ASTE 404 HW 10 Nikesh Persokar

Problem 1:

Assuming $\nabla D = 0$

$$\frac{\partial n}{\partial t} = -\nabla \cdot (n\vec{v}) + \nabla \cdot (D \nabla n) + R = -\nabla \cdot (n\vec{v}) + D \nabla^2 n + R$$

$$\nabla^2 n \equiv \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{\partial^2 f}{\partial z^2}$$ (in 2D, assuming $f = n$)

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) = \frac{1}{\rho}\left[\left(\frac{\partial}{\partial \rho}\rho\right)\frac{\partial f}{\partial \rho} + \rho \frac{\partial^2 f}{\partial \rho^2}\right] = \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{\partial^2 f}{\partial \rho^2}$$

$$\Rightarrow \boxed{\nabla^2 n \approx \frac{1}{\rho}\left(\frac{n_{\rho+1,z} - n_{\rho,z}}{\Delta \rho}\right) + \frac{n_{\rho-1,z} - 2n_{\rho,z} + n_{\rho+1,z}}{\Delta^2 \rho} + \frac{n_{\rho,z-1} - 2n_{\rho,z} + n_{\rho,z+1}}{\Delta^2 z}}$$

or, using central difference:

$$\boxed{\nabla^2 n \approx \frac{1}{\rho}\left(\frac{n_{\rho+1,z} - n_{\rho-1,z}}{2\Delta \rho}\right) + \frac{n_{\rho-1,z} - 2n_{\rho,z} + n_{\rho+1,z}}{\Delta^2 \rho} + \frac{n_{\rho,z-1} - 2n_{\rho,z} + n_{\rho,z+1}}{\Delta^2 z}}$$

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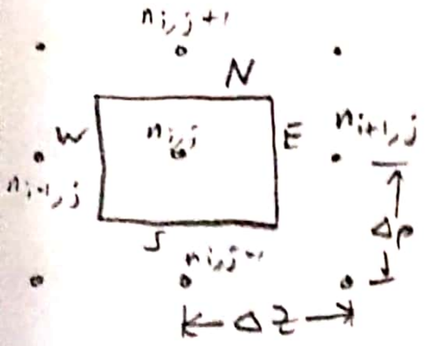
Problem 2:

$$D\nabla^2 n = ?\;;\quad \int_V D\nabla^2 n\, dV = D\oint_S (\nabla n)\cdot \hat{n}\, dA$$



$$(\hat{n}A)_N = \Delta z\left[2\pi(\rho+\Delta\rho)\right]$$

$$(\hat{n}A)_S = -\Delta z(2\pi\rho)$$

$$(\hat{n}A)_E = \pi(\rho+\Delta\rho)^2 - \pi\rho^2 = \pi(\rho^2+2\rho\Delta\rho+\cancel{\Delta\rho^2}) - \pi\rho^2 = 2\pi\rho\Delta\rho$$

$$(\hat{n}A)_W = -2\pi\rho\Delta\rho$$

$$D\oint_S(\nabla n)\cdot\hat{n}\,dA = D\overbrace{\frac{n_{i,j+1}-n_{i,j}}{\Delta\rho}\left[\Delta z(2\pi\rho+2\pi\Delta\rho)\right]}^{1} + D\overbrace{\frac{n_{i+1,j}-n_{i,j}}{\Delta z}(2\pi\rho\Delta\rho)}^{2} +$$

$$+ D\overbrace{\frac{n_{i,j}-n_{i,j-1}}{\Delta\rho}\left[-\Delta z(2\pi\rho)\right]}^{3} + D\overbrace{\frac{n_{i,j}-n_{i-1,j}}{\Delta z}(-2\pi\rho\Delta\rho)}^{4}$$

$$\text{Term } 1 = \nabla n_N \Delta z(2\pi\rho) + \frac{n_{i,j+1}-n_{i,j}}{\cancel{\Delta\rho}}2\pi\cancel{\Delta\rho}\Delta z$$

$$\Rightarrow D\oint_S(\nabla n)\cdot\hat{n}\,dA = D\overbrace{\left(\frac{n_{i,j+1}-2n_{i,j}+n_{i,j-1}}{\Delta\rho}\right)}^{4}\left[\Delta z(2\pi\rho)\right] + D\overbrace{\left(\frac{n_{i+1,j}-2n_{i,j}+n_{i-1,j}}{\Delta z}\right)}^{5}\left[\Delta\rho(2\pi\rho)\right]$$

$$+ D(n_{i,j+1}-n_{i,j})(2\pi\Delta z)\;;\quad \Delta V = 2\pi\rho\Delta\rho\Delta z$$

$$D\nabla^2 n = \frac{1}{\Delta V}D\oint_S(\nabla n)\cdot\hat{n}\,dA = \cancel{D}\,4\,\frac{\cancel{\Delta z}\cancel{2\pi\rho}}{\cancel{2\pi\rho}\Delta\rho\cancel{\Delta z}} + \cancel{D}\,5\,\frac{\cancel{\Delta\rho}\cancel{2\pi\rho}}{\cancel{2\pi\rho}\cancel{\Delta\rho}\Delta z} + \cancel{D}(n_{i,j+1}-n_{i,j})\frac{\cancel{2\pi}\cancel{\Delta z}}{\cancel{2\pi}\rho\Delta\rho\cancel{\Delta z}}$$

$$\Rightarrow \boxed{\nabla^2 n = \frac{1}{\rho}\left(\frac{n_{i,j+1}-n_{i,j}}{\Delta\rho}\right) + \left(\frac{n_{i,j+1}-2n_{i,j}+n_{i,j-1}}{\Delta^2\rho}\right) + \left(\frac{n_{i+1,j}-2n_{i,j}+n_{i-1,j}}{\Delta^2 z}\right)}$$