1. Use mathematical induction to prove that G(n) ≤ 2n − 4 for n ≥ 4. [Hint: In the inductive step, have a new person call a particular person at the start and at the end.]

CMPE310 HW6

Q1) Prove $G(n) \leq 2n-4$ for $n \geq 4$

let $P(n)$ be $G(n) \leq 2n-4$

$n=4$

when there are 4 people, suppose the first 2 and the last 2 call each other then person 1 and person 2 communicate and person 3 and 4 communicate

Therefore they all know 4 scandals.

$G(4) = 4 \leq 4$ shows that $P(1)$ is true

$\rightarrow G(k+1) = 1 + G(k) + 1 = G(k) + 2 \leq 2k - 4 + 2$
$= 2(k+1) - 4$

$G(k+1) \leq 2(k+1) - 4$

Therefore $P(k+1)$ is also true

→ By principle of mathematical induction, $P(n)$ is true for all positive integers n.

2. Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute rs. Show that no matter how you split the piles, the sum of the products computed at each step equals n(n − 1)/2.

Let P(n) be the sum of the products. When n=2, we can only split the 2 stones into two piles of 1 stone each. The product is then 1. Therefore P(2) is true.
Now, we assume P(1), P(2), P(k) are all true. Prove that P(k+1) is true. The first move consists of splitting the pile of k+1 stones into a pile with j and k - j + 1 stones.
The sum of the products for the pile of k+1 = (k+1)((k+1)-1) / 2
By the principle of strong induction P(n) is true for all positive integers n.

3. Suppose that P(n) is a propositional function. Determine for which positive integers n the statement P(n) must be true, and justify your answer, if

Q3) a) if P(1) is true, then P(n+2) is true

- P(3) = P(1+2) = true
  P(5) = P(3+2) is true
  but no information for P(2), P(4), P(6) even numbers
  ∴ statement is false

b) if P(n) and P(n+1) are true, then P(n+2) is true

- for P(1) is true
  P(3), P(5), P(7) also true
- for P(2) is true
  P(4), P(6), P(8) also true

Therefore statement is true for all pos ints n

c) if P(n) is true, then P(2n) is true

- P(2) = P(2·1) is true
- P(4) = P(2·2) is true
- P(8) = P(2·4) is true

But no info for odd numbers
P(3), P(5), P(7)
∴ statement is false

d) if P(n) is true, then P(n+1) is true

- The func P(n+1) is true for all pos ints n when P(n) is true

P(2) = P(1+1) true
P(3) = P(2+1) true
P(4) = P(3+1) true

∴ statement is true

4. Devise a recursive algorithm for finding xn (mod m) whenever n, x and m are positive integers. Also prove that your algorithm is correct. You should use the fact that xn (mod m) = (xn−1 (mod m) · x (mod m)) (mod m).

Input: Positive integers x, n, and m.
If n is 0, return 1 (base case)
If n is even, recursively calculate y = power(mod(x, n/2, m)) and return (y*y) %m
If n is odd, recursively calculate y = power(mod(x, (n-1)/2, m)) and return ((yy)%mx)%m
Output: the result of x^n mod m

5. How many positive integers between 100 and 999 inclusive

a)are divisible by 7?

Smallest number: (7*15) = 105
Largest: (7*142) = 994
994 - 105 = 889 / 7 = 127 +1 (for starting point) = 128 numbers

b)are odd?

999-100 +1 = 900/2 = 450

c)have the same three decimal digits?

9*1*1 = 9 numbers
111,222,333, etc

d)are not divisible by 4?

Check for integers divisible by 4 using the product rule,
|900|/4 =225. (which are divisible by 4)
900-225=675 numbers, not divisible

e)are divisible by 3 or 4?

Check for integers divisible by 3 using the product rule,
|900|/3 =300 .300 integers are divisible by 3.
225 integers are divisible by 4. (above proved)
Now integers divisible by both 3 and 4, thus divisible by 3 x4 =12 using the product rule
|900|/12 = 75 = 300 + 225 - 75 = 450 numbers divisible

f)are not divisible by either 3 or 4?

Total Number of integers - Number of Integers divisible by 3 or 4
= 900 - 450 = 450

g)are divisible by 3 but not by 4?

= integers divisible by 3 - integers divisible by both 3 and 4
=300 - 75 = 225

h)are divisible by 3 and 4?

Integers divisible by both 3 and 4 and thus divisible by 3 x4 =12 , using the product rule
|900|/12 = 75

6. How many different functions are there, from a set with 10 elements to sets with the following number of elements?

a)2

2^10= 1024

b)3

3^10= 59049

c)4

4^10 = 1048576

d)5

5^10 = 9765625

7. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

a)How many balls must she select to be sure of having at least three balls of the same color?

She must select 5 balls. Having only 4 balls, you either have at least 3 of one color or you are in a situation where you have 2 blue balls and 2 red ones. Selecting any additional ball gives you three balls of the same color. Therefore, selecting 5 balls is sufficient.

b)How many balls must she select to be sure of having at least three blue balls?

She must select 13 balls. To see this, consider the scenario that
the first ten balls she selects are all red. Then she still needs to pick 3 more balls to have three blue ones.

8. How many numbers must be selected from the set {1, 3, 5, 7, 9, 11, 13, 15} to guarantee that at least one pair of these numbers add up to 16?

The answer is 5. Divide the above numbers into the following 4 groups: {1, 15}, {3, 13}, {5, 11}, {7, 9}. If we choose 5 numbers out of 4 groups, then by Dirichlet's principle, we'll have at least 2 numbers in the same group, and their sum will be equal to 16.