Problem 1

Let χ_0 be the principal character mod 3, that is

$$\chi_0(n) = \begin{cases} 1, & \gcd(x,3) = 1 \\ 0, & \text{otherwise} \end{cases}$$

Define

$$\chi(n) = \begin{cases} 1, & n \equiv 1 \pmod 3 \\ -1, & n \equiv 2 \pmod 3 \\ 0, & \text{otherwise} \end{cases}$$

Prove that χ_0 and χ are completely multiplicative functions. Verify the identity

$$\mathbb{1} = \frac{1}{2}(\chi_0 + \chi)$$

Proof.

Let $m, n \in \mathbb{Z}$ be arbitrary integers. We consider the following cases:

1. At least one of m, n is divisible by 3. WLOG, we can assume $3 \mid m$. Then it is clear that $3 \mid mn$. By definition, we must have $\chi_0(m) = \chi(m) = 0$ and $\chi_0(mn) = \chi(mn) = 0$. Thus we have

$$\chi(m)\chi(n) = 0 \cdot \chi(n) = 0 = \chi(mn) \quad \text{and } \chi_0(m)\chi_0(n) = 0 \cdot \chi(n) = 0 = \chi(mn).$$

2. Both m, n are coprime to 3. There will be then two subcases

   - $m \equiv n \pmod 3$. Then clearly we have $\chi_0(m) = \chi_0(n) = 1$ and $\chi(m) = \chi(n)$. Moreover $mn \equiv 1 \pmod 3$, thus $\chi(mn) = \chi_0(mn) = 1$. In particular, we have

   $$\chi(m)\chi(n) = 1 \cdot 1 = 1 = \chi(mn) \quad \text{and } \chi_0(m)\chi_0(n) = 1 \cdot 1 = 1 = \chi(mn).$$

   - $m \not\equiv n \pmod 3$. We can further assume that $m \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$. clearly

   $$\chi_0(m)\chi_0(n) = 1 \cdot 1 = 1 = \chi_0(mn),$$

   as $\gcd(mn, 3) = 1$.
   On the other hand, $mn \equiv 1 \cdot (-1) \equiv 2 \pmod 3$, thus

   $$\chi(m)\chi(n) = 1 \cdot (-1) = -1 = \chi(mn)$$

In conclustion, χ and χ_0 are completely multiplicative. To verify the given identity, we consider the following cases

(a) $m \not\equiv 1 \pmod 3$: Then $\mathbb{1}(m) = 0$. On the other hand, we have

$$\frac{1}{2}\left(\chi(m) + \chi_0(m)\right) = \begin{cases} 0 + 0, & m \equiv 0 \pmod 3 \\ -1 + 1, & m \equiv 2 \pmod 3 \end{cases} = 0$$

(b) $m \equiv 1 \pmod 3$: Then $\mathbb{1}(m) = 1 = 1/2(\chi(m) + \chi_0(m))$

Thus we are done. $\square$

Problem 2

For $s > 1$, define the L function

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{k=0}^{\infty} \frac{1}{(3k+1)^s} - \sum_{k=0}^{\infty} \frac{1}{(3k+2)^s} = 1 - \frac{1}{2^x} + \frac{1}{4^x} - \frac{1}{5^x} + \dots$$

Show that

$$\frac{1}{2} \leq L(s,\chi) \leq 1$$

for $s > 1$, and hence show that

$$\frac{1}{2}\log L(s,\chi_0) + \frac{1}{2}\log L(s,\chi) \to \infty \quad \text{as } s \to 1^+,$$

where

$$L(s,\chi_0) := \sum_{n=1}^{\infty} \frac{\chi_0(n)}{n^s}.$$

Proof. It can be seen easily that

$$\sum_{k=1}^{\infty} \left| \frac{\chi(k)}{k^s} \right| < \sum_{k=1}^{\infty} \frac{1}{k^s} < \infty$$

for $s > 1$. Thus the series $L(s,\chi)$ converges absolutely. In particular, we can rearrange the term of the series without changing it values. Note that

$$L(s,\chi) = 1 - \frac{1}{2^x} + \frac{1}{4^x} - \frac{1}{5^x} + \dots = 1 - \sum_{k=0} \left(\frac{1}{(3k+2)^s} - \frac{1}{(3k+4)^s} \right) \leq 1,$$

and

$$L(s,\chi) = 1 - \frac{1}{2^x} + \frac{1}{4^x} - \frac{1}{5^x} + \dots = 1 - \frac{1}{2^s} + \sum_{k=1} \left(\frac{1}{(3k+1)^s} - \frac{1}{(3k+2)^s} \right) > 1 - \frac{1}{2^s} \geq \frac{1}{2},$$

where $s > 1$. In conclustion, we have

$$1/2 \leq L(s,\chi) \leq 1$$

Using formulae (14) for the principal character in the note, we get

$$L(s,\chi_0) \geq \frac{\Phi(3)}{3^x} \sum_{k=1}^{\infty} \frac{1}{k^s} \longrightarrow \infty,$$

as $s \to 1^+$. Moreover, $L(s,\chi)$ is bounded as $s \to 1^+$ as showed above, we can conclude that

$$L(s,\chi_0) \cdot L(s,\chi) \to \infty \quad \text{as } s \to 1^+$$

which clearly show that

$$\frac{1}{2}\log L(s,\chi_0) + \frac{1}{2}\log L(s,\chi) \to \infty \quad \text{as } s \to 1^+,$$

as desired. $\square$

Problem 3

Show that
$$\frac{1}{2}\log L(s,\chi_0) + \frac{1}{2}\log L(s,\chi) = \sum_p \sum_{k=1}^{\infty} \frac{\mathbb{1}(p^k)}{kp^{ks}}$$
for $s > 1$, where the sum extends over all primes p.

Proof. Since χ and χ_0 are real-valued, we can safely taking the logarithm without the need to taking care of complex logarithm. Using the Taylor's expansion for the $\log$, we get
$$\log L(s,\chi) = -\sum_p \log(1-\chi(p)p^{-s}) = \sum_p \sum_{k=1}^{\infty} \frac{\chi(p^k)p^{-ks}}{k}$$
and
$$\log L(s,\chi_0) = -\sum_p \log(1-\chi_0(p)p^{-s}) = \sum_p \sum_{k=1}^{\infty} \frac{\chi_0(p^k)p^{-ks}}{k}$$
As shown above, we have that $\mathbb{1} = \frac{\chi+\chi_0}{2}$, thus
$$\log L(s,\chi_0) + \log L(s,\chi) = \sum_p \sum_{k=1}^{\infty} \frac{[\chi(p^k)+\chi_0(p^k)]p^{-ks}}{k} = \sum_p \sum_{k=1}^{\infty} 2\cdot \frac{\mathbb{1}(p^k)p^{-ks}}{k},$$
which implies
$$\frac{1}{2}\log L(s,\chi_0) + \frac{1}{2}\log L(s,\chi) = \sum_p \sum_{k=1}^{\infty} \frac{\mathbb{1}(p^k)}{kp^{ks}},$$
for $s > 1$. $\square$

Problem 4

Put everything together to conclude that the series
$$\sum_{p\equiv 1 \pmod 3} \frac{1}{p}$$
diverges.

Proof. It can be seen easily that the double sum in problem 3 is the Taylor's expansion of the function $\log(L(s,\mathbb{1}))$ with $s > 1$. By problem 2 we have
$$\log(L(s,\mathbb{1})) \longrightarrow \infty \quad \text{as} \quad x \to 1^+$$
Taking the exponent of $\log(L(s,\mathbb{1}))$ and use the definition, we get
$$\log L(s,\mathbb{1}) = \sum_{p=3t+1} \frac{1}{p^s} + \underbrace{\sum_{p=3t+1}\sum_{k=2}^{\infty} \frac{1}{kp^{ks}}}_{E(x)}$$
If we can show that the latter double sum is bounded, then we are done. This sum can be estimated in the exactly the same way in the note as follows
$$\sum_{k=2}^{\infty} \frac{1}{kp^{ks}} < \sum_{k=2}^{\infty} \left(\frac{1}{p^s}\right)^s = \frac{1}{p^s(p-1)} < \frac{1}{p-1} - \frac{1}{p}$$

Summing over all primes $p \equiv 1 \pmod 3$, we get

$$E(x) = \sum_{p=3t+1} \sum_{k=2}^{\infty} \frac{1}{kp^{ks}} < \sum_{p=3t+1} \left(\frac{1}{p-1} - \frac{1}{p} \right) < \sum_{n \in \mathbb{Z}_{\geq 1}} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1$$

In particular, we have

$$\log(L(s, \mathbb{1})) = \sum_{p=3t+1} \frac{1}{p^s} + E(x) \longrightarrow \infty \quad \text{as} \quad x \to 1^+$$

As $E(x) \in (0,1)$ letting $s \to 1^+$ yields

$$\sum_{p \equiv 1 \pmod 3} \frac{1}{p} = \infty,$$

which means the series $\displaystyle\sum_{p \equiv 1 \pmod 3} \frac{1}{p}$ diverges, as desired. $\square$