Conformal equivalence between annuli

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Introduction First we give a definition for an annulus in complex plane:

Definition 1. *An annulus in* $\mathbb{C}$ *is the set*

$$A(r, R) = \{z \in \mathbb{C} : r < |z| < R\}$$

Given two annuli $A_1 = A(r_1, R_1)$ and $A_2 = A(r_2, R_2)$, one may ask under which conditions that two annuli are biholomorphic. It turns out that in a complex plane, the biholomorphic relation is defined using only the ratio r_1/r_2 and R_1/R_2. This is shown in the following theorem

Theorem 1. A_1 *is biholomorphic to* A_2 *if and only if* $\dfrac{R_1}{r_1} = \dfrac{R_2}{r_2}$.

Proof. First suppose that $\dfrac{R_1}{R_2} = \dfrac{r_1}{r_2} = k$. Then clearly the linear map $f(z) = kz$ is a biholomorphic map and $f(A_1) = A_2$. Thus A_1 is biholomorphic to A_2.
Conversely, assume that A_1 is biholomorphic to A_2. By scaling if necessary, we could further assume that $r_1 = r_2 = 1$. Thus now we need to show that $R_1 = R_2$. Fix some $1 < r < R_2$.

$\square$