Chapter 1

$\mathbf{SL}_2(\mathbb{R})$

In this chapter, I will give an exposition on the structure of $\mathrm{SL}_2(\mathbb{R})$ as the spaces of lattice, this space plays the role of a toy model before exploring the space of lattice in the higher rank. The exposition follows the paper [**?**] and [**?**] closely.

1.1 $\mathbf{SL}_2(\mathbb{R})$ and its action on the upper half plane $\mathfrak{H}$

A priori, the upper half plane

$$\mathfrak{H} = \{z : \Im z > 0\} \subset \mathbb{C}$$

has no group structure on its. However, we will show below that it can identify topologically with the space with the space of cosets $\mathrm{SO}_2(\mathbb{R})\, 2\backslash \mathrm{SL}_2(\mathbb{R})$, and thus we can study the spaces $\mathfrak{H}$ via the spac of lattices $\mathrm{SO}_2(\mathbb{R}) \backslash \mathrm{SL}_2(\mathbb{R})$. We define the action of $G = \mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{H}$ as follows

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ (z) = \frac{az+b}{cz+d}$$

Proposition 1.1.1. *The group $SL_2(\mathbb{R})$ stabilizes $\mathfrak{H}$ and acts transitively on it. In particular,*

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} (i) = x + iy \quad (\textit{for } x \in \mathbb{R},\ y > 0)$$

Further, for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ *and* $z \in \mathfrak{H}$,

$$\Im\, g(z) = \frac{\Im\, z}{|cz+d|^2}.$$

Proof. The first formula is clear. The second formula would imply that the upper half-plane is stabilized. Compute directly:

$$2i \cdot \Im \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) \right) = \frac{az+b}{cz+d} - \frac{d\overline{z}+b}{c\overline{z}+d} = \frac{(az+b)(c\overline{z}+d) - (a\overline{z}+b)(cz+d)}{|cz+d|^2}$$

$$= \frac{adz - bc\overline{z} - bcz + ad\overline{z}}{|cz+d|^2} = \frac{z - \overline{z}}{|cz+d|^2}$$

since $ad - bc = 1$. $\square$

The point $z = i$ is special, in the sense that its stability group is the orthogonal group $K = \mathrm{SO}_2(\mathbb{R})$. Indeed, for any $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ we have that

$$g \circ i = i \Leftrightarrow \frac{ai+b}{ci+d} = i \Leftrightarrow a = d \text{ and } b = -c$$

Combinining with the fact that $ad - bc = 1$, we must have $a^2 + b^2 = 1$. This implies that there is a θ such that $a = \cos\theta$ and $b = \sin\theta$. Since G acts on $\mathfrak{H}$ transitively, we know from group theory that there is a bijection between the collection of cosets of $\mathrm{Stab}(i)$ in G and the orbits of i. In particular

Proposition 1.1.2. *We have an isomorphism of $SL_2(\mathbb{R})$-spaces*

$$\mathrm{SO}_2(\mathbb{R}) \setminus SL_2(\mathbb{R}) \cong \mathfrak{H} \quad \textit{via} \quad SO(2)g \to g^{-1}(i)$$

That is, the map respects the action of $SL_2(\mathbb{R})$, in the sense that

$$(\mathrm{SO}_2(\mathbb{R})\, g) \cdot h \longrightarrow h^{-1}(g^{-1}i)$$

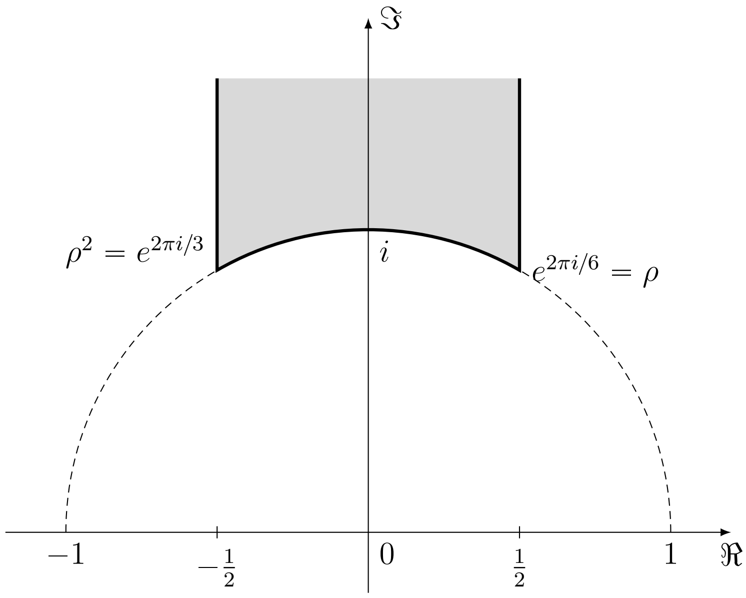
Proof. This is because of *associativity*:

$$(\mathrm{SO}_2(\mathbb{R})\, g) \cdot h = (\mathrm{SO}_2(\mathbb{R})) \cdot (gh) \longrightarrow (gh)^{-1}(i) = h^{-1}(g^{-1}(i))$$

giving the result. □

1.2 Fundamental domain for $\Gamma = SL_2(\mathbb{Z})$ on $\mathfrak{H}$

Here is a picture of the fundamental domain $\mathfrak{H}/\Gamma$.



The goal of this section is to prove that under the action of the $\Gamma = \mathfrak{sl}_2(\mathbb{R})\, z$, we can "move" every points on the upper half plane to a domain, under an equivalence given by a specific action. This is similar to the fundamental domain given by the translation action of $\mathbb{Z}$ to $\mathbb{R}$ is the half-open unit interval $[0, 1)$. In general, this give a simpler description to the homogenous space of lattice. Note that when we try to compute the fundamental domain of $\mathbb{Z}\backslash\mathbb{R}$, we have $\mathbb{Z}$ plays a role of *discrete* subset of $\mathbb{R}$. We give a precise definition of discreteness as follows

Definition 1.2.1. *Let a group G act continuously on a topological space X. A subset $\Gamma \subset G$ is called **discrete** if for any two compact subse A, B in X, there are only finitely many $g \in \Gamma$ such that $g \circ A \cap B \neq \varnothing$.*

We will prove that the set

$$\Gamma = \mathfrak{sl}_2(\mathbb{R})\, z = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\}$$

is a discrete subgroup of $G = \mathrm{SL}_2(\mathbb{R})$. To prove this, we first need the following lemma

Lemma 1.2.2. *Fix a real number $r > 0$ and $0 < \delta < 1$. We denote $R_{r,\delta}$ the rectangle*

$$R_{r,\delta} = \left\{z = x + iy : -r \leqslant x \leqslant r, 0 < \delta \leqslant y \leqslant \delta^{-1}\right\}$$

Then for any $\epsilon > 0$ and any fixed set $\mathbb{S}$ of coset representatives for $\Gamma_\infty \backslash \Gamma$, there are finitely many $g \in \mathbb{S}$ such that $\Im(g \circ z) > \epsilon$ for some $z \in R_{r,\delta}$.

In the above lemma, the notation Γ_∞ is defined to be the set

$$\Gamma_\infty = \left\{\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z}\right\}.$$

It can be seen easily that this is the stability group of ∞ in $\mathfrak{H}$.

Proof. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then for $z \in R_{r,\delta}$,

$$\mathrm{Im}(g \circ z) = \frac{y}{c^2y^2 + (cx+d)^2} < \epsilon$$

if $|c| > (y\epsilon)^{-\frac{1}{2}}$. On the other hand, for $|c| \leqslant (y\epsilon)^{-\frac{1}{2}} \leqslant (\delta\epsilon)^{-\frac{1}{2}}$, we have

$$\frac{y}{(cx+d)^2} < \epsilon$$

if the following inequalities hold:

$$|d| > |c|r + (y\epsilon^{-1})^{\frac{1}{2}} \geqslant |c|r + (\epsilon\delta)^{-\frac{1}{2}}.$$

Consequently, $\Im(g \circ z) > \epsilon$ only if

$$|c| \leqslant (\delta\epsilon)^{-\frac{1}{2}} \quad \text{and} \quad |d| \leqslant (\epsilon\delta)^{-\frac{1}{2}}(r+1),$$

and the total number of such pairs (not counting $(c,d) = (0, \pm 1), (\pm 1, 0)$) is at most $\frac{4(r+1)}{(\epsilon\delta)}$. This proves the lemma. ☐

It follows from Lemma 1.2.2 that $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ is a discrete subgroup of $SL(2,\mathbb{R})$. This is because:

1. It is enough to show that for any compact subset $A \subset \mathfrak{H}$ there are only finitely many $g \in SL(2,\mathbb{Z})$ such that $(g \circ A) \cap A \neq \phi$;
2. Every compact subset of $A \subset \mathfrak{H}$ is contained in a rectangle $R_{r,\delta}$ for some $r > 0$ and $0 < \delta < \delta^{-1}$;
3. $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$, except for finitely many $\alpha \in \Gamma_\infty$, $g \in \Gamma_\infty \backslash \Gamma$.

To prove (3), note that Lemma 1.2.2 implies that $(g \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$ except for finitely many $g \in \Gamma_\infty \backslash \Gamma$. Let $S \subset \Gamma_\infty \backslash \Gamma$ denote this finite set of such elements g. If $g \notin S$, then Lemma 1.2.2 tells us that it is because $\Im(g \circ z) < \delta$ for all $z \in R_{r,\delta}$. Since $\Im(\alpha g \circ z) = \Im(g \circ z)$ for $\alpha \in \Gamma_\infty$, it is enough to show that for each $g \in S$, there are only finitely many $\alpha \in \Gamma_\infty$ such that $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} \neq \phi$. This last statement follows from the fact that $g \circ R_{r,\delta}$ itself lies in some other rectangle $R_{r',\delta'}$, and every $\alpha \in \Gamma_\infty$ is of the form $\alpha = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$ $(m \in \mathbb{Z})$, so that

$$\alpha \circ R_{r',\delta'} = \{x + iy \mid -r' + m \leqslant x \leqslant r' + m,\ 0 < \delta' \leqslant \delta''^{-1}\},$$

which implies $(\alpha \circ R_{r',\delta'}) \cap R_{r,\delta} = \phi$ for $|m|$ sufficiently large. Now we are ready to describe the fundamental domain for $\mathfrak{sl}_2(\mathbb{R})\, z \backslash \mathfrak{H}$.

Proposition 1.2.3. *A fundamental domain for $\mathfrak{sl}_2(\mathbb{R})\, z\backslash\mathfrak{H}$ can be given as the region*

$$\mathfrak{D} = \{z = x + iy \in \mathfrak{H} : |z| \geqslant 1, -1/2 \leqslant x \leqslant 1/2\},$$

modulo the congruent boundary points symmetric with respect to the imaginary axis.

Proof. First we eliminated the repeated points on the boundary. Note that the line $x = -1/2$ is the same as the line $x = 1/2$ under the transformation $z \mapsto z + 1$. Similarly, given a point on the circle $\{|z| = 1\}$, the transformation $z \mapsto -|z|^{-1}$ satisfies

$$\frac{-1}{x+iy} = \frac{-x+iy}{x^2+y^2} = -x + iy,$$

which flips the sign of x. Thus it identifies the half circle on the right of the imaginary axis with that on the left.
Now we need to show two things:

1. For any $z \in \mathfrak{H}$ we can find an element $g \in \mathfrak{sl}_2(\mathbb{R})\, z$ such that $g \circ z \in \mathfrak{D}$.
2. If $z \equiv z' \in \mathfrak{D}$ modulor $\mathfrak{sl}_2(\mathbb{R})\, z$, then either $\Re(z) = \pm\frac{1}{2}$ and $z' = z \mp 1$, or $|z| = 1$ and $z' = \frac{-1}{z}$.

First we prove for (1): Fix $z \in \mathfrak{H}$. It follows from Lemma 1.2.2 that for every $\epsilon > 0$, there are at most finitely many $g \in \mathrm{SL}(2,\mathbb{Z})$ such that $g \circ z$ lies in the strip

$$D_\epsilon := \left\{ w \;\middle|\; -\frac{1}{2} \leqslant \mathrm{Re}(w) < \frac{1}{2},\ \epsilon \leqslant \mathrm{Im}(w) \right\}.$$

Let B_ϵ denote the finite set of such $g \in \mathrm{SL}(2,\mathbb{Z})$. Clearly, for sufficiently small ϵ, the set B_ϵ contains at least one element. We will show that there is at least one $g \in B_\epsilon$ such that $g \circ z \in D$. Among these finitely many $g \in B_\epsilon$, choose one such that $\Im(g \circ z)$ is maximal in D_ϵ. If $|g \circ z| < 1$, then for $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ we have, for any m,

$$\Im\left(T^m S g \circ z\right) = \Im\left(\frac{-1}{g \circ z}\right) = \frac{\Im(g \circ z)}{|g \circ z|^2} > \Im(g \circ z)$$

But we can choose m such that $T^m S g \circ z \in D_\epsilon$, which contradicts the maximality of $\Im(g \circ z)$.
Next we give a proof for (2):Let $z \in D$, $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, and assume that $g \circ z \in D$. Without loss of generality, we may assume that

$$\Im(g \circ z) = \frac{y}{|cz+d|^2} \geqslant \Im(z),$$

(otherwise just interchange z and $g \circ z$ and use g^{-1}). This implies that $|cz + d| \leqslant 1$ which implies that $1 \geqslant |cy| \geqslant \frac{1}{\sqrt{3}}|c|$. This is clearly impossible if $|c| \geqslant 2$. So we only have to consider the cases $c = 0, \pm 1$. If $c = 0$ then $d = \pm 1$ and g is a translation by b. Since $-\frac{1}{2} \leqslant \Re(z), \Re(g \circ z) \leqslant \frac{1}{2}$, this implies that either $b = 0$ and $z = g \circ z$ or else $b = \pm 1$ and $\Re(z) = \pm\frac{1}{2}$ while $\Re(g \circ z) = \mp\frac{1}{2}$. If $c = 1$, then $|z + d| \leqslant 1$ implies that $d = 0$ unless $z = e^{2\pi i/3}$ and $d = 0, -1$. The case $d = 0$ implies that $|z| \leqslant 1$ which implies $|z| = 1$. Also, in this case, $c = 1$, $d = 0$, we must have $b = -1$ because $ad - bc = 1$. Then $g \circ z = a - \frac{1}{z+1}$. It follows that $g \circ z = a - e^{2\pi i/3}$ and $d = 1$, then we must have $a - b = 1$. It follows that $g \circ z = a - \frac{1}{z+1} = a + e^{2\pi i/3}$, which implies that $a = 0$ or 1. A similar argument holds when $z = e^{\pi i/3}$ and $d = -1$. Finally, the case $c = -1$ can be reduced to the previous case $c = 1$ by reversing the signs of a, b, c, d. $\square$

1.3 Lattices and semi-stability in dimension 2

In this section, we investigate the notion of semi-stable lattices and how the upper half plane $\mathfrak{H}$ can be regard as a spaces of two dimensional lattices.
Now regard $\mathbb{C} \cong \mathbb{R}^2$ via $x + iy \mapsto (x, y)$, and the inner product is defined to be

$$\langle z_1, z_2 \rangle = x_1 x_2 + y_1 y_2,$$

where $z_i = x_i + iy_i$. Now for any $z \in \mathfrak{H}$, the pair $(1, z)$ can be identified with the lattice

$$L_z = \mathbb{Z}z \oplus \mathbb{Z}$$

First we prove the following statement

Proposition 1.3.1. *The upper half plane $\mathfrak{H}$ classifies similarity classes of two dimensional lattice.*

Proof. Let $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ be any lattice in $\mathbb{R}_2$. Then using the above identification, we can find two complex number z_1, z_2 such that $|z_1| = ||e_1||$ and $|z_2| = |e_2|$ □

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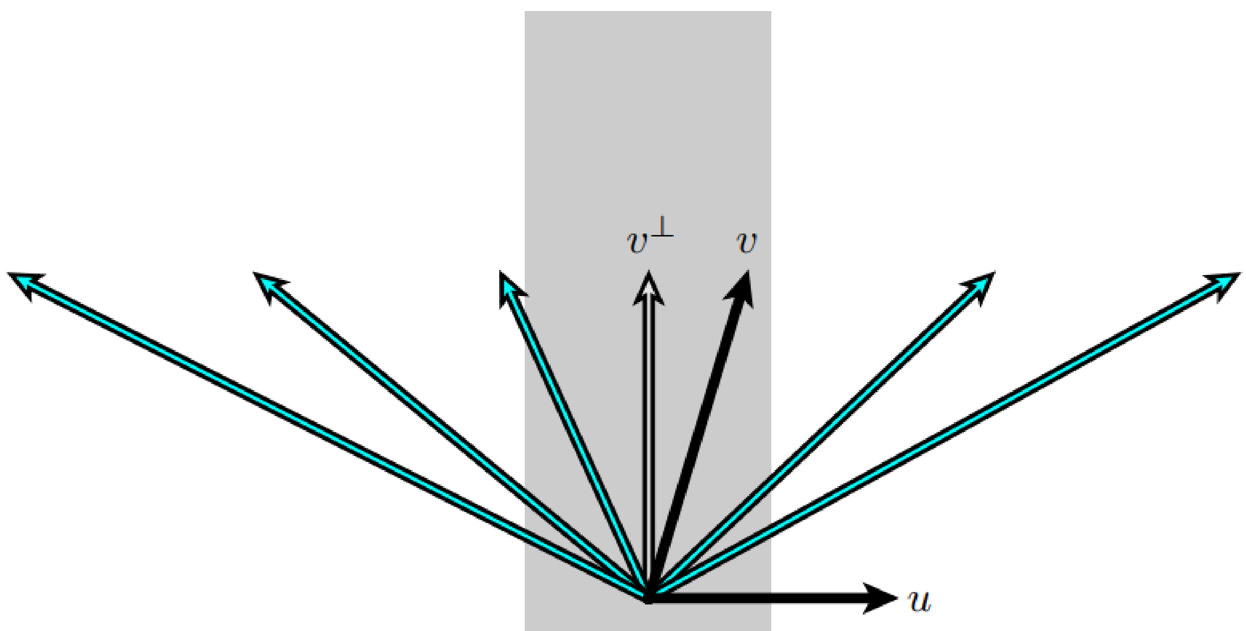
Clearly in each classes of similar lattice, there is a unique one that has unit covolume. The lattice spanned by z and 1 has volume y, so the corresponding unit lattice is the one spanned by $z/\sqrt{y}$ and $1/\sqrt{y}$.
Using Proposition 1.2.3, it is immediate that every lattice spanned by 1 and z is similar to lattice generated by 1 and a point z' inside ther region $\mathfrak{D}$.
Historically, in two dimension, Proposition 1.2.3 is first discovered by Lagrange, with the distribution of Gauss to solve for the shortest vector problem in two dimensional space. In the language of modern mathematics, it can be phrased as follows:

Proposition 1.3.2. *If L is any lattice, and u is a primitive vector in L, and v' is a vector in the sublattice $L' = L/\mathbb{Z}u$, then there exists a unique representative v of v' such that its projection onto u lies in the interval $(-u/2, u/2]$. Moreover, the following inequality holds*

$$||v||^2 \leqslant \frac{||u||^2}{4} + ||v'||^2,$$

where we identify v' with a vector $v^\perp$ in the orthogonal complement of u.

Remark. *Here the primitive vector is the vector such that it is not the multiple of any other vector in the lattice.*



To see why every point $z \in \mathfrak{H}$ can be transformed into a point inside $\mathfrak{D}$, we start with a lattice generated by the $\mathbb{Z}$-linear combination of $1, z$ and consider the shortest vector u. Applying the above lemma, we can find a vector v with the length as least as large as that of u. So if we rotate and scale to get $u = 1$, the vector v will lie in the strip $(-1/2, 1/2]$ and has the length at least 1. This clearly show that v is a point in the domain $\mathfrak{D}$.

Now Grayson - following a prior idea of Stuhler - associated every lattice to a sort of **Newton Polygon**. We will set up a graph coordinate in the following way:

1. First we construct a two dimensional coordinate, say Oxy
2. We highlight the origin.
3. If we are dealing with the lattice L, compute the area of the fundamental domain of L
4. Assign the point $(2, \log(\text{vol}(L)))$ to the line $x = 2$ in the coordinate.
5. If v is any primitive vector, we put the point $(1, \log(||v||))$ in the set.

Note that the lattice is discrete, so we can find a shortest vector v of the lattice L. This will correspond to the lowest point on the axis $x = 1$ in the diagram. Note the that x-coordinate of each of these point reflects its dimension.
As an example, let's consider the lattice of the following shape - with the shortest vector u has the length $||u|| < 1$. Now applying the above process, we get the figure on the left. If we further taking

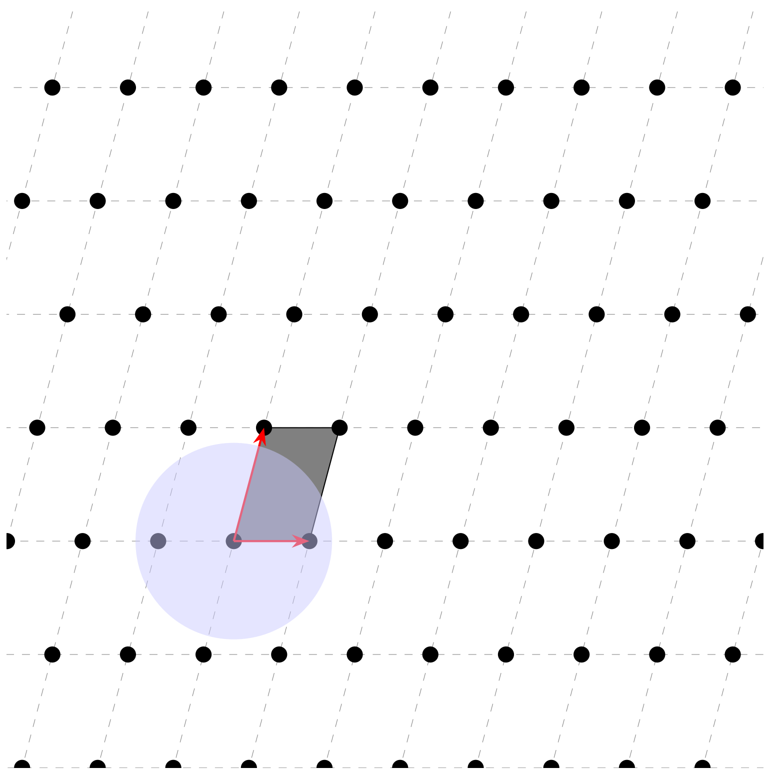
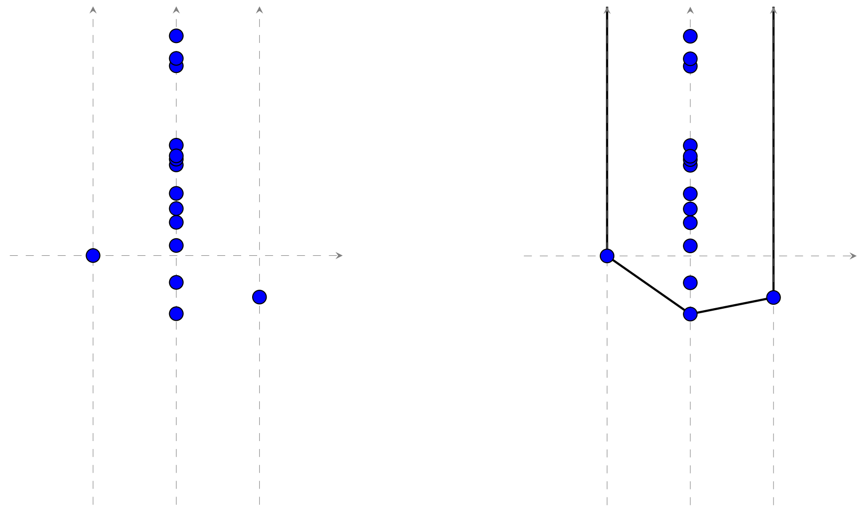


Figure 1.1: Example of a lattice

the convex hull of the diagram, we will get the figure on the right Clearly for each dimension, we have the corresponding lowest point, and so the convex hull of the plot is bounded from below. Grayson calls the plot on the left **canonical plot** of the lattice and the boundary of the convex hull of the canonical plot its **canonical polygon**. In the expository [**?**] of Bill Casselman, he instead calls the canonical polygon as **profile**. We will use the terminology of Casselman.
Now we will try to understand the profile of a lattice associated to a point $z \in \mathfrak{D}$. First we prove a simple observation



Lemma 1.3.3. *If $z \in \mathfrak{D}$ then the lattice $L_z = \mathbb{Z}z \bigoplus \mathbb{Z}$ admits 1 as the shortest vector.*

Proof. We identify $z = x + iy$ with $(x, y) \in \mathbb{R}^2$ and 1 with $(1, 0) \in \mathbb{R}^2$. Assume that 1 is not the shortest vector, then there exists $a, b \in \mathbb{Z}$ such that

$$|az + b|^2 < 1 \Leftrightarrow (ax + b)^2 + (ay)^2 < 1 \Leftrightarrow a^2|z|^2 + 2abx + b^2 < 1$$

Since $z \in \mathfrak{D}$, we clearly have $|x| \leqslant \frac{1}{2}$ and $|z| \geqslant 1$, thus the integers a, b must satisfy

$$a^2 - |ab| + b^2 < 1$$

Since the above expression are symmetric, we can assume $|a| \geqslant |b|$ and completing the square yields

$$\left(\frac{\sqrt{3}b}{2}\right)^2 \geqslant a^2 - ab + b^2 < 1 \Rightarrow b^2 < 4/3 \Rightarrow b \leqslant 1$$

Substituting $|b| = 1$ yields $|a|^2 - |a| < 0$. There is no non-zero integer a satisfying this condition. □

The area of the lattice L_z is given by $\det \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} = y$. Note that we can scale the basis by a factor $a = \sqrt{y}$ so that we get a lattice of volume 1. So the lowest points with respect to the axes $x = 0, 1, 2$ are $(0, 0), (1, -\log(a))$, and $(2, 0)$. The interesting part of $\mathfrak{D}$ is where $y \leqslant 1$. This corresponds to the lattice that has the canonical plot lying entirely on or above the x-axis. In particular, the profile of such a lattice only has the vertices at the origin and $(2, 0)$. Grayson and Stuhler call this kind of lattice **semi-stable**. If we don't normalize the area of such a lattice, then a semi-stable lattice has the bottom of the profile as a straight line.

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Conversely, the lattices assigned to the points $z \in \mathfrak{H}$ with $\Im(z) > 1$ correspond to lattices that have the canonical plot breaking at the lowest point on the axis $x = 1$. In the general case, this reflects the fact that a non semi-stable lattice has the shortest vector u satisfying $||u|| < \sqrt{\mathrm{vol}(L_z)}$. Following Casselman, we call such a lattice **unstable**. In some sense, we can see that *the degree of instability* is measured by the shortest vector compared to its volume. In the above lemma, we only find the semi-stable locus inside the fundamental domain. To find the semi-stable locus for the whole upper half plane $\mathfrak{H}$, we use the following lemma:

Lemma 1.3.4. *If L_z is semi-stable, then so is the lattice $L_{g \circ z}$, where $g \in SL_2(\mathbb{R})$.*

Proof. If we denote $L_z = \mathrm{span}_{\mathbb{Z}}\{1, z\}$, then $L_{\gamma \circ z} = cL_z$ for some complex number c. Indeed, we just need to check for γ being an inversion or translation, since these two transformations generate $\mathrm{SL}_2(\mathbb{R})(\mathbb{Z})$, but this is easy. Now let $c = re^{it}$. Multiplying by e^{it} doesn't change the length,

hence doesn't change the semi-stability. Multiplying by a positive number r will shift $(1, \log |u|)$ to $(1, \log |u| + \log r)$ and $(2, \log(\mathrm{vol}(A)))$ to $(2, \log(\mathrm{vol}(A)) + 2 \log r)$.
The line segment d connecting the origin with the final point intersects the line $x = 1$ at $(1, \log(\mathrm{vol}(A)) + \log r)$. By the semi-stability of the original lattice, the point $(1, \log |u| + \log r)$ is above the line segment d. $\square$

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From this lemma, we can see that the semi-stable locus is the complement of the Farey balls in the upper half plane, as illustrated in the following figure, where the blue part is the semi-stable locus and the gray part is the unstable one.

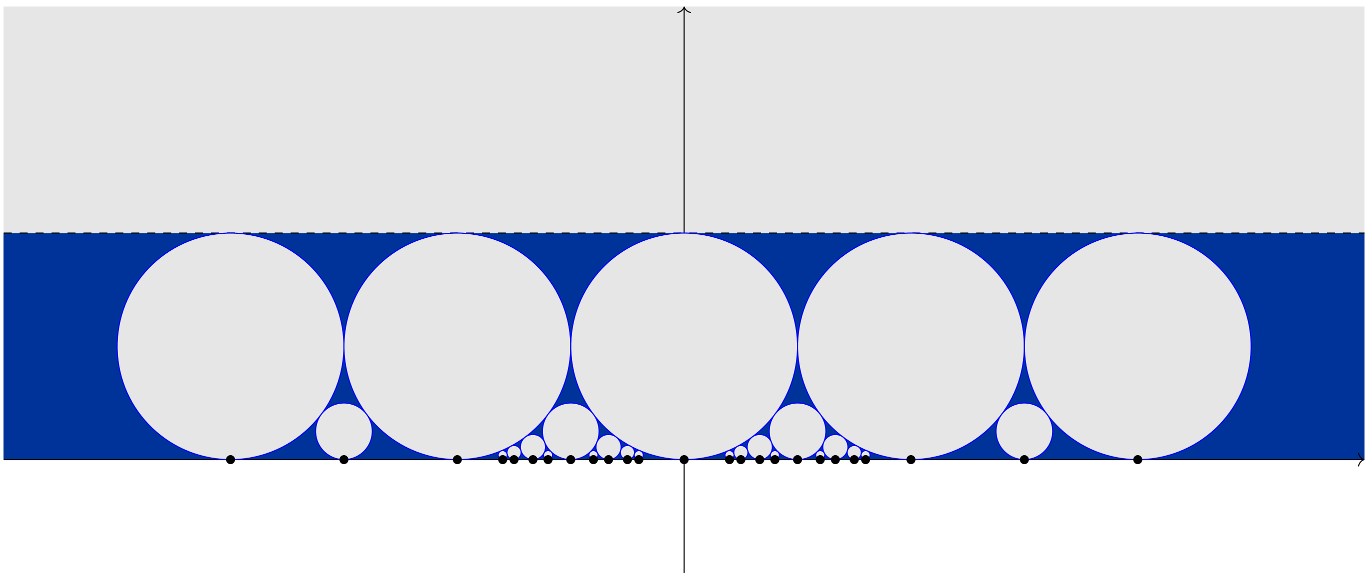

Figure 1.2: Semistable locus in upper half plane

1.4 ρ-semi-stability of lattice

The semi-stability can be defined in a more Lie-theoretic way. First we recall the Iwasawa decomposition for $\mathrm{SL}_2(\mathbb{R})$.

Proposition 1.4.1. *We have*

$$SL_2(\mathbb{R}) \cong K \times A \times N$$

where

- $K = \mathrm{SO}_2(\mathbb{R})$*: the special orthogonal group.*
- $A = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a > 0 \right\}$.
- $N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right\}$.

Combinining with Proposition 1.1.1, we have the following identification

$$\mathfrak{H} \cong A \times N$$

via the map

$$x + iy \mapsto \begin{bmatrix} 1/\sqrt{y} & 0 \\ 0 & \sqrt{y} \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = a\,(y)\,n(x)$$

Let's denote $\mathfrak{sl}_2(\mathbb{R})$ the Lie algebra of the Lie group $\mathrm{SL}_2(\mathbb{R})$ - the vector space of traceless matrices of size 2×2. We denote $\mathfrak{h} = \mathbb{R}H$ where $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ its standard Cartan subalgebra. We then have the map

$$H_B \colon \mathfrak{H} \to \mathfrak{h}, \quad z = x + iy \mapsto \log(a(y))H = \frac{-1}{2}\log(\Im(z))$$

Let $\alpha \colon \mathfrak{h} \to \mathbb{R}$ be the unique linear function such that $\alpha(H) = 2$. If we let $\rho = \frac{1}{2}\alpha$, then we define

$$\deg_{\mathrm{inst}}(z) := \min_{\gamma \in \Gamma/\Gamma \cap B} \langle \rho, H_B(x\gamma) \rangle$$

where B is the group of upper triangular matrices with invertible entries along the diagonal.

Definition 1.4.2. *The lattice L_z corresponds to the point $z \in \mathfrak{H}$ is called ρ-**semistable** or just **semi-stable** if* $\deg_{inst}(z) \geqslant 0$.

We shall use this definition to find the semi-stable locus in the upper half plane $\mathfrak{H}$.