Mechanics 8/31 ①

LL Ch. 1, 2
GPS Ch 1, 2

$$S(t_1, t_2) = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \equiv \text{"Action"}$$

Calculus of Variations & Functional derivatives

Recall: function $f(x)$

$f(x) \Rightarrow$ $x \Rightarrow f(x)$ (number $\Rightarrow$ number)

'Change of a function'

$x \to x + dx$; $f(x) \to f(x+dx)$

dx ; $f(x+dx) - f(x)$
$= \left(\frac{\partial f}{\partial x}\right) dx$

"const of proportionality"

Functional $S[f(x)]$: function $\to$ number
$q(t) \to S$

"Change of functional"

$q(t) \to q'(t) = q(t) + \delta q(t)$

so, $q'(t) - q(t) = \delta q(t)$

and $\delta S = S' - S = S[q'(t)] - S[q(t)]$

$= S[q(t) + \delta q(t)] - S[q(t)]$

Now, apply to S..

$$\delta S = \int_{t_1}^{t_2} dt \left[L(q(t) + \delta q(t), \dot{q}(t) + \delta\dot{q}(t), t) - L(q(t), \dot{q}(t), t) \right]$$

$$= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right]$$

$\Rightarrow$ since δq is infinitesimal change can say $\Delta L = \frac{\partial L}{\partial q} \delta q$.

Now, take 2^{nd} term..

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \delta \dot{q}\, dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \frac{d}{dt}(\delta q)\, dt = \int_{t_1}^{t_2} dt \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q\right) - \delta q \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right]$$

$$= \cancel{\left.\frac{\partial L}{\partial \dot{q}} \delta q \right|_{t=t_2} - \left.\frac{\partial L}{\partial \dot{q}} \delta q \right|_{t=t_1}}$$

0, if we say $\delta q = 0$ @ t_1 and t_2 (ie, fix endp

so, $\int \frac{\partial L}{\partial \dot{q}} \delta \dot{q}\, dt = -\int \delta q \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}\, dt \Rightarrow$

$$\delta S = \int_{t_1}^{t_2} dt \left[\delta q \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \right]$$

Now...

$$\delta S = \int_{t_1}^{t_2} dt\, \delta q(t) \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right]$$

For minimal action, claim that $\delta S = 0$

(for small deviation δq from "optimal path")

Since δq is "arbitrary," must be that

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0}$$ Lagrange's Equation!

~~very important?!~~

Consider $L(q, \dot{q}, t)$, and

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t)$$

just func. of q & t

so, $S = \int dt\, L$ and

$$S' = \int dt\, L + \underbrace{\int dt \frac{d}{dt} f(q,t)}_{=}$$

$$f(q,t)\Big|_{t_1} - f(q,t)\Big|_{t_2}$$

but, $\delta S' = \delta\left(\int dt\, L\right) + \underbrace{\delta\left(\int dt \frac{d}{dt} f(q,t)\right)}$

$$= \cancel{\delta\left[f(q,t)\Big|_{t_1} - f(q,t)\Big|_{t_2} \right]}$$

0, since $\delta q = 0$ @ endpoints!

Thus, if $L' = L + \underbrace{\frac{d}{dt} f(q,t)}$

Then $\delta S' = \delta S \Rightarrow$ same eq'n of motion.

This is a kind of "guage transf." for Lagrangians!