**Statistical Mechanics of Vortex Dipoles**

Consider a fluid consisting of well-separated dipoles on the complex plane. The $j$-th dipole comprises a point vortex of strength $+\Gamma$ located at $z_j + ip_j/2$ and a point vortex of strength $-\Gamma$ located at $z_j - ip_j/2$. Thus $z_j = x_j + iy_j$ is the centroid of the dipole, $p_j$ points in the direction of dipole self-propulsion, and $|p_j|$ is the distance between dipole members.

The Lagrangian for an arbitrary collection of point vortices located at $(x_\alpha, y_\alpha)$ is

$$L = \int dt \, \left( \sum_\alpha \Gamma_\alpha x_\alpha \dot{y}_\alpha - \sum_\alpha \sum_{\beta>\alpha} \frac{\Gamma_\alpha \Gamma_\beta}{2\pi} \ln r_{\alpha\beta} \right) \qquad (1) \quad \boxed{1}$$

where $r_{\alpha\beta} = |\mathbf{x}_\alpha - \mathbf{x}_\beta|$. (We use Greek subscripts to denote sums over individual point vortices, and Latin subscripts to denote sums over dipoles.) The contribution of dipole $j$ to the first term in (**??**) is

$$\Gamma \, \mathrm{Re}\,[z_j + ip_j/2]\, \mathrm{Im}\,[\dot{z}_j + i\dot{p}_j/2] - \Gamma \, \mathrm{Re}\,[z_j - ip_j/2]\, \mathrm{Im}\,[\dot{z}_j - i\dot{p}_j/2] \qquad (2) \quad \boxed{2}$$

After some algebra this becomes

$$-\frac{\Gamma}{2} \left( p_j \dot{z}_j^* + p_j^* \dot{z}_j \right) \qquad (3) \quad \boxed{3}$$

where $^*$ denotes the complex conjugate. The contribution of a single dipole to the second term in (**??**) is

$$-\frac{\Gamma\,(-\Gamma)}{2\pi} \ln |p_j| \qquad (4) \quad \boxed{4}$$

Thus for a single dipole on the complex plane,

$$L = \int dt \, \left( -\frac{\Gamma}{2} \left( p_j \dot{z}_j^* + p_j^* \dot{z}_j \right) + \frac{\Gamma^2}{2\pi} \ln |p_j| \right) \qquad (5) \quad \boxed{5}$$

The equations of motion for a single dipole are therefore

$$\delta z_j^* : \quad \dot{p}_j = 0 \qquad (6) \quad \boxed{6}$$

$$\delta p_j^* : \quad \dot{z}_j = \frac{\Gamma}{2\pi} \frac{p_j}{|p_j|^2} \qquad (7) \quad \boxed{7}$$

The dipole moves at constant speed in the direction of its ‘pointing vector’ $p_j$.

Now suppose that there are two dipoles, $j$ and $k$. Each dipole contributes terms of the form (**??**) to the Lagrangian, but we must also include terms from the double sum in (**??**) that represent the coupling between the two dipoles. Assuming that the dipoles are widely separated, we obtain (after considerable algebra)

$$\frac{\Gamma^2}{4\pi}\left(\frac{p_j p_k}{(z_j - z_k)^2} + \frac{p_j^* p_k^*}{(z_j^* - z_k^*)^2}\right) \qquad \boxed{\texttt{8}} \tag{8}$$

as the leading order approximation to the coupling between the dipoles. Thus the Lagrangian for a pair of dipoles is

$$\begin{aligned} L = \int dt\ \Big[ & -\frac{\Gamma}{2}\left(p_j \dot{z}_j^* + p_j^* \dot{z}_j\right) + \frac{\Gamma^2}{2\pi}\ln|p_j| \\ & -\frac{\Gamma}{2}\left(p_k \dot{z}_k^* + p_k^* \dot{z}_k\right) + \frac{\Gamma^2}{2\pi}\ln|p_k| \\ & +\frac{\Gamma^2}{4\pi}\left(\frac{p_j p_k}{(z_j - z_k)^2} + \frac{p_j^* p_k^*}{(z_j^* - z_k^*)^2}\right)\Big] \end{aligned} \qquad \boxed{\texttt{9}} \tag{9}$$

The resulting equations are

$$\delta z_j^* : \qquad \dot{p}_j = \frac{\Gamma}{\pi}\frac{p_j^* p_k^*}{\left(z_j^* - z_k^*\right)^3} \qquad \boxed{\texttt{10}} \tag{10}$$

$$\delta p_j^* : \qquad \dot{z}_j = \frac{\Gamma}{2\pi}\left(\frac{p_j}{|p_j|^2} + \frac{p_k^*}{\left(z_j^* - z_k^*\right)^2}\right) \qquad \boxed{\texttt{11}} \tag{11}$$

The generalization to an arbitrary number of dipoles is obvious: Sum over $k$ but exclude $k = j$. Thus

$$\dot{p}_j = \frac{\Gamma}{\pi}\sum_{k \neq j}\frac{p_j^* p_k^*}{\left(z_j^* - z_k^*\right)^3} \qquad \boxed{\texttt{12}} \tag{12}$$

$$\dot{z}_j = \frac{\Gamma}{2\pi}\left(\frac{p_j}{|p_j|^2} + \sum_{k \neq j}\frac{p_k^*}{\left(z_j^* - z_k^*\right)^2}\right) \qquad \boxed{\texttt{13}} \tag{13}$$

The sum in (**??**) is just the velocity field induced by distant dipoles at the location of the centroid $z_j$ of dipole $j$. Eqn (**??**) is equivalent to the equation for the displacement vector $\mathbf{p}$ between two infinitesimally separated fluid particles, viz.

$$\frac{D}{Dt}\mathbf{p} = (\mathbf{p} \cdot \nabla)\mathbf{u} \qquad (14)$$ 13a

where $\mathbf{u}$ is the fluid velocity induced by distant dipoles. Equations (**??**)-(**??**) are accurate only so long as the separation between dipoles greatly exceeds the distance between individual dipole members. That is the case for the application we have in mind.

Consider ocean swell propagating in the $x$-direction. Wave-breaking generates dipoles that propagate in the direction of the swell. If the breaking events are identical except for location, then each $p_j$ is initially equal to the same real constant. The dipoles initially propagate in the $x$-direction with the same self-propulsion. If the breaking events are widely separated, then the dipole interaction terms (the sums in (**??**)-(**??**)) are initially very small. On the basis of conservation laws (see below), it seems likely that these terms will remain small. Intuition based upon entropy considerations suggests that the two members of each dipole should tend to move apart. (In the absence of dipole interactions this cannot occur; see (**??**).) Since the dipole members are counter-rotating vortices, such separation contributes to an increase in energy. Since the total energy is conserved, this increase in 'separation energy' must be balanced by a decrease in dipole 'interaction energy.' The weakly interacting dipoles plausibly form a configuration that minimizes the interaction energy.

All the foregoing calculations are for dipoles in infinite geometry, but the geometry we have in mind is infinitely periodic. The equations must be modified accordingly. First consider a single dipole with pointing vector $p$ located at $(x, y) = (2\pi n, 2\pi m)$ where $n$ and $m$ range over all the integers. According to (**??**) the fluid velocity $(u(x, y), v(x, y))$ induced by this dipole is given by

$$u(x, y; p) + iv(x, y; p) = \frac{\Gamma p^*}{2\pi} \sum_{n,m=-\infty}^{+\infty} \frac{1}{(x - iy + 2\pi n + 2\pi i m)^2} \qquad (15)$$ 14

To evolve the system (**??**)-(**??**) we must sum over all the dipoles, and for each dipole we must sum over all its periodic images. To do this efficiently, we partially sum (**??**) to get a single, rapidly converging sum. Our result is similar to the result of Weiss et al for the general case of infinitely periodic point vortices (not configured as dipoles), but I have found a way to do it that is much less complicated than the method they describe.

The key step makes use of

$$\sum_{k=-\infty}^{\infty} \frac{1}{(x-k)^2} = \pi^2 \csc^2(\pi x) \qquad (16) \quad \boxed{15}$$

(Gradshteyn & Ryzhik, 1965, 1.422, p. 36). Other steps, which involve simple trig identities, are given in the Appendix. The result is

$$u(x,y;p) + iv(x,y;p) = \frac{\Gamma p^*}{4\pi} \sum_{m=-\infty}^{+\infty} \frac{(1-\cos x \cosh y_m) + i \sin x \sinh y_m}{(\cos x - \cosh y_m)^2} \qquad (17) \quad \boxed{16}$$

where $y_m = y + 2\pi m$. Singularities occur when $y_m = 0$ and $x = 2\pi n$, i.e. when $(x,y) = (2\pi n, 2\pi m)$, as expected. For dipoles located at $-\pi < x_j, y_j < +\pi$, the sum in (**??**) converges rapidly. That is, dipoles far outside the basic box have a negligible effect on dipoles within the basic box. The evolution eqn (**??**) becomes

$$\dot{z}_j = \frac{\Gamma}{2\pi} \frac{p_j}{|p_j|^2} + \sum_{k \neq j} \left[ u(x_j - x_k, y_j - y_k; p_k) + iv(x_j - x_k, y_j - y_k; p_k) \right] \qquad (18) \quad \boxed{17}$$

We also need the corresponding form of (**??**). This is easily obtained by noting that the sum in (**??**) is the derivative of the sum in (**??**).

A word about notation. So far we have favored the complex notation $z_j = x_j + iy_j$ and $p_j = p_j^x + ip_j^y$, but there is something to be said for the purely real variables $\mathbf{x}_j = (x_j, y_j)$ and $\mathbf{p}_j = (p_j^x, p_j^y)$. In terms of the complex variables, the Lagrangian for the infinite, *non-periodic*, system is

$$L = \int dt \, \left[ \sum_j \left( p_j \dot{z}_j^* + p_j^* \dot{z}_j \right) - H \right] \qquad (19) \quad \boxed{18}$$

where

$$H = \frac{\Gamma}{\pi} \sum_j \ln |p_j| + \frac{\Gamma}{2\pi} \sum_{j>k} \left( \frac{p_j p_k}{(z_j - z_k)^2} + \frac{p_j^* p_k^*}{(z_j^* - z_k^*)^2} \right) \qquad (20) \quad \boxed{19}$$

Compare (**??**)-(**??**) to (**??**). However, (**??**) may also be written

$$L = \int dt \, \left[ 2 \sum_j \mathbf{p}_j \cdot \dot{\mathbf{x}}_j - H \right] \qquad (21) \quad \boxed{20}$$

and

$$H = \frac{\Gamma}{4\pi}\left[\sum_j \ln(\mathbf{p}_j \cdot \mathbf{p}_j) + \sum_{j>k} \frac{2(\mathbf{p}_j \cdot \mathbf{p}_k)(\mathbf{r}_{jk} \cdot \mathbf{r}_{jk}) - 4(\mathbf{p}_j \times \mathbf{r}_{jk}) \cdot (\mathbf{p}_k \times \mathbf{r}_{jk})}{r_{jk}^4}\right] \tag{22} \qquad \boxed{21}$$

where $\mathbf{r}_{jk} = (\mathbf{x}_j - \mathbf{x}_k)$ and $r_{jk} = |\mathbf{r}_{jk}|$. Thus $(x_j, p_j^x)$ and $(y_j, p_j^y)$ form canonically conjugate pairs. This is very useful for applying equilibrium statistical mechanics. Of course, the double sums in (**??**) and (**??**) must be modified for the infinitely periodic case.

There is a strong temptation to jump directly into the stat mech computation, but this will be challenging because the energy takes such a complicated form. (Periodic geometry only makes it worse.) Account must also be taken of the conserved momentum,

$$\sum_j p_j \tag{23} \qquad \boxed{22}$$

Conservation of (**??**) and (**??**) suggest that, for our initial conditions, the dipoles will all move at about the same velocity for a very long time; dipole collisions should not occur. If they do occur, the dipole evolution equations are invalid, and we must treat the system as a system of (twice as many) monopoles. The latter system is notoriously stiff; there are great advantages to the dipole approximation if it holds.

Momentum and energy seem to be the only conservation laws for the infinitely periodic system; periodicity breaks the symmetry corresponding to angular momentum conservation.

Probably the best plan would be to write software that evolves the infinitely periodic dipoles. If that shows something interesting, the next step might be a Monte Carlo calculation of the equilibrium state for a given (low) energy and momentum. The final step would be an attempt to obtain the equilibrium state by analytic approximations.

**Appendix**

By (16)

$$\sum_{n,m=-\infty,+\infty} \frac{1}{(x - iy + 2\pi n + 2\pi i m)^2} = \frac{1}{4}\sum_{m=-\infty}^{+\infty} \csc^2(\pi\alpha_m) \tag{24}$$

where $\alpha_m \equiv (x - i(y + 2\pi m))\,/2\pi$. By standard identities, this is

$$\frac{1}{2}\sum_{m=-\infty}^{+\infty}\frac{1}{1-\cos(x - iy_m)} \tag{25}$$

where $y_m = y + 2\pi m$. Note that the sign of $m$ can be freely flipped because of the infinite summation limits. After some more identities, we obtain

$$\frac{1}{2}\sum_{m=-\infty}^{+\infty}\frac{(1-\cos x\cosh y_m) + i\sin x\sinh y_m}{(\cos x - \cosh y_m)^2} \tag{26}$$