

# SOME (VERY EASY) EXERCISES IN MATH 681

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## Problem 1

Let  $\mathfrak{k} = \{X \in \mathfrak{sl}_n(\mathbb{R}) : X^t = -X\}$  and  $\mathfrak{p} = \{X \in \mathfrak{sl}_n(\mathbb{R}) : X = X^t\}$ . Verify the following inclusion relation:

- $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ .
- $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ .
- $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ .

*Proof.*

Let  $X, Y \in \mathfrak{k}$  be arbitrary. Then we have

$$[X, Y]^t = (XY - YX)^t = Y^t X^t - X^t Y^t = YX - XY = -[X, Y]$$

Thus by definition,  $[X, Y] \in \mathfrak{k}$ .

Let  $U, V \in \mathfrak{p}$  be arbitrary. Similarly, we have

$$[U, V]^t = (UV - VU)^t = V^t U^t - U^t V^t = VU - UV = -[U, V]$$

which also implies that  $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ .

Lastly, we have

$$[X, U]^t = (XU - UX)^t = U^t X^t - X^t U^t = -UX + XU = [X, U]$$

which means  $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ . □

## Problem 2

This is a verification exercise about choosing the **right** basis. Let denote  $\kappa(., .)$  the Killing form. Follows the same notation in previous exercise, let

- $Y_1, \dots, Y_N$  be a basis of  $\mathfrak{p}$  and this basis is indexed by  $i, j, k, l$ .
- $Y_{N+1}, \dots, Y_n$  be a basis of  $\mathfrak{k}$  and this basis is indexed by  $a, b, c, d$ .

*Fact:*  $\kappa$  is positive definite on  $\mathfrak{p}$  and negative definite on  $\mathfrak{k}$ . Furthermore we chose the above basis such that

$$\begin{cases} \kappa(Y_i, Y_j) = \delta_{ij}, \\ \kappa(Y_a, Y_b) = -\delta_{ab} \end{cases}$$

By the above exercise, we have that

$$\begin{cases} [Y_i, Y_j] = \sum_a c_{ij}^a Y_a \\ [Y_a, Y_i] = \sum_j c_{ai}^j Y_j \end{cases}$$

Show that  $c_{ij}^a = c_{aj}^i$ .

*Proof.*

It is well-known that the Killing form is invariant in the following sense

$$\kappa([a, b], c) = \kappa(a, [b, c])$$

Using this invariant property, we will compute  $\kappa(Y_a, [Y_i, Y_j])$  in two ways

1. we have:

$$\kappa(Y_a, [Y_i, Y_j]) = \kappa\left(Y_a, \sum_b c_{ij}^b Y_b\right) = -c_{ij}^a$$

2. On the other hand we also have

$$\kappa(Y_a, [Y_i, Y_j]) = -\kappa([Y_a, Y_j], Y_i) = -\kappa\left(\sum_k c_{aj}^k Y_k, Y_i\right) = -c_{aj}^i$$

Comparing both results yield the desired equality. □

### Problem 3

Verify that

1.  $\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$ .
2.  $\kappa$  is positive definite on  $\mathfrak{p}$  and negative definite on  $\mathfrak{k}$ .

*Proof.* It is easy to check that for any traceless matrix  $X = [c_{ij}]$ , we can easily find two traceless matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$  such that

- $A$  is symmetric and  $B$  is skew-symmetric.
- $X = A + B$ .

Indeed, in order to determine the matrices  $A, B$ , it boils down to solve the the system

$$\begin{cases} a_{ij} + b_{ij} = c_{ij} \\ a_{ij} - b_{ij} = c_{ji} \end{cases},$$

for distinct indices  $i, j$ . This system is always solvable. It is obvious that the zero matrix is the only matrix that is both symmetric and skew-symmetric. Thus

$$\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$$

The positive/negative definite part follows directly from the definitions of the Killing form and the observation that

$$\kappa(X, X) = \mathbf{Tr}(\text{ad } X \circ \text{ad } X) = \mathbf{Tr}(\text{ad}(X)^t \text{ad } X) = \sum_{i,j} c_{ij}^2 \geq 0,$$

where  $X \in \mathfrak{p}$ . similarly for  $X \in \mathfrak{k}$

$$\kappa(X, X) = \mathbf{Tr}(\text{ad } X \circ \text{ad } X) = -\mathbf{Tr}(\text{ad}(X)^t \text{ad } X) = -\left(\sum_{i,j} c_{ij}^2\right) \leq 0.$$

Hence we are done. □

### Problem 4

Given a parabolic subgroup  $P \in \text{SL}_n(\mathbb{R})$  and assume that

$$P = M_P \times A_P \times U_P.$$

Recall that we have the Iwasawa decomposition

$$\text{SL}_n(\mathbb{R}) \cong K \times A \times N$$

Prove that  $K \cap P = K \cap M_P$ .

*Proof.* It is obvious that  $K \cap P \supset K \cap M_P$ , so we just need to prove the other inclusion. Let  $X \in K \cap P$  be arbitrary, then it has the form

$$X = \begin{bmatrix} A_{n_1} & \star & \dots & \star \\ 0 & A_{n_2} & \dots & \star \\ 0 & 0 & \dots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{n_k} \end{bmatrix},$$

and satisfies  $X^t = X^{-1}$ . By comparing the entries, all the entries off the diagonal block matrix  $A_{n_i}$  must be zero. So we have

$$X = \begin{bmatrix} A_{n_1} & 0 & \dots & 0 \\ 0 & A_{n_2} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{n_k} \end{bmatrix}$$

But the block matrices themselves also satisfy  $A_{n_i}^t = A_{n_i}^{-1}$ , thus implies  $\det(A_{n_i}) = \pm 1$ , which also means  $X \in M_P$ .  $\square$

#### Problem 5

Show that the map  $d_{q+1} \circ d_q$  in section 1.1 of the note is identically zero.

*Proof.* Recall that the map  $d_q$  is given by

$$\begin{aligned} d_q: C^q &\rightarrow C^{q+1} \\ f &\mapsto d_q f \end{aligned}$$

where  $(d_q f)(x_0, \dots, x_{q+1}) = \sum (-1)^i f(x_0, \dots, \hat{x}_i, \dots, x_{q+1})$ . Here the notation  $\hat{\phantom{x}}$  means we omit the corresponding variable. Now we have

$$\begin{aligned} (d_{q+1} \circ d_q)(f)(x_0, \dots, x_{q+2}) &= \sum (-1)^i (d_q f)((x_0, \dots, \hat{x}_i, \dots, x_{q+2})) \\ &= \sum_{i=0}^{q+2} (-1)^i \left[ \sum_{j \neq i} (-1)^{a_{i,j}} f(x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{q+1}) \right] \\ &= \sum (-1)^{i+a_{i,j}} f(x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{q+1}) \end{aligned}$$

Now we fix some nonnegative integers  $u, v$  and consider two cases

- $i = u > v = j$  : In this case  $a_{i,j} = u$ , thus the coefficients of  $f(x_0, \dots, \hat{x}_u, \dots, \hat{x}_v, \dots, x_{q+1})$  is  $(-1)^{u+v}$ .
- $i = v < u = j$  : In this case  $a_{i,j} = u - 1$ , thus the coefficients of  $f(x_0, \dots, \hat{x}_u, \dots, \hat{x}_v, \dots, x_{q+1})$  is  $(-1)^{u+v-1}$ .

This implies that for each pair of distinct  $(u, v)$ , the function  $f$  is evaluated at the same points with different signs, hence canceled out. In particular, the sum will vanish. Therefore  $d_{q+1} \circ d_q \equiv 0$ .  $\square$