1. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

a)How many balls must she select to be sure of having at least three balls of the same color?
She must select 5 balls. Having only 4 balls, you either have at least 3 of one color or you are in a situation where you have 2 blue balls and 2 red ones. Selecting any additional ball gives you three balls of the same color. Therefore, selecting 5 balls is sufficient.
b)How many balls must she select to be sure of having at least three blue balls?
She must select 13 balls. To see this, consider the scenario that
The first ten balls she selects are all red. Then she still needs to pick 3 more balls to have three blue ones.

2. How many numbers must be selected from the set {1, 3, 5, 7, 9, 11, 13, 15} to guarantee that at least one pair of these numbers add up to 16?

The answer is 5. Divide the above numbers into the following 4 groups: {1, 15}, {3, 13}, {5, 11}, {7, 9}. If we choose 5 numbers out of 4 groups, then by Dirichlet's principle, we'll have at least 2 numbers in the same group, and their sum will equal 16.

3. (10 points) A computer programming team has 15 members.

(a) How many ways can a group of seven be chosen to work on a project?
C(15,7) = 15!/ &!(15-7)! = 15!/ 7!8!
C(15,7) = 5*13*11 = 715 chosen groups possible
(b) Suppose nine team members are SE students and six are CPRE students.
i. How many groups of seven can be chosen that contain four SE and three CPRE students?
C(9,4) = 9!/4!(9-4)! = 9!/ 4!5!
C(6,3) = 6!/ 3!(6-3)! = 6!/3!3!
9*8*7*6/4*3*2*1 * 6*5*4/3*2*1 = 126 *20 = 2520 ways
ii. How many groups of seven that contain at least one SE student can be chosen?
Number of ways to choose 7 with at least one SE = total ways - ways without SE
C(15,7) - C(6,7) = C(15,7) - 0 = C(15,7)
We now from the first part that this is 715 ways to choose a group
iii. How many groups of seven can be chosen that contain at most four CPRE students?
Number of ways to choose 7 with at most 4 CPRE groups = sum from k=0 to k=4 C(6,k) * C(9,7-k)
C(6,0) =1 C(6,1) = 6
C(6,2) =15 C(6,3) =20 C(6,4) =15
1* C(9,7) + 6 * C(9,6) + 15 * C(9,5) + 20 *C(9,4) + 15 * C(9,3)
= 36 + 504 + 1890 + 2520 + 1260 = 6210 ways to choose a group

4. If there are 4 colors of jellybeans and you are trying to fill up a jar that holds 100 beans, how many different color combinations exist (assuming no restrictions on the distributions of the colors)?

C(n+ r - 1, r-1)
C(100 + 4 - 1, 4-1) = C(103,3)
=103!/ 3!*100! = 103*102*101/3*2*1 = 176,851 different color combinations

5. In how many ways can you place 2 identical rooks on an 8 × 8 chessboard such that they will not be able to capture each other (i.e., they do not share the same row or column).

2 rooks should be placed in 64 squares. So, the sample space is: 64C2 = 64! / (62! * 2!) = 64 * 63 / 2 = 2016

the number of ways to place 2 identical rooks on an 8x8 chessboard such that they will not be able to capture each other is: 64 * 49 / 2 = 3136/ 2 = 1568.

6. Here, we prove a deep result in number theory known as Fermat's Little Theorem. However, our proof will require very little knowl-edge of number theory! Instead, we construct a proof that purely uses combinatorics.

a) Suppose there are beads available in different colors for some integer > 1, and let p be a prime number. How many different length p sequences of beads can be strung together?

As there are 'p' different beads, the total number of ways of making such a string is
a*a* (p times) * a = a^p

two different colors? (Hint: Calculate how many beads contain exactly 1 color, and subtract from the first answer.)

Out of these, there are strings which are all of the same color. We can choose a color in 'a' ways and for each color, there is exactly one string of length 'p' having all beads of this color.
Therefore the number of beads with one color is = a.
The total number of strings is more than 1 color is a^p-a (total number of strings - total number of strings of one color only)

(c) Each string of p beads with at least two colors can be made into a bracelet by winding it around a circle in a clockwise manner and tying the two ends of the string together. Two bracelets are the same if one can be rotated to form the other. “Flipping” bracelets or reflecting them is not allowed. Argue that for every bracelet, there are exactly p distinct strings of beads yield it. (Here, you have to use the fact that p is a prime number.)

The bracelet is such that a bead in the 'k' position is the same color as the bead in position 'k+x'
But this is possible only if x divides p.
x=1 means every bead in position 'k' is the same as the bead in position 'k+1'
This effectively means the entire bracelet is the same color which is impossible since we have a bracelet that has at least 2 colors.
Therefore, all p rotations give us unique bracelets which is to say that p distinct strings give us the same bracelet

(d) Use the above result and the Division Rule to argue Fermat's Little Theorem: > for any integer a > 1 and any prime number p, ap − a is a multiple of p.

The number of distinct bracelets is a^p -a / p, which must be an integer (as it counts the different numbers of bracelets)