

# Conformal equivalence between annuli

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Disclaimer: This is just a rewritten version of the blog here: [see here](#).

First, we define an annulus in a complex plane

**Definition 1.** An annulus in  $\mathbb{C}$  is the set

$$A(r, R) = \{z \in \mathbb{C} : r < |z| < R\}$$

Given two annuli  $A_1 = A(r_1, R_1)$  and  $A_2 = A(r_2, R_2)$ , one may ask under which conditions that two annuli are biholomorphic. It turns out that in a complex plane, the biholomorphic relation is defined using only the ratio  $r_1/r_2$  and  $R_1/R_2$ . This is shown in the following theorem

**Theorem 1.**  $A_1$  is biholomorphic to  $A_2$  if and only if  $\frac{R_1}{r_1} = \frac{R_2}{r_2}$ .

*Proof.* First suppose that  $\frac{R_1}{R_2} = \frac{r_1}{r_2} = k$ . Then clearly the linear map  $f(z) = kz$  is a biholomorphic map and  $f(A_1) = A_2$ . Thus  $A_1$  is biholomorphic to  $A_2$ .

Conversely, assume that  $A_1$  is biholomorphic to  $A_2$  under a map  $f$ . By scaling, we could further assume that  $r_1 = r_2 = 1$ . Thus now we need to show that  $R_1 = R_2$ . Fix some  $1 < r < R_2$  and let  $C = \{z \in A_2 : |z| = r\}$ .

Since  $f^{-1}$  is a continuous map,  $f^{-1}(C)$  is a compact set. Thus we can find an  $m > 0$  such that  $|x| \geq m$  for all  $x \in f^{-1}(C)$ . Choose  $\delta > 0$  small enough such that  $1 + \delta < m$ . This implies the annulus  $A_3 = A(1, 1 + \delta) \cap f^{-1}(C) = \emptyset$ . Let  $V = f(A_3)$ . Since  $A_3$  is connected set, so is  $f(A_3)$ . Thus  $V$  is either inside  $A_2 \setminus A(1, r)$  or  $A(1, r)$ . By replacing  $f$  with  $R_2/f$ , we can reduce to consider the former case.

**Claim:**  $|f(z_n)| \rightarrow 1$  whenever  $|z_n| \rightarrow 1$ .

*Proof:* Clearly we have  $z_n = f^{-1}(f(z_n))$ . If  $f(z_n)$  converges to some points in  $A_2$ ,  $z_n$  must then converges to some points in  $A_1$ , contradicting the hypothesis that  $|z_n| \rightarrow 1$ . Thus  $|f(z_n)|$  must converge to either 1 or  $R_2$ , but the latter case is excluded as  $V \subset A(1, r)$ .

In the same manner, we also have the following claim:

**Claim:**  $|f(z_n)| \rightarrow R_2$  whenever  $|z_n| \rightarrow R_1$ .

Now set  $\alpha = \log(R_1)/\log(R_2)$  and define a new function  $g: A_1 \rightarrow \mathbb{R}$  by

$$g(z) = \log |f(z)|^2 - \alpha \log |z|^2 = 2(\log |f(z)| - \alpha \log |z|).$$

This is a harmonic function since  $\log$  is a harmonic over  $\mathbb{C}$ . Using the above claims,  $g$  can be extended continuously to  $\overline{A_1}$  and  $g(z) = 0$  for all  $z \in \partial A_1$ . By maximum modulus theorem for harmonic function,  $g$  must be identically zero on the whole disk  $\overline{A_1}$ . In particular

$$0 = \frac{\partial g}{\partial z} = \frac{f'(z)}{f(z)} - \frac{\alpha}{z}.$$

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Let  $\gamma = \gamma(t) = ce^{it}$  for some  $1 < c < R_1$ . Then we have

$$\alpha = \frac{1}{2\pi i} \int_{\gamma} \frac{\alpha}{z} = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)}$$

By the Argument Principle,  $\alpha$  must be an integer. A consequence is that

$$\frac{d}{dz}(z^{-\alpha}f(z)) = -\alpha z^{-\alpha-1}f(z) + z^{-\alpha}f'(z) = z^{-\alpha-1}(zf'(z) - \alpha f(z)) = 0.$$

This forces  $z^{-\alpha}f(z) = K$  for some constant  $K$  on  $A_1$ . Thus  $f(z) = Kz^{\alpha}$ . As  $f$  is injective,  $\alpha = 1$  by the fundamental theorem of Algebra.  $\square$