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Section Note 2

1 Logistics

- TA information (email, office hours, office numbers) can be found on the syllabus.
- Groups should have been assigned at this point. Please check out Homework 1 and work on it with your group. Only 1 member needs to submit.
- How to find group: On Canvas, when you look at the left hand side of the page, you'll see "Account - Dashboard - Courses - Groups - Calendar - etc." Go to Group/People then you'll see the members of your group.

2 Optimization (cont.)

2.1 Unconstrained optimization - single variable

At the end of the last section, we talked about an unconstrained optimization, in which you find the maximum value or minimum value of a function $f(x)$ on **all possible values of x** (no constraint on x). As mentioned before, there are generally two steps involved:

- First-order condition: Finding x^* by setting the first derivative $\frac{\partial f(x^*)}{\partial x} = 0$. We don't yet know if this solves for a local maximum or local minimum, so we need the second step below.
- Second-order condition: Calculate the second derivative at x^* and compare it to 0. this helps identifies if x^* maximizes or minimizes $f(x)$

2.2 Unconstrained optimization - multivariable

Now, instead of optimizing $f(x)$, we want to optimize $f(x, y)$ on all possible values of x and y. Similarly, we have two steps; however, we'll only talk about the First-order condition in this section.

At the optimum, first-order conditions require setting its partial derivatives with respect to each x and y to 0. This means:

$$\frac{\partial f(x^*, y^*)}{\partial x} = 0$$

$$\frac{\partial f(x^*, y^*)}{\partial y} = 0$$

This gives us a system of 2 equations for two unknowns (x^*, y^*) to solve.

Note: To calculate partial differential, says, with respect to x, we can apply the same rules in last section and treat the other variable, y, as a constant. For example, let $f(x, y) = 5x+4y+y^2$. Then, $\frac{\partial f(x,y)}{\partial x} = 5$ and $\frac{\partial f(x,y)}{\partial y} = 4 + 2y$.

2.3 Constrained optimization

Instead of solving for a maximum or a minimum on **all possible values of x and/or y**, we now impose **some constraints on x and/or y**.

Assume you are maximizing your utility given a limited budget of 5 dollars. There are two goods x, y in the market, and their prices are 2 and 1 dollar respectively. Your utility function is $f(x,y) = ln(x) + ln(y)$. Assume you spend all your money, then your problem can be written as:

$$\max_{x,y} f(x,y) = \max_{x,y} ln(x) + ln(y) \tag{1}$$

subjected to

$$2x + y = 5 \tag{2}$$

Here, we are trying to find a quantity of good x and y such that we maximize our utility $f(x,y)$. The constraint $2x + y = 5$ means that we spend $2 \times x$ dollar on x, and $1 \times y$ dollar on y, and their total must be equal to your income of 5 dollars.

There are multiple ways to solve this problem. The easiest way is to convert it into an unconstrained, single variable case:

From the constraint (2), we have

$$y = 5 - 2x \tag{3}$$

Replace this to (1) to get an unconstrained, single-variable problem:

$$\max_{x} f(x) = \max_{x} ln(x) + ln(5 - 2x) \tag{4}$$

Exercise: Solve (4) for x^* using the two-step in Section 2.1, then find y^* based on (3). **Hint:** An optimization problem was solved in the last discussion. Please check out last discussion note on Canvas.

3 Sequence (this section is optional)

A **sequence** is a set of things (usually numbers) that are in order. For example, let x_t be GDP of the US in year t, then

$$x_0, x_1, x_2, ...$$

is a sequence of GDP in year 0, year 1, year 2, etc.

A **geometric sequence** with a growth rate g is a sequence in which each term is found by multiplying its previous term by $(1 + g)$. In particular:

$$x_1 = (1 + g)x_0$$

$$x_2 = (1 + g)x_1$$

$$x_3 = (1+g)x_2$$

$$\dots$$

This means

$$x_t = (1+g)x_{t-1} = (1+g) \times (1+g)x_{t-2} = (1+g)^2 x_{t-2} = ... = (1+g)^t x_0$$

4 Lecture 1

4.1 What is Macroeconomics?

- The study of the economy as a whole. Questions include: Why are Americans so much richer than they were 100 years ago?
- Macroeconomic policies affect billions of people
- Model the world: simplified, tractable representation of the economy, uses math, and abstracts from many other features to focus on essential mechanisms

4.2 Long-Run Macroeconomics

- Countries grow substantially in the long run. We often use a ratio scale, i.e. taking the logarithm to look at this.
- USA has grown around 2 percent per year, evidence it might be slowing down though.
- Power of compounding-small growth leads to big change in long run. 7.7x increase in GDP per capita in North America over a century.
- World is getting richer.
- Free markets: property rights, incentives and competition, and creative destruction.

4.3 Short-Run Macroeconomics

- Business cycle: countries grow through periodic bouts of expansion and contraction.
- Recession: Two consecutive quarters of contraction
- Depression: Very large recession

5 Lecture 2

5.1 GDP

- What is GDP? **GDP is the market value of all final goods and services produced within an economy during a given period.**

- Three ways to measure GDP? Production, expenditure, income
- Expenditure is the main way. What are the things that make up GDP? Consumption, Investment, Government Spending, Net Exports
- Consumption: 60-70 percent of GDP in most countries. Includes consumer durable, non-durable goods, and services.
- Investment: 10-20 percent in most countries. Includes business fixed investment, residential investment, and inventory
- Government Spending: 15-20 percent in most countries. Includes government consumption and investment, no transfer payments.

5.2 Limitations of GDP

- Does not include environmental impact, value of leisure, or at home services, such as caring for children.
- Measured market prices: not always accurate capture of value.
- Possible income from abroad: this can be correct by GNP which is GDP plus net factor payments from abroad.

5.3 Nominal values and Inflation

- Nominal vs Real: Nominal is current dollars, real is the value adjusted for inflation.
- Inflation: Prices increase over time, as measured by CPI which is a fixed basket of goods and services and measuring their price.
- GDP Deflator: Nominal GDP / Real GDP

5.4 Interest Rates

- 'Nominal' measures the rate you see, while 'real' refers to actual increase in purchasing power.
- Fisher equation: $(1 + i) = (1 + r)(1 + \pi)$ where i is nominal, r is real, and π is inflation. Simplifies to: $i \approx r + \pi$

Section Note 3

1 Logistics

- Group members and emails are posted under Week 1 materials.

2 Calculating GDP

- Important terms and their relationship: GDP (C, G, I, NX), GNP, NFP, S (S_{pvt}, S_{govt}), CA.
- Real GDP is calculated using a base year price.
- Nominal GDP is calculated using price and quantity of each year.

3 Present value

In this section, we will learn how to price a future asset.

Assume you face two choices: receiving \$100 today, or \$100 one year from now. These choices are not equivalent. If you receive \$100 today, you can put it in a risk-free saving account and are guaranteed to get more than \$100 next year.

So what numbers would make the choices equivalent? What amount of money received *today* is equivalent to \$100 received *next year*? In other words, what is the **present value** of next year's \$100?

To answer this, we need to know about the interest rate i. Assume $i = 0.02$ (two percent). If you receive P_0 dollar right now and put in a saving account, in one year you'll get

$$P_0 \times (1 + 0.02)$$

in total (including your money and the interest).

You'll be indifferent between the two choices if this amount equals \$100. In other words, 'receiving \$100 in one year' would be equivalent to 'receive P_0 today, put it in a saving account, and get \$100 in one year'. Thus we have:

$$P_0 \times (1 + 0.02) = \$100$$

Or equivalently

$$P_0 = \frac{\$100}{1 + 0.02}$$

We end up with $P_0 \approx \$98$ dollars. This means today's \$98 is equivalent to next year's \$100. Or, **\$98 is the present value of next year's \$100**.

3.1 Generalization

Assume that the interest rate is i. The present value of receiving D dollars in 1 year is:

$$P_0 = \frac{D}{1+i}$$

More generally, the present value of receiving D in t years is:

$$P_0 = \frac{D}{(1+i)^t}$$

3.2 Application - Pricing a bond

A bond generally promises a stream of incomes over the year. But, since we are buying it *now*, we want to price it at present value.

Assume you want to price a bond with the following stream of payment:

Year	1	2	3
Payment ($)	500	500	1000

which means you receive 500 dollars in year 1, 500 in year 2, and 1000 in year 3. Assume interest rate is i (nominal interest rate).

We'll calculate the present value of each of these years separately:

Year	1	2	3
Payment ($)	500	500	1000
Present value ($)	$\frac{500}{(1+i)^1}$	$\frac{500}{(1+i)^2}$	$\frac{1000}{(1+i)^3}$

You then calculate present value of the bond by adding these up:

$$\frac{500}{(1+i)^1} + \frac{500}{(1+i)^2} + \frac{1000}{(1+i)^3}$$

In general form, if you receive D_t at year t until year T, then asset price is:

$$P_0 = \frac{D_1}{(1+i)^1} + \frac{D_2}{(1+i)^2} + ... + \frac{D_T}{(1+i)^T} = \sum_{t=1}^{T} \frac{D_t}{(1+i)^t}$$

3.3 Application - Pricing a stock

What makes a stock different from a bond? Many thing, but in this class, we'll focus about risk. Both bonds and stocks promise some income stream, but stocks' income stream (dividend) fluctuates more, and we incorporate this in the discount rate.

For bonds, discount rate is i, nominal interest rate.
For stocks, discount rate is higher, $\rho = i + \lambda$ to incorporate more risk.

In general form, if you receive D_t at year t, then asset price is:

$$P_0 = \frac{D_1}{(1+\rho)^1} + \frac{D_2}{(1+\rho)^2} + ... = \sum_{t=1}^{\infty} \frac{D_t}{(1+\rho)^t}$$

Note that here, stocks can potentially give you dividend forever instead of ending at time T like bonds do.

3.4 Interpretation

What would asset price change if:

- Nominal interest i increases:
- Payment D_t increases:
- High expectation for inflation:

4 Production function

What is a production function? A production function tells me that, for example, if I use 1 worker and 2 printers, how many flyers I can make a year. Or, if I use 2 workers and 10 printers, how many flyers I can make a year.

In this class, we use the following form:

$$Y = AK^a N^{1-a}$$

in which N is number of labor, K is capital, a is a given constant, normally between 0 and 1, A is a given productivity. This is called a Cobb-Douglas function.

In the flyer case, assume $Y = 5K^{0.8}N^{0.2}$. This means, if I use 1 worker and 2 printers, my return is:

$$Y = 5 \times 2^{0.8} \times 1^{0.2}$$

Or, if I use 2 workers and 10 printers, I can make:

5 Properties of the production function

Assume you own a business. You'll always have to think about expanding and downsizing. The production function gives you some ideas about that.

There are three different questions:

- If I scale up both labor and capital, what would happen?
- If I scale up capital only, what would happen?
- If I scale up labor only, what would happen?

All are asking about marginal change in production, and each refers to a different kind of answer, which we'll explore shortly.

5.1 Return to scale

Return to scale answers the 1st question: If I scale up both labor and capital, what happens?

Assume you increase both labor and capital by 2 times. Then, for a Cobb-Douglas function:

$$Y = AK^a N^{1-a}$$

the new production is:

$$Y' = A(2K)^a(2N)^{1-a} = ... = 2AK^a N^{1-a} = 2Y$$

This means, if you double both N and K, you double your output. Similarly, if you triple both N and K, you triple your output, and so on. This is result is a nice characteristic of the Cobb-Douglas production function, which is called a *constant return to scale.*

5.2 Marginal product of capital

Marginal product of capital (MPK) answers the 2nd question: If I scale up capital, what happens?

Mathematically, MPK is the partial differential of Y with respect to capital K:

$$MPK = \frac{\partial Y}{\partial K}$$

which means in Cobb-Douglas case:

$$MPK = \frac{\partial}{\partial K} AK^a N^{1-a} = aA(\frac{N}{K})^{1-a}$$

This tells you, for one additional unit of capital K, you can get $aA(\frac{N}{K})^{1-a}$ addition unit of output.

An interesting property of this is that, the more K you have, the lower MPK you can get. In other words, the higher capital level you have, the return from putting one more unit of capital in lowers. Thus, we have a *diminishing marginal product of capital.*

5.3 Marginal product of labor

Marginal product of labor (MPN) answers the 3rd question: If I scale up labor, what happens?

Mathematically, MPN is the partial differential of Y with respect to labor N:

$$MPN = \frac{\partial Y}{\partial N}$$

which means in Cobb-Douglas case:

$$MPN = \frac{\partial}{\partial N} AK^a N^{1-a} = (1-a)A(\frac{K}{N})^a$$

This tells you, for one additional unit of labor N, you can get $(1-a)A(\frac{K}{N})^a$ addition unit of output.

An interesting property of this is that, the more K you have, the lower MPN you can get. In other words, the higher labor level you have, the return from putting one more unit of labor in lowers. Thus, we have a *diminishing marginal product of labor*.

6 Graphical presentation of marginal product

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Section Note 5

1 Logistics

- Grades for HW1 are posted. Prelim incoming.
- Today:
  - Growth and growth calculation
  - Solow model
  - Optimization problems for labor demand and supply

2 Growth accounting (Optional)

2.1 Where does growth come from?

When we talk about growth, we are referring to the growth of output. And in this class, we assume that we can calculate outputs using:

$$Y = AK^aN^{1-a} \tag{1}$$

with Y as output (ie. number of flyers), K as capital (ie. number of printers), N as labor (ie. number of employees), and A as total factor productivity (TFP), which shows how productive your inputs are.

How do we increase output? By increase either capital K, labor N, or their productivity A. Now, assuming that we know K, or N, or A grow at some rate g (says, 2% every year), how do we calculate the growth rate of Y? In the next sections, we will answer the following questions:

- Given growth of N, K, A, what is the growth rate of Y?
- Given growth of capital per person, what is the growth rate of output per person?

2.2 Calculating growth rate of output

Let g_K be the growth rate of capital K, g_N for labor N, g_A for productivity A, and g_Y for output Y. From the production function (1), we can calculate growth of Y given growth of K, N, and A. In other words, we can express g_Y given g_N, g_K, and g_A.

For year t, your output is:

$$Y_t = A_tK_t^aN_t^{1-a} \tag{2}$$

Take log both sides:

$$\ln(Y_t) = \ln(A_t) + a\ln(K_t) + (1-a)\ln(N_t) \tag{3}$$

Doing the same for year $t+1$, we get

$$\ln(Y_{t+1}) = \ln(A_{t+1}) + a\ln(K_{t+1}) + (1-a)\ln(N_{t+1}) \tag{4}$$

Now subtract each side of (3) by (4), we get:

$$\ln(Y_{t+1}) - \ln(Y_t) = [\ln(A_{t+1}) - \ln(A_t)] + a[\ln(K_{t+1}) - \ln(K_t)] + (1-a)[\ln(N_{t+1} - \ln(N_t]) \tag{5}$$

From Discussion section 1, we know that the change of log from one year to another approximately equals growth rate (for example, $\ln(Y_{t+1}) - \ln(Y_t) \approx g_Y$). So, this means (5) is approximately equivalent to:

$$\boxed{g_Y \approx g_A + ag_K + (1+a)g_N}$$

Exercise 1 An economy produces with output $Y = AK^{0.2}N^{0.8}$. Assume population grows by 3% every year, while other things remain unchanged. What is the growth rate of output?

Exercise 2 Assume now productivity grows by 1%, capital by 1%, while population declines by 3% per year. What is the growth rate of output?

2.3 Calculating growth rate of output per capita

We don't just care about output but also output per capita (per person) growth. We'll answer the question: how does 'output per person' grow when 'capital per person' grow?

First, we need to transform our initial equation to 'per capita' terms. With $Y_t = A_t K_t^a N_t^{1-a}$, we can divide both sides by labor N and get:

$$\frac{Y_t}{N_t} = A_t(\frac{K_t}{N_t})^a$$

Let small $y_t = \frac{Y_t}{N_t}$ be output per capita, and small $k = \frac{K_t}{N_t}$ be capital per capita. Then this becomes:

$$y_t = Ak_t^a$$

Following the same steps in the previous section, we find a relationship for growth rate between per capita terms:

$$\boxed{g_y = g_A + ag_k}$$

Exercise 3 An economy produces with output $Y = AK^{0.2}N^{0.8}$. Assume capital per person grows by 5% every year, while other things remain unchanged. What is the growth rate of output per capita?

3 Solow model

In the previous section, we assumed the capital, labor, and productivity is growing, but we did not explain why and how they grow. In this section, we'll focus on the 'how' and 'why' for capital K.

The Solow model proposes an explanation for why an economy grows by accumulating capital. Simply put, in each period, we invest in new capital which then helps us produces more next period.

For any model, we make assumptions about how we produce and consume. In the Solow model, the assumptions are:

- The country produces with constant productivity: $Y_t = \bar{A}K_t^{1/3}L_t^{2/3}$
- Labor force remains unchanged overtime: $L_t = \bar{L}$
- A country either consume or invest only: $Y_t = C_t + I_t$
- We save at a constant proportion of income: $I_t = \bar{s}Y_t$

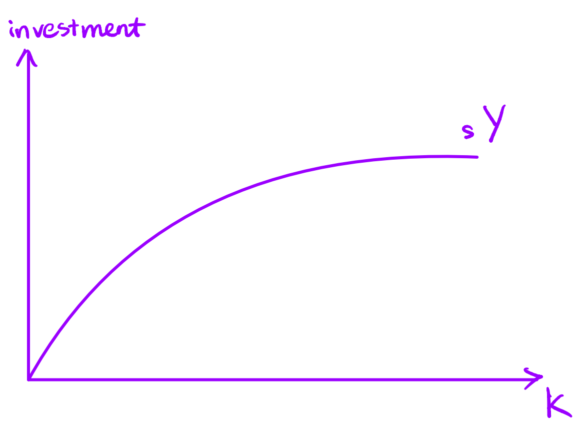
And finally, the most crucial assumption for capital:

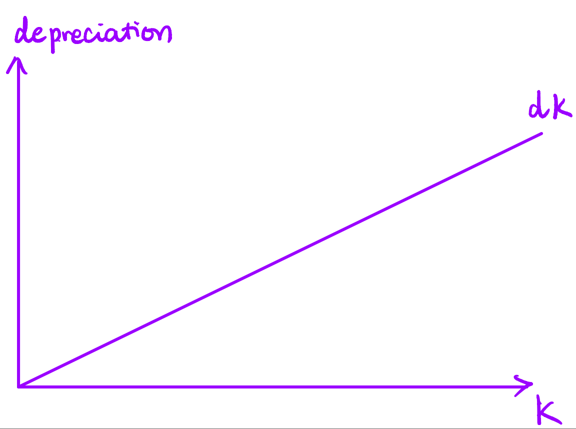
$$\Delta K_{t+1} = K_{t+1} - K_t = I_t - dK_t = \bar{s}Y_t - dK_t \tag{6}$$

The left-hand side (LHS) denotes the change in capital from one year to another. The right-hand side (RHS) shows the causes for that change. We increase capital by investing (hence the $+I_t$), and lose capital by depreciation (hence the $-dK_t$). Depreciation rate d shows how much of current capital K_t is lost each year (eg. if $d = 0.5$, half of our capital is depreciated).

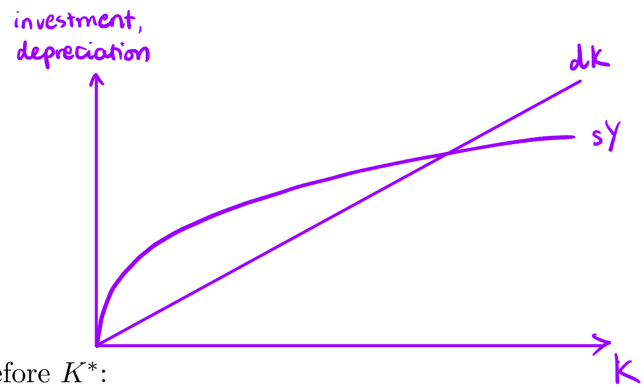
When ΔK is positive, we will have more capital and produce more next period (thus growth). If it is negative: less capital and fewer outputs. If it is zero: we are at steady state, and output is constant over time.

Now to visualize these three cases, we'll use the following graph:

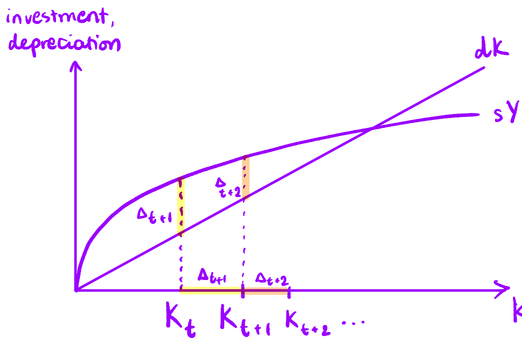




Together:

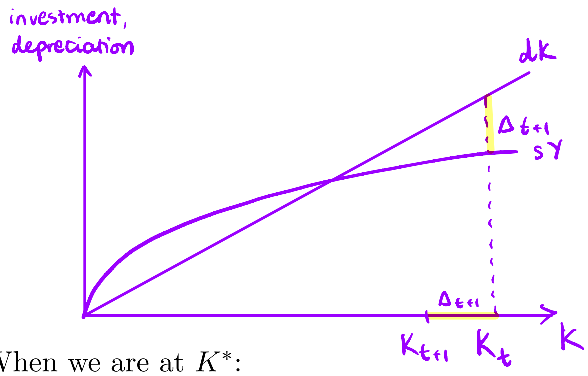


When we are before K^*:



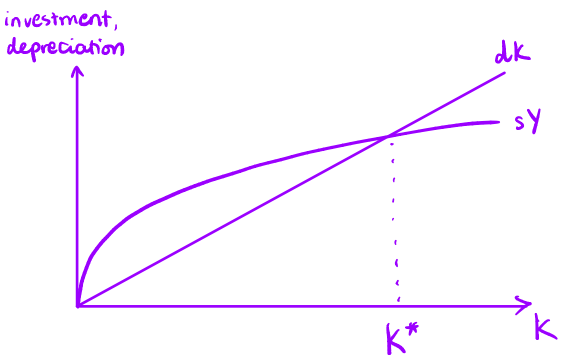
→ smaller ΔK after each period

When we are after K^*:



→ negative ΔK

When we are at K^*:



→ ΔK = 0, steady economy

Characteristics of steady state:

- We often use $*$ superscript for steady state.
- Steady state happens when $\Delta K = 0$, meaning there is no change of K from one period to another (no growth or decay).
- Steady state tends to happen eventually. This means the economy will stop growing at some point.
- From (6), steady state means $\bar{s}Y^* = dK^*$, or $\frac{K^*}{Y^*} = \frac{\bar{s}}{d}$

Exercise 4 Using the last graph, describe what happens at steady state (a) when $\bar{s}$ increases; (b) when d increases, (c) when TFP (A) increases.

4 Labor force

Often times we see vacancies and wages posted, and we decide whether we should apply to some of those jobs. What are the mechanism driving all this? Where does wages come from? How do firm determine their labor demand? How do the population determine how much labor to supply, ie. who goes to work and who stay at home? The following sections will explain these three questions.

4.1 Labor demand

Just like a market of apples or pears, there is a market of labor with demand and supply. The demand side is determined by firms, while the supply side is determined by the labor themselves.

In this economy, we assume that firms are price-takers, meaning that they take the price P of output, price UC of capital, as well as wage W (price of labor) as given. Each firm tries to maximize their profit π by choosing the optimal amount of labor N and capital K.

Given any production function $Y(N, K)$, a firm's problem is:

$$\max_{N\geq 0, K\geq 0} \pi = P \times Y(N, K) - W \times N - UC \times K \tag{7}$$

This is an unconstrained multivariable problem with two constraints.

Exercise 5 If you are given the numerical values of P, K, W, UC and the functional form of Y, then (7) becomes a single variable problem with no constraint. Assume the production is $Y = 5N^{0.5}K^{0.5}$ with output price $P = 10$, wage $W = 5$, capital rent $UC = 1$, and capital use $K = 9$, the problem becomes a simple single-variable problem:

$$\max_{N\geq 0}[10 \times (5 \times N^{0.5}9^{0.5}) - 5N - 9]$$

We can simply solve for N using the two-step (FOC, SOC) solution we've been using for the past sections. What is the optimal N in this case?

We will now find the general solution for (7). To find the optimal labor demand N, we find FOC with respect to labor N and set it to 0:

$$\frac{\partial}{\partial N}[P \times Y(N,K) - W \times N - UC \times K] = 0$$

This is equivalent to:

$$P \times \frac{\partial Y(N,K)}{\partial N} - W = 0$$

Note that $\frac{\partial Y(N,K)}{\partial N}$ is the marginal productivity of labor (talked about in Section 3), or MPN. MPN shows how many additional unit of output you get by having 1 more additional labor. So, the above equation can be written as:

$$\boxed{\text{P}\times MPN = W}$$

This expression shows that firms demand labor until marginal benefit equals marginal cost. On the left-hand side, we see how much more money we get by hiring one more labor (marginal benefit). On the right-hand side, we see how much money we lose by hiring one more labor (marginal cost).

Here, we have solved for labor demand using firms' optimization problem. Note that the demand depends on wage and price of outputs.

4.2 Labor supply

Instead of firms, we now solve the optimization problem for households (ie you) to determined the supply of labor.

In this economy, assume there is a representative household that tries to divide their time between working and resting. Denote their time spent to work as N, and their time spent resting as leisure l. They will have their own utility function $u(c,l)$ that depends their consumption c and leisure time l.

In our simple economy, there are two constraints that the household is facing: time and money. They can not rest and work more than 24 hours a day. They also cannot spend more than they make. Mathematically, these constraints can be written as:

- Budget constraint:

  $$c = wN$$

  On the LHS, we have the household's consumption. On the RHS, we have their hourly wage w times the number of hours worked N.

- Time constraint:

  $$l + N = B$$

  On the LHS, we have total numbers of hours worked and rested. On the RHS, we have B, the total number of hours available to you (can be 24 hours, 1 hour, 500 hours depending on what the question gives you).

We are done with the setup. Now, to find the optimal time allocation, household maximizes their utility:

$$\max_{c,l} u(c,l) \tag{8}$$

subjected to:

$$c = wN$$

$$l + N = B$$

This is a problem with two variables c, l and two constraints. However, we can convert this into a problem of one variable N. The constraints mean $c = wN$ and $l = B - N$. Substituting it back to (8), we get a simpler problem of one variable N with no constraint.

Exercise 6 Assume the above economy but with 24 hours constraint (meaning $B = 24$), wage is 2 (meaning $w = \cancel{1}$ 2). The utility function is $u(c,l) = l + ln(c)$. The problem becomes:

$$\max_{c,l}(l + ln(c))$$

subjected to $c = 2N$ and $l + N = 24$.
From the constraints, we have $c = 2N$ and $l = 24 - N$. Replace these into the maximization problem, we get the new simple problem with no constraint:

$$\max_{N}(24 - N + ln(2N))$$

What is the solution N^* for this? What is then, the optimal l^* and c^*?

Note that in this economy, there is no taxes, no saving, and no borrowing. If there are, we will change the budget constraint accordingly.

Exercise 7 Assume we are in a similar economy to Exercise 6, but with a 10% tax. What is the problem the household faces now?
Now, instead of bringing home wN and spend them all, we only bring home $0.9wN$ due to the 10% tax. Household's problem is:

$$\max_{c,l}(l + ln(c))$$

subjected to $c = 0.9 \times 2N$ and $l + N = 24$. Simplify this problem so it becomes a single variable problem with no constraint.

For the homework, you will need to know some terms and properties related to the utility functions. They are

- Marginal utility of consumption (MU_c): by spending 1 more dollar, how much happier I am (or, how much more utility I have)? Its formula is:

$$MU_c = \frac{\partial u(c,l)}{\partial c}$$

- Marginal utility of leisure (MU_l): by having one more hour resting, how much happier I am (or, how much more utility I have)? Its formula is:

$$MU_l = \frac{\partial u(c,l)}{\partial l}$$

- Marginal rate of substitution between c and l: what is the utility trade-off between having one more dollar to spend vs having one more hour to rest? Or, in other words, how many more dollars in consumption should I have to sacrifice one hour of rest? The formula is:

$$MRS_{c,l} = \frac{MU_l}{MU_c}$$

Note: In these sections, I described the simplest economy. There could be tax entering the equation, or changes in units (for example, days instead of hours, yen instead of dollars). These could affect how you set up problems (7) and (8).

5 Suggested answers

Exercise 1 An economy produces with output $Y = AK^{0.2}N^{0.8}$. Assume population grows by 2% every year, while other things remain unchanged. What is the growth rate of output?
Answer: In this case, $a = 0.2$ (given by the production function, with $g_N = 3\%$ and $g_A = g_K = 0$ (given by the setup). Thus, growth rate of output is: $g_Y = 0 + 0.2 \times 0 + 0.8 \times 3\% = 2.4\%$. So, output is growing by 2.4% annually.

Exercise 2 Assume now productivity grows by 1%, capital by 1%, while population declines by 3% per year. What is the growth rate of output?
Answer: $g_Y = 1\% + 0.2 \times 1\% + 0.8 \times (-3\%) = -1.2\%$. This economy has a decreasing GDP.

Exercise 3 An economy produces with output $Y = AK^{0.2}N^{0.8}$. Assume capital per person grows by 5% every year, while other things remain unchanged. What is the growth rate of output per capita?
Answer: Here, $a = 0.2$, $g_A = 0$, and $g_k = 3\%$. Thus, $g_y = 0 + 0.2 \times 3\% = 0.6\%$.

Exercise 4 Using the last graph, describe what happens at steady state (a) when $\bar{s}$ increases; (b) when d increases, (c) when TFP (A) increases.
Answer: See the end of this document.

Exercise 5 If you are given the numerical values of P, K, W, UC and the functional form of Y, then (7) becomes a single variable problem with no constraint. Assume the production is $Y = 5N^{0.5}K^{0.5}$ with output price $P = 10$, wage $W = 5$, capital rent $UC = 1$, and capital use $K = 9$, the problem becomes a simple single-variable problem:

$$\max_{N \geq 0} [10 \times (5 \times N^{0.5} 9^{0.5}) - 5N - 9]$$

We can simply solve for N using the two-step (FOC, SOC) solution we've been using for the past sections. What is the optimal N in this case? **Answer:** FOC: $f'(N) = 50 \times (0.5N^{0.5-1}) \times 3 - 5 = 75N^{-0.5} - 5 = 0$, so $N^* = 225$.
SOC: $f''(N) = -0.5(75N^{-1.5}) < 0$ for N^*, so N^* maximizes the problem.

Exercise 6 Assume the above economy but with 24 hours constraint (meaning $B = 24$), wage is 2 (meaning $w = 1$). The utility function is $u(c, l) = l + ln(c)$. The problem becomes:

$$\max_{c,l}(l + ln(c))$$

subjected to $c = 2N$ and $l + N = 24$.
From the constraints, we have $c = 2N$ and $l = 24 - N$. Replace these into the maximization problem, we get the new simple problem with no constraint:

$$\max_{N}(24 - N + ln(2N))$$

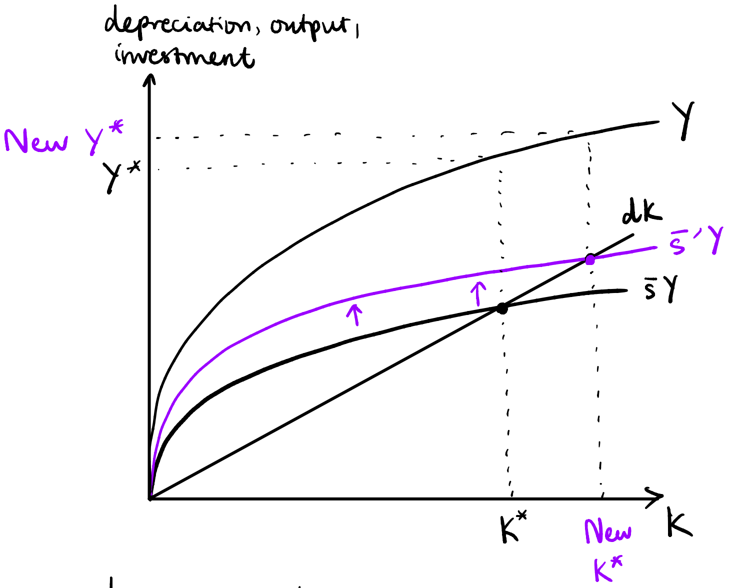
What is the solution N^* for this? What is then, the optimal l^* and c^*?
Answers: FOC: $f'(N) = -1 + \frac{1}{N} = 0$, so $N^* = 1$.
SOC: $f''(N) = -N^{-2} < 0$ for N^*, so N^* maximizes our problem.
$c^* = 2N^* = 2$, $l^* = 24 - N^* = 23$.

Exercise 7 Assume we are in a similar economy to Exercise 6, but with a 10% tax. What is the problem the household faces now?
Now, instead of bringing home wN and spend them all, we only bring home $0.9wN$ due to the 10% tax. Household's problem is:

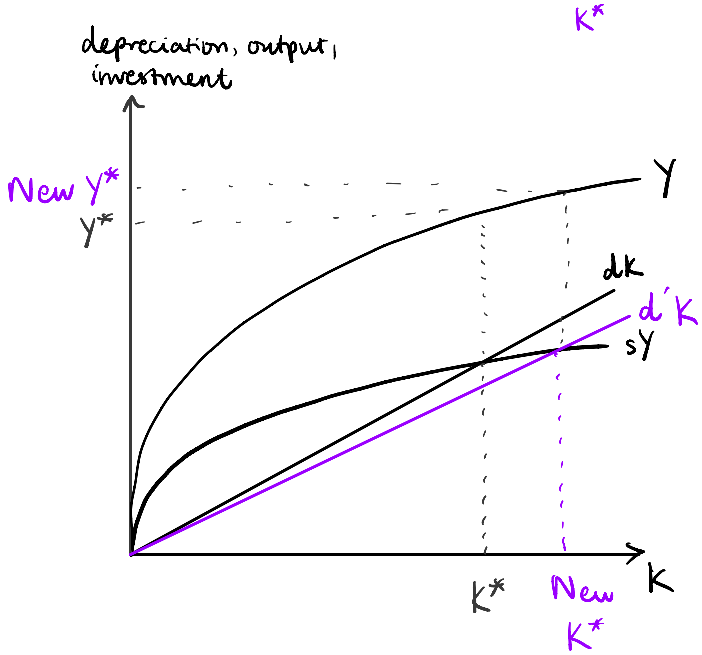
$$\max_{c,l}(l + ln(c))$$

subjected to $c = 0.9 \times 2N$ and $l + N = 24$. Simplify this problem so it becomes a single variable problem with no constraint.
Answers: Replace the constraints to the original problem to get:

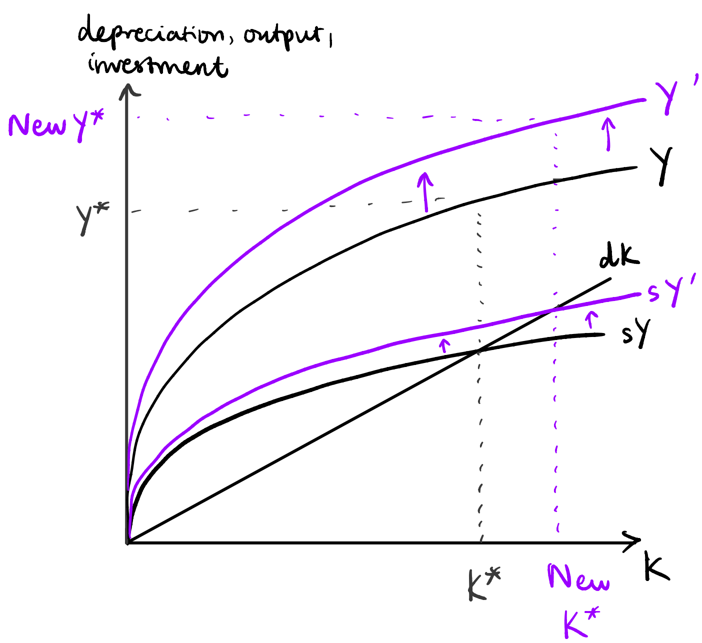
$$\max_{N}(24 - N + ln(1.8N))$$



$\bar{s}\uparrow \Rightarrow Y^*\uparrow, K^*\uparrow$



$d\downarrow \Rightarrow K^*\uparrow, Y^*\uparrow$



$A\uparrow \rightarrow K^*\uparrow, Y^*\uparrow\uparrow$

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Section Note 6 (draft)

1 Logistics

- Today:
  - Intertemporal consumption-savings problems
  - Interpretation of the Euler equation - functionally
  - Graphical solution for the consumption-saving problem

2 Consumption-saving problem

2.1 Set up

In this model, each household choose how much to consume every period. For simplification, they choose between two periods: 'now and future', or maybe 'now and next year'. We denote c and c^f as the consumption now and in the future, and we'll set up the household's problem to solve for (c, c^f).

Why do we have to choose both at the same time? Why not just write two separate problems and solve for c and c^f separately? The reason is because the household's happiness is determined by both c and c^f, and because the choice of consuming c now determines how much you save and have more to consume c^f in the future.

We define the household's total utility as:

$$f(c, c^f) = u(c) + \beta u(c^f)$$

which they wish to maximize by choosing the optimal (c, c^f). Here, we have $u(.)$ is a function of utility given any consumption level. In the first period, they consume c so their utility is $u(c)$. In the next period, they consume c^f so their utility is $u(c^f)$. Here, since the future matters less to us (psychologically for most people), we discount it by $\beta \in (0, 1)$.

Example 1 Assume you have a utility function $u(x) = ln(x)$, and the future is only half important as the present to you. This means your total (life-time) utility for now is $ln(c) + 0.5 \times ln(c^f)$.

A household cannot just consume an infinity amount of goods and services. There are two budget constraints for this problem: first-period and future-period constraints.

- First period: Assume a household is endowed with some initial wealth a, and their current income is y. They can save a^f this periods. Then, the first constraint is:

$$c + a^f = y + a$$

On the right-hand side (RHS), we have the total amount of money they have this period: their income plus endowment. On the LHS is how they choose to spend money: consumption plus saving.

- Future-period: Assume you earn y^f in the next period. That's not all the money you will have. You also have return on your saving from last year, which now gives $(1+r)a^f$ with r being the interest rate. Thus, the second constraint is:

$$c^f = y^f + (1+r)a^f$$

2.2 Household's problem

Given the setup, we have household's problem:

$$\max_{c,c^f} u(c) + \beta u(c^f)$$

subjected to

$$c + a^f = y + a$$
$$c^f = y^f + (1+r)a^f$$

with $u(.), a, y, y^f, r, \beta$ given to you as actual functions and numbers. You choose c, a^f, c^f to maximize your utility.

As usual, to solve for this, we will simplify the problem to make it a single-variable problem with no constraint. Note that since a^f shows up in both constraints, we can write $a^f = \frac{c^f - y^f}{1+r}$ (from the second constraint) and replace it to the first constraint as:

$$c + \frac{c^f - y^f}{1+r} = y + a$$

or equivalently:

$$\boxed{c + \frac{c^f}{1+r} = y + a + \frac{y^f}{1+r}}$$

We call this the **intertemporal budget constraint** ('intertempotal' because it relates current and future income, current and future consumption). Note that the RHS shows present value of all the income you have in your life-time: current income, endowment, and discounted future income. In out model, this is fixed (given to you) regardless of your choice to spend and save. Denote the RHS as **present value of lifetime resources** (PVLR), the above equation becomes:

$$c + \frac{c^f}{1+r} = PVLR$$

or equivalently,

$$c = PVLR - \frac{c^f}{1+r} \tag{1}$$

Replace this to the original problem, we get:

$$\max_{c^f} u(PVLR - \frac{c^f}{1+r}) + \beta u(c^f) \tag{2}$$

With everything else $(u(.), PVLR, \beta, r)$ taken as given, we have simplified the initial problem to a single-variable problem with no constraint.

2.3 Solving for the household problem

To solve for (2), we simply take the first order condition (FOC) and second order condition (SOC) with respect to c^f.

Exercise 1 Continue with Example 1.

Assume further that your income is \$10 each period, and you are endowed with \$200, and the bank's interest rate is 2%. Simplify and solve the problem.

Answer: This means $PVLR = y + a + \frac{y^f}{1+r} = 10 + 200 + \frac{10}{1+0.02} \approx 219.8$. The household's problem becomes:

$$\max_{c_f} ln(219.8 - \frac{c^f}{1.02}) + 0.5ln(c^f)$$

FOC and SOC give you the numerical answer for optimal c^f. With the optimal c^f, you can plug in (1) to get the optimal c.

2.4 Euler equation

Now we want a more general form of the FOC without the numerical values and the actual functional form of $u(.)$. FOC of the simplified problem (2) gives us:

$$u'(PVLR - \frac{c^f}{1+r}) \times (-\frac{1}{1+r}) + \beta u'(c^f) = 0$$

Replace $PVLR - \frac{c^f}{1+r} = c$ (due to (1)):

$$u'(c) \times (-\frac{1}{1+r}) + \beta u'(c^f) = 0$$

Move things around and multiply both sides with $(1+r)$, we'll get:

$$\boxed{u'(c) = \beta(1+r)u'(c^f)}$$

which is the frequently-used Euler equation that gives the connection between the *optimal* current consumption and future consumption. This connection depends on the slope of $u(.)$ (which is $u'(.)$), how much you subjectively discount your future happiness (β), and how much the interest rate (r) is. We'll look at how each of these affects the optimal choice of c and c^f in the next section.

Note that you can keep going with SOC. However, this equation alone is sufficient to do some important interpretations about how much you consumer now (c) and in the future (c^f).

3 Interpretation of the Euler equation

Rewrite the Euler equation as:

$$\frac{u'(c)}{u'(c^f)} = \beta(1+r) \tag{3}$$

What happens to the relationship between c and c^f if there are changes in r or β? We can't exactly tell at first glance, since c and c^f don't show up directly in the Euler equation, but only show up though $u'(.)$ (slope of u at c and c^f).

We need further assumption for the interpretation to work. Assume that $u(.)$ is a strictly concave function: the more money you spend (or the more you consume), the less additional happiness you gain per dollar. Mathematically, we have the slope of $u(c)$, or $u'(c)$, is higher when consumption c is lower. With this assumption, we now know:

- If $c > c^f$, then slope at c is lower than slope at c^f, or $u'(c) < u'(c^f)$, or $\frac{u'(c)}{u'(c^f)} < 1$
- If $c < c^f$, then $u'(c) > u'(c^f)$, or $\frac{u'(c)}{u'(c^f)} > 1$
- If $c = c^f$, then $u'(c) = u'(c^f)$, or $\frac{u'(c)}{u'(c^f)} = 1$

3.1 When household is very patient/impatient

Now back to our interpretation using (3). We know that $\beta \in (0, 1)$ shows how much a household care about their future. A low β means they don't care much about future happiness - for example, when $\beta = 0.01$, the future utility $u(c^f)$ is discounted greatly and is worth $0.01u(c^f)$ at the present. Then, household would want to consume a lot now instead of saving for the future. A high β, on the other hand, means the household is more patient and value the future utility greatly, and they would rather save and spend in the future that they care a lot about.

Mathematically, when β is very low (closer to 0) such that $\beta(1 + r) < 1$, this means $\frac{u'(c)}{u'(c^f)} < 1$ due to the Euler equation (3), which means $c > c^f$. An impatient household will spend more now compared to their future.

Exercise 2: Do the same mathematical interpretation when household is very patient.
Answer: When β is very high (closer to 1) such that $\beta(1 + r) > 1$, we have $\frac{u'(c)}{u'(c^f)} > 1$, which means $c < c^f$. A patient household will spend less now compared to their future.

3.2 When interest rate is very high/low

When interest rate is low, it will discourage households from saving, and they would just spend more now. On the other hand, if the interest rate is very high, it might make sense to save and get a lot of money to spend on more things next year.

Mathematically, when r is very high such that $\beta(1 + r) > 1$, Euler equation means $\frac{u'(c)}{u'(c^f)} > 1$, which means $c < c^f$. A household is motivated to save and thus have more money to spend next year (high c^f).

Exercise 3: Do the same mathematical interpretation when interest rate is very low and household is impatient, such that $\beta(1 + r) < 1$.

Answer: When r is very low such that $\beta(1+r) < 1$, we have $\frac{u'(c)}{u'(c^f)} < 1$, which means $c > c^f$. A household is discouraged from saving, and they consume more c at current year.

Exercise 4: Assume a household is considering how much to consumer now and next month. They know their preference that $\beta = 0.8$, and they can put their saving into a very-high yield saving account and get a 10% interest rate in a month. What can you say about their optimal consumption for this month c_1 and next month c_2?
Answers: From Euler equation: $\frac{u'(c_1)}{u'(c_2)} = \beta(1+r) = 0.8 * (1+0.1) = 0.88 < 1$. Therefore, $u'(c_1) < u'(c_2)$, which means $c_1 > c_2$. They still consume more now despite the high interest rate. Here, we have shown the relationship between c_1 and c_2 without knowing anything about their income this month and next month.

3.3 When interest rate gets higher/lower

Now, we don't just talk about the cases when interest rate r is very high or very low. What happens if interest rate increases/decreases by a bit? What would be the change in household's consumption?

The first effect is the **income effect**. When interest rate gets higher, you will get richer regardless of how much you save (assuming inflation stays the same in this simple economy, and assume you save instead of borrow). Therefore, with more income, a household will increases both c and c^f. This is called the income effect because the interest rate affect your life-time income, which then alter your behavior.

Exercise 5: What is the income effect on c and c^f when you borrow instead of save?
Answer: When you borrow, higher interest rate makes you poorer (because you have to pay more to the bank). Thus, your life-time income decreases, affecting both c and c^f negatively.

The second effect is **substitution effect**. With higher interest rate, you might want to save more in the first period instead of spending. This then increases future consumption c^f but also decrease current consumption c. This is called a substitution effect because it shows the trade-off between current and future consumptions (the substitution between them).

4 Graphical solution for the consumption-saving problem

1 Logistics

- Exams will be returned at the end. If you want a regrade request, please follow Julieta's instructions.
- Today: Building up on the current model in Section 6 with government and firms.

2 Household's consumption-saving problem with no tax (recap)

In Section note 6, I have demonstrated the household's problem. Simply put, they choose their consumption c today and c^f tomorrow so that their utility is maximized. The reason they could choose to allocate c, c^f is because saving/borrowing is possible (without it, they'll have to consume exactly what they make in each period).

Just to recap, the household's problem is:

$$\max_{c,c^f} u(c) + \beta u(c^f)$$

subjected to

$$\text{First period budget constraint: } c + a^f = y + a$$

$$\text{Second period budget constraint: } c^f = y^f + (1+r)a^f$$

From the two constraints, we can combine to get the intertemporal budget constraint:

$$c + \frac{c^f}{1+r} = a + y + \frac{y^f}{1+r}$$

On the LHS, we have present value of life-time consumption. On the RHS, we have present value of life-time income.

From now on, we'll try to build an economy with multiple sides of the economy and not just the household.

3 Introducing the government

3.1 The government's problem

The government is also maximizing their benefit subjected to their own budget constraint in the two periods. They can save, or borrow, too. The only difference is their income comes from tax T and T^f, not labor. Let G, G^f be their spending in each period, we have their constraints:

$$\text{First period budget constraint: } G + a^f = T$$

$$\text{Second period budget constraint: } G^f = T^f + (1+r)a^f$$

This is almost identical to the household's budget constraint, except that they don't have endowment. If, instead of saving, they borrow, then we can replace $a = -b$ and $a^f = -b^f$, with b, b^f

stands for borrowing. Then the equations will look identical to the one in class (Lecture 10, p.3).

Regardless of borrowing or lending, we can combine the two constraints to get:

$$G + \frac{G^f}{1+r} = T + \frac{T^f}{1+r} \tag{1}$$

Exercise 1: Explain how to get to (1) from the government's budget constraints. Explain how to get (1) if instead, we use the constraints in class (Lecture 10, page 3).
Answer: Take $a^f = ...$ out from second-period budget constraint, then replace to the first period budget constraint.
If using the constraints in class, take $b^f = ...$ out from second-period budget constraint, then replace to first period.

Exercise 2: Assume that the government spends \$100 each period. What is the present value of all the taxes you'll have to pay? Is it dependent on interest rate r, your preference β, the government's utility function, or your utility function?
Answer: PV of all taxes are $T + \frac{T^f}{1+r} = G + \frac{G^f}{1+r} = \$100 + \frac{\$100}{1+r}$ which is dependent on interest rate r but not anything else.

3.2 How does this affect the household's problem?

The government enters our current model with tax. Tax does not change the household's utility function. It also does not change things like your own discount of the future β, or the interest rate r. The only thing it changes is your take-home income in each period. Assuming you are taxed T today and T^f next period, we can simply modify the budget constraints and rewrite the household's problem as:

$$\max_{c,c^f} u(c) + \beta u(c^f)$$

subjected to

$$\text{First period budget constraint: } c + a^f = (y - T) + a$$

$$\text{Second period budget constraint: } c^f = (y^f - T^f) + (1+r)a^f$$

Note that here, nothing has changed, except that your income y, y^f are reduced slightly by the tax.

Using steps similar to Section 6 note, we can write a new intertemporal budget constraint:

$$c + \frac{c^f}{1+r} = a + (y - T) + \frac{y^f - T^f}{1+r} \tag{2}$$

This is almost identical to the no-tax case, except that each period's earning is reduced by some taxes! On the LHS, we have the present value of household's lifetime consumption. On the RHS, we have the present value of household's lifetime earning.

4 Introducing firms

4.1 Simple problem

We've been working with two periods model for household and the government. For firms, let's first consider a simple problem of one period. We have seen a similar problem before when solving for labor demand. Today, we'll solve for capital demand.

Assume that firms produce with production function $AF(K, N)$. Here, A is productivity, and F is some function of capital and labor.

Exercise 3: What kind of function F makes firms' production Cobb-Douglas?
Answer: If $F = K^\alpha N^{1-\alpha}$, we have total output equals $AK^\alpha N^{1-\alpha}$ for any input (K, N) used.

Assume firms also observe real wage w of labor and real rent uc of capital. They have to decide how much labor and capital to use to maximize profit. Firm's problem is thus:

$$\max_{K,N} A \times F(K, N) - w \times N - uc \times K$$

Note that we are not multiplying output $A \times F(K, N)$ with output price, because we are already using real input cost (wage and rent). This is a non-constraint two-variable problem. We tried to solve for labor before, now let's solve for capital K. Since it is a non-constraint problem, take FOC with respect to K to get:

$$A\frac{\partial F(K, N)}{\partial K} - uc = 0$$

Or equivalently,

$$A\frac{\partial F(K, N)}{\partial K} = uc$$

On the LHS, we have the marginal productivity of capital (we get MPK by differentiating the production function by K - review Section Note 3 for more. Here, differentiating the production function gives you precisely $A\frac{\partial F(K,N)}{\partial K}$). In other words, the LHS is the gain from using one more unit of capital. On the RHS, we have cost from using one more unit of capital. Firms choose K so that these two equals. We can rewrite the function as:

$$MPK = uc$$

Exercise 4: Assume that you are in an economy with only one firm that use all labors available. The production function is $y = 8K^{0.5}N^{0.5}$. Labor is supplied with $L = 100$. Real wage is $w = \$2$ per labor, real capital rent is $uc = \$5$. What is the optimal capital use?
Answer: Firm solves $\max_K 8K^{0.5}(100)^{0.5} - 2 \times 100 - 5 \times K$.
FOC gives $40K^{-0.5} - 5 = 0$, so the optimal K is 64 units.

4.2 Not-simple problem: Two periods

Let's go back to our two-period problem. Notation wise, for any value in the second period, simply add a f superscript (for example, labor in second period is N^f; usercost, or rent, is uc^f, etc.). We now assume that in the first period, firms are endowed with some capital K, and they have to make some investment decisions without actually producing. In the second period, they produce with the production function $A^f F(K^f, N^f)$. Firm maximizes second-period profit.

4.2.1 Solving for firm's problem

In the second period, firm chooses capital and labor to maximize profit. Firm's production function is the same as in Section 4.1, but with the future f superscript everywhere, since we are producing in the second period. Thus, the equality we solved earlier holds, which means:

$$MPK^f = uc^f \tag{3}$$

4.2.2 Calculating the usercost uc

We can think of the problem from a new angle, which then helps with relating the capital usercost to capital selling prices.

Firm has some capital endowment K that they can either use to produce in the next period, or sell now, put money in the bank account, and get the gain. (For example, you received a printer as a gift. You can sell and put the money in your saving account, or open a printing service to your classmates and earn the profit). Let the price of selling capital in the first period be p_k, and second period be p_k^f. This is different from the rent uc (or usercost to use capital).

What is the optimal choice here? Let's think in terms of gains for **each unit of capital**.

For option 1: use the unit of capital to produce. Here, by putting the unit into producing in the second period, firm gains MPK^f. This is because MPK^f is intuitively (and by definition) the gain when putting in an additional unit of capital into the production. Besides this gain, firm's asset also include the market value of the capital (because they didn't sell). In the second period, the 1 unit of asset depreciated to $1 \times (1-d)$, and since their price is now p_k^f, the market value of capital is $1 \times (1-d) \times p_k^f$.

Firm's total asset gained in the second period is thus:

$$MPK^f + (1-d)p_k^f$$

For option 2: sell the unit. If firm sells, then they will get $1 \times p_k$ dollars. Putting this into the bank account at an interest rate r, then, their total asset gained in the second period is:

$$p_k(1+r)$$

The firm will compare these two values of the two options. If any option is more attractive, firm will spend the unit in that option. They will do this same mental exercise for each unit of the entire capital K they have. Firm is at optimality when they cannot do anything to gain more, which means the two options have to be equally attractive (if any option is more attractive, firm would move one unit of capital from the other option to that option). Mathematically, this means:

$$MPK^f + (1-d)p_k^f = p_k(1+r)$$

Or equivalently,

$$MPK^f = p_k(1+r) - p_k^f(1-d) \tag{4}$$

4.3 Combining the two expressions (3) and (4)

Combining them, we can get:

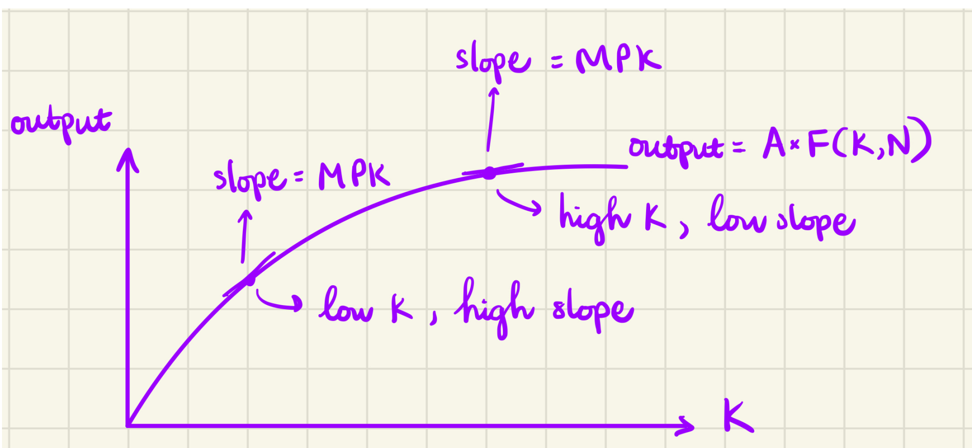
$$uc^f = p_k(1+r) - p_k^f(1-d) \tag{5}$$

which is the economy's relationship between capital rent price and selling price.

4.4 Optimal capital choices as the setup changes

Knowing the optimal choices in (3),(4) helps you interpret the change in firm's optimal capital choices when the setup changes.

Note that we have not solved for optimal K, but only found the optimal MPK. However, that is enough to do some interpretations! With a production function with diminishing return to capital (Cobb-Douglas is an example), as MPK gets higher, K gets lower, and vice versa. Visually, the function looks like this:



Note here, the slope is MPK.

Exercise 5: Assume the optimal MPK increases. What happens to optimal K? Assume the production function has diminishing return to capital.
Answer: Higher K, lower MPK. So, if optimal MPK increases, it means optimal K decreases.

Exercise 6: Assume r increases while everything else stays the same. What can you say about optimal K from (4)? What's the intuition?

Answer: r increases, so from (4), we know optimal MPK increases. If optimal MPK increases, it means optimal K decreases.

The intuition here is, when interest rate gets higher, firms might find selling capital in the first period and put in the bank account more profitable than producing. Thus, they reduce the optimal capital choice.

Exercise 7: Assume usercost decreases while everything else stays the same. What can you say about optimal K from (3)? What is the intuition here?

Answer: uc decreases, so from (3), we know optimal MPK decreases, so optimal K increases. When renting capital is cheaper, firms would like to boost production.

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Section Note 9

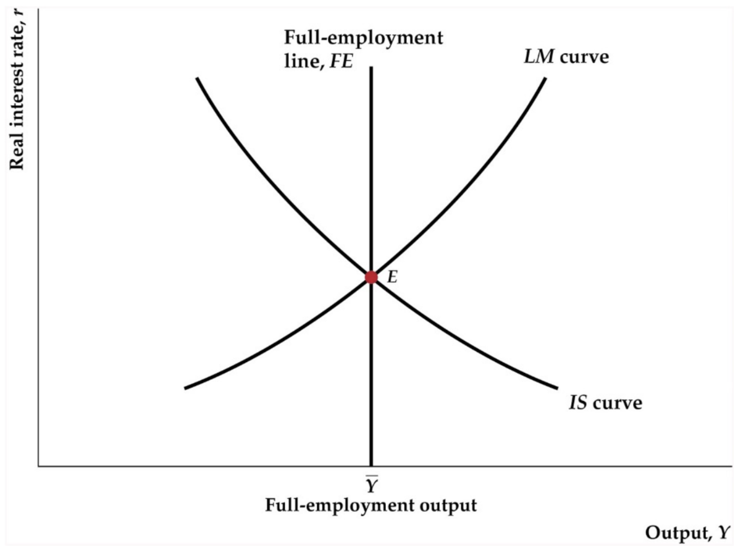
1 Logistics

- Homework 3 due soon
- Today: IS-LM-FE graph

2 Knowing your models

In economics, we care greatly about **the relationship between interest rate and output**. Would higher real interest rate r lead to a higher/lower output Y? Can a central bank manipulate the interest rate to boost/slow down an economy? The answer is not so straight-forward.

To learn about this, we construct the IS-LM-FE graph, which looks like this:



Note here, the vertical axis shows the real interest rate, and the horizontal one shows the output. These are also our two main focuses.

In this model, there are three markets that happens at the same time, and we would like to clear them all. The markets include:

- Labor market: When labor market clears, it results in a full-employment output level, $\bar{Y}$, regardless of interest rate.
- Goods market: For any level of output Y, there is a desired level of real interest rate r that clears the goods market that is consistent with investment and saving behaviours. We can express this by a single equation that relates r and Y together: the curve that show this is called the *IS curve*.
- Money market: When the money market clears, there will be some relationships between r and Y, shown by the LM curve.

The next sections will show the intuition and the math behind the construction of each curve.

3 The FE curve

3.1 Mathematical construction

We normally assume that the economy's output follows the function:

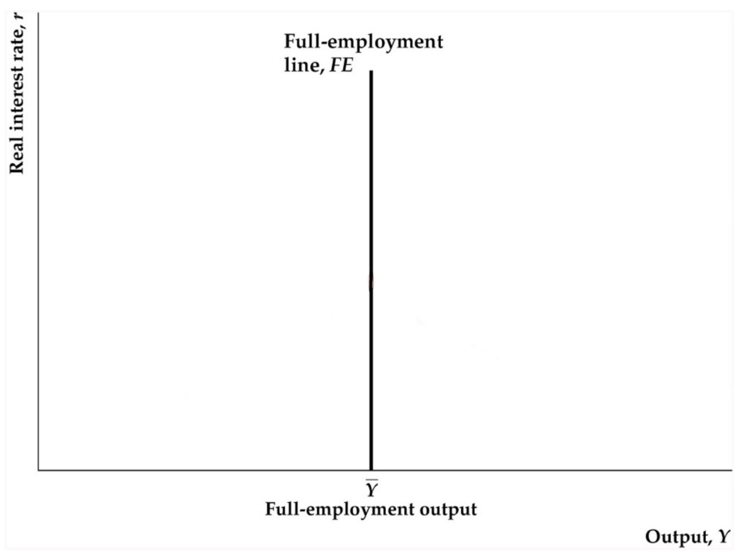
$$Y = A \times f(K, N)$$

Here, when employment is at full-employment level (call it $\bar{N}$), given technology and capital unchanged, we have the **FE equation**:

$$\boxed{\bar{Y} = A \times f(K, \bar{N})}$$

which is a constant. In other words, when the labor market clears and we are at full-employment, output Y is a constant that does not depend on interest rate r. We call the boxed equation the *FE equation*, and call $\bar{Y}$ the *full-employment output.*

Now, to graph the FE equation on the (interest rate, output) space is fairly simple (like graphing $x = 1$ on the (x, y) space):



Exercise 1: What can shift the FE curve? If I change interest rate r, how would it affect the FE curve?

Answer: Looking at the boxed equation (the FE equation), we can see three things:
+ Productivity shock: changes A.
+ Change in labor supply or population, which in turn changes the full-employment $\bar{N}$.
+ Change in capital stock, which changes K.
And a change in interest rate does not change $\bar{Y}$.

4 The IS curve

4.1 Mathematical construction

The IS curve is a result of equalizing investment to saving, so we want to first look at these two separately. Let's try a simple setup with no import or export, and a fixed tax rate (not dependent on output Y level).

(i) National saving is what's left after government consumption and private consumption:

$$S = Y - G - C$$

We assume that consumptions is related to disposable income (a pretty obvious statement that we talked about in intro to macroeconomics) such that $C = \underline{C} + bY_D$ (the more income you have, the more you spend). Here, $\underline{C}$ is a constant, which is what you consume even with no income; b is a constant and marginal propensity to spend. If there's some fixed tax T such that disposible income is $Y_D = Y - T$, then $C = \underline{C} + b(Y - T)$. Plug this in the saving equation, we get national saving:

$$S = Y - G - [\underline{C} + b(Y - T)] \tag{1}$$

(ii) National investment is assumed to be:

$$I = \underline{I} - d(r - r^*) \tag{2}$$

with r^* being the return from investment, which is the MPK of our production function. $\underline{I}$ and d are also constant. If the real interest rate r equals to the real return on investment r^*, then firm invest $I = \underline{I}$. If $r > r^*$, it means borrowing is very expensive for firms compared to their actual return, so they invest less, or $I < \underline{I}$.

(iii) To construct the IS curve, set saving in (1) equals investment in (2):

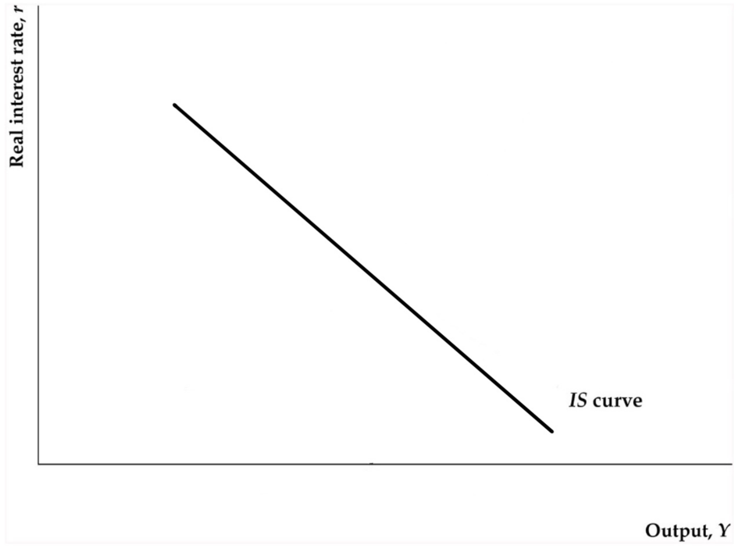
$$Y - G - [\underline{C} + b(Y - T)] = \underline{I} - d(r - r^*)$$

which is equivalent to the **IS equation**:

$$\boxed{r = [\frac{\underline{C} - bT + \underline{I} + G}{d} + r^*] - \frac{1 - b}{d}Y}$$

Here, we have shown that when investment equals saving, r relates to Y in a linear way (for instance, if we plug in all the constants, we might see something like $r = 5 - 0.3Y$).

Graphing the IS equation is as simple as graphing $y = 5 - 0.3x$ on the (x, y) space:



Exercise 2: At what point does the IS curve cross the vertical axis?

Answer: Assuming that the IS equation takes the form of the boxed equation above, then the IS curve crosses the vertical axis at $\frac{\underline{C}-bT+\underline{I}+G}{d}+r^*$.

Exercise 3: What is the slope of the IS curve? Is it negative or positive?

Answer: Assuming that the IS equation takes the form of the boxed equation above, then its slope is $-\frac{1-b}{d}$.
Since b is our propensity to consume, it is normally between 0 and 1 (if $b > 1$, it means we consume more than we have!). We also have $d > 0$, therefore, $-\frac{1-b}{d} < 0$, so the slope is negative. This also explains why the IS curve is downward-sloping.

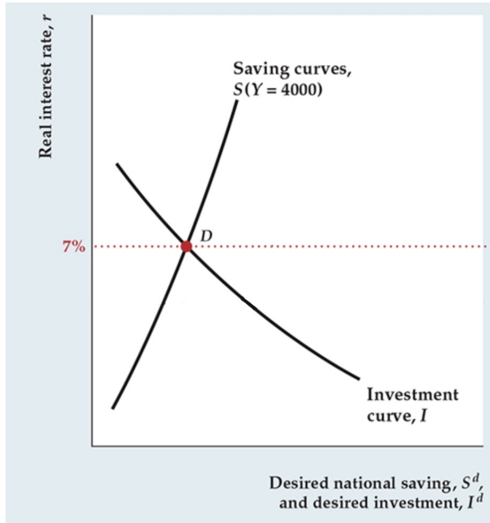
Exercise 4: Give an example when the IS equation does not look like the boxed equation.

Answer: If we change any of the assumption, the IS equation might look differently. For example, if tax is proportional to income (instead of a lumpsum tax), like $T = 0.1Y$, then saving in (1) will look different, and we end up with a different IS equation when setting $S = I$. This would change the slope and where the IS curve crosses the vertical axis.

4.2 Graphical construction

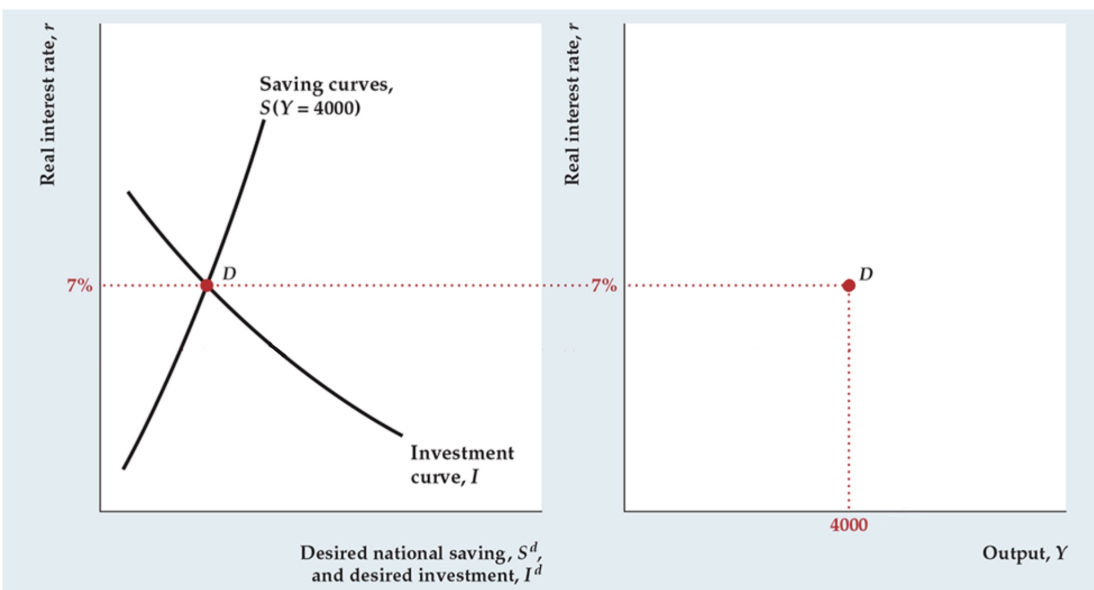
Now let's move on to a graphical construction of the IS curve. This might be more (or less) intuitive to you. Why do we need an additional way of constructing the IS curve? Because this way is relatively easy for interpretation: when the setup changes (something increases or decreases), we can tell right away from the graph. It also provides another angle to the problem, which is more intuitive.

First, let's set Investment equals Saving (again!), but on the (interest rate, money) space:



Here, we have a downward-sloping investment curve I: as real interest rate gets higher, borrowing gets more expensive, so firm borrow and invest less. We also have an upward-sloping saving curve: as real interest rate gets higher, nominal interest rate gets higher, so households would like to save more.

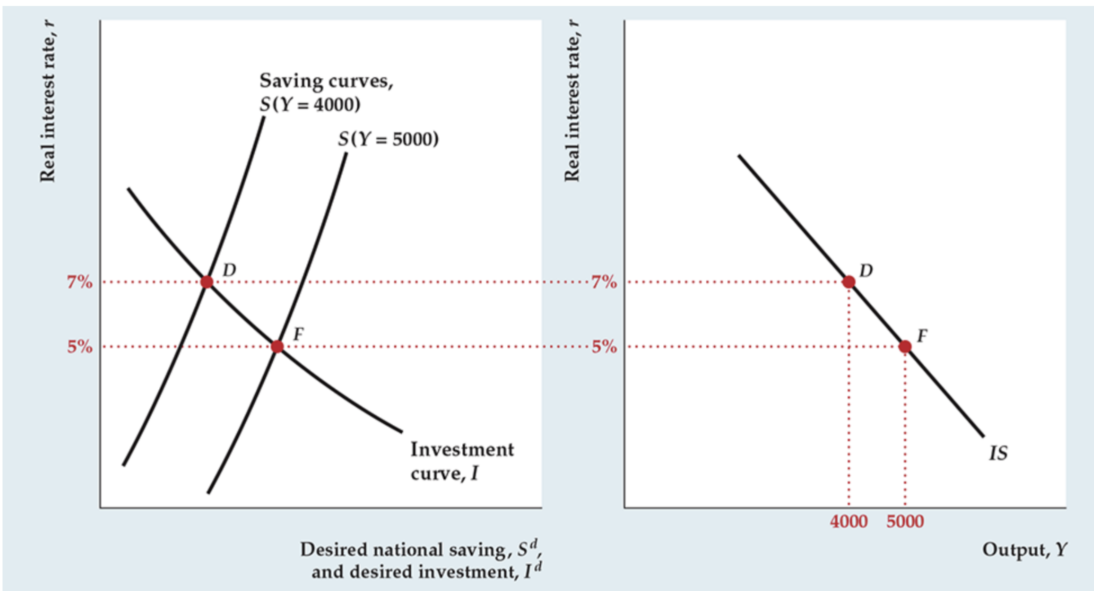
Note that our model cares about the relationship between interest rate and output Y (since IS curve is graphed on (r, Y) space), but we have not yet seen Y entering this graph. Therefore, we need to use another graph in the (r, Y) space and put it next to this graph, as they have the same vertical axis:



How do we mirror point D, the equilibrium, on the left graph to the right graph? It is clear that we need them on the same interest rate r level (here is 7%), as the two graphs have the same vertical r axis. But what about Y?

If you take a look at the saving equation (1), you'll see that each saving curve is associated with an output Y level. On the left graph, assume the Y associating with this particular S curve is 4000, then we could pin down point D on the right graph where $r = 7\%$ and $Y = 4000$. We have put down the first point on the IS curve.

Now, to graph a next point, pick another Y. Assume we want to see what happens at $Y = 5000$. When output increases, from (1), we can see that S increases, so S curve shifts to the right. We now have a new $(r = 5\%, Y = 5000)$ combination that can be mirrored to the right-hand-side graph:



You can keep trying all the Y possible on the left graph, and then put as many points down as possible on the right graph. In the end, all the points will connects and form a single line. That is our IS curve!

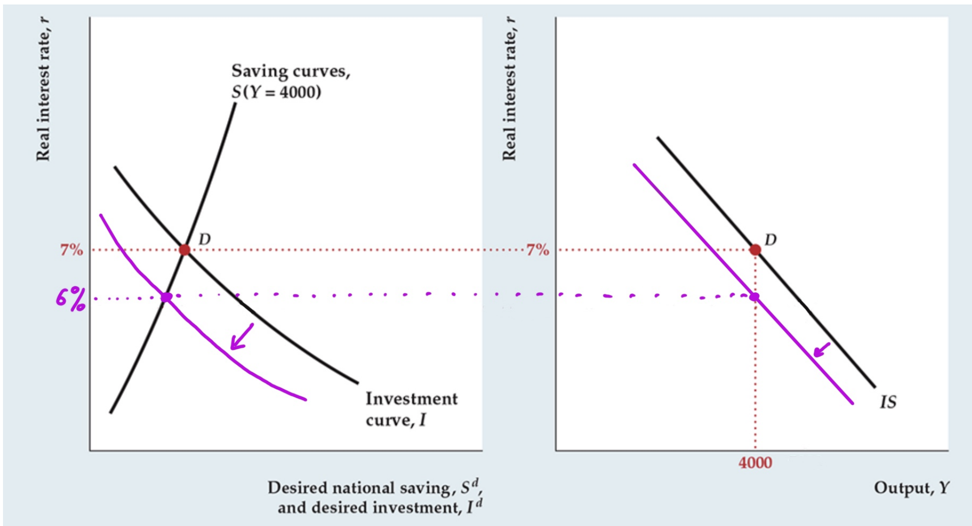
Note: You might be a little confused of why changing Y does not create a new line on the right graph, while change other things, like G (in class lecture) does. It is because we change Y to construct the IS graph (which we'll also do the same to construct LM), and then *after* constructing the graph, we try to see how the curve shifts by changing G, T, etc. The following exercises might help!

Exercise 5: Does changing Y shift any curve on the above graph?

> **Answer:** Yes and no. Y is exogenous on the left graph (not showing up on either axis), so it shifts S curve. On the right graph, however, Y is endogenous (showing up on the horizontal axis), so it only causes a movement along the IS curve. For example, when Y increased from 4000 to 5000, S shifts right (on the left graph) while we move down along the IS curve (on the right graph).

Exercise 6: Does changing effective tax rate on capital shift any curve on the above graph? Explain the shift/movement on both left-right graphs.

Answer: Yes and yes. Effective tax rate on capital is exogenous to both graphs (not showing up on either axis), so we expect shifting things on both graphs.



Let's assume capital tax increases. On the left graph: I curve shifts to the left, because capital is more expensive now, and so firms invest less. Now we have a new equilibirum: $r = 6\%$. Note that we are still at $Y = 4000$ level, so we now pin down point $(r = 6\%, Y = 4000)$ on the right graph (purple point).

On the right graph: The new point is where the IS curve shifts to. In the end, an increase in effective capital tax reduces real interest rate, reduce investment and saving, as well as shifting back the IS curve.

Exercise 7: What shifts the IS curve? Try to provide explanation for at least one of them in a similar way to Exercise 6.

Answer: Lecture 13 slide, page 11 and 12.

5 The LM curve

5.1 Mathematical construction

We have cleared two markets and show them on the (r, Y) space. Now we move onto the last one: the money market. As we set money supply equal to money demand, we'll also find some relationships between r and Y as in the boxed equations above.

Real money supply is assumed to be constant regardless of r. Real money supply is:

$$\frac{M}{P} = MS \tag{3}$$

with M being the nominal money amount, and P is price level (dividing the nominal term by price level gives you the real term). We assume MS is a constant number.

Nominal money demand follows the function:

$$M = P \times L(Y, i)$$

Under this assumption, a country's money demand is dictated by price P and liquidity preference $L(Y, i)$ (which is a function of output Y and nominal interest rate i; this function is affected by changes in Y and i). What's the intuition behind this construction?

- When price level P increases, we will need more money for transaction, affecting money demand, and vice versa
- When output Y increases, we will need more money for transaction, and vice versa
- When nominal interest rate i increases, it will be more costly to hold money. It is more attractive to just put your money in a saving account, thus reducing the need and demand for money used in day-to-day transaction. Therefore, money demand will decrease.

We would like to express this in real money term, so we divide both sides by P. We also care more about real interest rate, so we'll replace i by $r + \pi^e$. The above equation becomes the real money demand equation:

$$MD \equiv \frac{M}{P} = L(Y, r + \pi^e) \tag{4}$$

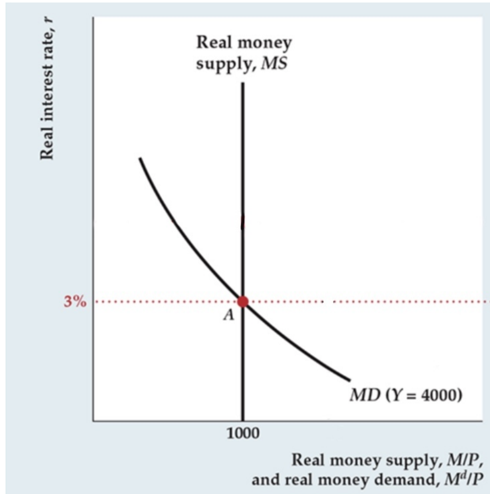
The term on the LHS is *real money demand*. Setting real money demand (4) equal to real money supply (3) to clear the money market, we get the **LM equation**:

$$\boxed{MS = L(Y, r + \pi^e)}$$

Exercise 8: Keeping all above assumptions, what does the MP curve look like, if *nominal* money supply is 2000 dollars, price level is 2, and $L = Y - r + \pi^e$, with expected inflation is 5%?
Answer: We have $2000/2 = Y - r + 0.05$, which means $r = Y + 0.05 - 1000$. On the graph with r on the vertical axis and Y on the horizontal axis, this is a straight, upward-sloping line.

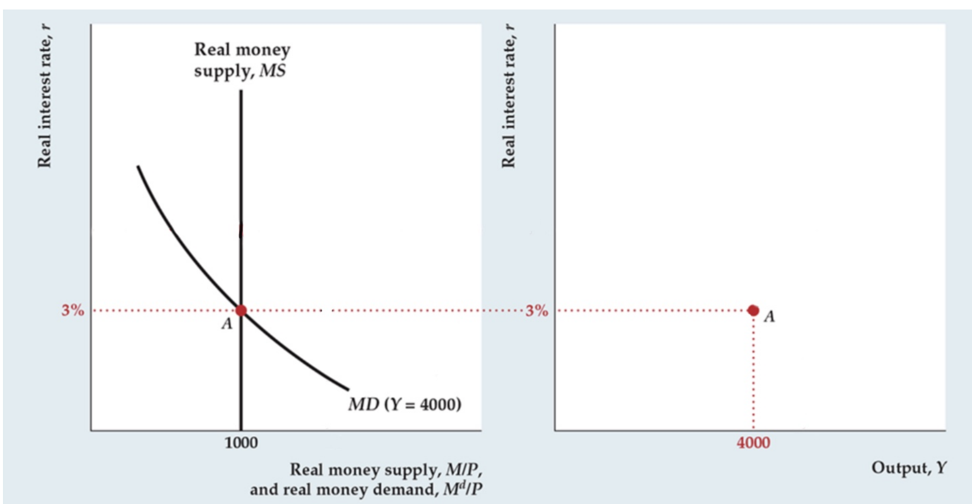
5.2 Graphical presentation

First, let's graph the M^s and M^d (money supply and demand) on the (real interest rate, money) graph.

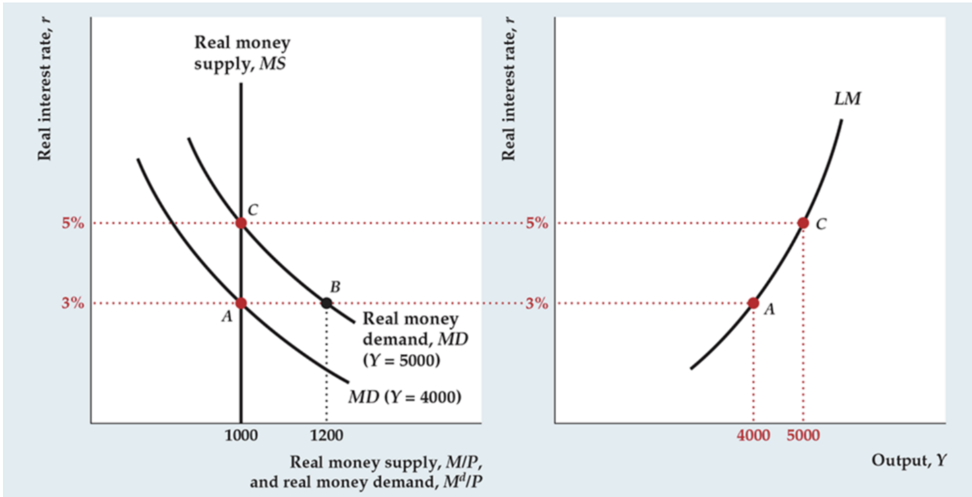


Here, real money supply in (3) is a constant, hence the vertical line. Real money demand (4) is a decreasing function of r, so the MD curve is downward-sloping.

How to construct the LM curve from this? We don't yet have a (r, Y) space yet, so let's put that space next to the above graph. Note that since the MD above is associated with output level $Y = 4000$, the equilibrium point A is associated with $(Y = 4000, r = 3\%)$. Mirror this point to the right graph:



To finish off the LM curve, pick other output levels. For example, pick $Y = 5000$. On the left graph, this will create a shift up in MD curve (higher output $\rightarrow$ more money demand). The new equilibrium is $(r = 5\%, Y = 5000)$. Plot this point to the right graph (point C):



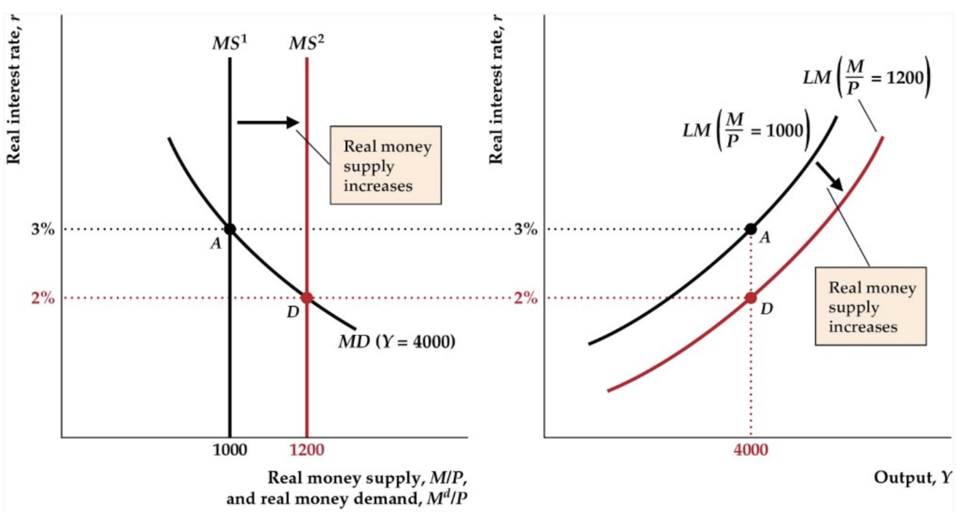
We can keep choosing different levels of Y, and plot as many points to the right graph as possible. In the end, they will create the LM curve!

Exercise 9: Does changing Y shift any curve on the above graph?

Answer: Yes and no. Y is exogenous on the left graph (not showing up on either axis), and since MD depends on Y, this change shifts MD curve. On the right graph, however, Y is endogenous (showing up on the horizontal axis), so it only causes a movement along the LM curve. For example, when Y increased from 4000 to 5000, MD shifts right (on the left graph) while we move up along the LM curve (on the right graph).

Exercise 10: What happens to the graphs if real money supply increases? Explain the shift/movement on both left-right graphs.

Answer: We'll see shiftings in both graphs.



Assume MS increases. Real money supply $\frac{M}{P} = MS$ now increases, so we shifts the MS curve to the right. We get a new equilibrium point D, which corresponds to $(Y = 4000, r =$

2%). Plotting this point on the right graph gives you where the LM curve is shifting to (new LM curve going through point D).

Exercise 11: What shifts the LM curve? Try to provide explanation for at least one of them in a similar way to Exercise 10.
Answer: Lecture 13 slide, page 19.