On Semi-stable lattices in higher rank

by

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Chapter 1

$\mathrm{SO}_2(\mathbb{R}) \setminus \mathrm{SL}_2(\mathbb{R})$

In this chapter, I will give an exposition on the structure of $\mathrm{SL}_2(\mathbb{R})$ as the spaces of lattice, this space plays the role of a toy model before exploring the space of lattice in the higher rank. The exposition follows [5, Chapter I] closely.

1.1 $\mathrm{SL}_2(\mathbb{R})$ and its action on the upper half plane $\mathfrak{H}$

Consider the upper half plane

$$\mathfrak{H} = \{z : \Im z > 0\} \subset \mathbb{C}.$$

We define the action of $G = \mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{H}$ as follows

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ (z) := \frac{dz - b}{-cz + a}$$

We need to check that this is indeed an action. It is routine to check that

$$\begin{bmatrix} \frac{1}{\sqrt{y}} & 0 \\ 0 & \sqrt{y} \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \circ (i) = x + iy \quad (\text{for } x \in \mathbb{R},\, y > 0)$$

Moreover, we have

$$2i \cdot \Im\left(\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \circ (z)\right) = \frac{az+b}{cz+d} - \frac{d\overline{z}+b}{c\overline{z}+d} = \frac{(az+b)(c\overline{z}+d) - (a\overline{z}+b)(cz+d)}{|cz+d|^2}$$

$$= \frac{adz - bc\overline{z} - bcz + ad\overline{z}}{|cz+d|^2} = \frac{z - \overline{z}}{|cz+d|^2}$$

since $ad - bc = 1$. In particular, we have the following Proposition

Proposition 1.1.0.1. *The group* $\mathrm{SL}_2(\mathbb{R})$ *acts transitively on* $\mathfrak{H}$*. In particular,*

$$\begin{bmatrix} \frac{1}{\sqrt{y}} & 0 \\ 0 & \sqrt{y} \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \circ (i) = x + iy \quad (\text{for } x \in \mathbb{R},\ y > 0)$$

Further, for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ *and* $z \in \mathfrak{H}$,

$$\Im\,(g \circ z) = \frac{\Im\, z}{|cz + d|^2}.$$

Remark. *In many texts, the above action is defined as*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ (z) := \frac{az + b}{cz + d}$$

*The side of the action doesn't really matter here, as we can recover the whole theory by switching between the left/right action. However, the result of this thesis follows the recent paper [**?**], which use the inverse action. That's why we follow the same notations set up in that paper.*

The point $z = i$ is special, in the sense that its stability group is the orthogonal group $K = \mathrm{SO}_2(\mathbb{R})$. Indeed, for any $g = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \in SL_2(\mathbb{R})$ we have that

$$g \circ i = i \Leftrightarrow \frac{ai + b}{ci + d} = i \Leftrightarrow a = d \text{ and } b = -c$$

Combinining with the fact that $ad - bc = 1$, we must have $a^2 + b^2 = 1$. This implies that there is a θ such that $a = \cos\theta$ and $b = \sin\theta$. Since G acts on $\mathfrak{H}$ transitively, we know from group theory that there is a bijection between the collection of cosets of $\mathrm{Stab}(i)$ in G and the orbits of i. It is also useful to consider the group that stabilizes the point at infinity, which we include in the following proposition

Proposition 1.1.0.2.

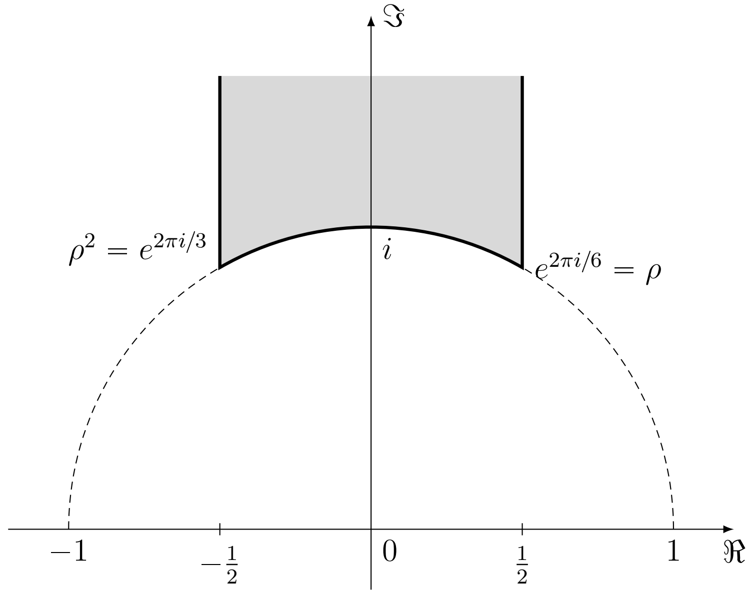
1. *Stab*$(i) = \mathrm{SO}_2(\mathbb{R})$.
2. *We have a bijection between the left cosets spaces and the upper half plane given by*

$$\mathrm{SO}_2(\mathbb{R}) \setminus \mathrm{SL}_2(\mathbb{R}) \cong \mathfrak{H} \quad \textit{via} \quad SO(2)g \to g^{-1}(i)$$

3. *Stab*$(\infty) := \Gamma_\infty := \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\}$

1.2 Fundamental domain

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and consider its action on $\mathfrak{H}$.



The goal of this section is to prove that under the action of the $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, we can "move" every points on the upper half plane into a domain pictued above. This is similar to the fundamental domain given by the translation action of $\mathbb{Z}$ to $\mathbb{R}$ is the half-open unit interval $[0, 1)$. In general, this give a simpler description to the homogenous space of lattice.
Note that when we try to compute the fundamental domain of $\mathbb{Z}\backslash\mathbb{R}$, we have $\mathbb{Z}$ plays a role of *discrete* subset of $\mathbb{R}$. We give a precise definition of discreteness as follows

Definition 1.2.0.1. *Let a group G act continuously on a topological space X. A subset $\Gamma \subset G$ is called **discrete** if for any two compact subse A, B in X, there are only finitely many $g \in \Gamma$ such that $g \circ A \cap B \neq \varnothing$.*

We will prove that the set

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\}$$

is a discrete subgroup of $G = \mathrm{SL}_2(\mathbb{R})$. To prove this, we first need the following lemma

Lemma 1.2.0.2. *Fix a real number $r > 0$ and $0 < \delta < 1$. We denote $R_{r,\delta}$ the rectangle*

$$R_{r,\delta} = \left\{ z = x + iy : -r \leqslant x \leqslant r, 0 < \delta \leqslant y \leqslant \delta^{-1} \right\}$$

Then for any $\epsilon > 0$ and any fixed set $\mathbb{S}$ of coset representatives for $\Gamma_\infty\backslash\Gamma$, there are finitely many $g \in \mathbb{S}$ such that $\Im(g \circ z) > \epsilon$ for some $z \in R_{r,\delta}$.

Proof. Let $g = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Then for $z \in R_{r,\delta}$,

$$\mathrm{Im}(g \circ z) = \frac{y}{c^2y^2 + (cx+d)^2} < \epsilon$$

if $|c| > (y\epsilon)^{-\frac{1}{2}}$. On the other hand, for $|c| \leqslant (y\epsilon)^{-\frac{1}{2}} \leqslant (\delta\epsilon)^{-\frac{1}{2}}$, we have

$$\frac{y}{(cx+d)^2} < \epsilon$$

if the following inequalities hold:

$$|d| > |c|r + (y\epsilon^{-1})^{\frac{1}{2}} \geqslant |c|r + (\epsilon\delta)^{-\frac{1}{2}}.$$

Consequently, $\Im(g \circ z) > \epsilon$ only if

$$|c| \leqslant (\delta\epsilon)^{-\frac{1}{2}} \quad \text{and} \quad |d| \leqslant (\epsilon\delta)^{-\frac{1}{2}}(r+1),$$

and the total number of such pairs (not counting $(c,d) = (0, \pm 1), (\pm 1, 0)$) is at most $\frac{4(r+1)}{(\epsilon\delta)}$. This proves the lemma. □

It follows from Lemma 1.2.0.2 that $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ is a discrete subgroup of $SL(2, \mathbb{R})$. This is because:

1. It is enough to show that for any compact subset $A \subset \mathfrak{H}$ there are only finitely many $g \in SL(2, \mathbb{Z})$ such that $(g \circ A) \cap A \neq \phi$;

2. Every compact subset of $A \subset \mathfrak{H}$ is contained in a rectangle $R_{r,\delta}$ for some $r > 0$ and $0 < \delta < \delta^{-1}$;

3. $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$, except for finitely many $\alpha \in \Gamma_\infty$, $g \in \Gamma_\infty \backslash \Gamma$.

To prove (3), note that Lemma 1.2.0.2 implies that $(g \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$ except for finitely many $g \in \Gamma_\infty \backslash \Gamma$. Let $S \subset \Gamma_\infty \backslash \Gamma$ denote this finite set of such elements g. If $g \notin S$, then Lemma 1.2.0.2 tells us that it is because $\Im(g \circ z) < \delta$ for all $z \in R_{r,\delta}$. Since $\Im(\alpha g \circ z) = \Im(g \circ z)$ for $\alpha \in \Gamma_\infty$, it is enough to show that for each $g \in S$, there are only finitely many $\alpha \in \Gamma_\infty$ such that $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} \neq \phi$. This last statement follows from the fact that $g \circ R_{r,\delta}$ itself lies in some other rectangle $R_{r',\delta'}$, and every $\alpha \in \Gamma_\infty$ is of the form $\alpha = \begin{bmatrix} 1 & -m \\ 0 & 1 \end{bmatrix}$ $(m \in \mathbb{Z})$, so that

$$\alpha \circ R_{r',\delta'} = \{x + iy \mid -r' + m \leqslant x \leqslant r' + m,\ 0 < \delta' \leqslant \delta''^{-1}\},$$

which implies $(\alpha \circ R_{r',\delta'}) \cap R_{r,\delta} = \phi$ for $|m|$ sufficiently large. Now we are ready to describe the domain for $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}$. First we need to specify what we mean by the term fundamental domain

Definition 1.2.0.3. *Given a group G acting on the set X. A **fundamental domain** for this action is the set $D \subset X$ such that*

1. *For any $x \in X$, there exists a $d \in D$ such that $xg = d$.*

2. *if xg_1 and xg_2 are both in the domain D, then $g_1 = g_2$.In particular, the representative for the orbit of x under the action of G in D is unique.*

Example 1.2.0.4. *Let $\mathbb{Z}$ acts on $\mathbb{R}$ by translation, namely $nx := n + x$. Then it is easy to see that the fundamental domain for $\mathbb{R}/\mathbb{Z}$ is the half interval $[0, 1)$.*

Proposition 1.2.0.5. *A fundamental domain for $\mathfrak{H} / \mathrm{SL}_2(\mathbb{Z})$ can be given as the region*

$$\mathfrak{D} = \{z = x + iy \in \mathfrak{H} : |z| \geqslant 1, -1/2 \leqslant x < 1/2\} \cup \{|z| = 1, x \leqslant 0\}.$$

Proof. First we eliminated the repeated points on the boundary. Note that the line $x = -1/2$ is the same as the line $x = 1/2$ under the transformation $z \mapsto z + 1$. Similarly, given a point on the circle $\{|z| = 1\}$, the transformation $z \mapsto -|z|^{-1}$ satisfies

$$\frac{-1}{x + iy} = \frac{-x + iy}{x^2 + y^2} = -x + iy,$$

which flips the sign of x. Thus it identifies the half circle on the right of the imaginary axis with that on the left.
Now we need to show two things:

1. For any $z \in \mathfrak{H}$ we can find an element $g \in \mathrm{SL}_2(\mathbb{Z})$ such that $g \circ z \in \mathfrak{D}$.

2. If $z \equiv z' \in \mathfrak{D}$ modulor $\mathrm{SL}_2(\mathbb{Z})$, then either $\Re(z) = \pm\frac{1}{2}$ and $z' = z \mp 1$, or $|z| = 1$ and $z' = \frac{-1}{z}$.

First we prove for (1): Fix $z \in \mathfrak{H}$. It follows from Lemma 1.2.0.2 that for every $\epsilon > 0$, there are at most finitely many $g \in \mathrm{SL}(2, \mathbb{Z})$ such that $g \circ z$ lies in the strip

$$D_\epsilon := \left\{ w \;\middle|\; -\frac{1}{2} \leqslant \mathrm{Re}(w) < \frac{1}{2},\ \epsilon \leqslant \mathrm{Im}(w) \right\}.$$

Let B_ϵ denote the finite set of such $g \in \mathrm{SL}(2, \mathbb{Z})$. Clearly, for sufficiently small ϵ, the set B_ϵ contains at least one element. We will show that there is at least one $g \in B_\epsilon$ such that $g \circ z \in D$. Among these finitely many $g \in B_\epsilon$, choose one such that $\Im(g \circ z)$ is maximal in D_ϵ. If $|g \circ z| < 1$, then for $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ we have, for any m,

$$\Im\left(T^m S g \circ z\right) = \Im\left(\frac{-1}{g \circ z}\right) = \frac{\Im(g \circ z)}{|g \circ z|^2} > \Im(g \circ z)$$

But we can choose m such that $T^m S g \circ z \in D_\epsilon$, which contradicts the maximality of $\Im(g \circ z)$. Next we give a proof for (2):Let $z \in D$, $g = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, and assume that $g \circ z \in D$. Without loss of generality, we may assume that

$$\Im(g \circ z) = \frac{y}{|cz + d|^2} \geqslant \Im(z),$$

(otherwise just interchange z and $g \circ z$ and use g^{-1}). This implies that $|cz + d| \leqslant 1$ which implies that $1 \geqslant |cy| \geqslant \frac{1}{\sqrt{3}}|c|$. This is clearly impossible if $|c| \geqslant 2$. So we only have to consider the cases $c = 0, \pm 1$. If $c = 0$ then $d = \pm 1$ and g is a translation by b. Since $-\frac{1}{2} \leqslant \Re(z), \Re(g \circ z) \leqslant \frac{1}{2}$, this implies that either $b = 0$ and $z = g \circ z$ or else $b = \pm 1$ and $\Re(z) = \pm\frac{1}{2}$ while $\Re(g \circ z) = \mp\frac{1}{2}$. If $c = 1$, then $|z + d| \leqslant 1$ implies that $d = 0$ unless $z = e^{2\pi i/3}$ and $d = 0, -1$. The case $d = 0$ implies that $|z| \leqslant 1$ which implies $|z| = 1$. Also, in this case, $c = 1$, $d = 0$, we must have $b = -1$ because $ad - bc = 1$. Then $g \circ z = a - \frac{1}{z+1}$. It follows that $g \circ z = a - e^{2\pi i/3}$ and $d = 1$, then we must have $a - b = 1$. It follows that $g \circ z = a - \frac{1}{z+1} = a + e^{2\pi i/3}$, which implies that $a = 0$ or 1. A similar argument holds when $z = e^{\pi i/3}$ and $d = -1$. Finally, the case $c = -1$ can be reduced to the previous case $c = 1$ by reversing the signs of a, b, c, d. □

1.3 Lattices, upper half plane and $\mathrm{SL}_2(\mathbb{R})$

In this section, we investigate the notion of semi-stable lattices and how the upper half plane $\mathfrak{H}$ can be regard as a space of two-dimensional lattices. First we need to define what a lattice is.

Definition 1.3.0.1. *A **lattice** L in $\mathbb{C}$ is an additive group of the form*

$$L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$$

*where ω_1, ω_2 are linearly independent over $\mathbb{R}$. We call the set $\{\omega_1, \omega_2\}$ a **base** for L.*

We want to classify 2-dimensional lattices, given they are "the same" under certain conditions. Note that, from the definition 1.3.0.1, a lattice is specified by giving a base. Let us denote BAS the set of of $\mathbb{R}$-bases for $\mathbb{C}$ and BAS^+ the set of all bases $\{z, w\}$ such that $\Im(z/w) > 0$. We clearly have a map

$$\Phi \colon \mathrm{BAS}^+ \to \{\text{lattices in } \mathbb{C}\}$$

where

$$\Phi(\{z, w\}) = \mathbb{Z}z \oplus \mathbb{Z}w$$

The maps is surjective, as we can always choose the sign of the elements in the base so that it is an element in BAS^+. Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$ acts on BAS^+ in the following way

$$\gamma\{z, w\} = \{az + bw, cz + dw\}$$

It is clear that this is indeed an action. Two lattices in the same orbit under this action are called similar. Modulo this action, we can define a bijective map as in the below lemma

Lemma 1.3.0.2. *Two bases in BAS^+ are mapped to the same lattice if they are in the same orbit under the action of* $\mathrm{SL}_2(\mathbb{R})$. *In particular, the quotient set* $\mathrm{SL}_2(\mathbb{R})\backslash BAS^+$ *are in bijection with the set of lattices in* $\mathbb{C}$.

Proof. Let $\{z, w\}$ and $\{z', w'\}$ be two bases such that

$$\begin{cases} \Im(z/w) > 0 \\ \Im(z'/w') > 0 \\ \Phi(z, w) = \Phi(z', w') = L \end{cases}$$

Since z', w' are elements in the lattice L spanned by $\{z, w\}$, there exists $a, b, c, d \in \mathbb{Z}$ such that

$$\begin{bmatrix} z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}$$

By the same argument, there must be $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ such that

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} z' \\ w' \end{bmatrix}$$

Thus we must have

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has integral entries, which implies that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm 1$. Note that

$$\Im\left(\frac{z'}{w'}\right) = \Im\left(\frac{az + bw}{cz + dw}\right) = \Im\left(\frac{a(z/w) + b}{b(z/w) + d}\right) = (ad - bc)\Im\left(\frac{z}{w}\right)$$

Hence $ad - bc = 1$ or $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. $\square$

We also note that, each element $\{z, w\} \in \mathrm{BAS}^+$ can be rescaled to get a base of the form $\{\tau, 1\}$ where $\Im(\tau) = \Im(z/w) > 0$. This mean we can also define another action - the resclaing action on BAS^+. This action commutes with the action of $\mathrm{SL}_2(\mathbb{R})$, hence induces an action on the space of similar lattices $\mathrm{SL}_2(\mathbb{R}) \backslash \mathrm{BAS}^+$. In particular, we can choose a representative for $\mathrm{SL}_2(\mathbb{R}) \backslash \mathrm{BAS}^+ / \mathbb{C}^\times$ to be $\{\tau, 1\}$ with $\Im(\tau) > 0$. We just proved

Proposition 1.3.0.3. *The upper half plane $\mathfrak{H}$ classifies similarity classes of two dimensional lattices.*

Now regard $\mathbb{C} \cong \mathbb{R}^2$ via $x + iy \mapsto (x, y)$, and the inner product is defined to be

$$\langle z_1, z_2 \rangle = x_1 x_2 + y_1 y_2,$$

where $z_i = x_i + iy_i$. With this inner product, we can define the volume of the lattice

Definition 1.3.0.4. *Via the identification $\mathbb{C} \cong \mathbb{R}^2$ and let e_1, e_2 to be the usual standard basis for $\mathbb{R}^2$, the volume of a lattice $L = \mathbb{Z}z \bigoplus \mathbb{Z}w$ is given by*

$$\begin{bmatrix} \langle z, e_1 \rangle & \langle z, e_2 \rangle \\ \langle w, e_1 \rangle & \langle w, e_2 \rangle \end{bmatrix}$$

Now for any $z \in \mathfrak{H}$, the pair $(1, z)$ can be identified with the lattice

$$L_z = \mathbb{Z}z \bigoplus \mathbb{Z}.$$

The lattice spanned by z and 1 has volume y, so the corresponding unit lattice is the one spanned by $z/\sqrt{y}$ and $1/\sqrt{y}$.
Using Proposition 1.2.0.5, it is immediate that every lattice spanned by 1 and z is similar to lattice generated by 1 and a point z' inside ther region $\mathfrak{D}$.

Definition 1.3.0.5. *A primitive vector is a vector such that it is not the multiple of any other vector in the lattice.*

Proposition 1.3.0.6. *If L is any lattice, and u is a primitive vector in L, and v' is a vector in the sublattice $L' = L/\mathbb{Z}u$, then there exists a unique representative v of v' such that its projection onto u lies in the interval $(-u/2, u/2]$. Moreover, the following inequality holds*

$$||v||^2 \leqslant \frac{||u||^2}{4} + ||v'||^2,$$

where we identify v' with a vector $v^\perp$ in the orthogonal complement of u.

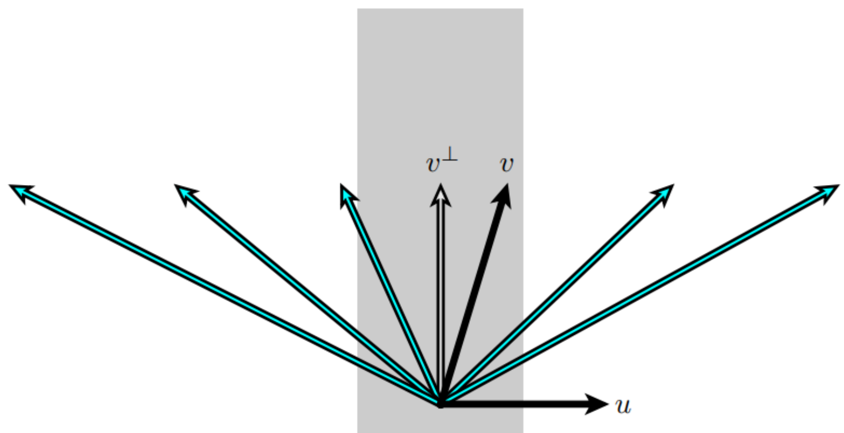


Figure 1.1: Reduction in two dimensional space

To see why every point $z \in \mathfrak{H}$ can be transformed into a point inside $\mathfrak{D}$, we start with a lattice generated by the $\mathbb{Z}$-linear combination of $1, z$ and consider the shortest vector u. Applying the above lemma, we can find a vector v with the length as least as large as that of u. So if we rotate and scale to get $u = 1$, the vector v will lie in the strip $(-1/2, 1/2]$ and has the length at least 1. This clearly show that v is a point in the domain $\mathfrak{D}$.

Now Grayson - following a prior idea of Stuhler - associated every lattice to a sort of **Newton Polygon**. We will set up a graph coordinate in the following way:

1. First we construct a two dimensional coordinate, say Oxy
2. We highlight the origin.
3. If we are dealing with the lattice L, compute the area of the fundamental domain of L
4. Assign the point $(2, \log(\mathrm{vol}(L)))$ to the line $x = 2$ in the coordinate.
5. If v is any primitive vector, we put the point $(1, \log(||v||))$ in the set.

Note that the lattice is discrete, so we can find a shortest vector v of the lattice L. This will correspond to the lowest point on the axis $x = 1$ in the diagram. Note the that x-coordinate of each of these point reflects its dimension.
As an example, let's consider the lattice of the following shape - with the shortest vector u has the length $||u|| < 1$. Now applying the above process, we get the figure on the left. If we further taking the convex hull of the diagram, we will get the figure on the right Clearly for each dimension, we have the corresponding lowest point, and so the convex hull of the plot is bounded from below. Grayson calls the plot on the left **canonical plot** of the lattice

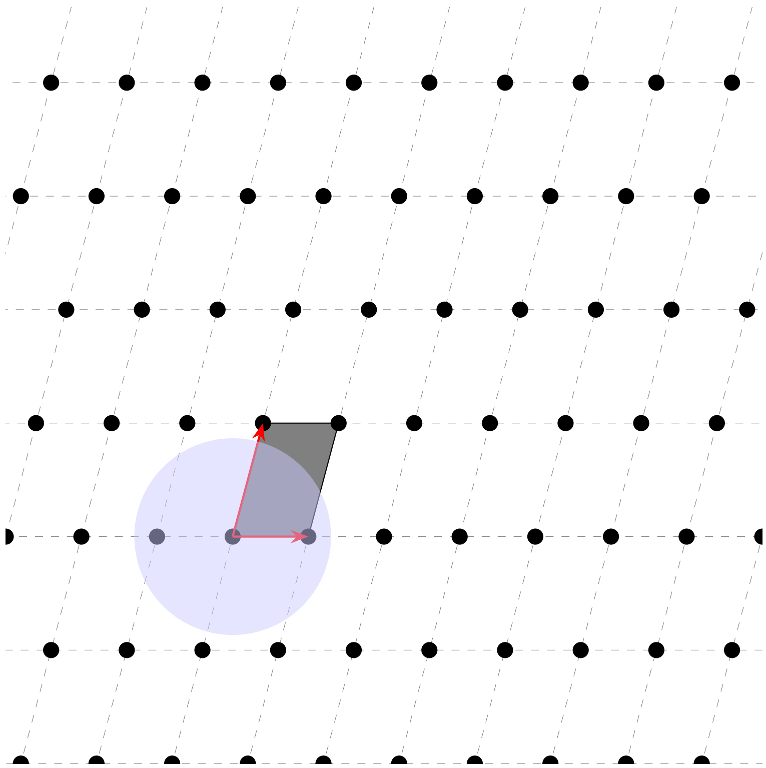


Figure 1.2: Example of a lattice

and the boundary of the convex hull of the canonical plot its **canonical polygon**. In the expository [6] of Bill Casselman, he instead calls the canonical polygon as **profile**. We will use the terminology of Casselman.

Now we will try to understand the profile of a lattice associated to a point $z \in \mathfrak{D}$. First we prove a simple observation

Lemma 1.3.0.7. *If $z \in \mathfrak{D}$ then the lattice $L_z = \mathbb{Z}z \oplus \mathbb{Z}$ admits 1 as the shortest vector.*

Proof. We identify $z = x + iy$ with $(x, y) \in \mathbb{R}^2$ and 1 with $(1, 0) \in \mathbb{R}^2$. Assume that 1 is not the shortest vector, then there exists $a, b \in \mathbb{Z}$ such that

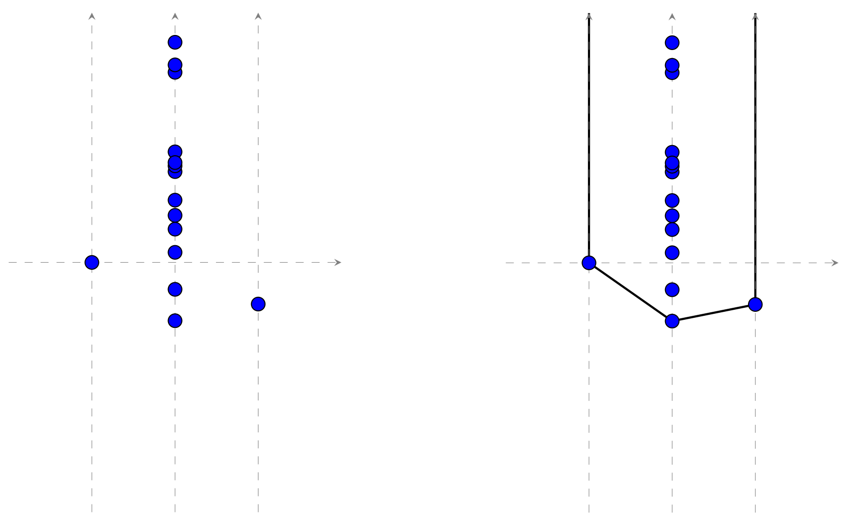
$$|az + b|^2 < 1 \Leftrightarrow (ax + b)^2 + (ay)^2 < 1 \Leftrightarrow a^2|z|^2 + 2abx + b^2 < 1$$

Since $z \in \mathfrak{D}$, we clearly have $|x| \leqslant \frac{1}{2}$ and $|z| \geqslant 1$, thus the integers a, b must satisfy

$$a^2 - |ab| + b^2 < 1$$

Since the above expression are symmetric, we can assume $|a| \geqslant |b|$ and completing the square yields

$$\left(\frac{\sqrt{3}b}{2}\right)^2 \geqslant a^2 - ab + b^2 < 1 \Rightarrow b^2 < 4/3 \Rightarrow b \leqslant 1$$



Substituting $|b| = 1$ yields $|a|^2 - |a| < 0$. There is no non-zero integer a satisfying this condition. $\square$

The area of the lattice L_z is given by $\det \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} = y$. Note that we can scale the basis by a factor $a = \sqrt{y}$ so that we get a lattice of volume 1. So the lowest points with respect to the axes $x = 0, 1, 2$ are $(0,0), (1, -\log(a))$, and $(2,0)$. The interesting part of $\mathfrak{D}$ is where $y \leqslant 1$. This corresponds to the lattices that have the canonical plots lying entirely on or above the x-axis. In particular, the profile of such a lattice only has the vertices at the origin and $(2,0)$. Grayson and Stuhler call this kind of lattice **semi-stable**. If we don't normalize the area of such a lattice, then a semi-stable lattice has the bottom of the profile as a straight line.

Conversely, the lattices assigned to the points $z \in \mathfrak{H}$ with $\Im(z) > 1$ correspond to lattices that have the canonical plot breaking at the lowest point on the axis $x = 1$. In the general case, this reflects the fact that a non semi-stable lattice has the shortest vector u satisfying $||u|| < \sqrt{\mathrm{vol}(L_z)}$. Following Casselman, we call such a lattice **unstable**. In some sense, we can see that *the degree of instability* is measured by the shortest vector compared to its volume. In the above lemma, we only find the semi-stable locus inside the fundamental domain. To find the semi-stable locus for the whole upper half plane $\mathfrak{H}$, we use the following lemma:

Lemma 1.3.0.8. *If L_z is semi-stable, then so is the lattice $L_{g \circ z}$, where $g \in \mathrm{SL}_2(\mathbb{R})$.*

Proof. If we denote $L_z = \mathrm{span}_{\mathbb{Z}} \{1, z\}$, then $L_{\gamma \circ z} = cL_z$ for some complex number c. Indeed, we just need to check for γ being an inversion or translation, since these two transformations generate $\mathrm{SL}_2(\mathbb{R})$, but this is easy. Now let $c = re^{it}$. Multiplying by e^{it} doesn't change the length, hence doesn't change the semi-stability. Multiplying by a positive number r will shift $(1, \log|u|)$ to $(1, \log|u| + \log r)$ and $(2, \log(\mathrm{vol}(A)))$ to $(2, \log(\mathrm{vol}(A)) + 2\log r)$. The line segment d connecting the origin with the final point intersects the line $x = 1$ at $(1, \log(\mathrm{vol}(A)) + \log r)$. By the semi-stability of the original lattice, the point $(1, \log|u| + \log r)$ is above the line segment d. $\square$

I think in 2-dimensiona case, c is 1

From this lemma, we can see that the semi-stable locus is the complement of the Farey balls in the upper half plane, as illustrated in the following figure, where the blue part is the semi-stable locus and the gray part is the unstable one.

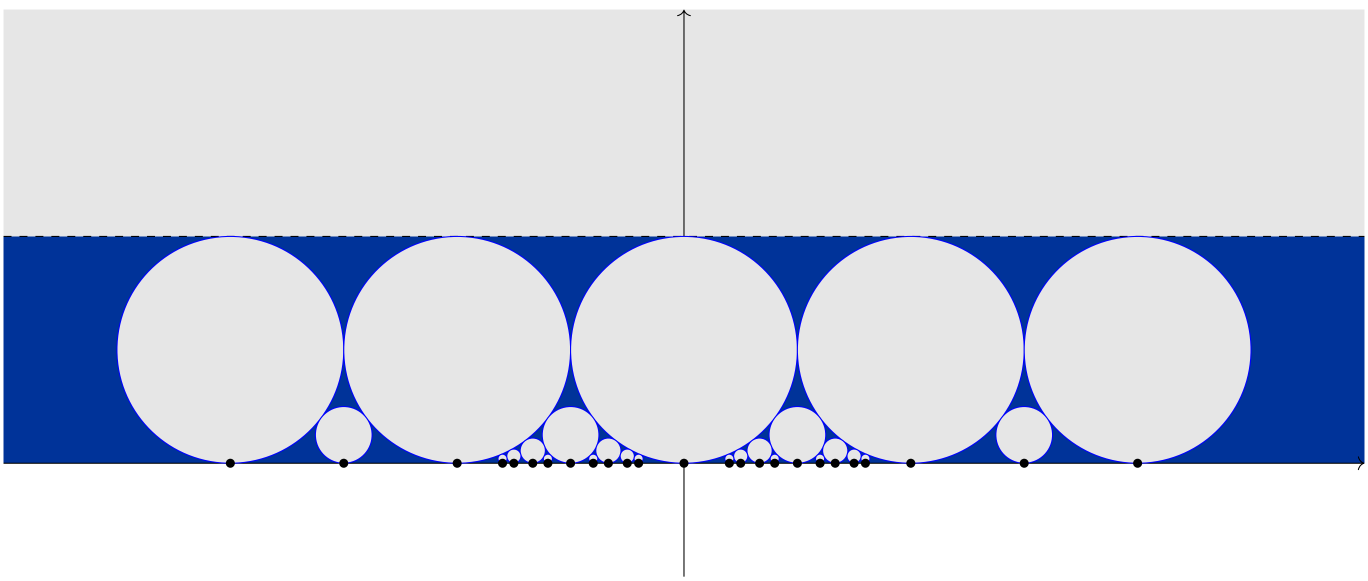


Figure 1.3: Semistable locus in upper half plane

1.4 ρ-semi-stability of lattices

The semi-stability can be defined in a more Lie-theoretic way. First we recall the Iwasawa decomposition for $\mathrm{SL}_2(\mathbb{R})$.

Proposition 1.4.0.1. *We have*

$$\mathrm{SL}_2(\mathbb{R}) \cong K \times A \times N$$

where

- $K = \mathrm{SO}_2(\mathbb{R})$*: the special orthogonal group.*
- $A = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a > 0 \right\}.$

- $N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right\}$.

Combinining with Proposition 1.1.0.1, we have the following identification

$$\mathfrak{H} \cong A \times N$$

via the map

$$x + iy \mapsto \begin{bmatrix} 1/\sqrt{y} & 0 \\ 0 & \sqrt{y} \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} = a\,(y)\,n(x)$$

Let's denote $\mathfrak{sl}_2(\mathbb{R})$ the Lie algebra of the Lie group $\mathrm{SL}_2(\mathbb{R})$ - the vector space of traceless matrices of size 2×2. We denote $\mathfrak{h} = \mathbb{R}H$ where $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ its standard Cartan subalgebra. We then have the map

$$H_B \colon \mathfrak{H} \to \mathfrak{h}, \quad z = x + iy \mapsto \log(a(y))H = \frac{-1}{2}\log(\Im(z))$$

Let $\alpha \colon \mathfrak{h} \to \mathbb{R}$ be the unique linear function such that $\alpha(H) = 2$. If we let $\rho = \frac{1}{2}\alpha$, then we define

$$\deg_{\text{inst}}(z) := \min_{\gamma \in \Gamma/\Gamma \cap B} \langle \rho, H_B(z\gamma) \rangle$$

where B is the group of upper triangular matrices with invertible entries along the diagonal.

Definition 1.4.0.2. *The lattice L_z corresponds to the point $z \in \mathfrak{H}$ is called ρ-**semistable** or just **semi-stable** if $\deg_{inst}(z) \geqslant 0$.*

We shall use this definition to find the semi-stable locus in the upper half plane $\mathfrak{H}$.

Proposition 1.4.0.3. *The locus of ρ-semistable points in the upper half plane $\mathfrak{H}$ is the complement of the Farey balls.*

Proof. We first make simple observation: If $\deg_{\text{inst}}(z)$ is achieved at some $\gamma_0 \in \Gamma$ and z is ρ-semistable, then

$$\langle \rho, H_B(z\gamma) \rangle \geqslant 0 \text{ for all } \gamma \in \Gamma$$

From this observation, the ρ-semistable locus must be the set

$$\{z \in \mathfrak{H} : \langle \rho, H_B(z\gamma) \rangle \geqslant 0 \text{ for all } \gamma \in \Gamma\}$$

We identify $z = x + iy$ with the product $a(y)n(x)$ as above. Under the identification 1.1.0.2, we must have

$$\big\langle \rho, H_B(z\gamma) = -\log(a(\gamma^{-1} \circ z)) \big\rangle$$

Assume that $\gamma^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\gamma^{-1} \circ z$ satisfies

$$\Im(\gamma^{-1} \circ z) = \frac{\Im(z)}{(cx+d)^2 + (cy)^2}$$

Thus, z is semistable if and only if it satifies the inequalities

$$y \leqslant (cx+d)^2 + (cy)^2 \Leftrightarrow \left(x + \frac{c}{d}\right)^2 + \left(y - \frac{1}{2c^2}\right)^2 \geqslant \left(\frac{1}{2c^2}\right)^2$$

for any c, d coprime. Since the above equation is exactly the equation for Farey balls, we are done. $\square$

In particular, we just proved that the ρ-semistablility and the semistablility in Grayson's sense are equivalent.

1.4.1 Unstable lattices

Consider an unstable lattice L with a shortest vectur u, then the bottom of the profile of L has a break at the point $(1, \log(||u||))$. Clearly u is a primitive vector, so $L_1 = \mathbb{Z}u$ is a sublattice of L. This determines a **lattice flag**

$$\mathcal{F} : 0 \subset L_1 \subset L_2 = L$$

And this is called the **canonical flag** associated to L. By tensoring with $\mathbb{R}$ we get a flag of rational subspaces

$$0 \subset V_1 = L_1 \otimes \mathbb{R} \subset V_2 = L_2 \otimes \mathbb{R}$$

Conversely, for any rational flag $\mathcal{F}$ in $\mathbb{R}^2$, we can denote $\mathcal{H}_{\mathcal{F}}$ the set of all unstable lattices that gives rise to the flag $\mathcal{F}$. What can we say about the set $\mathcal{H}_{\mathcal{F}}$? It turns out that the can be parametrized by the Farey balls in the complex plane.

Proposition 1.4.1.1. *Given a flag of rational vector spaces*

$$\mathcal{F} : 0 \subset V_1 \subset V_2 = \mathbb{R}^2$$

Assume that $V_1 = span(pf_1 + qf_2)$ for some $\mathbb{R}$- linearly independent set $\{f_1, f_2\}$. Then the set $\mathcal{H}_{\mathcal{F}}$ corresponds to the Farey balls that is tangent to the $x-axis$ at the fraction p/q.

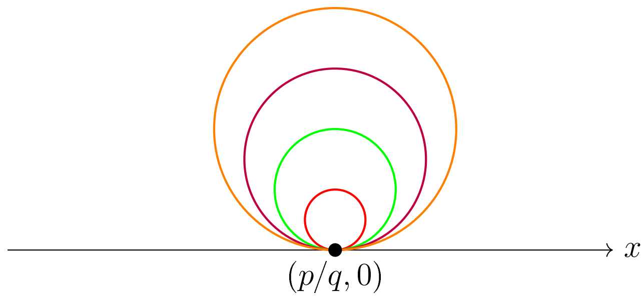


Figure 1.4: The Farey Circles correspond to the same rational flag

Proof. Consider the lattice $L = L_z \in \mathcal{H}_{\mathcal{F}}$ for $z = x + iy$. This lattice L has a basis

$$\begin{cases} f_1 = \dfrac{e_1}{\sqrt{y}} \\ f_2 = \dfrac{-xe_1}{\sqrt{y}} + e_2\sqrt{y} \end{cases}$$

Assume that $u = pf_1 + qf_2$ is the shortest vector of L. Since u is primitive, we must have $\gcd(p, q) = 1$. Since the lattice L is unstable, it follows that $\mu(L_1) \leqslant \mu(L/L_1)$ for $L_1 = \mathbb{Z}u$. In particular, we have

$$\frac{|-qz+p|}{\sqrt{y}} \leqslant 1 \Leftrightarrow (-qx+p)^2 + (qy)^2 \leqslant y$$

If $q \neq 0$, this is the equation for the Farey balls that tangent to the horizonal axis at $(p/q, 0)$. If $p = 0$ then $q = 1$, the equation $\dfrac{|-qz+p|}{\sqrt{y}} \leqslant 1$ degenerates to $y \geqslant 1$. Hence we are done. $\square$

Remark. *Note that for every flag in $\mathbb{R}^2$ corresponds to a parabolic subgroup. For example, the subgroup*

$$B = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

stabilizes the flag $0 \subset \mathbb{R}e_1 \subset \mathbb{R}^2$ for some choice of basis $\{e_1, e_2\}$ of $\mathbb{R}^2$. Any parabolic subgroup of $\mathrm{SL}_2(\mathbb{R})$ is conjugate with B. So there is a correspondence between rational parabolic subgroups of $\mathrm{SL}_2(\mathbb{R})$, that is, the parabolic subgroup that stabilizes a rational flag $\mathcal{F}$ - and the collections $\mathcal{H}_{\mathcal{F}}$ of lattices gives rise to the flag $\mathcal{F}$.

1.5 Lattices in higher rank

Since lattices is the central object for studying in this thesis, we will define the notion of lattice of rank at least 3 in the last section in chapter I. The definitions and properties of a general lattice will be used extensively in the later part of this thesis. Note that, in many reference, such as [6], the lattice is defined to be a module over a ring of algebraic integers, but we can always restrict to a $\mathbb{Z}$-module.

1.5.1 Abstract lattices

Definition 1.5.1.1 (Abstract $\mathbb{Z}$-lattices)**.** *Let L be a finitely generated $\mathbb{Z}$-module. In particular, it is a free $\mathbb{Z}$-module of finite rank. Suppose that L is endowed with a real-valued positive definite*[1] *quadratic form $Q\colon L \to \mathbb{R}$, such that the set*

$$\{x \in L : Q(x) \leqslant r\}$$

*is finite for any real number r. We will call the pair (L, Q) a **abstract $\mathbb{Z}$-lattice**.*

An easy example is to take $L = \mathbb{Z}^n$ and choose our quadratic form to be the standard one. namely

$$\langle x, y\rangle = x \cdot y = \sum_{i=1}^{n} x_i y_i$$

Here the multiplication is just the usual dot product between 2 vectors. In term of matrix, this quadratic form is assigned to the identity matrix I_n.
If there is no further confusion, we can just denote a Euclidean lattice by L, without specifying the bilinear form Q. The lattice L determines a full-rank lattice inside $L_{\mathbb{R}}$, namely, the rank of the lattice L is equal to the dimension of $L_{\mathbb{R}}$.

1.5.2 An alternative definition of lattices

For the sake of computation, we also usually adopt another definition of the lattice. In particular, we view lattice as a free $\mathbb{Z}-$ module of rank n that is isomorphic to $\mathbb{R}^n$ via base changing.

Definition 1.5.2.1. *A lattice in $\mathbb{R}^n$ is a subset $L \subset \mathbb{R}^n$ such that there exists a basis $b_1, \ldots, b_n$ of $\mathbb{R}^n$ such that*

$$L = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \ldots \mathbb{Z}b_n$$

If we put the vector $b_1, b_2, \ldots, b_n$ in columns, with respect to the standard basis, namely

$$g = [b_1|b_2|\ldots|b_n],$$

then $L = g\mathbb{Z}^n$.

[1] The non-degenerate implicity state that rank L is the same as dim$L_{\mathbb{R}}$

In the second definition, we can just identify L with the standard lattice $\mathbb{Z}^n$ and the symmetric positive definite form is $g^t g$. So an Euclidean $\mathbb{Z}$-lattice is an abstract lattice with the standard positive definite quadratic form.

1.5.3 Equivalence between two definitions of lattices

In this subsection, we will show that every abstract $\mathbb{Z}$- lattice is isomorphic to an Euclidean $\mathbb{Z}-$ lattice. This will be helpful in visualizing the abstract lattices, as we are just looking at concrete lattices with deformation by a linear transformation.
First we need to specify the notion of isomorphic lattices - in the first definition

Definition 1.5.3.1. *A map $f\colon (L,Q) \to (L',Q')$ is an **isomorhism** between lattices if it is a group isomorphism and for all $x \in L$, we have*

$$Q(x) = Q'(f(x))$$

Proposition 1.5.3.2. *Any abstract lattice is isomorphic to a Euclidean $\mathbb{Z}-$ lattice.*

Proof. Let (L,Q) be an arbitrary lattice. We define a bilinear form as

$$\langle x,y\rangle := \frac{Q(x+y)-Q(x-y)}{4}$$

We will show that this bilinear form defines an inner product over the real vector space $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$. Clearly we have $\langle x,x\rangle = 4Q(x)/4 = Q(x) \geqslant 0$ for all $x \in L\backslash\{0\}$. Now the extended bilinear form is defined as

$$\begin{aligned}\langle\cdot,\cdot\rangle : L_{\mathbb{R}} \times L_{\mathbb{R}} &\to \mathbb{R}\\ (x\otimes a, y\otimes b) &\mapsto ab\langle x,y\rangle\end{aligned}$$

It is immediate that the extended bilinear form is inner product. So we have proved that $L_{\mathbb{R}}$ is a Euclidean space containing L. Moreover, L is embedded injectively in $L_{\mathbb{R}}$ as $\mathbb{R}$ is a flat $\mathbb{Z}$ module. The condition that

$$\#\{x \in L : Q(x) \leqslant r\} < \infty$$

implies L can be identified with a discrete in $L_{\mathbb{R}}$. But this implies that there exists a basis $\{b_1,\ldots,b_n\} \subset L_{\mathbb{R}}$ such that

$$L = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \ldots \mathbb{Z}b_n$$

Hence we are done. □

fix the proof so that we use

1.5.4 Covolume of a lattice

Now that for every abstract lattice L we can find an invertible matrix g such that

$$L \cong g\mathbb{Z}^n$$

The number n is called the **rank** of the lattice L.
Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of $L_{\mathbb{R}} \cong \mathbb{R}^n$ and

$$g = [b_1|b_2|\ldots|b_n].$$

The covolume of the lattice L is defined as

Definition 1.5.4.1. *The covolume of L is given by the formulae*

$$\mathrm{vol}(L) = |\det(b_i \cdot e_j)|$$

The rank and covolume are invariant numerical values of L, as they don't depend on the choice of basis. Indeed, two bases of a rank n lattice L are related by a transformation $g \in \mathrm{GL}_n(\mathbb{Z})$. Clearly this preserves the volume and the rank as a $\mathbb{Z}-$module.

1.5.5 Sublattices

To work with semi-stable lattice L, we need to consider all the sublattices contained inside L.

Definition 1.5.5.1 (sublattice)**.** *Let (L, Q) be a Euclidean $\mathbb{Z}$-lattice. We say that a $\mathbb{Z}-$submodule M of L a **sublattice** if and only if L/M is torsion free.*

From this definition, we can prove that M is a sublattice of L if it satisfies one of the following equivalent properties:

1. M is a summand of L.
2. every basis of M can be extended to a basis of L.
3. The group M is an intersection of L with a rational subspace of $L_{\mathbb{R}}$.

We refer to the [9, Section 1.4] for a proof of these equivalences.

Example 1.5.5.2. *If $L = \mathbb{Z}^2$, then any sublattice of L is a primitive vector $u = (a, b)$, i.e* $\gcd(a, b) = 1$. *Indeed, $u = (a, b)$ is a sublattice of $\mathbb{Z}^2$ if and only if there exists a vector $v \in \mathbb{Z}^2$ such that $L = \mathbb{Z}u \oplus \mathbb{Z}v$. With respect to the usual inner product on $\mathbb{R}^2$, we have*

$$1 = \mathrm{vol}(\mathbb{Z}^2) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

This happens if and only if $\gcd(a, b) = 1$.

Definition 1.5.5.3. *Given a lattice L and a subgroup $M \subset L$, then we call the M* ***primitive*** *if L/M is torsion free. Equivalently, $M \subset L$ is a primitive sublattice if no non-zero multiples of non-members of M are in M. In formula*

$$(M \otimes \mathbb{Q}) \cap L = M$$

Example 1.5.5.4. *We consider the subgroup M generated by a single element $v \in L$. Then $\mathbb{Z}v$ is primitive if and only if there is no $w \in L\backslash\{v\}$ such that*

$$w = c \cdot v,$$

for some integer c.

Chapter 2

BASIC LIE THEORY FOR $\mathbf{SL}_{\mathrm{n}}(\mathbb{R})$ & $\mathbf{GL}_{\mathrm{n}}(\mathbb{R})$

In this chapter, we review the basic theory of roots and weights for the Lie groups $\mathrm{SL}_{\mathrm{n}}(\mathbb{R})$ and $\mathrm{GL}_{\mathrm{n}}(\mathbb{R})$.

2.1 Lie algebras

Throughout this section, the ground field is $k = \mathbb{R}$.

2.1.1 Definition

A Lie algebra $\mathfrak{g}$ is a vector space over k such that the Lie bracket

$$\begin{aligned} [\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (x, y) &\mapsto [x, y] \end{aligned}$$

satisfies the following axioms

1. $[x, y]$ is bilinear.
2. $[x, x] = 0$ for all $x \in \mathfrak{g}$.
3. $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$

The last property is called *Jacobi's identity.*

Example 2.1.1.1. *The first example is the set of all matrices of size $n \times n$, denoted by $\mathfrak{gl}_{\mathrm{n}}(\mathbb{R})$. We define the Lie bracket by*

$$[A, B] := AB - BA,$$

for $A, B \in \mathfrak{gl}_\mathrm{n}(\mathbb{R})$. Therefore $\mathfrak{gl}_\mathrm{n}(\mathbb{R})$ is a Lie algebra.

Example 2.1.1.2. *Another example is the subspace $\mathfrak{sl}_\mathrm{n}(\mathbb{R})$ of $\mathfrak{gl}_\mathrm{n}(\mathbb{R})$, defined by*

$$\mathfrak{sl}_\mathrm{n}(\mathbb{R}) = \{A \in \mathfrak{gl}_\mathrm{n}(\mathbb{R}) : \mathrm{tr}(A) = 0\}$$

The Lie bracket over $\mathfrak{sl}_\mathrm{n}(\mathbb{R})$ is just the restriction of the Lie bracket over $\mathfrak{gl}_\mathrm{n}(\mathbb{R})$. Indeed, for any $A, B \in \mathfrak{sl}_\mathrm{n}(\mathbb{R})$ we have

$$tr[A, B] = \mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$$

2.1.2 Cartan Subalgebras

First we need the notion of Cartan subalgebras

Definition 2.1.2.1. *For any Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be a Cartan algebra if it is*

- *$\mathfrak{h}$ is a nilpotent subalgebra, i.e. $\mathfrak{h}^{(n)} = \underbrace{[\mathfrak{h}, [\mathfrak{h}, \ldots, [\mathfrak{h}, \mathfrak{h}]]]}_{n-times} = 0$ for some integer $n > 0$.*
- *It is self normalizing. Namely, we have $\mathfrak{h} = \{x \in \mathfrak{g} : [x, \mathfrak{g}] \subset \mathfrak{g}\}$.*

A priori, it is not clear whether such a subalgebra exists for any Lie algebras. However, we have the following theorem

Theorem 2.1.2.2. *[1, Corollary 1.2] Let L be a Lie algebra of dimension n over a field F of at least $n - 1$ elements. Then L has a Cartan subalgebra.*

In particular, any finite dimensional Lie algebra over an infinite field has a Cartan subalgebra.

Example 2.1.2.3. *For the simple Lie algebra $\mathfrak{sl}_\mathrm{n}(\mathbb{R})$, one choice of Cartan subalgebra $\mathfrak{h}$ is the set*

$$\mathfrak{h} = \left\{ H = \begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}, a_1 + a_2 + \ldots + a_n = 0 \right\} \tag{2.1}$$

Proof. Indeed, it is obvious that $[\mathfrak{h}, \mathfrak{h}] = 0$, thus the Lie algebra $\mathfrak{h}$ is nilpotent. To show the self-normalizing property, we pick an any element $\sum_{i,j} a_{ij} E_{ij}$ such that

$$\left[\sum_{i,j} a_{ij} E_{ij}, \mathfrak{h} \right] \subset \mathfrak{h}$$

Here E_{ij} denotes the matrix that has 1 at the ij entry and 0 otherwise. Clearly the matrix $E_{pp} - E_{qq} \in \mathfrak{h}$ for $p \neq q$. We then have

$$\left[\sum_{i,j} a_{ij} E_{ij}, E_{pp} - E_{qq}\right] \in \mathfrak{h}$$

or equivalently

$$\sum_i a_{ip} E_{ip} - \sum_i a_{iq} E_{iq} - \sum_j a_{pj} E_{pj} - \sum_j a_{qj} E_{qj} \in \mathfrak{h}$$

The coefficients of E_{pq} in the above sum is $-2a_{pq}$. By our choice of $\mathfrak{h}$ it must be the case that $a_{pq} = 0$. Thus, only the coefficients of E_{pq} for $p = q$ survive. Hence $\sum_{i,j} a_{ij} E_{ij} \in \mathfrak{h}$ and $\mathfrak{h}$ is then a Cartan subalgebra of $\mathfrak{sl}_{\mathrm{n}}(\mathbb{R})$. □

2.1.3 Root space decomposition

Consider a semisimple Lie algebra $\mathfrak{g}$. With respect to a choice of Cartan subalgebra $\mathfrak{h}$, we have a root space decomposition. In particular, there is a finite set $\Phi \subset \mathfrak{h}^*$ of linear forms on $\mathfrak{h}$, whose elements are called **roots**, such that

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right),$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$ for any $\alpha \in \Phi$. We refer to [7, Chapter 8] for a detail analysis of this decomposition in general case. For this thesis, most of the time, we stick with the following example

Example 2.1.3.1. *Let $I = \{1, 2, \ldots, n-1\}$. For the simple Lie algebra $\mathfrak{sl}_{\mathrm{n}}(\mathbb{R})$, one choice of Cartan subalgebra $\mathfrak{h}$ is as in (2.1). With respect to this Cartan subalgebra, we can define the linear function*

$$L_i \colon \mathfrak{h} \to \mathbb{R}, \quad H \mapsto a_i$$

Then the roots are given by $\alpha_{ij} := L_i - L_j$ for distinct $i, j \in I$. We have the root space decomposition for $\mathfrak{sl}_{\mathrm{n}}(\mathbb{R})$ as follows

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{i \neq j} \mathfrak{g}_{\alpha_{ij}} \right) = \mathfrak{h} \oplus \left(\bigoplus_{i \neq j} \mathbb{R} E_{ij} \right)$$

*For the sake of brevity, we will denote $\alpha_{i,i+1}$ by α_i - these are called **simple roots**. We can decompose the set of roots $\Phi = \{\alpha_{ij}, i \neq j\}$ into disjoint subsets, namely*

$$\Phi = \Phi_+ \coprod \Phi_-$$

where the positive set of roots Φ_+ comprises α_{ij} for $i < j$ and the remaining roots are in the negative set of roots Φ_-.

Lemma 2.1.3.2. *Each $\alpha \in \Phi$ can be written uniquely as a linear combination*

$$\alpha = m_1\alpha_1 + \ldots + m_d\alpha_d$$

with all $m_i \in \mathbb{Z}_{\geqslant 0}$ or $m_i \in \mathbb{Z}_{\leqslant 0}$. If $\alpha \in \Phi_+$ then all $m_i \geqslant 0$, otherwise $m_i \leqslant 0$ for all i.

2.1.4 Killing form

To work with the geometry of the root system, we need the notion of the Killing form.

Definition 2.1.4.1. *Let $\mathfrak{g}$ be any Lie algebra and $x, y \in \mathfrak{g}$. We define the map*

$$\begin{aligned} \mathrm{ad}(x)\colon \mathfrak{g} &\to \mathfrak{g} \\ y &\mapsto \mathrm{ad}(x)(y) := [x, y] \end{aligned}$$

The bilinear map $\kappa(x, y) := \mathrm{Tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y))$ is called **Killing form**. If there is no confusion, we write $(x, y) := \kappa(x, y)$. When $\mathfrak{g}$ is semisimple, the Killing form has many nice properties, as recorded in the following theorem

Theorem 2.1.4.2. *[7, Chapter 5] A Lie algebra $\mathfrak{g}$ is semi-simple if and only if the Killing form is non-degenerate.*

It can also be showed that, given the root space decomposition as in section 2.1.3, if $\alpha, \beta \in \Delta$ satisfy $\alpha + \beta \neq 0$, then the space $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ with respect to the Killing form. A corollary of this is

Theorem 2.1.4.3. *[7, Chapter 8] Given a semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then the restriction of the Killing form to $\mathfrak{h}$ is non-degenerate.*

By Theorem 2.1.4.3, there exists a bijective linear map

$$\begin{aligned} f\colon \mathfrak{h} &\to \mathfrak{h}^* \\ x &\mapsto f_x \end{aligned}$$

where $f_x(y) := \kappa(x, y)$ for all $y \in \mathfrak{h}$. We may thus define a bilinear map

$$\begin{aligned} (\cdot,\cdot)\colon \mathfrak{h}^* \times \mathfrak{h}^* &\to \mathbb{R} \\ (f_x, f_y) &\mapsto \kappa(x, y) \end{aligned}$$

It this notation, it can be observed that $(f_x, f_y) = (x, y)$. A more thorough treatment of the Killing form can be found in [7, Chapter 5,8] .

Example 2.1.4.4. *Consider the lie algebra $\mathfrak{g} = \mathfrak{sl}_{\mathrm{n}}(\mathbb{R})$ and $\mathfrak{h}$ as in (2.1). It can be shown that*

$$\kappa(x, y) = 2n\mathrm{tr}(xy)$$

where xy is just the usual matrix multiplication and tr *is the trace of the matrix. We refer to [4, Section 4.3] for a detailed computation.*

Claim:

$$2\frac{E_{i,i} - E_{i+1,i+1}}{(E_{i,i} - E_{i+1,i+1}, E_{i,i} - E_{i+1,i+1})} = \frac{E_{i,i} - E_{i+1,i+1}}{2n}$$

gives rise to the root $\alpha_i = L_i - L_{i+1}$ via the identification using Killing form.
Indeed, by direct computation, we have

$$\left(\frac{E_{i,i} - E_{i+1,i+1}}{2n}, E_{j,j} - E_{j+1,j+1}\right) = \begin{cases} 2, \; \textit{if } i = j \\ -1 \; \textit{if } |i-j| = 1 \\ 0 \; \textit{otherwise} \end{cases} \tag{2.2}$$

On the other hand we also have

$$\alpha_i\left(E_{j,j} - E_{j+1,j+1}\right) = \left(L_i - L_{i+1}\right)\left(E_{j,j} - E_{j+1,j+1}\right) \begin{cases} 2, & \textit{if } i = j \\ -1 & \textit{if } |i-j| = 1 \\ 0 & \textit{otherwise} \end{cases}$$

Since $\{E_{j,j} - E_{j+1,j+1}\}$ where $j \in I$ forms a basis for $\mathfrak{h}$, we must have

$$\frac{(E_{i,i} - E_{i+1,i+1}, x)}{2n} = (L_i - L_{i+1})(x)$$

for all $x \in \mathfrak{h}$. This proves the desired claim.

2.1.5 Co-roots

In previous section, we showed that the Killing form gives rise to an inner product over the dual space $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{R})$. For each root $\alpha \in \mathfrak{h}^*$, we define the corresponding co-root $\alpha^\vee \in \mathfrak{h}$ to be the element such that

$$\langle \beta, \alpha^\vee \rangle := \beta(\alpha^\vee) = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

for all $\beta \in \mathfrak{h}^*$. It is clear from this definition that $\langle \alpha, \alpha^\vee \rangle = 2$.

Example 2.1.5.1. *This continues the example 2.1.4.4. We want to determine the co-root $\alpha_i^\vee$. It is then sufficient to compute $\langle \alpha_j, \alpha_i^\vee \rangle$ for $j \in I$. By definition*

$$\langle \alpha_j, \alpha_i^\vee \rangle = \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)}$$

By example 2.1.4.4, we have

$$(\alpha_j, \alpha_i) = \frac{1}{4n^2}(E_{j,j} - E_{j+1,j+1}, E_{i,i} - E_{i+1,i+1}) = \begin{cases} \frac{1}{n}, & \textit{if } i = j \\ \frac{-1}{2n}, & \textit{if } |i-j| = 1 \\ 0, & \textit{otherwise} \end{cases}$$

In particular,

$$\alpha_j(\alpha_i^\vee) = \begin{cases} 2, & \textit{if } i = j \\ -1 & \textit{if } |i-j| = 1 \\ 0 & \textit{otherwise} \end{cases}$$

This proves that $\alpha_i^\vee = E_{i,i} - E_{i+1,i+1}$.

2.1.6 Weights and co-weights

Fix $i \in I$, define $\lambda_i \in \mathfrak{h}^*$ by specifying

$$\left\langle \lambda_i, \alpha_j^\vee \right\rangle = \delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

Example 2.1.6.1. *Let again consider the Lie algebra $\mathfrak{sl}_{\mathrm{n}}(\mathbb{R})$ with the co-root $\alpha_{ij}^\vee = E_i - E_j$. Explicitly, it can be checked that $\lambda_i = L_1 + L_2 + \ldots + L_i$. Indeed, we have*

$$(L_1 + L_2 + \ldots + L_i)(E_{j,j} - E_{j+1,j+1}) = 1 \Leftrightarrow i = j \quad \textit{and } 0 \textit{ otherwise}$$

which uniquely characterize λ_i.

In the same manner we define the co-weight $\lambda_i^\vee$ such that

$$\left\langle \alpha_i, \lambda_j^\vee \right\rangle = \delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

Example 2.1.6.2. *We work with $\mathfrak{sl}_{\mathrm{n}}(\mathbb{R})$. Note that*

$$(L_i - L_{i+1})(E_{1,1} + \ldots + E_{j,j}) = 1 \Leftrightarrow i = j, \quad \textit{and } 0 \textit{ otherwise}$$

we can see that $\lambda_i^\vee = E_{1,1} + \ldots + E_{i,i}$

From these definitions, we can see that $\widehat{\Delta} = \{\lambda_i, i = 1, 2, \ldots, n-1\}$ is a basis for the the space $\mathfrak{h}^*$. We call $\widehat{\Delta}$ the set of **fundamental weights** . Similarly the set $\widehat{\Delta}^\vee = \{\lambda_i^\vee, i = 1, 2, \ldots, n-1\}$ is a basis for the Cartan subalgebra $\mathfrak{h}$. We have the following

Lemma 2.1.6.3. *We can write*

$$\lambda_i := r_1\alpha_1 + r_2\alpha_2 + \ldots + r_d\alpha_d$$

where r_i's are rational number such that $r_i \geqslant 0$.

This coefficients r_i's is determined by inverting the Cartan matrix of $\mathfrak{sl}_\mathrm{n}(\mathbb{R})$. We refer to [10] for an explicit formula for $r_i's$.

2.1.7 Weyl group

We define the map

$$s_\alpha(\beta) := \beta - \langle \beta, \alpha^\vee \rangle \alpha = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$$

where $\alpha, \beta \in \mathfrak{h}^*$ are roots. The space $\mathfrak{h}^*$ along with the bilinear map $(\cdot, \cdot)$ have the structure of a Euclidean space, so we can think of s_α as reflection transformation. When $\alpha = \alpha_i$ - the simple root - we call s_α the **simple reflection** and denote $s_i := s_{\alpha_i}$.

Definition 2.1.7.1. *The Weyl group of $\mathfrak{sl}_\mathrm{n}(\mathbb{R})$ is defined to be the set generated by the simple reflections*

$$W = \langle s_i, i = 1, 2, \ldots n-1 \rangle$$

Example 2.1.7.2. *We continue the computation as in example 2.1.5.1. For any $i \in \{1, 2 \ldots, n-1\}$ we have*

$$s_i(L_{i+1}) = L_{i+1} - \langle L_{i+1}, \alpha_i^\vee \rangle \alpha_i = L_{i+1} - (-1)(L_i - L_{i+1}) = L_i$$

and

$$s_i(L_i) = L_i - \langle L_i, \alpha_i^\vee \rangle \alpha_i = L_i - (L_i - L_{i+1}) = L_i$$

Thus, if we consider the $l+1$ dimensional space generated by $L_i, i = 1, 2, \ldots, n$, then the Weyl group is generated by the set of transposition $(i, i+1)$. Hence $W \cong S_{l+1}$ - the symmetric group of $l+1$ letters.

2.2 Lie groups

We only define linear Lie groups.

2.2.1 General linear group and special linear group

Definition 2.2.1.1. *A general linear group over the real field, denoted $GL_{\mathrm{n}}(\mathbb{R})$, is defined to be the set*

$$GL_{\mathrm{n}}(\mathbb{R}) = \{A \in \mathfrak{gl}_{\mathrm{n}}(\mathbb{R}) : \det(A) \neq 0\}$$

An linear Lie group is a closed subgroup of $\mathrm{GL}_{\mathrm{n}}(\mathbb{R})$. The key example is the following group

$$\mathrm{SL}_{\mathrm{n}}(\mathbb{R}) = \{A \in \mathrm{GL}_{\mathrm{n}}(\mathbb{R}) : \det(A) = 1\}$$

2.2.2 Relation with Lie algebras

We first need to define exponential map

Definition 2.2.2.1. *Given a matrix $X \in \mathfrak{gl}_{\mathrm{n}}(\mathbb{R})$, we define the exponential of X is series*

$$e^X := I + X + \frac{X^2}{2!} + \ldots = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

For each linear Lie group $G \subset \mathrm{GL}_{\mathrm{n}}(\mathbb{R})$, we associate the set

$$\mathrm{Lie}(G) := \{X \in \mathfrak{gl}_{\mathrm{n}}(\mathbb{R}) : e^{tX} \in G, \forall t \in \mathbb{R}\}$$

It can be proven, for example see [4, Theorem 3.2.1], that $\mathrm{Lie}(G)$ is a vector spaces. We will try to compute $\mathrm{Lie}(\mathrm{GL}_{\mathrm{n}}(\mathbb{R}))$ and $\mathrm{Lie}(\mathrm{SL}_{\mathrm{n}}(\mathbb{R}))$.

Example 2.2.2.2. *We need the following result*

Theorem 2.2.2.3. *[4, Section 2.1] For any $X \in M_n(\mathbb{R})$, we have*

$$\det(e^X) = e^{\mathrm{tr}(X)},$$

where $\mathrm{tr}(X)$ is just the sum of entries along the diagonal of X.

Using Theorem 2.2.2.3, it is clearly that for any $X \in M_n(\mathbb{R})$, e^X is an invertible matrix, thus $\mathrm{Lie}(GL_{\mathrm{n}}(\mathbb{R})) = \mathfrak{gl}_{\mathrm{n}}(\mathbb{R})$. Let $X \in \mathfrak{sl}_{\mathrm{n}}(\mathbb{R})$ so that $\mathrm{tr}(X) = 0$, and apply Theorem 2.2.2.3 again, we have

$$\det(e^{tX}) = e^{t\mathrm{tr}(X)} = 1,$$

, which implies $\mathrm{Lie}(SL_{\mathrm{n}}(\mathbb{R})) \supset \mathfrak{sl}_{\mathrm{n}}(\mathbb{R})$. Conversely, if $X \in SL_{\mathrm{n}}(\mathbb{R})$, then

$$e^{tX} \in SL_{\mathrm{n}}(\mathbb{R}), \forall t \in \mathbb{R} \Leftrightarrow e^{t\mathrm{tr}(X)} = \det(e^{tX}) = 1, \forall t \in \mathbb{R}$$

Hence

$$\mathrm{tr}(X) = \left(\frac{d}{dt} e^{t\mathrm{tr}(X)}\right)_0 = 0$$

which implies $\mathrm{Lie}(SL_{\mathrm{n}}(\mathbb{R})) \subset \mathfrak{sl}_{\mathrm{n}}(\mathbb{R})$. We then proved $\mathrm{Lie}(SL_{\mathrm{n}}(\mathbb{R})) = \mathfrak{sl}_{\mathrm{n}}(\mathbb{R})$.

2.2.3 Roots at group level

We want to understand how the roots behave at Lie group level. The analog for a Cartan subalgebra is a maximal torus given by the set

$$T = \left\{ t = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} : \prod_{i=1}^{n} a_i = 1 \right\},$$

Then T acts on $\mathfrak{g}$ by conjugation: $\mathrm{Ad}(t)(X) := tXt^{-1}$ for any $t \in T$ and $X \in \mathfrak{g}$. Explicitly, we can check that

$$\mathrm{Ad}(t)(E_{ij}) = a_i a_j^{-1} E_{ij}$$

Since every element of T are diagonal and commutative, they are simultaneously diagonal. If we let $\mathfrak{g}_\chi := \{X \in G : \mathrm{Ad}(t)(X) = \chi(t)X, \forall t \in T\}$, then we have the decomposition

$$\mathfrak{g} = \bigoplus_{\chi \in X(T)} \mathfrak{g}_\chi,$$

where $X(T)$ is just the set of multiplicative character from the torus T above to $\mathbb{R}^*$. This is the **root space decomposition** at Lie group level. Therefore, at the group level, the character $\chi_{ij}(\mathrm{diag}(t_1, \dots, t_n)) = t_i t_j^{-1}$ is a root whenever $i \neq j$. We define the set of **simple roots** as

$$\Delta = \{\chi_i \mid i = 1, \dots, n\}$$

where

$$\chi_i : T \mapsto \mathbb{R}, t \mapsto \frac{t_i}{t_{i+1}}.$$

Remark. *If we define the logarith map over the torus T by*

$$t = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \mapsto \log(t) := \begin{bmatrix} \log(a_1) & 0 & \dots & 0 \\ 0 & \log(a_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \log(a_n) \end{bmatrix}$$

Then it is clear that $\log$ *is a map from T to $\mathfrak{h}$. In this way, we can think χ_{ij} as $\alpha_{ij} \circ \log$. When there are no confusion, we use the symbol α_{ij} to indicate the root at both Lie algebra level and Lie group level.*

2.2.4 Weights

Similarly, at Lie group level, we also have the notion of **fundamental weights**. For each $i \in I$, we define a character $\lambda_i \in \mathrm{Hom}(T, \mathbb{R}^*)$ where T is the torus defined in 2.2.3

$$\omega_i \colon T \to \mathbb{R}^*, \quad \omega_i(t) = a_1 \dots a_i$$

Remark. *It can be seen that ω_i can be thought of $\lambda_i \circ \log$, so we will also use λ_i for the notation of simple weight in both Lie algebra and Lie group level as in the case of root.*

Example 2.2.4.1. *When $n = 3$, we have the following relations*

$$\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \quad \lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$$

We refer to the figure 2.1 for an illustration.

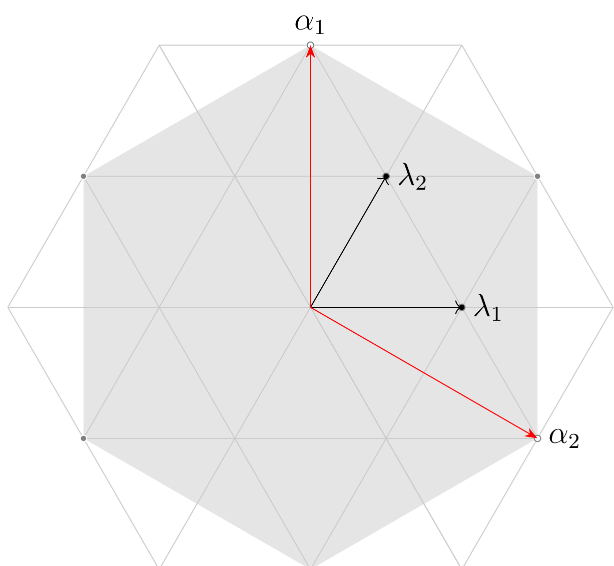


Figure 2.1: Roots and weights for the Lie group $\mathrm{SL}_3(\mathbb{R})$

Definition 2.2.4.2. *A weight λ is called **dominant** if it satisfies $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geqslant 0}$ for all α.*

Clearly by lemma 2.1.3.2, the weight λ is dominant if and only if $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geqslant 0}$ for all fundamental root α_i. It is also clear that the set of dominant weights is given by

$$\Lambda^+ := \{c_1\lambda_1 + \ldots + c_d\lambda_d \mid c_i \in \mathbb{Z}_{\geqslant 0}\}$$

The set of dominant weights is denoted Λ^+. A weight $\lambda = \sum n_i\lambda_i$ is called strongly dominant if $n_i > 0$ for all i. One important example is the minimal strongly dominant weight given by

$$\rho = \sum \lambda_i$$

This is called **Weyl vector** and is characterized in several ways:

1. $\langle \rho, \alpha_i^\vee \rangle = 1$ for all i.

2. $$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

To prove the last equation we use the action of the Weyl group W. Let $\mu = \frac{1}{2}\sum \alpha$. Apply the simple reflection s_i given by

$$s_i(x) = x - \langle x, \alpha_i^\vee \rangle \alpha_i$$

We know that s_i sends α_i to $-\alpha_i$ and permutes the other positive roots. So:

$$s_i(\mu) = \mu - \langle x, \alpha_i^\vee \rangle \alpha_i$$

Therefore, $(\mu, \alpha_i) = \mu(h_i) = 1$ for all i. So, $\mu = \rho$.
We of course also have an action of W on the weights. For example, one can verify that

$$s_i(\lambda_i) = \lambda_i - \alpha_i \quad \text{and} \quad s_j(\lambda_i) = \lambda_i \text{ for } i \neq j \ .$$

Unlike Lemma 2.1.6.3, if we try to express the fundamental weights in terms of the fundamental roots, we do not always get positive coefficients. However, it is true that all the coefficients must be integer. In particular, we have

$$\alpha_j = \sum_{n_j} n_j \lambda_j, \quad n_j \in \mathbb{Z}.$$

To put it another way, the root lattice

$$\mathbb{Z}\Delta = \{m_1\alpha_1 + \ldots + m_d\alpha_d, m_i \in \mathbb{Z}\}$$

is contained inside the weight lattice $\mathbb{Z}\Pi = \{n_1\lambda_1 + \ldots + n_d\lambda_d, n_i \in \mathbb{Z}\}$.

2.2.5 Cartan matrix

We fix a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_d\}$ is defined to be the matrix

$$A = \left[\left\langle \alpha_i, \alpha_j^\vee \right\rangle\right]$$

If we let $a_{ij} = \left\langle \alpha_i, \alpha_j^\vee \right\rangle$ then the Cartan matrix has the following simple properties:

Lemma 2.2.5.1.

- *For any i, we have $a_{ii} = 2$.*

- *For any $i \neq j$, a_{ij} is a non-positive integer, i.e. $a_{ij} \in \mathbb{Z}_{\leqslant 0}$.*

We give an explicit example for $\mathfrak{sl}_{\mathrm{n}}(\mathbb{R})$, which has the root system A_n. The corresponding Cartan matrix is

$$A_n = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix}$$

The Cartan matrix provides a linear combination presentation of the fundamental roots α_j's in terms of fundamental weights λ_i, as stated in the following lemma

Lemma 2.2.5.2. *Fix a number n and consider the set of simple roots α_j as well as the set of weights λ_i of $SL_{\mathrm{n}}(\mathbb{R})$, we have the following relations*

$$\alpha_i = \begin{cases} 2\lambda_1 - \lambda_2, & \text{if } i = 1 \\ -\lambda_{i-1} + 2\lambda_i - \lambda_{i+1}, & \text{if } 1 < i < n \\ -\lambda_{n-1} + 2\lambda_n, & \text{if } i = n \end{cases}.$$

Proof. Note that we have

$$\left\langle \alpha_i, \alpha_j^\vee \right\rangle = \begin{cases} 2, \ \text{if } i = j \\ -1, \ \text{if } |i - j| = 1 \\ 0, \ \text{otherwise} \end{cases}.$$

So if we let $\alpha_j = \sum_{i=1}^n a_{ij}\lambda_i$ and compute $\left\langle \alpha_i, \alpha_j^\vee \right\rangle$, the result follows immediately. □

2.3 Parabolic subgroups and parabolic subalgebras

We shall provide two equivalent viewpoints on parabolic subgroups. They will play different roles in defining different notions of semi-stability in the next chapter.

2.3.1 Parabolic subgroups I: An explicit description

For our purposes, it is enough to define the standard parabolic subgroups. There exists a bijection between standard parabolic subgroups of $\mathrm{SL}_{\mathrm{n}}(\mathbb{R})$ and partitions of n, and partition $n = n_1 + \ldots + n_k$ defines the standard parabolic subgroup corresponding to the partition as follows

Definition 2.3.1.1. *The standard parabolic subgroup P of $GL_{\mathrm{n}}(\mathbb{R})$ associated to the partition $n = n_1 + n_2 + \cdots + n_r$ is denoted $P_{n_1,\ldots,n_r}$ and is defined to be the group of all matrices of the form*

$$\begin{bmatrix} M_{n_1} & 0 & \ldots & 0 \\ 0 & M_{n_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M_{n_k} \end{bmatrix},$$

*where $M_{n_i} \in \mathrm{GL}(n_i, \mathbb{R})$ for $1 \leqslant i \leqslant r$. The integer $r-1$ is called the rank of the parabolic subgroup $P_{n_1,\ldots,n_r}$. We denote the collection of such subgroups **ParSt**.*

We denote $\{e_i\}$ the standard basis of $\mathbb{R}^n$. Let

$$d_i = n_1 + n_2 + \ldots + n_i$$

Then any element in $P_{n_1,\ldots,n_r}$ stabilizes the chain of vector spaces

$$0 \subset \mathbb{R}^{d_1} \subset \mathbb{R}^{d_2} \subset \ldots \subset \mathbb{R}^{d_k} = \mathbb{R}^n$$

Here $\mathbb{R}^k$ is generated by the linearly independent subset $\{e_1, \ldots, e_k\}$ of the standard basis. We call such chain of vector spaces a **flag** of type $(n_1, \ldots, n_k)$.

Definition 2.3.1.2. *The maximal standard parabolic subgroups in $GL_n(\mathbb{R})$ corresponds to the stabilizer of the flag of type $\rho_i = (i, n-i)$, where $i = 1, \ldots, n-1$ of n. We will further denote $Q_i = P_{\rho_i}$ and **MaxParSt** the collection of such maximal parabolic subgroups.*

Example 2.3.1.3. *Below we list all the standard parabolic subgroup in $GL_3(\mathbb{R})$. For $GL_3(\mathbb{R})$, there are three standard parabolic subgroups corresponding to three partitions of 3, namely*

$$3 = 1+1+1, \quad 3 = 1+2, \quad 3 = 2+1$$

For a partition $(r_1, \ldots, r_{s+1})$, we denote $P_{(r_1,\ldots,r_{s+1})}$ the corresponding parabolic subgroups. Thus, we have

$$P_{1,1,1} = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\}, \quad P_{1,2} = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\}$$

$$P_{2,1} = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$$

*Clearly **MaxParSt** $= \{P_{2,1}, P_{1,2}\}$.*

2.3.2 Parabolic subgroups II: Using BN-pairs

There is another way to look at the parabolic subgroups

Definition 2.3.2.1 (BN-pairs)**.** *A* ***BN-pairs*** *is a 4-tuple* (G, B, N, S) *where* G *is a group generated by subgroups* B *and* N*. The subgroup* $H = B \cap N$, S *is a finite set of involutions which generate the Weyl group* $W = N/H$*. Moreover, the following axioms holds*

- *If* $r \in R$ *and* $w \in W$*, then* $rBw \subset BwB \cup BrwB$*.*
- *If* $r \in R, rBr \neq B$*.*

Let $J \subset R$, we define W_J to be the subgroup of W generated by the involutions $r \in J$. We call it **standard parabolic subgroup** of W. Set $P_J = BW_JB$ as in the notation of BN-pairs. We have the following theorem

Theorem 2.3.2.2.

- P_J *is a subgroup if* G*. In particular, we have*

$$G = BWB,$$

which is called ***Bruhat decomposition*** *of* G*.*

- *If* $P_I = P_J$ *then we have* $I = J$*.*
- *All subgroups of* G *containing* B *arises in this way.*

For our purpose, it is enough to concentrate on the following example

Example 2.3.2.3. *Let* $G = GL_{\mathrm{n}}(\mathbb{R})$*, then the sets* B, N, R *are given explicitly as follows*

- $B =$ *upper triangular matrices*
- N = *monomial matrices, namely, matrices that have exactly one non-zero entry in each row and column.*
- *From the above, it is clear that* $H = B \cap N$ *is the diagonal group, and this group is normal in* N*.*
- *It can be shown that* $W = N/H \cong S_n$*, and we can choose* $R = \{(i, i-1)\}$ - *the set of transpositions.*

The above theorem leads to the following definition of parabolic subgroups

Definition 2.3.2.4 (Parabolic subgroups). *Using the same notation in the previous theorem, we call the subgroups P_J with $J \subset R$ standard parabolic subgroups of group G.*

We would like to explicitly describe the parabolic subgroup for $\mathrm{SL_n}(\mathbb{R})$.

Example 2.3.2.5. *As introduced in previous section, the set $\Phi = \{\alpha_{ij}\}$ forms a root system for $SL_{\mathrm{n}}(\mathbb{R})$. The Borel subgroup is just $B \cap SL_{\mathrm{n}}(\mathbb{R})$, the group of upper triangular matrices with determinant 1. We also consider the set N of monomial matrices with determinant 1. Then it is easy to check that N/H is the set of all permutation matrices. That is, N/H is generated by the matrices of the form*

$$r_i := \begin{bmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{n-i-1} \end{bmatrix}$$

Via the identification $s_i \mapsto (i, i+1)$, we identify the Weyl group $W = N/H \cong S_n$. So $R = \{(i, i+1) | i = 1, 2, \ldots, n\}$. We consider the set $I = R \backslash \{r_i\}$ for some $i < n$. The associated parabolic subgroup of W is

$$W_I = \langle r_1, r_2, \ldots r_{i-1}, r_{i+1}, \ldots, r_{n-1} \rangle \cong S_i \times S_{n-i}$$

The corresponding parabolic subgroup is

$$P_{r_i} := P_I = \left\{ \begin{bmatrix} A & * \\ & B \end{bmatrix} \in SL_{\mathrm{n}}(\mathbb{R}) : A \in GL_i, B \in GL_{n-i} \right\}$$

2.3.3 Langlands decomposition

We fix a partition of n as

$$n = n_1 + n_2 + \ldots + n_k$$

and consider the parabolic subgroup of this type, i.e. the subgroup

$$P_{n_1,\ldots,n_k} = \left\{ \begin{bmatrix} M_1 & * & \ldots & * \\ 0 & M_2 & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M_k \end{bmatrix} \right\}$$

where $M_i \in \mathrm{GL}_{n_i}(\mathbb{R})$ is invertible of size $n_i \times n_i$.
This group can be factored as

$$P_{n_1,\ldots,n_k} = M_{n_1,\ldots,n_k} \ltimes N_{n_1,\ldots,n_k}$$

where

$$N_{n_1,\dots,n_k} = \left\{ \begin{bmatrix} I_1 & * & \dots & * \\ 0 & I_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_k \end{bmatrix} \right\} \quad (I_k \text{ is the } n_k \times n_k \text{ identity matrix})$$

and

$$M_{n_1,\dots,n_k} = \left\{ \begin{bmatrix} M_1 & 0 & \dots & 0 \\ 0 & M_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_k \end{bmatrix} \right\}$$

The subgroup $M_{n_1,\dots,n_k}$ is called **Levi component**. We can further factor this subgroup as

$$M_{n_1,\dots,n_k} = M^1_{n_1,\dots,n_k} \cdot A_{n_1,\dots,n_k}$$

with $A_{n_1,\dots,n_k}$ plays the role of the connected center of $M_{n_1,\dots,n_k}$:

$$A_{n_1,\dots,n_k} = \left\{ \begin{bmatrix} t_1 I_1 & 0 & \dots & 0 \\ 0 & t_2 I_k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_k I_k \end{bmatrix} : t_i > 0 \right\}$$

and

$$M^1_{n_1,\dots,n_k} = \left\{ \begin{bmatrix} \mathfrak{m}'_1 & 0 & \dots & 0 \\ 0 & \mathfrak{m}'_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathfrak{m}'_k \end{bmatrix} \right\},$$

where $\det(\mathfrak{m}'_i) = \pm 1$.

Definition 2.3.3.1. *For a given parabolic subgroup P, the factorization*

$$P = M^1_P \times A_P \times N_P$$

as above is called ***Langlands decomposition****.*

2.3.4 Iwasawa decomposition and P-horospherical decomposition

We introduce two important ways to decompose a matrix in $\mathrm{GL_n}(\mathbb{R})$ into simpler parts. This will be frequently used in the remaining of this thesis.

Proposition 2.3.4.1. *Let*

$$K = \mathrm{O_n}(\mathbb{R}), \quad A = \left\{ \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix} : a_i > 0 \right\}, \quad U = \left\{ \begin{bmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \right\}$$

Then the natural product map

$$p\colon K \times A \times U \to GL_{\mathrm{n}}(\mathbb{R}), \quad p(k,a,n) = kan$$

is a bijection.

Thus, the following decomposition makes sense

Definition 2.3.4.2. *The decomposition of $g = kan$ for $k \in K, a \in A$ and $n \in N$ is called* ***Iwasawa*** *decomposition.*

Using the Langlands decomposition as well as the Iwasawa decomposition $G = KAN = KB$, we obtain the decomposition

$$G = KP = KM_P A_P N_P \cong KM_P \times A_P \times N_P$$

As a consequence, we get

$$X_G = K\backslash G = X_{M_P} \times A_P \times N_P$$

for $X_{M_P} := (K \cap M_P)\backslash M_P$.
This is called the P**-horospherical** decomposition of the lattice space X_G. In this way, we can identify $x \in X_G$ with the triple $(m_P(x), a_P(x), n_P(x))$.

Example 2.3.4.3. *When $n = 2$, the only nontrivial standard parabolic subgroup is the Borel subgroup $P = B$, so the P-horospherical decomposition is*

$$X_G = K\backslash G = A_B \times N_B$$

which is in fact just the Iwasawa decomposition.

Furthermore, if $Q \supset P$ is also a parabolic subgroup, it gives rise to a parabolic subgroup $*P := P \cap M_Q$ of M_Q. The space X_{M_Q} itself has the decomposition

$$X_{M_Q} = X_{*P} \times A(Q)_{*P} \times N(Q)_{*P}$$

where

$$X_{*P} = X_{M_P}, \quad A(Q)_{*P} A_Q = A_P, \quad N(Q)_{*P} N_Q = N_P.$$

This is called **relative** P-horospherical decomposition with respect to Q.

Example 2.3.4.4.

2.3.5 Parabolic sets and parabolic subalgebras

Definition 2.3.5.1. *Given a root system Φ, a* ***parabolic subset*** *$\Phi(P)$ is a subset of Φ that satisfies the following conditions:*

1. *$\Phi(P) \cup -\Phi(P) = \Phi$.*

2. *It is closed, in the sense that, for any two root $\alpha, \beta \in \Phi(P)$ such that $\alpha + \beta$ is a root, then $\alpha + \beta \in \Phi(P)$.*

The parabolic set parametrizes the corresponding parabolic subalgebra with a choice of base Δ of the root system Φ as given in the following theorem

Theorem 2.3.5.2. *Consider a semisimple Lie algebra $\mathfrak{g}$ with the root system Φ and a base Δ. We denote Φ^+ the set of positive roots with respect to this base. There exists a correspondence between parabolic subset $\Phi(P)$ of Φ and subalgebras of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{b}$. The correspondence is given by*

$$\Phi(P) \longleftrightarrow \mathfrak{p} := \mathfrak{b} \oplus \bigoplus_{\alpha \in \Phi(P) \setminus \Phi^+} \mathfrak{g}_\alpha$$

Proof. We refer to [3, Chapter VI, §1.7, Proposition 20] for a proof of this fact. □

Example 2.3.5.3. *We consider the case $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$. Denote $\Delta = \{\alpha, \beta\}$ base for the root system of $\mathfrak{g}$, so that*

$$\Phi^+ = \{\alpha, \beta, \alpha + \beta\}$$

There are 4 parabolic sets, corresponding to 4 parabolic subalgebras given as follows

$$\begin{aligned} \Phi(P) = \Phi^+ &\longleftrightarrow \mathfrak{p} = \mathfrak{b} \\ \Phi(P) = \Phi^+ \cup \{-\beta\} &\longleftrightarrow \mathfrak{p} = \mathfrak{b} \oplus \mathfrak{g}_{-\beta} \\ \Phi(P) = \Phi^+ \cup \{-\alpha\} &\longleftrightarrow \mathfrak{p} = \mathfrak{b} \oplus \mathfrak{g}_{-\alpha} \\ \Phi(P) = \Phi^+ \cup -\Phi^+ &\longleftrightarrow \mathfrak{p} = \mathfrak{g} \end{aligned}$$

2.3.6 On the function H_P

Recall that for the Lie group $\mathrm{SL_n}(\mathbb{R})$, we attach to it a root system $\Phi = \{\alpha_{i,j}\}$ with

$$\Delta = \{\alpha_i \mid i \in I\}, \quad I = \{1, 2, \ldots, n-1\}$$

as the set of fundamental roots. It is a fact that the Weyl group W satisfies

$$W = \langle r_{\alpha_i} : i \in I \rangle$$

For the sake of brevity, we denote $r_{\alpha_i} := r_i$. For a standard parabolic subgroup $P = P_J$ where $J \subset I$, we define the following subset of a fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_n(\mathbb{R})$

$$\mathfrak{a}(P) := \operatorname{span}\{\alpha_i^\vee : i \in J\}$$
$$\mathfrak{a}_P := \{H \in \mathfrak{h} : \langle \alpha_i, H \rangle = 0 \text{ for } i \in J\}$$

By the definition of weights, i.e. $\left\langle \lambda_i, \alpha_j^\vee \right\rangle = \delta_{ij}$, we can see that the subspace $\mathfrak{a}_P$ of $\mathfrak{h}$ contains the subspace

$$\operatorname{span}\{\lambda_i^\vee : i \notin J\}$$

It is also clear that the dimension of $\mathfrak{a}(P)$ is $|J|$ as the set $\{\alpha_i^\vee : i \in J\}$ is a linearly independent set. Assume that

$$\sum_{i \notin J} c_i \lambda_i^\vee = 0$$

Pairing both sides with α_i, we deduce that $c_i = 0$ for all $i \notin J$. In particular, the set $\{\lambda_i^\vee : i \notin J\}$ contains linearly independent vectors and thus spans a subspace of $\mathfrak{a}_P$ of dimension $|I| - |J|$. Therefore

$$\mathfrak{a}_P = \operatorname{span}\{\lambda_i^\vee : i \notin J\}$$

To put it another way, we have a direct sum decomposition

$$\mathfrak{h} = \mathfrak{a}(P) \oplus \mathfrak{a}_P$$

We also write $\lambda_{j,J}^\vee$ for the basis of $\mathfrak{a}(P)$ containing the fundamental weight, namely $\left\langle \alpha_k, \lambda_{j,J}^\vee \right\rangle = \delta_{kj}$ for $j, k \in J$. We have the following lemma

Lemma 2.3.6.1. *[8, Subsection 2.3.5] Let $H \in \mathfrak{g}$ be arbitrary. For $J \subsetneq I$ we have the decomposition*

$$H = H(J) + H_J,$$

where $H(J) \in \mathfrak{a}(P)$ and $H_J \in \mathfrak{a}_P$. If $H = \sum_{i=1}^n p_i \lambda_i^\vee$ then

$$H(J) = \sum_{i \in J} p_i \lambda_{i,J}^\vee$$

Proof. Let $H(J) = \sum_{i \in J} c_i \lambda_{i,J}^\vee$, then for $i \in J$

$$p_j = \langle \alpha_i, H \rangle = \langle \alpha_i, H_J + H(J) \rangle = \sum_{i \in J} \left\langle \alpha_i, c_i \lambda_{i,J}^\vee \right\rangle = c_i$$

Hence $H(J) = \sum_{i \in J} p_i \lambda_{i,J}^\vee$. ∎

We now define the function H_P as follows:

Definition 2.3.6.2. *Assume that for $x \in X$, we have a P-horospherical decomposition*

$$x = (m_P(x), a_P(x), n_P(x))$$

Then we define

$$\begin{aligned} H_P \colon X &\to \mathfrak{a}_P \\ x &\mapsto H_P(x) = \log(a_P(x)), \end{aligned}$$

where log *means we take the logarithm of the diagonal entries of $a_P(x)$.*

This definition is well-defined since $\log(a_P(x))$ is an element of $\mathfrak{a}_P$ by definition. We prove the following lemma

Lemma 2.3.6.3. *Let $P \subset Q$ be standard parabolic subgroups of G, and $x \in K\backslash G$. Then Then we have*

$$H_P(x) = H_Q(x) + H_{*P}(pr_{M_Q}(x))$$

where $pr_{M_Q}(x)$ stands for the natural projection of $x \in X$ on $M_Q/K \cap M_Q$.

Proof. This follows immediately from the observation that

$$A_P = A_Q A(Q)_{*P}$$

and the definition 2.3.6.2 of the function H_P for parabolic subgroup P. □

2.3.7 Some additional notations

We define some important set that will be used in the later part of this thesis. Given a standard parabolic subgroup $P = P_J$, let

$$\Delta(P) := \{\alpha_i^{\cdot} \, i \in J\} \quad \text{and} \quad \Delta_P := \{\alpha_{i|\mathfrak{a}_P} : i \notin J\}$$

Similarly, we define

$$\widehat{\Delta}(P) := \{\lambda_{i|\mathfrak{a}(P)} : i \in J\} \quad \text{and} \quad \widehat{\Delta}_P := \{\lambda_i, i \notin J\}$$

Example 2.3.7.1. *Let $G = \mathrm{SL}_3(\mathbb{R})$. We can choose a base $\Delta = \{\alpha_1, \alpha_2\}$ as before, thus $I = \{1, 2\}$. Assume that $J = \{2\}$. Then $P_1 := P_{I\backslash J}$. We have the following sets*

$$\Delta_{P_1} = \{\alpha_{1|\mathfrak{a}_{P_1}}\}, \quad \widehat{\Delta}_{P_1} = \{\lambda_1\}$$

Remark. *Over $\mathfrak{a}(P)$ we can also define the notion of weight $\lambda_{i,J}$ by the relations*

$$\left\langle \omega_j^J, \alpha_i^\vee \right\rangle = \delta_{ij}$$

for $j \in J$. It is known that we have the relation

$$\omega_i^J = \sum_{i \in J} c_i \alpha_j,$$

where $c_j \geqslant 0$. Thus we can naturally define $\lambda_{i,J}$ on the Cartan subalgebra $\mathfrak{h}$ by letting $\lambda_{i,J}$ to be identically zero on $\mathfrak{a}_P$. For example, the weights in $\mathfrak{h}(J)$ are different from that of $\mathfrak{h}$. In example 2.3.7.1, we have the identification

$$\lambda_{1|\mathfrak{a}(P)} = \frac{1}{2}\alpha_1$$

But they extend differently over $\mathfrak{h}$.

2.3.8 The linear functionals ρ_P and ρ_Q^P

We define an analog for Weyl vector ρ, which is essential in studying the semi-stablility in the later of this thesis.

Definition 2.3.8.1. *Given a standard parabolic subgroup $P = P_J$ of $G = SL_{\mathrm{n}}(\mathbb{R})$ or $G = GL_{\mathrm{n}}(\mathbb{R})$, we define*

$$\rho(P) := \sum_{\omega \in \hat{\Delta}(P)} \omega \quad \text{and} \quad \rho_P := \rho - \rho(P),$$

where ρ is the Weyl vector in section 2.2.4.

If we are given another standard parabolic subgroup $Q \subset P$ then we also set

$$\rho_Q^P := \rho(P) - \rho(Q) = \rho_Q - \rho_P$$

We have a more concrete description for ρ_P as follows

Lemma 2.3.8.2. *[8, Subsection 4.1.3] Fix a standard parabolic subgroup P and define $d_P(\alpha) := \langle \rho(P), \alpha^\vee \rangle$ for each $\alpha \in \Delta_P$. Then $d_P(\alpha)$ is non-positive and*

$$\rho_P = \sum_{\alpha \in \Delta_P} (1 - d_P(\alpha))\lambda_\alpha$$

Proof. We write $\rho_P = \sum_{i \in I} c_i \lambda_i$ and evaluate at α_i. Then

$$c_i = \langle \rho_P, \alpha_i^\vee \rangle = \langle \rho, \alpha_i^\vee \rangle - \langle \rho(P), \alpha_i^\vee \rangle = \begin{cases} 0, & \text{if } i \in J \\ 1 - d_P(\alpha), & \text{otherwise} \end{cases}$$

Since $\rho(P)$ is a non-negative linear combination of roots in $\Delta(P)$, says

$$\rho(P) = \sum_{i \in J} m_i \alpha_i$$

we have

$$d_P(\alpha) = \sum_{i \in J} m_i \left\langle \alpha_i, \alpha^\vee \right\rangle$$

where $\alpha \in \Delta_P$. It is known in Lie theory that the angle between two simple roots are obtuse, hence $\left\langle \alpha_i, \alpha^\vee \right\rangle \leqslant 0$, which implies $d_P(\alpha)$ is non-positive. $\square$

Chapter 3

SEMI-STABLE LATTICE IN HIGHER RANK

In this chapter, we will establish the notion of semi-stable lattice. Heuristically, this is the lattice that achieve all the successive minima at the same time, see [2].
We will provide two different definitions of semi -stable lattice: one is geometric - which follows Grayson's idea of utilizing the canonical plot, and one is purely algebraic, which make use of the maximal standard parabolic subgroups. The toy model will be the moduli space of 2-dimensional lattice, which is essential the upper half plane in the complex field. At the end, we will show that the two definitions coincide.

3.1 Grayson's definition of semi-stability

3.1.1 Grayson's definition

In this section, we introduce the idea of Grayson in defining *semi-stable* lattices, as first introduced in [6]. In particular, he associates every lattices a plot and its convex hull - called *profiles*. An easy observation is that, if $M \subset L$ is a sublattice, then the space $M_{\mathbb{R}} = M \otimes \mathbb{R}$ is a subspace of $L_{\mathbb{R}}$, equipped with the restriction of the positive definite symmetric form Q of L, hence M is also a lattice of rank not exceeding rank of L.

Definition 3.1.1.1 (slope)**.** *The slope of a non-zero lattice L is the number*

$$\mu(L) = \frac{\log \operatorname{vol}(L)}{\dim L}$$

Definition 3.1.1.2. *Suppose we have a lattice L. For any sublattice $M \subset L$, we assign M to a point*

$$\ell(M) = (\dim M, \log \operatorname{vol}(M))$$

in the plane $\mathbb{R}^2$. The collection of all points $\ell(M)$ where M ranges over all sublattices of L is called ***the canonical plot*** *of the lattice L. By convention, we assign the lattice of zero rank to the origin of the plane.*

The following lemma asserts that, for each vertical axis $x = i$, there is a lowest point.

Lemma 3.1.1.3. *Given a lattice L and a number c, there exists only a finite number of sublattices $M \subset L$ such that $\operatorname{vol}(M) < c$.*

Proof. We will prove by induction on the rank of the sublattices.

- For $r = 1$, the collection of all rank 1 sublattices of L is just the set of all vectors in L. So we reduce to show that for any $c > 0$, the set $B(0, c) \cap L$ has finitely many elements. But this follows immediately from the fact that L is a discrete subset of $L_{\mathbb{R}}$.

- Assume that the lemma holds for $r > 1$. Assume that

$$M = \mathbb{Z}m_1 \oplus \ldots \mathbb{Z}m_r$$

is a sublattice of the lattices L of rank n. Consider the wedge product $\bigwedge^r L$, then clearly $m_1 \wedge m_2 \ldots \wedge m_r$ is a vector in the lattice $\bigwedge^r L$. By the previous case, there are finitely many vectors with bounded length inside lattice. So we only need to show that the map

$$M \mapsto \bigwedge^r M$$

is finite to one, then we are done. But this is due to the observation of $\bigwedge^r M$ determines the lattice $L \cap (M \otimes \mathbb{Q})$ and the index of M inside this lattice.

So the canonical plot is bounded below. □

Definition 3.1.1.4. *The boundary polygon of the convex hull of the canonical plot is called* ***profile*** *of the lattice L.*

In theory, we can compute the profile by searching for the shortest vector in each of its exterior product, but this computation is infeasible when the dimension of the lattice grows. Since there are lattices with arbitrarily large volume of any rank smaller than that of L, we add to the side the point $(0, \infty)$ and (n, ∞). The sides of the profile are therefore two vertical lines. The bottom is just the convex polygonal connecting the origin with the point $\ell(L) = (n, \log \operatorname{vol}(L))$, where n is the rank of L.

Definition 3.1.1.5. *If the bottom of the profile contains only two points $(0, 0)$ and $(n, \log \operatorname{vol} L)$, then the lattice L is said to be* ***semi-stable****. Otherwise L is said to be* ***unstable****.*

Below are the picture of two lattices. Visually, a lattice is called **semi-stable** if it satisfies the other equivalent conditions: If M is an arbitrary sublattice of L then $\mu(M) \geqslant \mu(L)$.

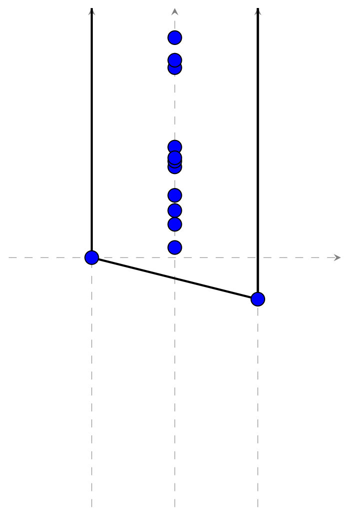


Figure 3.1: A semi-stable lattice

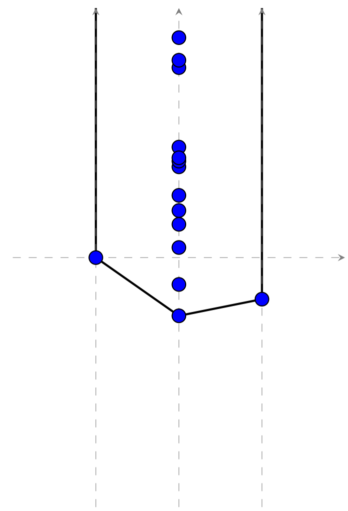


Figure 3.2: An unstable lattice

3.1.2 Canonical filtration

Given a lattice L and a sublattice $M \subset L$, the quotient group L/M have the structure of a lattice. Indeed, consider the exact sequence of lattices

$$0 \to M \to L \to L/M \to 0$$

By tensoring with $\mathbb{R}$ we get a short exact sequence of $\mathbb{R}$-vector subspaces

$$0 \to M_{\mathbb{R}} \to L_{\mathbb{R}} \to (L/M)_{\mathbb{R}} \to 0,$$

which is split. Thus we have the isomorphisms

$$(L/M)_{\mathbb{R}} \cong L_{\mathbb{R}}/M_{\mathbb{R}} \cong M_{\mathbb{R}}^{\perp}$$

Therefore, by restricting of the inner product over $L_{\mathbb{R}}$ to $M_{\mathbb{R}}^{\perp}$, we clearly see that L/M also inherits an inner product. In particular, it is a lattice.

Definition 3.1.2.1. *Given a lattice L containing a sublattice M, then L/M is a lattice. We call this lattice* ***quotient lattice****.*

Lemma 3.1.2.2. *If L is a lattice and $M \subset L$ is a sublattice, we have*

$$\mathrm{vol}(L) = \mathrm{vol}(M) \cdot \mathrm{vol}(L/M)$$

and if N is any sublattice of L that satisfies $N + M = L$ then

$$\mathrm{vol}(N) \geqslant \mathrm{vol}(L/M)$$

Proof. Assume that $\{m_i\}$ is a basis for the lattice M and $\{e_i\}$ be an orthonormal basis for the vector space $M_{\mathbb{R}}$. Since M is a sublattice of L, we can extend the basis $\{m_i\}$ to get a basis $\{m_i\} \cup \{n_j\}$ for the lattice L. Similarly, we can extend $\{e_i\}$ to get an orthonormal basis $\{e_i\} \cup \{f_j\}$ for the vector space $L_{\mathbb{R}}$. In particular, we would have $\langle m_i, f_j \rangle = 0$ for all i, j. By definition, we have

$$\begin{aligned} vol(L) &= \det \begin{bmatrix} \langle m_i, e_i \rangle & \langle n_j, e_i \rangle \\ \langle m_i, f_J \rangle & \langle n_j, f_j \rangle \end{bmatrix} \\ &= \det \begin{bmatrix} \langle m_i, e_i \rangle & \langle n_j, e_i \rangle \\ 0 & \langle n_j, f_j \rangle \end{bmatrix} \\ &= \mathrm{vol}(M) \cdot \mathrm{vol}(L/M) \end{aligned}$$

Hence we are done. The latter inequality follows from the fact that the volume decrease under the orthogonal projection and the observation that $N_{\mathbb{R}} \supset (L/M)_{\mathbb{R}}$. □

In the canonical plot, the import of lemma 3.1.2.2 is that the slope of the quotient lattice L/M appears as the slope of the line segment connecting the points corresponding to the sublattice M and the lattice L. This is due to the geometry fact that

$$(\mathrm{rank}(M), \log(\mathrm{vol}(M)) + (\mathrm{rank}(L/M), \log \mathrm{vol}(L/M))) = (\mathrm{rank}(L), \log \mathrm{vol}(L))$$

Given two sublattices $M_1, M_2 \subset L$, if we apply lemma 3.1.2.2 to the sublattices $M_1/M_1 \cap M_2$

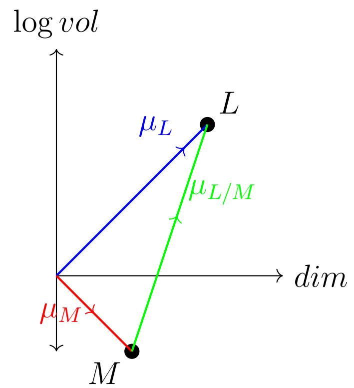


Figure 3.3: Geometric meaning of slope in canonical plot

and $M_2/M_1 \cap M_2$ in $M_1 + M_2/M_1 \cap M_2$, we get

$$\mathrm{vol}(M_1/M_1 \cap M_2)\,\mathrm{vol}(M_2/M_1 \cap M_2) \geqslant \mathrm{vol}(M_1 + M_2/M_1 \cap M_2)$$

or equivalently

Lemma 3.1.2.3.

$$\operatorname{vol}(M_1 + M_2)\operatorname{vol}(M_1 \cap M_2) \leqslant \operatorname{vol}(M_1)\operatorname{vol}(M_2)$$

Grayson used the logarithm to express the above inequality in additive terms:

Proposition 3.1.2.4. *Suppose M_1, M_2 are sublattices of L then*

$$\log\operatorname{vol}(M_1) + \log(\operatorname{vol}(M_2)) \geqslant \log(\operatorname{vol}(M_1 + M_2)) + \log\operatorname{vol}(M_1 \cap M_2)$$

Proof. It follows immediately from the fact that log is an increasing function over $(0, \infty)$. □

Grayson called this parallelogram rule, as the geometric meaning of proposition 3.1.2.4 is illustrated in the figure 3.4. The **vertices** of the profile are the extremal/lowest points of the

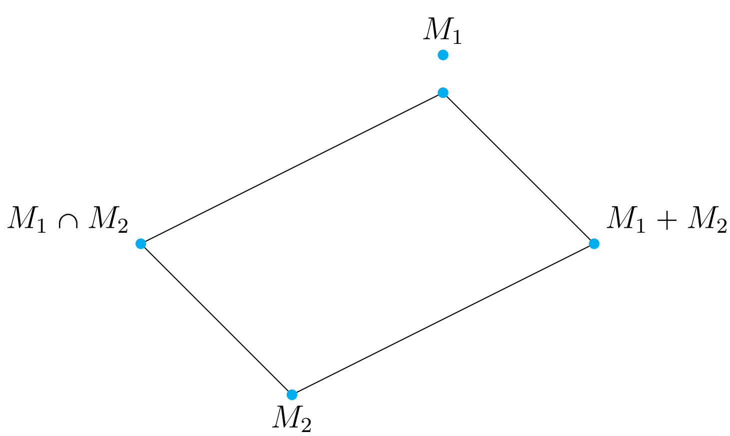


Figure 3.4: Grason's parallelogram rule

plot. Clearly the two points $(0,0)$ and $(n, \log\operatorname{vol}(L))$ are vertices of the plot of the lattices L with $\operatorname{rank}(L) = n$. The following lemma states the situation for other vertices of the profile

Lemma 3.1.2.5. *Suppose that M_* is a lattice with with the point $\ell(M_*)$ as a vertex of the profile. If M is any other sublattice such that $\ell(M)$ is also a vertex of the profile, then we must have either $M \subset M_*$ or $M_* \subset M$.*

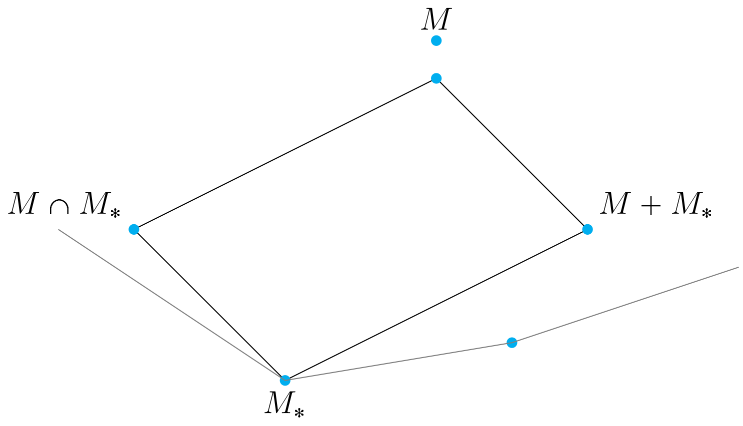


Figure 3.5: Illustration for lemma 3.1.2.5

Proof. We refer to figure 3.5. Clearly we can see that the point $\ell(M \cap M_*)$ lies somewhere on the left of both $\ell(M)$ and $\ell(M_*)$. Similarly, the point $\ell(M + M_*)$ lies somewhere inclusively to the right of $\ell(M)$ and $\ell(M_*)$. By the paralellogram rule, the point $ell(M)$ will therefore be above the boundary points $\ell(M \cap M_*)$, $\ell(M_*)$ and $\ell(M + M_*)$. If M is also a vertex of the profile, the parallelogram must degenerate. In particular we either have $M \cap_M * = M$, in which $M \subset M_*$, or $M + M_* = M$, in which $M_* \subset M$. □

An immediate consequence is that

Theorem 3.1.2.6. *The vertices of the profile of a lattice L are represented by unique sublattices, and they form a chain.*

For a given lattice L, we call the chain of sublattices in theorem 3.1.2.6 the **canonical filtration** of L. Assume that the canonical filtration for L is

$$\mathcal{F} : 0 = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_{k-1} \subset L_k = L$$

then it can be seen from the diagram that

1. L_i/L_{i-1} is semi-stable for all $1 \leqslant i \leqslant k$.
2. $\mu(L_i/L_{i-1}) \leqslant \mu(L_{i+1}/L_i)$ for all $1 \leqslant i \leqslant k-1$.

These two conditions is also sufficent for a chain to be a canonical filtration

Theorem 3.1.2.7. *Suppose*

$$\mathcal{F} : 0 = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_{k-1} \subset L_k = L$$

is a chain of lattices such that L_i/L_{i-1} is semi-stable and the slope L_i/L_{i-1} is not larger than the slope of L_{i+1}/L_i. Then this chain is the canonical filtration.

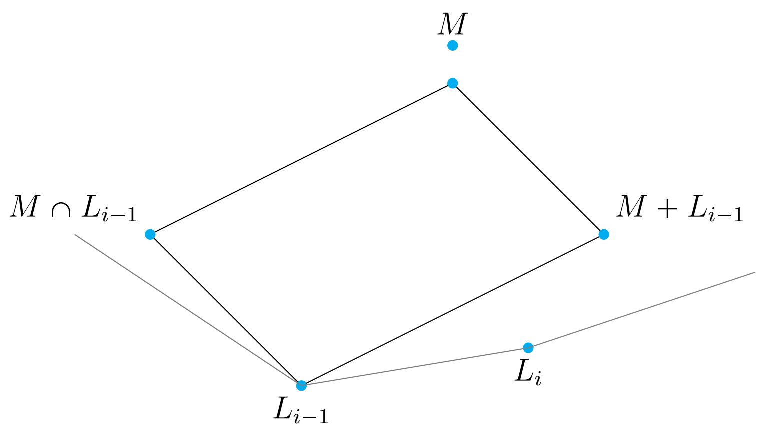


Figure 3.6: Canonical filtration

Proof. Suppose M to be any other sublattice of L. We want to know that $\ell(M)$ lies above the plot P of the $\ell(L_i)$. We prove by induction on the index i that if $M \subset L_i$, then $\ell(M)$ lies above the plot P.

- If $i = 1$ then $M \subset L_1$. Since $L_1$is semi-stable, we must have $\ell(M)$ lies above the line connecting $(0, 0)$ and $\ell(L_1)$. Hence $\ell(M)$ lies above the plot P.

- Suppose that $M \subset L_i$ for $i > 1$. Then $M + L_{i-1} \subset L_i$ contains L_{i-1}. Therefore, it corresponds to the point lies above the line connecting $\ell(L_{i-1})$ and $\ell(L_i)$, thus lies above P. By induction, the fact that $(M \cap L_{i-1}) \subset L_{i-1}$ implies $\ell(M \cap L_{i-1})$ lies above the plot P. Using the parallelogram rule, the point $\ell(M)$ must then also lie above the plot P.

Thus any chain satisfies the conditions given in the theorem is a canonical filtration. $\square$

3.2 ρ- definition of semi-stability

3.2.1 ρ-definition

We are now ready to define the ρ-definition of semi-stable lattice. Recall that we define the space of lattices of rank n by $X_n := K\backslash \mathrm{GL}_n(\mathbb{R})$, where K is the orthogonal subgroup.

Definition 3.2.1.1 (ρ-definition)**.** *Let $x \in X_n$ be an arbitrary lattice, then the lattice x is called ρ-**semi-stable** if and only if its degree of instability* $\deg_{inst}(x) \geqslant 0$*, where*

$$\deg_{inst}(x) := \min_{Q \in ParSt, \gamma \in GL(\mathbb{Q})/Q_i(\mathbb{Q})} \langle \rho_Q, H_Q(x\gamma) \rangle$$

3.2.2 Some basis properties

We first have an elementary lemma

Lemma 3.2.2.1. *Given $x \in X_n$, then the following are equivalent*

1. $\deg_{inst}(x) \geqslant 0$.
2. *For every stsandard parabolic subgroup $P \subset G$ and $\omega \in \widehat{\Delta}_P$ we have*

$$\langle \omega, H_P(x\delta) \rangle \geqslant 0$$

for each $\delta \in G(\mathbb{Q})/Q_i(\mathbb{Q})$.

3. *For every maximal parabolic subgroup $Q \subset G$ and $\omega \in \widehat{\Delta}_Q$ we have*

$$\langle \omega, H_Q(x\delta) \rangle \geqslant 0$$

for each $\delta \in G(\mathbb{Q})/Q_i(\mathbb{Q})$.

Proof.
First we prove $1 \Rightarrow 3$: This follows immediately as $\rho_Q = c\omega$ for some positive number c and $\omega \in \widehat{\Delta}_Q$, by lemma 2.3.8.2. In particular

$$\langle \omega, H_Q(x\delta) \rangle = \frac{1}{c} \langle \rho_Q, H_Q(x\delta) \rangle \geqslant \frac{1}{c} \deg_{\mathrm{inst}}(x) \geqslant 0$$

For $2 \Rightarrow 3$: We can choose a maximal standard parabolic Q such that $P \subset Q \subset G$. But then

$$\langle \omega, H_P(x\delta) \rangle = \langle \omega, H_Q(x\delta) \rangle \geqslant 0$$

Finally, we have $2 \Rightarrow 1$ as ρ_P is a positive linear combination of elements contained in $\widehat{\Delta}_P$, again by lemma 2.3.8.2 □

3.2.3 Check for semi-stability

From lemma 3.2.2.1, to check whether $x \in G$ is semi-stable, we just need to verify whether

$$\langle \omega, H_Q(x\delta) \rangle \geqslant 0.$$

A simple observation is that - a lattice x is semi - stable if for all maximal standard parabolic subgroups Q_i, we have

$$\min_{\gamma \in \mathrm{GL}_n(\mathbb{Q})/Q_i(\mathbb{Q})} \langle \rho_Q, H_Q(x\gamma) \rangle \geqslant 0$$

From lemma 2.3.6.1

$$H_B = H_Q + H(B)$$

where H_Q is a scalar multiple of $\lambda_i^\vee$ for $Q = Q_i$ and $H(B)$ is a linear combination of $\alpha_j^\vee$ for $j \neq i$. On the other hand, since Q is a maximal standard parabolic subgroup of G, ρ_Q is proportional to λ_i. Thus

$$\langle \rho_Q, H(B)(x\gamma) \rangle = 0.$$

In particular, we can replace H_Q by H_B in verifying the semi-stability. This implies that, if

$$x\gamma = kan, \quad k \in K, a \in A, n \in N,$$

as in Iwasawa decomposition, then $H_B(x) = H$ where $H = \exp(a)$. In particular, if

$$a = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}$$

then

$$\langle \rho_{Q_i}, H_B(x\gamma) \rangle = \frac{n}{2} \log(a_1 a_2 \dots a_i) \tag{3.1}$$

Thus, to check for the semi-stability of a lattice x, we just need to look at the A-coordinate of $x\gamma$ for every $\gamma \in G(\mathbb{Q})/Q_i(\mathbb{Q})$, and verify whether the following system holds

$$\begin{cases} a_1 \geqslant 1 \\ a_1 a_2 \geqslant 1 \\ \dots \\ a_1 a_2 \dots a_n \geqslant 1 \end{cases}$$

3.3 Canonical pair

Consider a pair (P, δ) of a standard parabolic subgroup $P \subsetneq G$ and $\delta \in G(\mathbb{Q})/P(\mathbb{Q})$. Such a pair is called **destabilizing** for x if

$$\langle \rho_P, H_P(x\delta) \rangle = \deg_{\text{inst}}(x)$$

Such a pair is called **extremal** for x if for any standard parabolic $Q \subset P$ such that

$$\langle \rho_Q, H_Q(x\delta) \rangle = \langle \rho_P, H_P(x\delta) \rangle$$

then $Q = P$.

Definition 3.3.0.1. *A pair (P, δ) that is both destabilizing and extremal for x is called a* ***canonical pair*** *for x.*

A canonical pair for x, if exists, will be denoted by $\mathbf{cp}(\mathrm{x}) := (P, \delta)$. A priori, it is not clear whether $\mathbf{cp}(\mathrm{x})$ exists or not. We will show that it is in fact equivalent to the notion of canonical filtration introduced in the previous sections, and deduce that $\mathbf{cp}(\mathrm{x})$ must exist and it is unique.
We obtain the following lemma as a consequence of the definition of canonical pair

Lemma 3.3.0.2. *Let $x \in X_n$ and (P, δ) be a pair with $P \subsetneq G$ be standard parabolic subgroup and $\delta \in G(\mathbb{Q})/P(\mathbb{Q})$. Then*

1. *If (P, δ) is destabilizing for x, then $\deg^P_{inst}(x\delta) \geqslant 0$.*
2. *If (P, δ) is extremal for x, then $\langle \alpha, H_P(x\delta) \rangle \leqslant 0$ for any $\alpha \in \Delta_P$.*

Proof. For the first part, let $Q \subset P$ be any standard parabolic, then

$$\rho_Q = \rho_P + \rho_Q^P$$

Thus, for any $\eta \in P(\mathbb{Q})/Q(\mathbb{Q})$

$$\begin{aligned}\left\langle \rho_Q^P, H_Q(x\gamma\eta) \right\rangle &= \langle \rho_Q, H_Q(x\gamma\eta) \rangle - \langle \rho_P, H_P(x\gamma\eta) \rangle \\ &= \langle \rho_Q, H_Q(x\gamma\eta) \rangle - \deg_{inst}(x) \geqslant 0\end{aligned}$$

For the second part, we can pick a standard parabolic $Q \subset G$ containing P such that P is maximal in Q. In particular, we have $\Delta_P^Q = \{\alpha\}$. Then

$$\begin{aligned}\left\langle \rho_P^Q, H_P(x\delta) \right\rangle &= \langle \rho_P, H_P(x\gamma) \rangle - \langle \rho_Q, H_Q(x\gamma\eta) \rangle \\ &= \deg_{inst}(x) - \langle \rho_Q, H_Q(x\gamma\eta) \rangle < 0\end{aligned}$$

Since ρ_P^Q and α are proportional by a positive number, the result follows immediately. $\square$

Chapter 4

Equivalence between Grayson's semistability and ρ-semistability

4.1 The equivalence between definitions of semi-stable lattices

Recall that we have two distinct definitions of semi-stability, as defined in definition 3.1.1.5 and definition 3.2.1.1. The following theorem asserts their equivalence:

Theorem 4.1.0.1. *Let $x \in X_n = K\backslash GL_n(\mathbb{R})$ and L_x be the corresponding Euclidean lattice. Then the following conditions on x are equivalent*

1. *The bottom of the profile of L_x is a line connect solely two points: the origin and $(n, 0)$.*

2. *The degree of instability of x is nonnegative, namely, $\deg_{inst}(x) \geqslant 0$.*

4.1.1 A useful lemma

Lemma 4.1.1.1. *There exists a bijection*

$$GL_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap GL_n(\mathbb{Z})) \longleftrightarrow \{ \textit{ sublattices of rank } i \textit{ of } \mathbb{Z}^n\}$$

Proof. We define the map from the collection of sublattices of rank i to the cosets space as follows: For any sublattice $M \subset \mathbb{Z}^n$, there exists a basis of M, denoted by

$$\mathfrak{C} = \{v_1, v_2, \ldots, v_i\}$$

we can extend this basis to get a basis of $\mathbb{Z}^n$

$$\mathfrak{C}_1 = \{v_1, v_2, \ldots, v_n\}$$

In $\mathbb{Z}^n$ we have the standard basis $\mathfrak{B} = \{e_1, e_2, \ldots, e_n\}$. Clearly there exists a map $\gamma \in \mathrm{GL_n}(\mathbb{Z})$ such that

$$\gamma \cdot e_k = v_k \quad \forall k = 1, 2, \ldots, n$$

So we define the map

$$\begin{aligned} \varphi\colon \{\text{sublattices of rank } i \text{ of } \mathbb{Z}^n\} &\to \mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})) \\ M &\mapsto [\gamma] \end{aligned}$$

where $[\gamma]$ denoted the equivalent class of γ in the quotient space. This is a well-defined map. Indeed, assume that we have a different basis

$$\mathfrak{C}_2 = \{w_1, \ldots, w_n\}$$

where

$$\mathfrak{C}' = \{w_1, \ldots, w_i\}$$

is a basis for sublattice M. As above, there also exists $\gamma' \in \mathrm{GL_n}(\mathbb{Z})$ such that

$$\gamma' e_k = w_k \quad \forall k$$

Since $\mathfrak{C}$ and $\mathfrak{C}'$ are basis for a sublattice of $\mathbb{Z}^n$ of rank i, we can find an element $g \in \mathrm{GL}_i(\mathbb{Z})$ such that

$$g v_k = w_k \quad \forall k \leqslant i$$

Similarly, there also exists $h \in \mathrm{GL}_{n-i}(\mathbb{Z})$ such that

$$h v_k = w_k \quad \forall k \geqslant i,$$

as $\mathbb{Z}^n/M$ is the quotient lattice, hence a lattice. In particular, we can immediately see that

$$\gamma' \cdot (\gamma)^{-1} v_k = \gamma' e_k = w_k \quad \forall k$$

Thus

$$\gamma' \cdot (\gamma)^{-1} = \begin{bmatrix} g & * \\ 0 & h \end{bmatrix} \in Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$$

This is exactly the desired result. □

4.1.2 Proof of the main theorem

Now we give a proof for theorem 4.1.0.1

Proof.
If we can prove there is a correspondence between $\gamma \in \mathrm{GL}_n(\mathbb{Q})/Q_i(\mathbb{Q})$ and a sublattice of rank i of x, then we are done. We first need a slight reduction - we identify the quotient $\mathrm{GL}_n(\mathbb{Q})/Q_i(\mathbb{Q})$ with the quotient $\mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z}))$. Now let x be an arbitrary lattice of rank n. Lemma 4.1.1.1 generalizes in the obvious way for lattice $x = g\mathbb{Z}^n$ for some $g \in \mathrm{GL}_n(\mathbb{R})$. Indeed, we just define the map

$$\phi_g \colon \{\text{sublattices of rank } i \text{ of } x\mathbb{Z}^n\} \to \mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z}))$$

$$gM = g\bigoplus_{k=1}^{i} \mathbb{Z}v_i \mapsto [\gamma]$$

where $\gamma e_k = v_k$ in $\mathbb{Z}^n$ for all $k \leqslant i$ and $\phi_g^{-1}([\gamma]) = g\bigoplus_{k=1}^{i} \mathbb{Z}(\gamma \cdot e_i)$.
Now apply lemma 3.2.2.1, we have that $\deg_{\text{inst}}(x) \geqslant 0$ if and only if, for every maximal parabolic subgroup $Q \subset G$ and $\omega \in \widehat{\Delta}_Q$ we have

$$\langle \omega, H_Q(x\delta) \rangle \geqslant 0$$

for each $\delta \in G(\mathbb{Q})/Q_i(\mathbb{Q})$. Let $M = \phi_g([\delta])$. Note that for $\omega \in \widehat{\Delta}_Q$, we have

$$\langle \omega, H_Q(x\delta) \rangle = \langle \omega, H_B(x\delta) \rangle = \frac{n}{2}\log(a_1 \cdots a_i)$$

by equation 3.1. Hence

$$\langle \omega, H_B(x\delta) \rangle = c\langle \rho_{Q_i}, H_B(x\gamma) \rangle = \frac{n}{2}\log(a_1 a_2 \ldots a_i) = c\,\mathrm{vol}(M)$$

for some positive integer c. Using the remark as in section 3.2.3, we have the desired conclusion. □

4.2 Canonical pair and Canonical filtration

4.2.1 Equivalence between Canonical pair and canonical filtration

We can further prove that, the equivalence between different notions of semistability comes from the equivalence between the notion of canonical pair and the bottom of the profile of a lattice - the canonical filtration. The main result is the following proposition

Proposition 4.2.1.1. *Given an $x \in X_n = K\backslash G$. Then (P, δ) is the canonical pair for x if and only if P is the stabilizer of the canonical filtration for the lattice $L_x = x\mathbb{Z}^n = x\delta\mathbb{Z}^n$.*

Recall that, from theorem 3.1.2.7 and lemma 3.3.0.2, both canonical pair and canonical filtration are characterized by two conditions. We will show that these two conditions are equivalent.

4.2.2 Equivalence between chain condition and condition 2

Throughout this section, we fix an $x \in X_n$ and the corresponding lattice $L_x = x\mathbb{Z}^n$.

Lemma 4.2.2.1. *The the following conditions are equivalent for $x \in X_n$.*

1. *$\langle \alpha, H_P(x\delta) \rangle < 0$ for any $\alpha \in \Delta_P$ and for some $\delta \in G_{\mathbb{Q}}/P_{\mathbb{Q}}$.*

2. *L_x has a chain of lattices*

$$\mathcal{F} : 0 = L_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{k-1} \subset M_k = L_x$$

such that $\mu(M_i/M_{i-1}) < \mu(M_{i+1}/M_i)$ for all i.

Remark. *We also attach to the flag*

$$\mathcal{F} \otimes \mathbb{R} \colon L_0 \subset M_1 \otimes \mathbb{R} \subset M_2 \otimes \mathbb{R} \subset \cdots \subset M_{k-1} \otimes \mathbb{R} \subset M_k \otimes \mathbb{R} = \mathbb{R}^n$$

a rational standard parabolic subgroup $P_{\mathbb{Q}}$.

Proof. We first prove (1) implies (2). Assume that (P, δ) is the canonical pair for x and we further assume that the standard parabolic P in the canonical pair is of type $(n_1, \ldots, n_k)$ as defined in subsection 2.3.1, and set

$$d_i = n_1 + n_2 + \ldots + n_i$$

In particular, the A_P-coordinate of $x\delta$ in the P-horospherical decomposition

$$a_P(x\delta) = \begin{bmatrix} a_1 I_{n_1} & 0 & \ldots & 0 \\ 0 & a_2 I_{n_2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & a_k I_{n_k} \end{bmatrix} \tag{4.1}$$

Then from the chain of standard lattice

$$0 \subset \mathbb{Z}^{d_1} \subset \mathbb{Z}^{d_2} \subset \cdots \subset \mathbb{Z}^{d_k}$$

we obtain a chain of sublattices of L_x as follows

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_k$$

where

$$M_i := \bigoplus_{m=1}^{d_i} \mathbb{Z} x\delta \cdot e_m$$

.

Note that for $M_k = x\delta\mathbb{Z}^{d_k} = x\mathbb{Z}^n = L_x$.
From the P-horospherical coordinate 2.3.4

$$x\delta = km_P(x\delta)a_P(x\delta)n_P(x\delta),$$

where $k \in \mathrm{SO_n}(\mathbb{R})$, $n_P(x\delta) \in N_P$, $m_p(x\delta) \in M_P$ as in section 2.3.4 and $a_P(x)$ is defined as in equation (4.1)

$$\mathrm{vol}(M_i/M_{i-1}) = \left| km_P(x)a_P(x)n_P(x) \bigwedge_{j=d_i+1}^{d_{i+1}} e_j \right| = \left| a_P(x) \bigwedge_{j=d_i+1}^{d_{i+1}} e_j \right| = a_i^{n_i}.$$

The second equality is because $k \in \mathrm{SO_n}(\mathbb{R})$ preserves the length, $n_p(x\delta)$ stabilizes the flag $0 \subset \mathbb{Z}^{d_1} \subset \mathbb{Z}^{d_2} \subset \cdots \subset \mathbb{Z}^{d_k}$ and M_P is semisimple, so the $m_p(x\delta)$ acts trivially on the length function. Thus,

$$\begin{aligned}\mu(M_i/M_{i-1}) - \mu(M_{i+1}/M_i) &= \frac{\log \mathrm{vol}(M_i/M_{i-1})}{n_i} - \frac{\log \mathrm{vol}(M_{i+1}/M_i)}{n_{i+1}} \\ &= \log(a_i) - \log(a_{i+1})\end{aligned}$$

Claim:

$$\begin{aligned}H_P(x\delta) &= m_1\lambda_{d_1}^\vee + m_2\lambda_{d_2}^\vee + \ldots + m_k\lambda_{d_k}^\vee \textit{where} \\ m_i &= \log(a_i) - \log(a_{i+1})\end{aligned} \tag{4.2}$$

If we can prove this then

$$m_i = \langle \alpha_{d_i}, H_P(x\delta) \rangle < 0 \Leftrightarrow \mu(M_i/M_{i-1}) - \mu(M_{i+1}/M_i) < 0$$

By the definition of (H_P-function), we know that

$$H_P(x\delta) = \log a_P(x\delta) = \begin{bmatrix} \log(a_1)I_{n_1} & 0 & \ldots & 0 \\ 0 & \log(a_2)I_{n_2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \log(a_k)I_{n_k} \end{bmatrix}$$

The set $(\widehat{\Delta}_P)^\vee = \{\lambda_{d_i}\}$ is a basis for $\mathfrak{a}_P$, which clearly yields the expression (4.2). Thus we just need to evaluate the value for m_i, but

$$m_i = \langle \alpha_{d_i}, H_P(x\delta) \rangle = \log(a_i) - \log(a_{i+1})$$

This is the desired result. □

Remark. *From the proof the lemma 4.2.2.1, we explicitly construct an equivalence*

$$xG_{\mathbb{Q}}/P_{\mathbb{Q}} \longleftrightarrow \{ \textit{ flag of sublattices of } x\mathbb{Z}^n \textit{ of the same type of partition type of } P\}$$

4.2.3 P-semistability and M_P-semistability

Before proving the remaining equivalence, we need the following lemma

Lemma 4.2.3.1. *Let P be a standard parabolic subgroup of $GL_{\mathrm{n}}(\mathbb{R})$. For $x \in X = K\backslash GL_{\mathrm{n}}(\mathbb{R})$, we let $m := pr_{M_P}(x)$, the projection of x on the space $X_{M_P} := M_P/M_P \cap K$. Then*

$$\deg^P_{inst}(x) = \deg^{M_P}_{inst}(m).$$

In particular, x is P-semi-stable if and only if its projection m is M_P-semi-stable.

Proof. The identity follows essentially from the definition:

$$\deg^P_{inst}(x) = \min_{R\subset P, \delta\in P_{\mathbb{Q}}/R_{\mathbb{Q}}} \left\langle \rho^P_R, H_R(x\delta)\right\rangle \quad \text{and} \quad \deg^{M_P}_{inst}(m) = \min_{*R\subset M_P, \overline{\delta}\in M_{P,\mathbb{Q}}/R_{\mathbb{Q}}} \left\langle \rho^P_R, H_R(m\overline{\delta})\right\rangle$$

where R ranges over all standard parabolic subgroup of P. Note that $*R = R \cap M_P$ is also a standard parabolic subgroup of M_P. Moreover, the map

$$\begin{aligned} \mathrm{pr}_{M_P} \colon P_{\mathbb{Q}}/R_{\mathbb{Q}} &\to M_{P,\mathbb{Q}}/ * R_{\mathbb{Q}} \\ \delta &\mapsto \overline{\delta} \end{aligned}$$

is a bijection. This implies that one takes the minimum over the same set in evaluating the degree of P-instability and M_P-instability. By 2.3.6.3, For any $R \subset P$ a standard parabolic subgroup, we have

$$H_R(x\delta) = H_P(x\delta) + H_{*R}(\mathrm{pr}_{M_P}(x\delta)) = H_P(x\delta) + H_{*R}(m\overline{\delta})$$

Note that, by definition, we have

$$\rho^P_R = \rho(P) - \rho(R)$$

which vanishing identically on $\mathfrak{a}_P$. Thus

$$\begin{aligned} \deg^P_{inst}(x) &= \min_{R\subset P, \delta\in P_{\mathbb{Q}}/R_{\mathbb{Q}}} \left\langle \rho^P_R, H_R(x\delta)\right\rangle \\ &= \min_{R\subset P, \delta\in P_{\mathbb{Q}}/R_{\mathbb{Q}}} \left\langle \rho^P_R, H_P(x) + H_{*R}(m\overline{\delta})\right\rangle \\ &= \min_{*R\subset M_P, \overline{\delta}\in M_{P,\mathbb{Q}}/R_{\mathbb{Q}}} \left\langle \rho^P_R, H_P(x) + H_{*R}(m\overline{\delta})\right\rangle \\ &= \deg^{M_P}_{inst}(m) \end{aligned}$$

This is what we want. $\square$

4.2.4 Equivalence between increasing chain condition and condition 1

Lemma 4.2.4.1. *Fixed the notations as in lemma 4.2.2.1, the following conditions are equivalent*

1. *M_i/M_{i-1} is semi-stable for all i, where the flag*

$$\mathcal{F} : 0 \subset M_1 \subset M_2 \subset \cdots \subset M_{k-1} \subset M_k = L_x$$

   corresponds to $x\delta$.

2. *$x\delta$ is $P-$semi-stable for some $\delta \in G_{\mathbb{Q}}/P_{\mathbb{Q}}$. It is also clear*

Proof. By lemma 4.2.3.1, the condition $x\delta$ is $P-$semi-stable is equivalent to $m\overline{\delta}$ is M_P-semi-stable. Observe that, if P is of type $(n_1, \dots, n_k)$, then

$$m\overline{\delta} = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & A_k \end{bmatrix}$$

where A_i is a matrix of size $n_i \times n_i$ with determinant ± 1. We can then construct a flag of lattices

$$\mathcal{F}' : 0 \subset M_1' \subset M_2' \ldots \subset M_k' = L_x'$$

where L_x' is just the normalization of lattice L_x, i.e. we rescale L_x to get a similar lattice L_x' of volume 1. The sublattice M_i' is constructed by setting

$$M_i' := \bigoplus_{m=1}^{d_i} \mathbb{Z} m\overline{\delta} \cdot e_m$$

It is clear that $M_i'/M_{i-1}' \cong A_i \mathbb{Z}^{n_i}$. It is also clear that, $m\overline{\delta}$ is semi-stable if and only if each quotient M_i'/M_{i-1} is semi-stable. Rescaling does not change the semistability, so we can rescale the flag $\mathcal{F}'$ by a suitable factor to get the desired flag $\mathcal{F}$

$$\mathcal{F} : 0 \subset M_1 \subset M_2 \subset \cdots \subset M_{k-1} \subset M_k = L_x$$

where M_i/M_{i-1} is semi-stable for all i.

4.2.5 Proof of proposition 4.2.1.1

By [8, Subsection 4.3.4], the two conditions in lemma 3.3.0.2 are sufficient to guarantee that $\mathrm{cp}(x) = (P, \delta)$. Thus, by lemma 4.2.4.1 and lemma 4.2.2.1, $\mathrm{cp}(x) = (P, \delta)$ if and only if there exists a flag of sublattices

$$\mathcal{F}\colon 0 = M_0 \subset M_1 \subset \ldots \subset M_k = L_x$$

stabilized by P, such that

- $\mu(M_i/M_{i-1}) < \mu(M_{i+1}/M_i)$.
- M_i/M_{i-1} is semi-stable for all i.

By theorem 3.1.2.7, $\mathcal{F}$ satisfies the above conditions if and only if $\mathcal{F}$ is the canonical filtration for L_x. □

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