# Semi-stable lattices in higher rank

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# Outline

# Historical motivation

Serre and Quillen used the notion of semistable vector bundle on an algebraic curve to study $\mathrm{SL}_n(\mathcal{O})$ when $\mathcal{O}$ is a Dedekind domain finitely generated over a finite field. Stuhler then realized he can used the same method to adapt some work of Harder and Narasimhan on stable vector bundles to yields new facts about lattices in a Euclidean space.

# Definition of two-dimensional lattices

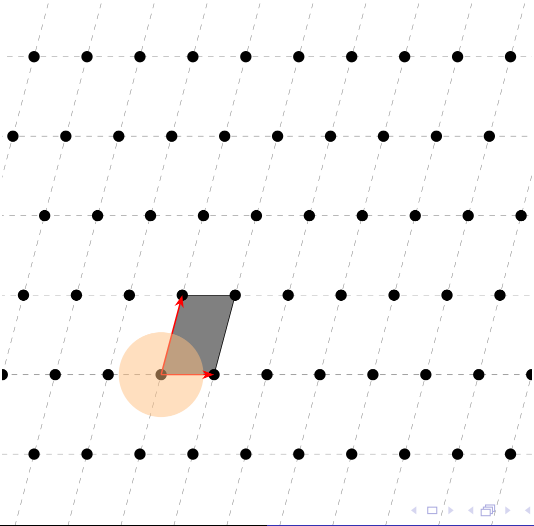
## Lattice

A lattice $L \subset \mathbb{R}^2$ is a set of the form

$$L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$$

where $e_1, e_2 \in \mathbb{R}^2$ are linearly independent over $\mathbb{R}$.

# Example of a 2-dim lattice



# Classification of lattices

Do we know all the possible 2 dimensional "lattice shapes"?

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**Answer:** Up to rescaling, rotation and change of basis, the answer is yes.

# Fundamental domain

Up to rotations and rescaling, we can reduce a lattice

$$L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$$

to a lattice of the form

$$L_z = \mathbb{Z}z \oplus \mathbb{Z}, \quad \Im(z) > 0$$

So the upper half-plane parametrizes the 2 dimensional lattices.

## Classification of unit lattices

The map $z \mapsto L_z$ induces a bijection

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \cong \{\text{ lattices}\} /\mathbb{C}^\times$$

# Fundamental domain

So we reduce the study of the space of lattices by looking the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane. Geometrically, the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ is given by

$$\mathfrak{D} = \{z = x + iy \in \mathbb{H} : |z| \geq 1, -1/2 \leq x \leq 1/2\}$$

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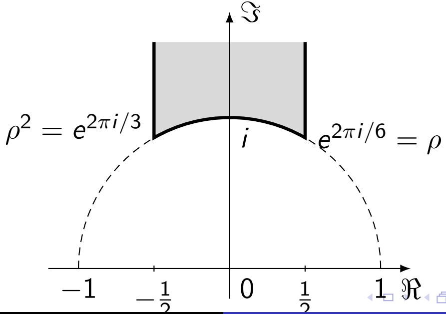

# Canonical plot

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1. Put $(0,0)$ in the plot.
2. For each primitive vector $v \in L$, he assigns the point $(1, \log(||v||))$ to the plot.
3. Put the point $(2, \log(vol(L)))$ in the plot.

# Canonical plot

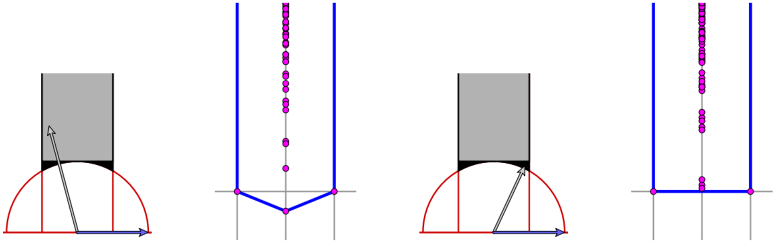

Figure: Casselman - The figure on the left corresponds to $z = -2/5 + 3i/2$ and on the right corresponds to $z = 7/16 + 15i/16$

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Since the lattice is discrete, there is a shortest primitive vector - on the plot we have the lowest point on the vertical line $x = 1$.

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Grayson called the set of points plotted above as **canonical plot**. The convex hull of the collection of the plot points is called **profile**.

For any $z \in \mathbb{H} = \{\mathrm{Im}(z) > 0\}$, we can assign to it a lattice of covolume 1 as follows

$$z \mapsto L_z = \mathbb{Z}\frac{e_1}{\sqrt{y}} + \mathbb{Z}\left(\frac{x}{\sqrt{y}}e_1 + \sqrt{y}e_2\right)$$

The shortest vector is then $e_1/\sqrt{y}$, with length $\dfrac{1}{\sqrt{y}}$. So for $y < 1$, the lowest point is above the horizontal axis.

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The element $z$ corresponding to a lattice $L_z$ such that its lowest point on the vertical line $x = 1$ lies above the $x$-axis is called **semi-stable**, otherwise $z$ is called **unstable**.

# Semi-stable locus in fundamental domain

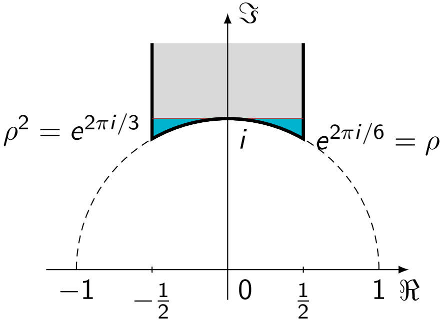


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Figure: The blue part is the semistable locus in the fundamental domain

Since the semi-stability is preserved under the action of $\mathrm{SL}_2(\mathbb{Z})$, the semi-stable locus in the upper half plane $\mathbb{H}$ is as follows

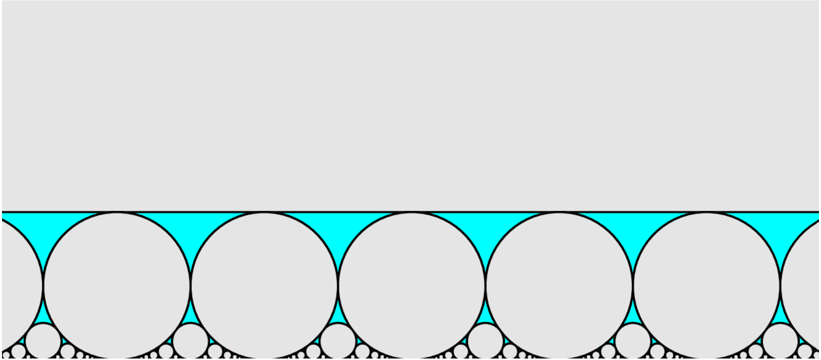

Figure: Semi-stable locus over $\mathbb{H}$ - it is the complement of the union of the gray area

# In higher dimensions

We work with the lattices of the form $g\mathbb{Z}^n$ for $g \in \mathrm{GL}_n(\mathbb{R})$ or $g \in \mathrm{SL}_n(\mathbb{R})$. The latter yields lattices with unit volumes. Even if we want to work with unit lattices, we still need to consider the sublattices of arbitrary volumes.

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## Sublattice

A discrete subgroup $M$ of the lattice $L$ is called **sublatice** if it satisfies one of the the following equivalent conditions:

1. $L/M$ is torsion-free.
2. $M$ is a direct summand in $L$.
3. Every basis of $M$ can be extended to a basis in $L$.
4. The quotient $L/M$ is a free $\mathbb{Z}-$module.

# Volume of lattice

The volume of $L = g\mathbb{Z}^n$ is just $\det(g)$. Assume that $M$ is a sublattice of $L$ of rank $k \leq n$ with a basis

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Let $e_1, e_2, \ldots, e_n$ be the standard basis in $L \otimes \mathbb{R} \cong \mathbb{R}^n$. We can form a matrix of size $k \times n$

$$A = \left[\langle v_i, e_j \rangle\right]$$

The volume of $M$ is defined to be the square root of the sum of the squares of the determinants of the $k \times k$ minor matrices in the matrix $A$.

# Canonical plot in higher dimension

Grayson assigns to the lattice $L$ a canonical plot as follows:

1. Put the point $(0, 0)$ in the plot.
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As before, we call the convex hull of this plot its **profile**.

We have the following proposition:

**Lemma - Grayson**

Fix a lattice $L$ of rank $n$ and a positive number $c$. For each $k \leq n$, there are only finitely many sublattices $M \subset L$ such that $\mathrm{vol}(M) < c$.

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**Semi-stable lattice**

A lattice $L$ is called **semi-stable** if the bottom of the profile is just a line.

# Example of a higher rank profile

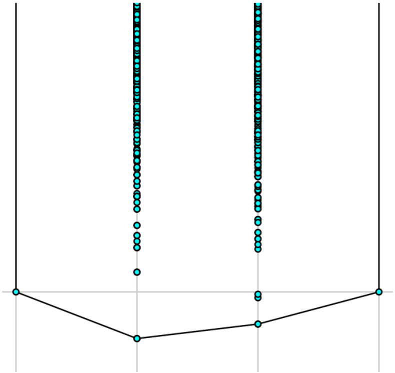

Figure: An unstable lattice

# Iwasawa decomposition

We recall the Iwasawa decomposition for $G = \mathrm{GL}_n$:

$$G = K \times A \times N$$

where:

1. $K$ is the orthogonal subgroup.
2. $A$ is the group of diagonal matrices with positive entries along the diagonal.
3. $N$ is the unipotent subgroup.

# Parabolic subgroups

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## Standard Parabolic subgroups of $\mathrm{GL}_n$

For each partition

$$n = n_1 + n_2 + \ldots + n_k$$

We denote $P_{n_1, n_2, \ldots, n_k}$ the standard parabolic subgroup of type $(n_1, \ldots, n_k)$ to be the subgroup of matrices of the form

$$P_{n_1,\ldots,n_k} = \left\{ \begin{bmatrix} \mathfrak{m}_1 & * & \ldots & * \\ 0 & \mathfrak{m}_2 & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathfrak{m}_k \end{bmatrix} \right\}$$

where $\mathfrak{m}_i$ is invertible of size $n_i \times n_i$.

# Degree of instability

Now we are ready to define the degree of instability

**Degree of instability, Chaudouard**

For each $x \in G$, we define its degree of instability to be

$$\deg_{\mathrm{inst}}(x) := \min_{P \in \mathrm{ParSt}, \gamma \in G(\mathbb{Q})/P(\mathbb{Q})} \langle \rho_P, H_B(x\gamma) \rangle$$

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We define the notion of $\rho$-semistable as follows

## $\rho$-semistable

A point $x \in G$ is called **semi-stable** iff $\deg_{\mathrm{inst}}(x) \geq 0$.

# Equivalent between two notions of semi-stable

We have

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where

$$x\gamma = k_x a_x n_x \in K \times A \times N,$$

in which

$$a_x = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}$$

# Equivalent between two notions of semi-stable lattices

We have the following Lemma

**Lemma - Chaudouard**

The following are equivalent:

1. $\deg_{\text{inst}}(x) \geq 0$;
2. For every maximal parabolic subgroup $P \subset G$, every $\delta \in G(\mathbb{Q})/P(\mathbb{Q})$ , and every $\varpi \in \hat{\Delta}_P^G$, we have:

$$\langle \varpi, H_B(x\delta) \rangle \geq 0.$$

This suggests that there should be a connection between the maximal parabolic subgroups of $G$ and sublattices of $L$. Indeed we have

$$\mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})) \longleftrightarrow \{ \text{ sublattices of rank } i \text{ of } \mathbb{Z}^n\}$$

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So we have the main theorem

**Main theorem**

Let $x \in X_n = K\backslash \mathrm{GL}_n(\mathbb{R})$ - the space of lattices. Then $x$ is semi-stable if one of the following equivalent conditions holds

1. The bottom of the profile of the lattice corresponding to $x$ is a straight line that connects the origin and $(n, \log(\mathrm{vol}(L)))$.
2. The degree of instability of $x$ is nonnegative, namely, $\deg_{\mathrm{inst}}(x) \geq 0$.

# References

[1] Bill Casselman. Stability of lattices and the partition of arithmetic quotients. 2004.

[2] Pierre-Henri Chaudouard. Sur une variante des troncatures d'arthur. In *Simons Symposium on the Trace Formula*, pages 85–120. Springer, 2016.

[3] Daniel R Grayson. Reduction theory using semistability. *Commentarii Mathematici Helvetici*, 59(1):600–634, 1984.

*THANK YOU FOR YOUR ATTENTION.*