$U = (1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)$$

$$D = (41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)$$

$$F = (17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11)$$

$$B = (33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27)$$

Let's formally prove that the Rubik's cube forms indeed a group.

Theorem 2. *The set of all* 3×3 *Rubik's cube permutations form a group.*

Proof. Given the set of all Rubik's cube permutations, assume this forms a group. To show that this is a group we have to show that permutations are associative, the set contains the identity, every permutation has an inverse and we also have closure.

Our operation is associative since

$$(R\ U)\ R' = R\ (U\ R').$$

We also have an identity element, that we can denote as $\varepsilon = ()$ and it is the Rubik's cube when it is in a solved position and we have performed no permutations.

For each permutation α in G, there exists a permutation β also in G such that the cube goes back to a solved state. In other words, the cube can always be solved. For the basic permutations we have already defined their inverses, but the inverse of more complex concatenated permutations can be found as well. If we have a specific scramble, we will always be able to find it's inverse, for example,

$$(U2\ L'\ D\ B2\ U\ L2\ D2\ L2\ U2\ F2\ U'\ B2\ L\ D'\ R2\ F\ D2\ L'\ R'\ F) *$$
$$(F'\ R\ L\ D2\ F'\ R2\ D\ L'\ B2\ U\ F2\ U2\ L2\ D2\ L2\ U'\ B2\ D'\ L\ U2) = \varepsilon$$

can be found by starting at the end of the element and finding the inverse of each individual permutation.

The cube also has closure, since we can concatenate all Rubik's cube permutations and it will always create a different element of the permutation group. For example,

R U R', and *U2 L' D B2 U L2 D2 L2 U2 F2 U' B2 L D' R2 F D2 L' R' F.*

Therefore, the set of all permutations of the Rubik's cube forms a group since it follows the properties of associativity, identity element, inverses, and closure. ∎

The list of elements that belong to the Rubik's cube permutation group is very large. Some elements are very simple, like the identity (ε), or all the simple basic moves (*R, L, U, D, F, B*) and their inverses (*R', L', U', D', F', B'*), but then we can combine all these moves to create exactly 43,252,003,274,489,856,000 elements that belong to the Rubik's Cube Group. Therefore, we can represent the Rubik's cube group as $(G, *)$ where the operator $(*)$ is the concatenation of permutations and for the sake of simplification we will avoid it when presenting a scramble.

Now that we understand groups more, we have the tools to understand another way we can get to the 43 quintillion possibilities that were discussed in the first chapter. As you can see, the group we just described is no other than a subgroup of S_{48}, generated by the 6 basic moves. So,

$$G = \langle R, L, U, D, F, B \rangle.$$

Similarly, the edges form their own symmetric subgroup and the corners form their own symmetric subgroup as well.

5.3. God's Number

We have gotten to a point where we know the order of the Rubik's cube group,

$$|G| = 43,252,003,274,489,856,000.$$

But what is the maximum order that an element of this group can have? In other words, if we were to scramble the Rubik's cube non-stop to try to get the "most difficult" combination. What would the minimum maximum amount of permutations be to get back to a solved state? There is a simple way to visualize this using graph theory.

5.3.1. Cayley Graphs

Turns out that the cube also has some implications and ties to graph theory and discrete mathematics. The following explanation has been adapted from Rokicki (2014).

A **Cayley graph** is defined by a group and a particular generating set. Each of the vertices of the graph correspond to an element (or a cube scramble), and each of the edges are the application of an element of the generating set (In our case, $\langle R, L, U, D, F, B \rangle$). Under that construction, each path from a given position to the identity element (solved) is a solution to the cube.

With that in mind, the minimal, or optimal solution is the shortest path in the graph, and the distance of a position from solved is the length of the shortest path.

Since we defined that the group is closed and this means that the generating set ($\langle R, L, U, D, F, B \rangle$) will also generate all the inverse moves (R', L', U', D', F', B'), the graph will be undirected as the moves can be executed clockwise or counterclockwise. This graph would have a total of 43 quintillion elements, and we could imagine it as a big three-dimensional mesh. God's number is also equivalent to the diameter of this Cayley graph. For the sake of understanding and simplification, take a look at Figure 7 which recreates what a small portion of this graph would look like.

In the following chapter, we will be using this and other knowledge to show that the diameter of this graph is in fact 20, and how different investigators were able to arrive at such a conclusion.

5.3.2. Cube 20

In July 2010, a group of 4 researchers, proved that the number is in fact 20 in Rokicki, Kociemba, Davidson, and Dethridge (2014). They needed 35-CPU years of computer time donated by Google and essentially solved each position as well as showed that no position needs more than 20 moves.

The way they did this was by using symmetries and partitioning. They took a problem that seemed pretty much impossible to solve and broke it down into a much smaller one.

One of the main concepts that they used to achieve such an accomplishment was recoloring-equivalence and symmetry-equivalence. Recoloring equivalence is when two different cube scrambles have a one-to-one mapping.

Symmetry equivalence is when you can reorient the first cube in some manner and then it becomes color-equivalent to the second. To put it another way, if you scramble the cube and put it upside down it will still take you the same amount of moves to solve it, therefore, although it initially seems like a different scramble we can create an equivalence

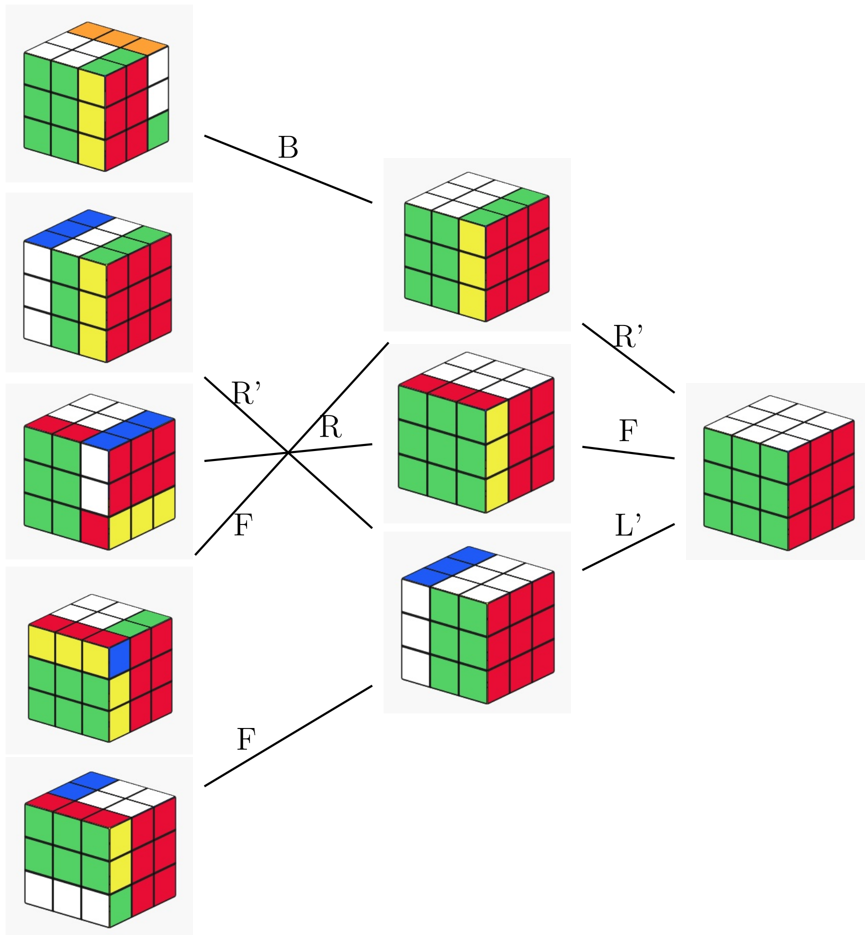
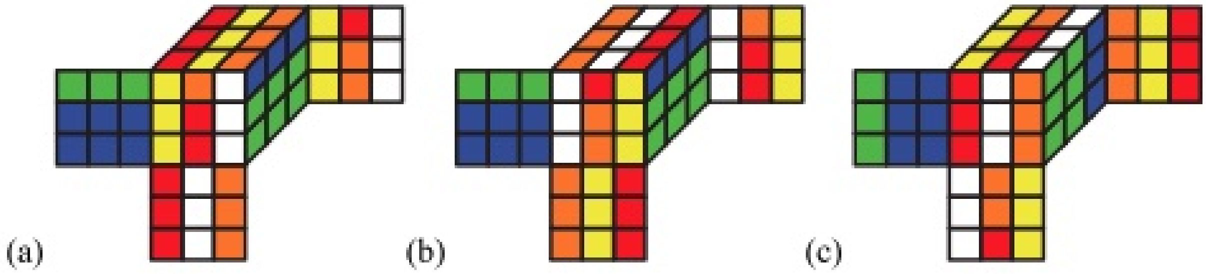


Figure 7
Cayley Graph Example

between them. There are a total of 24 ways that you can orient a cube in space and you can get another factor of two using a mirror so that you get a total of 48 different symmetries out of one scramble. See Figure 8 for extra clearance.

Partitioning is its own different beast and we could have done an entire project on partitioning only. For the sake of conciseness be aware that they partitioned the positions into 2,217,093,120 sets of 19,508,428,800 positions each. They used the cosets of the group generated by $U, F2, R2, D, B2, L2$, which allowed them to solve each set more rapidly.

Using that for the Cube 20 project, Rokicki et al. (2014), they were able to narrow down the scrambles that they needed to solve from 43,252,003,274,489,856,000 to 55,882,296. That simplification allowed them to write the computer program that solved



(a) (b) (c)

Figure 8

from Rokicki (2014). Positions (a) and (b) are recoloring-equivalent. Positions (a), (b), and (c) are all symmetry-equivalent.

those over 55 million using 35 years of computing power. Achieving such a breakthrough did not only help the Rubik's cube but also other areas of mathematics as well since they found the diameter of a Cayley graph that had over 43 quintillion elements, something that initially seemed impossible.

6 Future Research

As you have been able to see there are a lot of advanced mathematics that can be covered within the realm of the Rubik's cube. During this thesis paper, we have only barely scratched the surface. We could have dived deeper into symmetries, cosets, and homomorphisms. We could have also explored solution strategies and how to use technology and mathematics to our advantage when it comes to solving the 3×3 cube. Further research can also explore different cube shapes that also belong to the permutation puzzle family.

All in all, cubes are magnificent toys that have many hidden gems and secrets related to mathematics. Advances in understanding how this puzzle family works can also help advance other areas of mathematics and technology and it can help deepen the abstract algebra knowledge. It can awaken the curiosity of others like myself in this wonderful mathematical world.

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In conclusion, I am deeply grateful to God, who has been evident in every aspect of my life, has brought me to where I am today, and granted me all these wonderful opportunities. I also extend my thanks to mathematics itself for its infinite beauty, elegance, and limitless potential, which have made the journey of exploration both challenging and rewarding.

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