Name: Kyra Xian
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Problem 1

Let $q \geq 2$ be an integer and let χ be a Dirichlet character mod q. Consider the L

$$L(z; \chi) := \sum_{i=1}^{\infty} \frac{\chi(n)}{n^z}$$

which converges absolutely for $\Re(z) =: x > 1$ since

$$\sum_{n \geq 1} |\chi(n)n^{-z}| \geq \sum_{n \geq 1} n^{-x} < \infty,$$

by the $p-$test from FPM. Show that $L(z; \chi)$ defines a holomorphic function on $X = \{z \in \mathbb{C} : \Re(z) > 1\}$ and $L(z; \chi) \neq 0$ for all $z \in X$. Furthermore, show that

$$\frac{L'(z; \chi)}{L(z; \chi)} = -\sum_{p} \frac{\chi(p) \log(p)}{p^z - \chi(p)};$$

for $z \in X$.

Proof. Using Weierstrass M-test as in the statement in the problem, we have proved that the sequence

$$L_N(z; \chi) = \sum_{n=1}^{N} \frac{\chi(n)}{n^z}$$

converges uniformly to the function $L(z; \chi)$. In particular, $L(z; \chi)$ is continuous over the domain X. Clearly the sequence of functions L_N are holomorphic on X, as it is just a finite sum of exponential functions, which are entire. Thus we have

$$\int_{\gamma} L_N(z; \chi) dz = 0,$$

for piecewise smooth closed curve γ inside X, by Cauchy's theorem. Furthermore, we have

$$\int_{\partial \Delta} L(z; \chi) = \lim_{N \to \infty} \int_{\partial \Delta} L_N(z, \chi) = 0$$

By Morera's theorem, we can conclude that $L(z, \chi)$ is holomorphic over the domain X. Here I used the notation $\partial \Delta$ to indicate the boundary of a triangle.

The domain X is simply connected as it is convex. Using proposition 2.3 in the note, we can rewrite the $L-$ function as

$$L(z; \chi) = \sum_{i=1}^{\infty} \frac{\chi(n)}{n^z} = \prod_{p} \frac{1}{1 - \chi(p)p^{-z}}$$

Here proposition 2.3 is just a rephrase of lemma 2.2, so I don't write down the proof again. Note that over $X = \{z \in \mathbb{C} : \Re(z) > 1\}$, we have

$$\frac{1}{1 - \chi(p)p^{-z}} = 1 + \chi(p)p^{-z} + (\chi(p)p^{-z})^2 + \ldots = 1 + \sum_{k \geq 1} \chi(p^k)p^{-kz}$$

where

$$\sum_{k \geq 1} |\chi(p^k)p^{-kz}| \leq \sum_{n \geq 1} n^{-\Re z} < \infty$$

Let denote $u_p(z) = \sum_{k\geq 1} \chi(p^k) p^{-kz}$, then

$$L(z;\chi) = \prod_p (1 + u_p(z))$$

Then $L(z;\chi) = 0$ if and only if $u_p(z) = -1$ for some p. But this is impossible as

$$|u_p(z)| \leq \sum_{k\geq 1} |\chi(p^k) p^{-kz}| = \sum_{k\geq 1} p^{-k\Re z} = \frac{p^{-\Re(z)}}{1 - p^{-\Re z}} = \frac{1}{p^{\Re z} - 1} < 1$$

as the smallest prime p is 2. Thus $L(z;\chi)$ never vanishes on the domain X.

Lastly, we will prove the identity for $\frac{L'(z;\chi)}{L(z;\chi)}$. Similarly the discussion of the note, we will prove that there exists a branch of the logarithm $\log L(z,\chi)$ on X and satisfies

$$\log L(z;\chi) = \sum_p \mathrm{Log}(1 - \chi(p)p^{-z})$$

We then define the partial sum $S_m = \sum_{p\leq m} \mathrm{Log}(1 - \chi(p)p^{-z})$. Clearly this partial sum is holomorphic and S_m converges uniformly to a holomorphic function $F(z)$ as $m \to \infty$. Using the remark - please see below - for $w = \chi(p)p^{-z}$ with $p \geq 5$, we have that

$$\left|\mathrm{Log}(1 + \chi(p)p^{-z})\right| \leq 2|-\chi(p)p^{-z}| = 2p^{\Re z} \leq 1/2$$

and similarly to above, we shown $\sum_p 2p^{\Re z} < 2\sum_{n\geq 1} n^{\Re z} < \infty$. By Weierstrass M-test, S_m converges uniformly to a holomorphic function $F(z)$ on X as desired.

This function $F(z)$ is then a branch of the $\log L(z;\chi)$.

Indeed, first we can check that S_m defines a brach of the function $P_m = \prod_{p\leq m}(1 - \chi(p)p^{-z})$. As shown above, we already had $P_m \to L(z;\chi)$ as $m \to \infty$. Thus the fact that $S_m \to F$ implies $e^F = L(z;\chi)$. In particular, $F(z)$ defines a branch of the logarithm $\log L(z;\chi)$, namely

$$\log L(z;\chi) = \sum_p \mathrm{Log}(1 + u_p(z)) = \lim S_m(z)$$

Thus

$$S'_m(z) = -\sum_{p\leq m} \frac{\chi(p)\log(p)}{p^z - \chi(p)} \to \frac{d}{dz}\left[\log(L(z;\chi))\right] = \frac{L'(z;\chi)}{L(z;\chi)}$$

In particular, for $z \in X$, we have

$$\frac{L'(z;\chi)}{L(z;\chi)} = -\sum_p \frac{\chi(p)\log(p)}{p^z - \chi(p)},$$

where the sum extends over all prime number p, as we wanted.

Remark: In order to prove that S_m converges uniformly to a holomorphic function, we actually prove that following exercise: Consider the analytic function $f(w) = \mathrm{Log}(1 - w)$, show that $|\mathrm{Log}(1 - w)| \leq 2|w|$ given $|w| < 1/2$. Using the Taylor's series, we have

$$|\mathrm{Log}(1 - w)| = \left|-\sum_{n=1}^{\infty} \frac{w^n}{n}\right| \leq |w| \cdot \sum_{n=1}^{\infty} \frac{|w|^{n-1}}{n} \leq |w| \left(\sum_{n=1}^{\infty}(1/2)^{n-1}\right) = 2|w|$$

$\square$

Problem 2

Let G be a finite abelian group. Show that $\#\hat{G} \leq \#G$.

Proof. First we will prove that the set S of maps $f\colon G \to \mathbb{C}$ has a vector space structure. The sum and the scalar multiplication of maps are defined as follows

- $(\chi_1 + \chi_2)(a) := \chi_1(a) + \chi_2(a)$ for all $a \in G$.

- $(c\chi)(a) := c \cdot \chi(a)$ for any $\chi \in S$ and $c \in \mathbb{C}$.

But we can check all the axioms for a set being a vector space pointwisely, with the zero beging the zero map. We can verify as follows

1. Additive closure: this is clear from the way we defined the sum of two functions in S.

2. Commutative: For any $a \in G$, we have $\chi_1(a) + \chi_2(a) = \chi_1(a) + \chi_2(a)$. Since this holds for any $a \in G$, we have $\chi_1 + \chi_2 = \chi_2 + \chi_1$.

3. Associative: Similarly, for all $a \in G$

$$(\chi_1(a) + \chi_2(a)) + \chi_3(a) = \chi_1(a) + (\chi_2(a) + \chi_3(a))$$

   In general

$$(\chi_1 + \chi_2) + \chi_3 = \chi_1 + (\chi_2 + \chi_3)$$

4. The zero map is just the function $0\colon G \to \mathbb{C}$ that sends everything to 0.

5. For a map $f \in S$, we can define its additive inverse to be $(-f)\colon G \to \mathbb{C}$ such that $(-f)(a) := -f(a)$. Clearly

$$(f + (-f))(a) = f(a) + (-f(a)) = 0, \forall a \in G$$

   Thus $f + (-f)$ is the zero map.

6. The other axioms on the scalar multiplication can be checked pointwisely in the exact same way.

So S have a a structure of vector space.

Let's compute the dimension of S as as $\mathbb{C}-$ vector space. Since G is finite, we can assume that, as a set

$$G = \{a_1, \ldots, a_n\}$$

Then we can define the map f_i to be the "dual" of a_i in the following sense: $f_i(a_j) = \delta_{ij}$, i.e. $f_i(a_i) = 1$ and vanishes at $a_j \neq a_i$. We claim that $\mathfrak{B} = \{f_i\}$ forms a basis of S. Indeed, let $f\colon G \to \mathbb{C}$ be arbitrary. Since G is finite, f is determined by its value at each $a_i \in G$. Assume that $c_i = f(a_i)$. Then for any $1 \leq i \leq n$, we have

$$f(a_i) = c_i = \sum_{i=1}^{n} c_i f_i(a_i)$$

In particular, we can rewrite the following identity as

$$f = c_1 f_1 + c_2 f_2 + \ldots + c_n f_n,$$

which implies $\mathfrak{B}$ spans S. On the other hand, if

$$0 = c_1 f_1 + c_2 f_2 + \ldots + c_n f_n$$

Applying both sides to a_i yields

$$0 = c_i f_i(a_i) = c_i$$

which means $\mathfrak{B}$ is a set of linearly independent vectors. Thus $\dim S = \#\mathfrak{B} = n$. In the note, we proved that the character $\chi_1, \chi_2, \ldots, \chi_m$ are mutually orthogonal. In particular, they are a subset of S comprising of linearly independent vectors. Thus

$$m = \#\hat{G} \leq n = \#G.$$

Remark: In fact, if we assume the theorem about the structure of finite abelian group, we can prove that the equality always happens, namely $\#\hat{G} = \#G$ $\qquad\qquad\square$

Problem 3

Consider the finite abelian group G_9 consisting of the invertible elements of $\mathbb{Z}/9\mathbb{Z}$. Find all the characeres of G_9. Make sure you prove that the characters of G_9 you find are all distinct and there are no others.

Proof. As a set, we can list the elements of G_9 based on the definition as

$$G_9 = \{1, 2, 4, 5, 7, 8\},$$

where we excluded $3, 6$ as they are not coprime with 9. Here I use a to indicate the class $[a]$ mod 9.

Using the Remark in the previous exercise, we expect 6 characters in total. We will check that 2 generates G_9, so it is a cyclic group of order 6. Indeed, we have

$$2^1 \equiv 2 \pmod 9$$
$$2^2 \equiv 4 \pmod 9$$
$$2^3 \equiv 8 \pmod 9$$
$$2^4 \equiv 7 \pmod 9$$
$$2^5 \equiv 5 \pmod 9$$
$$2^6 \equiv 1 \pmod 9$$

So the character is defined solely by determining where it sends $[2]$ in $\mathbb{C}$. Let $\chi \in \hat{G}_9$ be any character, we then have

$$\chi(2)^6 = \chi(2^6) = \chi(1) = 1 \in \mathbb{C}$$

thus implies $\chi(2)$ must be a $6-th$ root of unity. There are 6 choices in total, namely

$$\chi(2) = e^{\frac{2i\pi k}{6}}, 1 \leq k \leq 6$$

Clearly these are 6 distinct characters - as they have different values at residue class $[2]$ - and they are all possible characters of G_9.

We can compute a character table as follows: $\chi(n) \pmod 9$

n	1	2	4	5	7	8
$\chi_1(n)$	1	1	1	1	1	1
$\chi_2(n)$	1	ω	ω^2	$-\omega^2$	$-\omega$	-1
$\chi_3(n)$	1	ω^2	$-\omega$	$-\omega$	ω^2	1
$\chi_4(n)$	1	-1	1	-1	1	-1
$\chi_5(n)$	1	$-\omega$	ω^2	ω^2	$-\omega$	1
$\chi_6(n)$	1	$-\omega^2$	$-\omega$	ω	ω^2	-1

where $\omega = e^{i\pi/3}$. $\qquad\square$

Problem 4

Show that there is exactly one real, non-principal Dirichlet character $\chi \pmod 9$. Find $\chi(916)$.

Proof. Continuing from the previous exercise, with a note that a character is called real character if its image lies entirely in $\mathbb{R}$, namely we have a group homomorphism $\chi \colon G_9 \to \mathbb{R}$. Since we know that all character $\chi \in \hat{G}$ satisfy

$$|\chi(a)| = 1, \text{ for all } a \in G_9,$$

we can deduce that a real character χ must satisfy $\chi(G_9) \subset \{\pm 1\}$. Since we want to find a non-principal character, the only choices is that $\chi(G_9) = \{\pm 1\}$. As shown above, χ is defined solely by its value at 2, so we must choose $\chi(2) = -1$. This implies that there is only one non-principal, real Dirichlet character over G_9.

Note that $916 \equiv 16 = 2^4 \pmod 9$. Thus

$$\chi(916) = \chi(2^4) = (\chi(2)^4) = (-1)^4 = 1.$$

And we are done.

Remark: Note that if we already have the full character table as computed in the previous exercise, it can be deduced immediately that there is a unique non principal real Dirichlet character mod 9. In the given table, it is the character χ_4 - corresponds to the highlighted row. $\qquad\square$

We have that

$$5^{7x} = 7^{5x} \Leftrightarrow \log_5(5^{7x}) = \log_5(7^{5x}) \Leftrightarrow 7x = 5x \cdot \log_5(7) \Leftrightarrow x(7 - \log_5(7)) = 0 \Leftrightarrow x = 0$$