CHAPTER :SEMI-STABLE LATTICE IN HIGHER RANK

March 30, 2025

In this chapter, we will establish the notion of semi-stable lattice. Heuristically, this is the lattice that achieve all the successive minima at the same time, see [**?**].
We will provide two different definitions of semi -stable lattice: one is geometric - which follows Grayson's idea of utilizing the canonical plot, and one is purely algebraic, which make use of the maximal standard parabolic subgroups. The toy model will be the moduli space of 2-dimensional lattice, which is essential the upper half plane in the complex field. At the end, we will show that the two definitions coincide.

1 Lattices in higher rank

For each z with $\Im(z) > 0$, we can attach to z a lattice structure $L_z = \mathbb{Z}z \oplus \mathbb{Z}$. Roughly speaking a lattice is a discrete subgroup that is generated by a $k-$ basis of the k-space V. In particular, we will only work with the real vector space V. Grayson works with lattice over a ring of algebraic integers, but we will restrict to just the lattice that has the underlying structure as a $\mathbb{Z}-$ module. The precise definition of a lattice is as follows:

Definition 1.1 (Euclidean $\mathbb{Z}$-lattices)**.** *Let L be a finitely generated $\mathbb{Z}$-module. In particular, it is a free $\mathbb{Z}$-module of finite rank. Suppose that P is endowed with a real-valued symmetric positive definite*[1] *bilinear form, called Q. Then the space $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$ equipped with the bilinear form Q forms a real inner product space. We will call the pair (L, Q) a **Euclidean $\mathbb{Z}$-lattice**.*

add proof showing L is a lattice in the second definition

If there is no further confusion, we can just denote a Euclidean lattice by L, without specifying the bilinear form Q. The lattice l determines a full-rank lattice inside $L_{\mathbb{R}}$, namely, the rank of the lattice L is equal to the dimension of $L_{\mathbb{R}}$. We first recall the definition of discrete subgroup

Definition 1.2. *Let V be a finite-dimensional vector space over $\mathbb{R}$, endowed with the natural topology. A subgroup L of the additive group underlying the vector space V is said to be discrete if each point y in L has a neighbourhood in V whose intersection with L is $\{y\}$ or, equivalently, if, given a bounded set C in V, the set $C \cap L$ is finite.*

Thus, using the following Proposition, L has a structure of a discrete subgroup $V = L_{\mathbb{R}}$.

Proposition 1.3. *Given a finite-dimensional vector space V over $\mathbb{R}$, let L be a subgroup of the additive group V, and let m be the dimension of the $\mathbb{R}$-span of L in V. Then L is a discrete subgroup if and only if L is a free abelian group of rank m.*

A proof can be found in [**?**]. We now can define the notion of covolume of a lattice:

Definition 1.4 (Volume)**.** *Let's assume that L is a full-rank lattice and has a basis*

$$L = \mathbb{Z}l_1 \oplus \ldots \oplus \mathbb{Z}l_n$$

[1] The non-degenerate implicity state that rank L is the same as $dimL_{\mathbb{R}}$

Then the volume of this lattice is defined to be the volume of the fundamental parallepiped. In particular, let $\{e_i\}$ *be any orthonormal basis of the vector space* $V = L_{\mathbb{R}}$*. Then*

$$\mathrm{vol}(L) := |\det Q(l_i, e_j)|$$

However, for the sake of computation, we also usually adopt another definition of the lattice. In particular, we view lattice as a free $\mathbb{Z}-$ module of rank n that is isomorphic to $\mathbb{R}^n$ via base changing. In more detail

Definition 1.5. *A lattice in* $\mathbb{R}^n$ *is a subset* $L \subset \mathbb{R}^n$ *such that there exists a basis* $b_1, \ldots, b_n$ *of* $\mathbb{R}^n$ *such that*

$$L = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \ldots \mathbb{Z}b_n$$

If we put the vector $b_1, b_2, \ldots, b_n$ *in columns, with respect to the standard basis, namely*

$$g = [b_1|b_2|\ldots|b_n],$$

then $L = g\mathbb{Z}^n$*.*

In the second sense, we can just identify L with the standard lattice $\mathbb{Z}^n$ and the symmetric positive definite form is $g^t g$.
Now, the basic problem we want to deal with is to classify "isomorphic" classes of lattice. Here we say two lattices L_1 and L_2 are isomorphic if and only if there is a map $\gamma \in \mathrm{GL_n}(\mathbb{Z})$ such that

$$\gamma \cdot g_1 = g_2,$$

From the first point of view, we identify L_i with the $\mathbb{Z}-$ module $\mathbb{Z}^n$ associated to the form $g_i^t g_i$. If we define X_n the space of all symmetric positive definite bilinear forms, then we are looking at the space $\mathrm{GL_n}(\mathbb{Z}) \backslash X_n$. We can also regard $L_i \otimes \mathbb{R} \cong \mathbb{R}^n$. From this point of view, the problem of classification isomorphic classes of lattices is the same as looking for discrete subgroups of $\mathbb{R}^n$ of rank n, modulo rotation. We will interchange these equivalent points of view depend on the situation.
As Bill Casselman note in his expository, even if we normalize the lattice to get a unimodular lattice, we will still have to work with arbitrary lattices in the smaller rank. This means we are embedding several copies of $\mathrm{GL}_m(\mathbb{Z})$ along the diagonal of $\mathrm{SL_n}(\mathbb{Z})$. Therefore it is not necessary to normalize the volume of the lattice.

2 Semi-stable lattices: two definitions

2.1 Grayson's definition of semi-stable lattice

In this section, we introduce the idea of Grayson in defining *semi-stable* lattices. In particular, he associates every lattices a plot and its convex hull - called *profiles.* To understand what this means, we must first introduce the notation of *sublattice.*

Definition 2.1 (sublattice)**.** *Let* (L, Q) *be a Euclidean* $\mathbb{Z}$*-lattice. We say that a* $\mathbb{Z}-$*submodule* M *of* L *a* ***sublattice*** *if and only if* L/M *is torsion free.*

From this definition, we can prove that M is a sublattice of L if it satisfies one of the following equivalent properties:

1. M is a summand of L.

2. every basis of M can be extended to a basis of L.

3. L/M is torsion free.

4. The group M is an intersection of L with a rational subspace of $L_{\mathbb{R}}$.

Example 2.2. *If $L = \mathbb{Z}^2$, then a sublattice of L is a primitive vector $u = (a, b)$, i.e $\gcd(a, b) = 1$.*

add proof

An easy observation is that, if $M \subset L$ is a sublattice, then the space $M_{\mathbb{R}} = M \otimes \mathbb{R}$ is a subspace of $L_{\mathbb{R}}$, equipped with the restriction of the positive definite symmetric form Q of L,hence M is also a lattice of rank not exceeding rank of L.
As stated in definition 1.4, we can computed a volume of a lattice by base changing and choose an orthonormal basis. However, if we view the lattice L as $\mathbb{Z}^n$ under an action of $g \in \mathrm{GL}_n(\mathbb{R})$ as in definition 1.5, it is more convenient to define volume use wedge product. Suppose L has rank n, then L has a basis $b_1, b_2, \ldots, b_n$ such that

$$b_i = g \cdot e_i, \quad g \in \mathrm{GL}_n(\mathbb{R}),$$

where $e_1, e_2, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. let $\bigwedge^* \mathbb{R}^n$ denote the corresponding exterior algebra. If $\bigwedge^p \mathbb{R}^n$ denotes the pth exterior power of $\mathbb{R}^n$, the products $f_v := e_{v_1} \wedge \cdots \wedge e_{v_p}$, where v ranges over the ordered p-tuples $(v_1, \ldots, v_p)$ subject to the condition $1 \leqslant v_1 < \cdots < v_p \leqslant n$, then form a basis $\{f_v\}$ of $\bigwedge^p \mathbb{R}^n$. In a natural way, $\bigwedge^p \mathbb{R}^n$ permits the Euclidean norm, $\|\cdot\|$ defined by $\|f_v\| = 1$ and $\|\sum_v \lambda_v f_v\| = (\sum_v \lambda_v^2)^{1/2}$.
The group $\mathrm{GL}_n(\mathbb{R})$ operates on the exterior power $\bigwedge^p \mathbb{R}^n$, $p = 1, \ldots, n$, via

$$g(e_{v_1} \wedge \cdots \wedge e_{v_p}) = g(e_{v_1}) \wedge \cdots \wedge g(e_{v_p})$$

and linear extension. So if M is a sublattice of L with basis $l_1, \ldots, l_m$ then the volume of M is the length of the vector $l_1 \wedge \ldots \wedge l_m$. In other words, it is the square root of the sum of the squares of the determinants of the $m \times m$ minor matrices in the $n \times m$ matrix whose columns are the coordinate of the l_i with respect to any orthonormal basis of $L_{\mathbb{R}}$. Now we are ready to define the **canonical plot.**

add a numerical example.

Definition 2.3 (slope)**.** *The slope of a non-zero lattice L is the number*

$$\mu(L) = \frac{\log \mathrm{vol}(L)}{\dim L}$$

Definition 2.4. *Suppose we have a lattice L. For any sublattice $M \subset L$, we assign M to a point*

$$l(M) = (\dim M, \log \mathrm{vol}(M))$$

*in the plane $\mathbb{R}^2$. The collection of all points $l(M)$ where M ranges over all sublattices of L is called **the canonical plot** of the lattice L. By convention, we assign the lattice of zero rank to the origin of the plane.*

For example, if M is of rank 1, then the volume is just the length of the generator. The following lemma asserts that, for each vertical axis $x = i$, there is a lowest point.

Lemma 2.5. *Given a lattice L and a number c, there exists only a finite number of sublattices $M \subset L$ such that $\mathrm{vol}(M) < c$.*

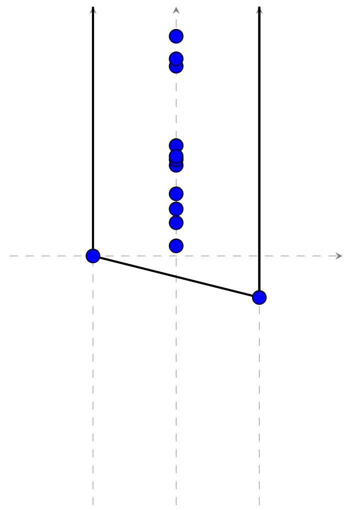
add a proof

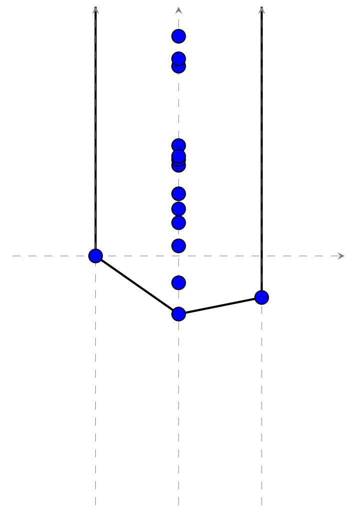
Definition 2.6. *The boundary polygon of the convex hull of the canonical plot is called **profile** of the lattice L.*

In theory, we can compute the profile by searching for the shortest vector in each of its exterior product, but this computation is infeasible when the dimension of the lattice grows. Since there are lattices with arbitrarily large volume of any rank smaller than that of L, we add to the side the point $(0, \infty)$ and (n, ∞). The sides of the profile are therefore two vertical lines. The bottom is just the convex polygonal connecting the origin with the point $l(L) = (n, \log \mathrm{vol}(L))$, where n is the rank of L.

Definition 2.7. *If the bottom of the profile contains only two points $(0, 0)$ and $(n, \log \mathrm{vol}\, L)$, then the lattice L is said to be **semi-stable**. Otherwise L is said to be **unstable**.*

Here are the picture of two lattices. The one on the left is semi-stable while the one on the right is unstable.





Visually, a lattice is called **semi-stable** if it satisfies the other equivalent conditions: If M is an arbitrary sublattice of L then $\mu(M) \geqslant \mu(L)$.

2.2 ρ-definition of semi-stability

There is another, more algebraic way to determine whether a given lattice is semi-stable. This definition ultilize the notion of parabolic subgroups. We will first recall what is a $k-$ parabolic subgroups for the general linear group GL_n for $n \geqslant 2$, over an arbitrary field k. Let $e_1, e_2, \ldots, e_n$ be a standard basis for the vector space k^n. From linear algebra, we know that each linear map $T\colon k^n \to k^n$ can be identified with a $n \times n$ matrix. In particular we obtain an identification between the group $\mathrm{GL}_n(k)$ with $\mathrm{GL}(k^n)$ of $k-$ automorphisms of k^n.

Definition 2.8. *A flag $\mathcal{F}$ of k^n is a chain of linear subspaces*

$$\mathcal{F}\colon 0 \subset F_1 \subset F_2 \subset \ldots \subset F_r \subset k^n$$

Let $d_i = \dim F_i$, then we call the ordered r-tuple $(d_1, d_2, \ldots, d_r)$ the type of the flag $\mathcal{F}$.

A parabolic subgroup of $\mathrm{Gl}(k^n)$ is the stabilizer $P_{\mathcal{F}} = P$ of a flag $\mathcal{F}$. A parabolic subgroup P is call *minimal* if it stabilizes a flag of type $(1, 2, \ldots, n)$.
Let $e_1, \ldots, e_n$ be the standard basis for the k-vector space k^n. For any $1 \leqslant i \leqslant n$, define V_i to be $e_1 + \cdots + e_i$. We call a flag $\mathcal{V}$ by the chain

$$\mathcal{V} : 0 \subset V_{d_1} \subset V_{d_2} \subset \cdots \subset V_{d_r} \subset k^n,$$

a standard flag in k^n. Let $d_0 = 0$ and $d_{r+1} = n$. We define $r_j := d_j - d_{j-1}$, where $j = 1, \ldots, r+1$. Then $\rho = (r_1, \ldots, r_{r+1})$ is an ordered partition of n into positive integers, i.e., an ordered sequence of positive integers so that $r_1 + \cdots + r_{r+1} = n$. The corresponding standard parabolic subgroup $P_{\mathcal{V}} := P_\rho$ consists of all matrices in $\mathrm{GL}_n(k)$ admitting a block decomposition whose diagonal blocks are $(r_j \times r_j)$-matrices in $\mathrm{GL}_{r_j}(k)$, $j = 1, \ldots, r+1$, the lower entries are 0, and the other entries are arbitrary. Every parabolic subgroup of $\mathrm{GL}_n(k)$ is conjugate to a subgroup of this type. The maximal standard parabolic subgroups in $\mathrm{GL}_n(k)$ corresponds to the stabilizer of the flag of type $\rho_i = (i, n-i)$, where $i = 1, \ldots, n-1$ of n. We will further denote $Q_i = P_{\rho_i}$ and **MaxParSt** the collection of such maximal parabolic subgroups. We are now ready to define the ρ-definition of semi-stable lattice. Recall that we define the space of lattices of rank n by $X_n := K\backslash\mathrm{GL}_n(\mathbb{R})$, where K is the orthogonal subgroup.

Definition 2.9 (ρ-definition)**.** *Let $x \in X_n$ be an arbitrary lattice, then the lattice x is called **semi-stable** if and only if its degree of instability $\deg_{inst}(x) \geqslant 0$, where*

$$\deg_{inst}(x) := \min_{Q \in MaxParSt, \gamma \in GL(\mathbb{Q})/Q_i(\mathbb{Q})} \langle \rho_Q, H_Q(x\gamma) \rangle$$

A simple observation is that - a lattice x is semi - stable if for all maximal standard parabolic subgroups Q_i, we have

$$\min_{\gamma \in \mathrm{GL}_n(\mathbb{Q})/Q_i(\mathbb{Q})} \langle \rho_Q, H_Q(x\gamma)\rangle \geqslant 0$$

The above definition in some sense "measures" the volume of the sublattices of the lattice x. We will illustrate with using Iwasawa's decomposition for $n = 3$. Recall that for any $g \in \mathrm{GL}_n(\mathbb{R})$ we have

$$g = k_g a_g n_g \in K \times A \times N,$$

where K is the orthogonal subgroup, A is the set of all diagonal matrices with all positive entries along the diagonal and N is the subgroup of unipotent matrices. If we let $x\gamma = g = k_g a_g n_g$ and

$$a_g = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

Then

$$\min_{\gamma \in \mathrm{GL}_n(\mathbb{Q})/Q_1(\mathbb{Q})} \langle \rho_{Q_1}, H_{Q_1}(x\gamma)\rangle = \log(a_1)$$

$$\min_{\gamma \in \mathrm{GL}_n(\mathbb{Q})/Q_2(\mathbb{Q})} \langle \rho_{Q_2}, H_{Q_2}(x\gamma)\rangle = \log(a_1 a_2)$$

So a lattices x in $K\backslash\mathrm{GL}_3(\mathbb{R})$ is semi-stable if and only if the $\log(a_1)$ and $\log(a_1a_2)$ defined as above are nonnegative. So far we have two distinct definitions of semi-stability. The following theorem asserts that they are equivalent:

Theorem 1. *Let $x \in X_n = K\backslash GL_n(\mathbb{R})$. Then x is semi-stable if one of the following equaivalent conditions holds*

1. *The bottom of the profile of x is a line connect solely two points: the origin and $(n, \log \mathrm{vol}(x))$.*

2. *The degree of instability of x is nonnegative, namely, $\deg_{inst}(x) \geqslant 0$.*

Remark. *Let denote $v_i = e_1 \wedge \ldots \wedge e_i$. Clearly we have*

$$||k_g(v)|| = ||v|| \quad \text{and} \quad n_g(v_i) = v_i$$

for all $v \in \bigwedge^i \mathbb{R}^n, i = 1, \ldots n$. Thus, in term of volume, the only relevant factor is the A-component. In particular, we have

$$||g(v_i)|| = ||a_g(v_i)|| = ||a_g e_i \wedge \ldots a_g e_n|| = a_{11} \ldots a_{ii}$$

The above remark suggests that for each maximal standard parabolic subgroups, the degree of instability detects for the sublattices with the smallest volume. So if we can prove there is a correspondence between $\gamma \in \mathrm{GL}_n(\mathbb{Q})/Q_i(\mathbb{Q})$ and a sublattice of rank i of x, then we are done.

Proof.
We first need a slight reduction - we identified the quotient $\mathrm{GL}_n(\mathbb{Q})/Q_i(\mathbb{Q})$ with the quotient $\mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z}))$. Now let x be an arbitrary lattice of rank n.
We will show the following correspondence

$$\gamma(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})) \longleftrightarrow \{ \text{ sublattice of rank } i \text{ of } \mathbb{Z}^n\}$$

Indeed, let $M \subset \mathbb{Z}^n$ be a sublattice with a basis $v_1, v_2, \ldots, v_i$. Since M is a sublattice, we can extend the basis to $\{v_1, \ldots, v_i, v_{i+1}, \ldots, v_n\}$ to get a basis of $\mathbb{Z}^n$. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{Z}^n$, there clearly there exsists a $\gamma \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$\gamma \cdot e_i = v_i \quad \forall i = 1, \ldots, n$$

If we identify M with the wedge product $v_1 \wedge \ldots \wedge v_n$, then $\gamma \in \mathrm{GL}_n(\mathbb{Z})$ acts on the exterior product via

$$\gamma(e_1 \wedge \ldots \wedge e_n) = v_1 \wedge \ldots \wedge v_n$$

□

So the set of rank i sublattice of $\mathbb{Z}^n$ is the orbit of $e_1 \wedge \ldots \wedge e_n$ under the action of $\mathrm{GL}_n(\mathbb{Z})$, with the stabilizer as $(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z}))$.
An immediate consequence of the above correspondence is that

$$x\gamma(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})) \longleftrightarrow \{ \text{ sublattice of rank } i \text{ of } x\mathbb{Z}^n\}$$

So we constructed a bijection between the maximal parabolic subgroups Q_i's and sublattices of rank i in any lattice.