SOLUTION FOR THE BONUS QUESTION

April 3, 2025

In this report, I will give a proof for the bonus question for the class MATH 538. First we recall the question

Problem 1

Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ viewed as a real Lie algebra. We will denote this Lie algebra by $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$. Show that

1. $\mathfrak{g}$ is simple as a real Lie algebra.
2. $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ is not a split real Lie algebra, i.e. it has no split Cartan subalgebra.

Proof.

1. First we will show that $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ is semisimple. Indeed, we have

$$\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}} \otimes \mathbb{C} = (\mathfrak{sl}(2, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}(2, \mathbb{R}) \otimes_{\mathbb{R}} (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$$

   Now note that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$ as rings, so we clearly reduce to

$$\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}} \otimes \mathbb{C} \cong \mathfrak{sl}(2, \mathbb{R}) \otimes_{\mathbb{R}} (\mathbb{C} \oplus \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$$

   This clearly shows that $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ is semi-simple. Now we show that it is in fact simple. Assume not, then $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ has a nontrivial simple ideal I. Then for any $\lambda \in \mathbb{C} \setminus \{0\}$, clearly $\lambda(I)$ is also an ideal as

$$I \supset [I, \lambda(I)] = \lambda[I, I] = \lambda(I) \neq 0$$

   Clearly $\dim I = \dim \lambda(I)$, this implies that $I = \lambda(I)$. That means when we allow complex scalar multiplication the ideal I stays the same. Thus I is in fact an ideal of $\mathfrak{sl}(2, \mathbb{C})$. But since $\mathfrak{sl}(2, \mathbb{C})$ is simple as a complex Lie algebra, this implies $I = \mathfrak{sl}(2, \mathbb{C})$. The restriction of scalar doesn't change anything, which implies $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ is simple.

2. Assume not, then $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ is a split real Lie algebra. From the previous part, we regconized that $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ is a real form of the semisimple Lie algebra $\mathfrak{g}$. Assume that the real Lie algebra $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ is split, then we have a root space decomposition

$$\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

   where Δ is the set of root with respect to the split Cartan subalgebra $\mathfrak{h}$. We know that the root space $\mathfrak{g}_\alpha$ has dimension 1, so we can rewwrite the above as

$$\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathbb{R} x_\alpha$$

   Under the base changing, we get

$$\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}} \otimes \mathbb{C} = \left(\mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathbb{R} x_\alpha \right) \otimes \mathbb{C} = (\mathfrak{h} \otimes \mathbb{C}) \oplus \bigoplus_{\alpha \in \Delta} \mathbb{C} x_\alpha$$

In particular, the root system doesn't change after the base changing, and we can see that the corresponding root system to the given Lie algebra is of type $A_1 \times A_1$. But this is a reducible root system, so the corresponding split real Lie algebra must also reducible. We will show that if a root system corresponding to a semisimple Lie algebra is reducible, then the corresponding Lie algebra is not semisimple.
Indeed, assume that $\Delta = \Delta_1 \sqcup \Delta_2$, then clearly the subalgebra

$$\mathfrak{g}_1 = \bigoplus_{\alpha \in \Delta_1^+} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}])$$

is a proper ideal of $\mathfrak{g}$, as

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \begin{cases} 0 & \text{if } \alpha + \beta \notin \Delta, \text{ in particular } \alpha \in \Delta_1, \beta \in \Delta_2 \\ \mathfrak{g}_{\alpha+\beta}, & \text{otherwise} \end{cases}.$$

So the real Lie algebra $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$, if it is split, must be semisimple and has a nontrivial simple ideal. But as we proved above $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ is a simple real Lie algebra. $\square$