Conformal equivalence between annuli

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Disclaimer: This is just a rewritten version of the blog here: see here.
First, we define an annulus in a complex plane

Definition 1. *An annulus in* $\mathbb{C}$ *is the set*

$$A(r, R) = \{z \in \mathbb{C} : r < |z| < R\}$$

Given two annuli $A_1 = A(r_1, R_1)$ and $A_2 = A(r_2, R_2)$, one may ask under which conditions that two annuli are biholomorphic. It turns out that in a complex plane, the biholomorphic relation is defined using only the ratio r_1/r_2 and R_1/R_2. This is shown in the following theorem

Theorem 1. A_1 *is biholomorphic to* A_2 *if and only if* $\frac{R_1}{r_1} = \frac{R_2}{r_2}$.

Proof. First suppose that $\frac{R_1}{R_2} = \frac{r_1}{r_2} = k$. Then clearly the linear map $f(z) = kz$ is a biholomorphic map and $f(A_1) = A_2$. Thus A_1 is biholomorphic to A_2.
Conversely, assume that A_1 is biholomorphic to A_2 under a map f. By scaling, we could further assume that $r_1 = r_2 = 1$. Thus now we need to show that $R_1 = R_2$. Fix some $1 < r < R_2$ and let $C = \{z \in A_2 \colon |z| = r\}$.
Since f^{-1} is a continuous map, $f^{-1}(C)$ is a compact set. Thus we can find an $m > 0$ such that $|x| \geqslant m$ for all $x \in f^{-1}(C)$. Choose $\delta > 0$ small enough such that $1 + \delta < m$. This implies the annulus $A_3 = A(1, 1 + \delta) \cap f^{-1}(C) = \varnothing$. Let $V = f(A_3)$. Since A_3 is connected set, so is $f(A_3)$. Thus V is either inside $A_2 \backslash A(1, r)$ or $A(1, r)$. By replacing f with R_2/f, we can reduce to consider the former case.
Claim: $|f(z_n)| \to 1$ whenever $|z_n| \to 1$.
Proof: Clearly we have $z_n = f^{-1}(f(z_n))$. If $f(z_n)$ converges to some points in A_2, z_n must then converges to some points in A_1, contradicting the hypothesis that $|z_n| \to 1$. Thus $|f(z_n)|$ must converge to either 1 or R_2, but the latter case is excluded as $V \subset A(1, r)$.
In the same manner, we also have the following claim:
Claim: $|f(z_n)| \to R_2$ whenever $|z_n| \to R_1$.
Now set $\alpha = \log(R_1)/\log(R_2)$ and define a new function $g \colon A_1 \to \mathbb{R}$ by

$$g(z) = \log|f(z)|^2 - \alpha \log|z|^2 = 2(\log|f(z)| - \alpha \log|z|).$$

This is a harmonic function since $\log(|f|^2)$ is a harmonic over $\mathbb{C}$, where f is a non vanishing analytic function. Indeed, we have

$$\Delta\left(\log(|f|^2)\right) = \Delta\left(\log(f) + \log(\overline{f})\right) = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\log(f) + 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}\log(\overline{f}) = 0$$

Using the above claims, g can be extended continuously to $\overline{A_1}$ and $g(z) = 0$ for all $z \in \partial A_1$. By maximum modulus theorem for harmonic function, g must be identically zero on the whole disk $\overline{A_1}$. In particular

$$0 = \frac{\partial g}{\partial z} = \frac{f'(z)}{f(z)} - \frac{\alpha}{z}.$$

Let $\gamma = \gamma(t) = ce^{it}$ for some $1 < c < R_1$. Then we have

$$\alpha = \frac{1}{2\pi i}\int_\gamma \frac{\alpha}{z} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}$$

By the Argument Principle, α must be an integer. A consequence is that

$$\frac{d}{dz}(z^{-\alpha}f(z)) = -\alpha z^{-\alpha-1}f(z) + z^{-\alpha}f'(z) = z^{-\alpha-1}(zf'(z) - \alpha f(z)) = 0.$$

This forces $z^{-\alpha}f(z) = K$ for some constant K on A_1. Thus $f(z) = Kz^{\alpha}$. As f is injective, $\alpha = 1$ by the fundamental theorem of Algebra. $\square$