CHAPTER II :ROOTS AND WEIGHTS FOR $\mathrm{SL_n}(\mathbb{R})$

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In this chapter, we review some basis theory of roots and weight. We will first recall the general theory and compute explicitly the exammples for $\mathrm{SL_n}(\mathbb{R})/\mathrm{GL_n}(\mathbb{R})$.

1 Structure theory

1.1 The Cartan subalgebra

First we need the notion of Cartan subalgebra

Definition 1.1. *For any Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be Cartan algebra if it is*

- $\mathfrak{h}$ *is a nilpotent subalgebra.*
- *It is self normalizing. In particular, we have* $\mathfrak{h} = \{x \in \mathfrak{g} : [x, \mathfrak{g}] \subset \mathfrak{g}\}$.

when $\mathfrak{g}$ is a semisimple Lie algebra, we have the following theorem

Theorem 1.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field k of charateristic 0 with a subalgebra $\mathfrak{h}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ if and only if it is a maximal toral subalgebra, i.e. is is maximal among all subalgebras contanining only semisimple elements.*

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1.2 Root space decomposition

With respect to some choice of Cartan subalgebra, we have a root space decomposition. In particular, there is a finite set $\Phi \subset \mathfrak{h}^*$ of linear forms on H, whose elements are called **roots**, such that

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right),$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \forall h \in \mathfrak{h}\}$ for any $\alpha \in \Phi$.

1.3 A specific example: root space decomposition for $\mathfrak{sl}_n(\mathbb{R})$

For the semisimple Lie algebra $\mathfrak{sl}_\mathrm{n}(\mathbb{R})$, a typical choice of the Cartan subalgebra is the set

$$\mathfrak{h} = \left\{ H = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}, a_1 + a_2 + \ldots + a_n = 0 \right\}$$

With respect to this Cartan subalgebra, we can define the linear function

$$L_i \colon \mathfrak{h} \to \mathbb{R}, \quad H \mapsto L_i(H) = a_i$$

Then the roots are given by $\alpha_{ij} := L_i - L_j$ for distinct i, j. We have the root space decomposition for $\mathfrak{sl}_\mathrm{n}(\mathbb{R})$ as follows

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus \mathfrak{g}_{\alpha_{ij}} \right).$$

For the sake of brevity, we will denote $\alpha_{i,i+1}$ by α_i.