CHAPTER I : $\mathrm{SL}_2(\mathbb{R})$

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In this chapter, I will give an exposition on the structure of $\mathrm{SL}_2(\mathbb{R})$ as the spaces of lattice, this space plays the role of a toy model before exploring the space of lattice in the higher rank. The exposition follows the paper [?] and [?] closely.

1 $\mathbf{SL}_2(\mathbb{R})$ and its action on the upper half plane $\mathfrak{H}$

A priori, the upper half plane

$$\mathfrak{H} = \{z : \Im z > 0\} \subset \mathbb{C}$$

has no group structure on its. However, we will show below that it can identify topologically with the space with the space of cosets $\mathrm{SO}_2(\mathbb{R}) \backslash \mathrm{SL}_2(\mathbb{R})$, and thus we can study the spaces $\mathfrak{H}$ via the spac of lattices $\mathrm{SO}_2(\mathbb{R}) \backslash \mathrm{SL}_2(\mathbb{R})$. We define the action of $G = \mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{H}$ as follows

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ (z) = \frac{az+b}{cz+d}$$

Proposition 1.1. *The group $SL_2(\mathbb{R})$ stabilizes $\mathfrak{H}$ and acts transitively on it. In particular,*

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} (i) = x + iy \quad (\textit{for } x \in \mathbb{R},\ y > 0)$$

Further, for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathfrak{H}$,

$$\Im\, g(z) = \frac{\Im\, z}{|cz+d|^2}.$$

Proof. The first formula is clear. The second formula would imply that the upper half-plane is stabilized. Compute directly:

$$2i \cdot \Im\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z)\right) = \frac{az+b}{cz+d} - \frac{d\bar{z}+b}{c\bar{z}+d} = \frac{(az+b)(c\bar{z}+d) - (a\bar{z}+b)(cz+d)}{|cz+d|^2}$$
$$= \frac{adz - bc\bar{z} - bcz + ad\bar{z}}{|cz+d|^2} = \frac{z - \bar{z}}{|cz+d|^2}$$

since $ad - bc = 1$. □

The point $z = i$ is special, in the sense that its stability group is the orthogonal group $K = \mathrm{SO}_2(\mathbb{R})$. Indeed, for any $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ we have that

$$g \circ i = i \Leftrightarrow \frac{ai+b}{ci+d} = i \Leftrightarrow a = d \text{ and } b = -c$$

Combinining with the fact that $ad - bc = 1$, we must have $a^2 + b^2 = 1$. This implies that there is a θ such that $a = \cos\theta$ and $b = \sin\theta$. Since G acts on $\mathfrak{H}$ transitively, we know from group theory that there is a bijection between the collection of cosets of $\mathrm{Stab}(i)$ in G and the orbits of i. In particular

Proposition 1.2. *We have an isomorphism of $SL_2(\mathbb{R})$-spaces*

$$\mathrm{SO}_2(\mathbb{R}) \setminus SL_2(\mathbb{R}) \approx \mathfrak{H} \quad \textit{via} \quad SO(2)g \to g^{-1}(i)$$

That is, the map respects the action of $SL_2(\mathbb{R})$, in the sense that

$$(\mathrm{SO}_2(\mathbb{R})\, g) \cdot h \longrightarrow h^{-1}(g^{-1}i)$$

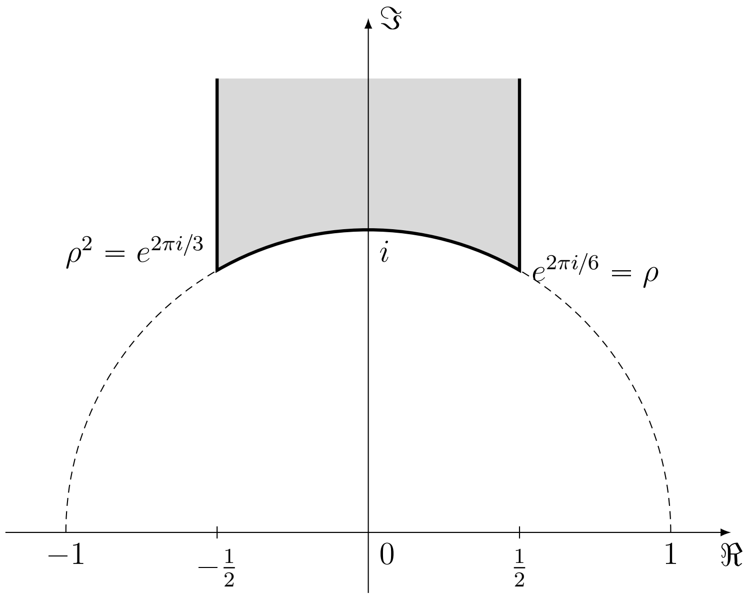
Proof. This is because of *associativity*:

$$(\mathrm{SO}_2(\mathbb{R})\, g) \cdot h = (\mathrm{SO}_2(\mathbb{R})) \cdot (gh) \longrightarrow (gh)^{-1}(i) = h^{-1}(g^{-1}(i))$$

giving the result. □

2 Fundamental domain for $\Gamma = SL_2(\mathbb{Z})$ on $\mathfrak{H}$

Here is a picture of the fundamental domain $\mathfrak{H}/\Gamma$.



The goal of this section is to prove that under the action of the $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, we can "move" every points on the upper half plane to a domain, under an equivalence given by a specific action. This is similar to the fundamental domain given by the translation action of $\mathbb{Z}$ to $\mathbb{R}$ is the half-open unit interval $[0, 1)$. In general, this give a simpler description to the homogenous space of lattice. Note that when we try to compute the fundamental domain of $\mathbb{Z}\backslash\mathbb{R}$, we have $\mathbb{Z}$ plays a role of *discrete* subset of $\mathbb{R}$. We give a precise definition of discreteness as follows

Definition 2.1. *Let a group G act continuously on a topological space X. A subset $\Gamma \subset G$ is called **discrete** if for any two compact subse A, B in X, there are only finitely many $g \in \Gamma$ such that $g \circ A \cap B \neq \varnothing$.*

We will prove that the set

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\}$$

is a discrete subgroup of $G = \mathrm{SL}_2(\mathbb{R})$. To prove this, we first need the following lemma

Lemma 2.1. *Fix a real number $r > 0$ and $0 < \delta < 1$. We denote $R_{r,\delta}$ the rectangle*

$$R_{r,\delta} = \left\{ z = x + iy : -r \leqslant x \leqslant r, 0 < \delta \leqslant y \leqslant \delta^{-1} \right\}$$

Then for any $\epsilon > 0$ and any fixed set $\mathbb{S}$ of coset representatives for $\Gamma_\infty\backslash\Gamma$, there are finitely many $g \in \mathbb{S}$ such that $\Im(g \circ z) > \epsilon$ for some $z \in R_{r,\delta}$.

In the above lemma, the notation Γ_∞ is defined to be the set

$$\Gamma_\infty = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}.$$

It can be seen easily that this is the stability group of ∞ in $\mathfrak{H}$.

Proof. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then for $z \in R_{r,\delta}$,

$$\mathrm{Im}(g \circ z) = \frac{y}{c^2y^2 + (cx+d)^2} < \epsilon$$

if $|c| > (y\epsilon)^{-\frac{1}{2}}$. On the other hand, for $|c| \leqslant (y\epsilon)^{-\frac{1}{2}} \leqslant (\delta\epsilon)^{-\frac{1}{2}}$, we have

$$\frac{y}{(cx+d)^2} < \epsilon$$

if the following inequalities hold:

$$|d| > |c|r + (y\epsilon^{-1})^{\frac{1}{2}} \geqslant |c|r + (\epsilon\delta)^{-\frac{1}{2}}.$$

Consequently, $\Im(g \circ z) > \epsilon$ only if

$$|c| \leqslant (\delta\epsilon)^{-\frac{1}{2}} \quad \text{and} \quad |d| \leqslant (\epsilon\delta)^{-\frac{1}{2}}(r+1),$$

and the total number of such pairs (not counting $(c,d) = (0, \pm 1), (\pm 1, 0)$) is at most $\frac{4(r+1)}{(\epsilon\delta)}$. This proves the lemma. $\square$

It follows from Lemma 2.1 that $\Gamma = SL(2,\mathbb{Z})$ is a discrete subgroup of $SL(2,\mathbb{R})$. This is because:

1. It is enough to show that for any compact subset $A \subset \mathfrak{H}$ there are only finitely many $g \in SL(2,\mathbb{Z})$ such that $(g \circ A) \cap A \neq \phi$;

2. Every compact subset of $A \subset \mathfrak{H}$ is contained in a rectangle $R_{r,\delta}$ for some $r > 0$ and $0 < \delta < \delta^{-1}$;

3. $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$, except for finitely many $\alpha \in \Gamma_\infty$, $g \in \Gamma_\infty \backslash \Gamma$.

To prove (3), note that Lemma 2.1 implies that $(g \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$ except for finitely many $g \in \Gamma_\infty \backslash \Gamma$. Let $S \subset \Gamma_\infty \backslash \Gamma$ denote this finite set of such elements g. If $g \notin S$, then Lemma 2.1 tells us that it is because $\Im(g \circ z) < \delta$ for all $z \in R_{r,\delta}$. Since $\Im(\alpha g \circ z) = \Im(g \circ z)$ for $\alpha \in \Gamma_\infty$, it is enough to show that for each $g \in S$, there are only finitely many $\alpha \in \Gamma_\infty$ such that $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} \neq \phi$. This last statement follows from the fact that $g \circ R_{r,\delta}$ itself lies in some other rectangle $R_{r',\delta'}$, and every $\alpha \in \Gamma_\infty$ is of the form $\alpha = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$ $(m \in \mathbb{Z})$, so that

$$\alpha \circ R_{r',\delta'} = \{x + iy \mid -r' + m \leqslant x \leqslant r' + m,\ 0 < \delta' \leqslant \delta''^{-1}\},$$

which implies $(\alpha \circ R_{r',\delta'}) \cap R_{r,\delta} = \phi$ for $|m|$ sufficiently large. Now we are ready to describe the fundamental domain for $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}$.

Proposition 2.1. *A fundamental domain for* $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}$ *can be given as the region*

$$\mathfrak{D} = \{z = x + iy \in \mathfrak{H} : |z| \geqslant 1, -1/2 \leqslant x \leqslant 1/2\}$$

, modulo the congruent boundary points symmetric with respect to the imaginary axis.

Proof. First we eliminated the repeated points on the boundary. Note that the line $x = -1/2$ is the same as the line $x = 1/2$ under the transformation $z \mapsto z + 1$. Similarly, given a point on the circle $\{|z| = 1\}$, the transformation $z \mapsto -|z|^{-1}$ satisfies

$$\frac{-1}{x+iy} = \frac{-x+iy}{x^2+y^2} = -x+iy,$$

which flips the sign of x. Thus it identifies the half circle on the right of the imaginary axis with that on the left.
Now we need to show two things:

1. For any $z \in \mathfrak{H}$ we can find an element $g \in \mathrm{SL}_2(\mathbb{Z})$ such that $g \circ z \in \mathfrak{D}$.

2. If $z \equiv z' \in \mathfrak{D}$ modulor $\mathrm{SL}_2(\mathbb{Z})$, then either $\Re(z) = \pm\frac{1}{2}$ and $z' = z \mp 1$, or $|z| = 1$ and $z' = \frac{-1}{z}$.

First we prove for (1): Fix $z \in \mathfrak{H}$. It follows from Lemma 2.1 that for every $\epsilon > 0$, there are at most finitely many $g \in \mathrm{SL}(2,\mathbb{Z})$ such that $g \circ z$ lies in the strip

$$D_\epsilon := \left\{ w \;\middle|\; -\frac{1}{2} \leqslant \mathrm{Re}(w) < \frac{1}{2},\ \epsilon \leqslant \mathrm{Im}(w) \right\}.$$

Let B_ϵ denote the finite set of such $g \in \mathrm{SL}(2,\mathbb{Z})$. Clearly, for sufficiently small ϵ, the set B_ϵ contains at least one element. We will show that there is at least one $g \in B_\epsilon$ such that $g \circ z \in D$. Among these finitely many $g \in B_\epsilon$, choose one such that $\Im(g \circ z)$ is maximal in D_ϵ. If $|g \circ z| < 1$, then for $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ we have, for any m,

$$\Im\left(T^m S g \circ z\right) = \Im\left(\frac{-1}{g \circ z}\right) = \frac{\Im(g \circ z)}{|g \circ z|^2} > \Im(g \circ z)$$

But we can choose m such that $T^m S g \circ z \in D_\epsilon$, which contradicts the maximality of $\Im(g \circ z)$.
Next we give a proof for (2):Let $z \in D$, $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, and assume that $g \circ z \in D$. Without loss of generality, we may assume that

$$\Im(g \circ z) = \frac{y}{|cz+d|^2} \geqslant \Im(z),$$

(otherwise just interchange z and $g \circ z$ and use g^{-1}). This implies that $|cz+d| \leqslant 1$ which implies that $1 \geqslant |cy| \geqslant \frac{1}{\sqrt{3}}|c|$. This is clearly impossible if $|c| \geqslant 2$. So we only have to consider the cases $c = 0, \pm 1$. If $c = 0$ then $d = \pm 1$ and g is a translation by b. Since $-\frac{1}{2} \leqslant \Re(z), \Re(g \circ z) \leqslant \frac{1}{2}$, this implies that either $b = 0$ and $z = g \circ z$ or else $b = \pm 1$ and $\Re(z) = \pm\frac{1}{2}$ while $\Re(g \circ z) = \mp\frac{1}{2}$. If $c = 1$, then $|z + d| \leqslant 1$ implies that $d = 0$ unless $z = e^{2\pi i/3}$ and $d = 0, -1$. The case $d = 0$ implies that $|z| \leqslant 1$ which implies $|z| = 1$. Also, in this case, $c = 1$, $d = 0$, we must have $b = -1$ because $ad - bc = 1$. Then $g \circ z = a - \frac{1}{z+1}$. It follows that $g \circ z = a - e^{2\pi i/3}$ and $d = 1$, then we must have $a - b = 1$. It follows that $g \circ z = a - \frac{1}{z+1} = a + e^{2\pi i/3}$, which implies that $a = 0$ or 1. A similar argument holds when $z = e^{\pi i/3}$ and $d = -1$. Finally, the case $c = -1$ can be reduced to the previous case $c = 1$ by reversing the signs of a, b, c, d. $\square$

3 Lattices and semi-stability in dimension 2

In this section, we investigate the notion of semi-stable lattices and how the upper half plane $\mathfrak{H}$ can be regard as a spaces of two dimensional lattices.
Now regard $\mathbb{C} \cong \mathbb{R}^2$ via $x + iy \mapsto (x, y)$, and the inner product is defined to be

$$\langle z_1, z_2 \rangle = x_1 x_2 + y_1 y_2,$$

where $z_i = x_i + iy_i$. Now for any $z \in \mathfrak{H}$, the pair $(1, z)$ can be identified with the lattice

$$L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$$

First we prove the following statement

Proposition 3.1. *The upper half plane $\mathfrak{H}$ classifies similarity classes of two dimensional lattice.*

Proof. Let $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ be any lattice in $\mathbb{R}_2$. Then using the above identification, we can find two complex number z_1, z_2 such that $|z_1| = ||e_1||$ and $|z_2| = |e_2|$ add this later, refer to the book by Anton Deitmar □

Clearly in each classes of similar lattice, there is a unique one that has unit covolume. The lattice spanned by z and 1 has volume y, so the corresponding unit lattice is the one spanned by $z/\sqrt{y}$ and $1/\sqrt{y}$.
Using Proposition 2.1, it is immediate that every lattice spanned by 1 and z is similar to lattice generated by 1 and a point z' inside ther region $\mathfrak{D}$.