ASSIGNMENT 1

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Problem 1

Let $\mathfrak{k} = \{X \in \mathfrak{sl}_{\mathrm{n}}(\mathbb{R}) : X^t = -X\}$ and $\mathfrak{p} = \{X \in \mathfrak{sl}_{\mathrm{n}}(\mathbb{R}) : X = X^t\}$. Verify the following inclusion relation:

- $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$.
- $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$.
- $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

Proof.
Let $X, Y \in \mathfrak{k}$ be arbitrary. Then we have

$$[X, Y]^t = (XY - YX)^t = Y^t X^t - X^t Y^t = YX - XY = -[X, Y]$$

Thus by definition, $[X, Y] \in \mathfrak{k}$.
Let $U, V \in \mathfrak{p}$ be arbitrary. Similarly, we have

$$[U, V]^t = (UV - VU)^t = V^t U^t - U^t V^t = VU - UV = -[U, V]$$

which also implies that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.
Lastly, we have

$$[X, U]^t = (XU - UX)^t = U^t X^t - X^t U^t = -UX + XU = [X, U]$$

which means $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. □

Problem 2

This is a verification exercise about choosing the **right** basis. Let denote $\kappa(.,.)$ the Killing form. Follows the same notation in previous exercise, let

- $Y_1, \ldots, Y_N$ be a basis of $\mathfrak{p}$ and this basis is indexed by i, j, k, l.
- $Y_{N+1}, \ldots, Y_n$ be a basis of $\mathfrak{k}$ and this basis is indexed by a, b, c, d.

Fact: κ is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. Furthermore we chose the above basis such that

$$\begin{cases} \kappa(Y_i, Y_j) = \delta_{ij}, \\ \kappa(Y_a, Y_b) = -\delta_{ab} \end{cases}$$

By the above exercise, we have that

$$\begin{cases} [Y_i, Y_j] = \sum_a c_{ij}^a Y_a \\ [Y_a, Y_i] = \sum_j c_{ai}^j Y_j \end{cases}$$

Show that $c_{ij}^a = c_{aj}^i$.

Proof.
It is well-known that the Killing form is invariant in the following sense

$$\kappa([a,b],c) = \kappa(a,[b,c])$$

Using this invariant property, we will compute $\kappa(Y_a,[Y_i,Y_j])$ in two ways

1. we have:
$$\kappa\left(Y_a,[Y_i,Y_j]\right) = \kappa\left(Y_a, \sum_b c_{ij}^b Y_b\right) = -c_{ij}^a$$

2. On the other hand we also have
$$\kappa(Y_a,[Y_i,Y_j]) = -\kappa([Y_a,Y_j],Y_i) = -\kappa\left(\sum_k c_{aj}^k Y_k, Y_i\right) = -c_{aj}^i$$

Comparing both results yield the desired equality. □

Problem 3

Verify that

1. $\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$.
2. κ is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$.

Proof. It is easy to check that for any traceless matrix $X = [c_{ij}]$, we can easily find two traceless matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ such that

- A is symmetric and B is skew-symmetric.
- $X = A + B$.

Indeed, in order to determine the matrices A, B, it boils down to solve the the system

$$\begin{cases} a_{ij} + b_{ij} = c_{ij} \\ a_{ij} - b_{ij} = cji \end{cases},$$

for distinct indices i, j. This system is always solvable. It is obvious that the zero matrix is the only matrix that is both symmetric and skew-symmetric. Thus

$$\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$$

The positive/negative definite part follows directly from the definitions of the Killing form and the obvervation that

$$\kappa(X,X) = \mathbf{Tr}(\operatorname{ad} X \circ \operatorname{ad} X) = \mathbf{Tr}(\operatorname{ad}(X)^t \operatorname{ad} X) = \sum_{i,j} c_{ij}^2 \geqslant 0,$$

where $X \in \mathfrak{p}$. similarly for $X \in \mathfrak{k}$

$$\kappa(X,X) = \mathbf{Tr}(\operatorname{ad} X \circ \operatorname{ad} X) = -\mathbf{Tr}(\operatorname{ad}(X)^t \operatorname{ad} X) = -\left(\sum_{i,j} c_{ij}^2\right) \leqslant 0.$$

Hence we are done. □

Problem 4

Given a parabolic subgroup $P \in \mathrm{SL}_n(\mathbb{R})$ and assume that

$$P = M_P \times A_P \times U_P.$$

Recall that we have the Iwasawa decomposition

$$\mathrm{SL}_n(\mathbb{R}) \cong K \times A \times N$$

Prove that $K \cap P = K \cap M_P$.

Proof. It is obvious that $K \cap P \supset K \cap M_P$, so we just need to prove the other inclusion. Let $X \in K \cap P$ be arbitrary, then it has the form

$$X = \begin{bmatrix} A_{n_1} & \star & \dots & \star \\ 0 & A_{n_2} & \dots & \star \\ 0 & 0 & \dots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{n_k} \end{bmatrix},$$

and satisfies $X^t = X^{-1}$. By comparing the entries, all the entries off the diagonal block matrix A_{n_i} must be zero. So we have

$$X = \begin{bmatrix} A_{n_1} & 0 & \dots & 0 \\ 0 & A_{n_2} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{n_k} \end{bmatrix}$$

But the block matrices themselves also satisfy $A_{n_i}^t = A_{n_i}^{-1}$, thus implies $\det(A_{n_i}) = \pm 1$, which also means $X \in M_P$. $\square$

Problem 5

Show that the map $d_{q+1} \circ d_q$ in section 1.1 of the note is identically zero.

Proof. Recall that the map d_q is given by

$$\begin{aligned} d_q \colon C^q &\to C^{q+1} \\ f &\mapsto d_q f \end{aligned}$$

where $(d_q f)(x_0, \dots, x_{q+1}) = \sum (-1)^i f(x_0, \dots \hat{x_i}, \dots, x_{q+1})$. Here the notation $\hat{}$ means we omit the corresponding variable. Now we have

$$\begin{aligned} (d_{q+1} \circ d_q)(f)(x_0, \dots, x_{q+2}) &= \sum (-1)^i (d_q f)((x_0, \dots \hat{x_i}, \dots, x_{q+2})) \\ &= \sum_{i=0}^{q+2} (-1)^i \left[\sum_{j \neq i} (-1)^{a_{i,j}} f(x_0, \dots \hat{x_i}, \dots, \hat{x_j}, \dots, x_{q+1}) \right] \\ &= \sum (-1)^{i + a_{i,j}} f(x_0, \dots \hat{x_i}, \dots, \hat{x_j}, \dots, x_{q+1}) \end{aligned}$$

Now we fixes some nonnegative integers u, v and consider two cases

- $i = u > v = j$: In this case $a_{i,j} = u$, thus the coefficients of $f(x_0, \dots \hat{x_u}, \dots, \hat{x_v}, \dots, x_{q+1})$ is $(-1)^{u+v}$.
- $i = v < u = j$: In this cases $a_{i,j} = u - 1$, thus the coefficients of $f(x_0, \dots \hat{x_u}, \dots, \hat{x_v}, \dots, x_{q+1})$ is $(-1)^{u+v-1}$.

This implies that for each pair of distinct (u, v), the function f is evaluated at the same points with different signs, hence canceled out. In particular, the sum will vanish. Therefore $d_{q+1} \circ d_q \equiv 0$. $\square$