

Name:

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### Problem 1

Let  $q \geq 2$  be an integer and let  $\chi$  be a Dirichlet character mod  $q$ . Consider the  $L$

$$L(z; \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^z}$$

which converges absolutely for  $\Re(z) =: x > 1$  since

$$\sum_{n \geq 1} |\chi(n)n^{-z}| \geq \sum_{n \geq 1} n^{-x} < \infty,$$

by the  $p$ -test from FPM. Show that  $L(z; \chi)$  defines a holomorphic function on  $X = \{z \in \mathbb{C} : \Re(z) > 1\}$  and  $L(z; \chi) \neq 0$  for all  $z \in X$ . Furthermore, show that

$$\frac{L'(z; \chi)}{L(z; \chi)} = - \sum_p \frac{\chi(p) \log(p)}{p^z - \chi(p)};$$

for  $z \in X$ .

*Proof.* Using Weierstrass M-test as in the statement in the problem, we have proved that the sequence

$$L_N(z; \chi) = \sum_{n=1}^N \frac{\chi(n)}{n^z}$$

converges uniformly to the function  $L(z; \chi)$ . In particular,  $L(z; \chi)$  is continuous over the domain  $X$ . Clearly the sequence of functions  $L_N$  are holomorphic on  $X$ . Thus we have

$$\int_{\gamma} L_N(z; \chi) dz = 0,$$

for piecewise smooth closed curve inside  $X$ , by Cauchy's theorem. Furthermore, we have

$$\int_{\partial \Delta} L(z; \chi) = \lim_{N \rightarrow \infty} \int_{\partial \Delta} L_N(z; \chi) = 0$$

By Morera's theorem, we can conclude that  $L(z; \chi)$  is holomorphic over the domain  $X$ .

Clearly the domain  $X$  is simply connected. Using proposition 2.3 in the note, we can rewrite the  $L$ -function as

$$L(z; \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^z} = \prod_p \frac{1}{1 - \chi(p)p^{-z}}$$

Note that over  $X = \{z \in \mathbb{C} : \Re(z) > 1\}$ , we have

$$\frac{1}{1 - \chi(p)p^{-z}} = 1 + \chi(p)p^{-z} + (\chi(p)p^{-z})^2 + \dots = 1 + \sum_{k \geq 1} \chi(p^k)p^{-kz}$$

where

$$\sum_{k \geq 1} |\chi(p^k)p^{-kz}| \leq \sum_{n \geq 1} n^{-\Re z} < \infty$$

Let denote  $u_p(z) = \sum_{k \geq 1} \chi(p^k) p^{-kz}$ , then

$$L(z; \chi) = \prod_p (1 + u_p(z))$$

Then  $L(z; \chi) = 0$  if and only if  $u_p(z) = -1$  for some  $p$ . But this is impossible as

$$|u_p(z)| \leq \sum_{k \geq 1} |\chi(p^k) p^{-kz}| = \sum_{k \geq 1} p^{-k\Re z} = \frac{p^{-\Re(z)}}{1 - p^{-\Re z}} = \frac{1}{p^{\Re z} - 1} < 1$$

Thus  $L(z; \chi)$  never vanishes on the domain  $X$ .

Lastly, we will prove the identity for  $\frac{L'(z; \chi)}{L(z; \chi)}$ . From the discussion of the note, there exists a branch of the logarithm  $\log L(z, \chi)$  on  $X$  and satisfies

$$\log L(z; \chi) = \sum_p \text{Log}(1 - \chi(p)p^{-z})$$

We can then define the partial sum  $S_m = \sum_{p \leq m} \text{Log}(1 - \chi(p)p^{-z})$ . Clearly this partial sum converges uniformly to  $\log L(z; \chi)$  as  $m \rightarrow \infty$  and are holomorphic. Thus

$$S'_m(z) = - \sum_{p \leq m} \frac{\chi(p) \log(p)}{p^z - \chi(p)} \rightarrow [\log(L(z; \chi))] = \frac{L'(z; \chi)}{L(z; \chi)}$$

In particular, for  $z \in X$ , we have

$$\frac{L'(z; \chi)}{L(z; \chi)} = - \sum_p \frac{\chi(p) \log(p)}{p^z - \chi(p)},$$

where the sum extends over all prime number  $p$ , as desired. □

### Problem 2

Let  $G$  be a finite abelian group. Show that  $\#\hat{G} \leq \#G$ .

*Proof.* First we will prove that the set  $S$  of maps  $f: G \rightarrow \mathbb{C}$  has a vector space structure. The sum and the scalar multiplication of maps are defined as follows

- $(\chi_1 + \chi_2)(a) := \chi_1(a) + \chi_2(a)$  for all  $a \in G$ .
- $(c\chi)(a) := c \cdot \chi(a)$  for any  $\chi \in S$  and  $c \in \mathbb{C}$ .

But we can check all the axioms for a set being a vector space pointwisely, with the zero being the zero map. So  $S$  have a a structure of vector space. Let's compute the dimension of  $S$  as as  $\mathbb{C}$ - vector space. Since  $G$  is finite, we can assume that, as a set

$$G = \{a_1, \dots, a_n\}$$

Then we can define the map  $f_i$  to be the "dual" of  $a_i$  in the following sense:  $f_i(a_j) = \delta_{ij}$ , i.e.  $f_i(a_i) = 1$  and vanishes at  $a_j \neq a_i$ . We claim that  $\mathfrak{B} = \{f_i\}$  forms a basis of  $S$ . Indeed, let  $f: G \rightarrow \mathbb{C}$  be arbitrary. Since  $G$  is finite,  $f$  is determined by its value at each  $a_i \in G$ . Assume that  $c_i = f(a_i)$ . Then for any  $1 \leq i \leq n$ , we have

$$f(a_i) = c_i = \sum_{i=1}^n c_i f_i(a_i)$$

In particular, we can rewrite the following identity as

$$f = c_1 f_1 + c_2 f_2 + \dots + c_n f_n,$$

which implies  $\mathfrak{B}$  spans  $S$ . On the other hand, if

$$0 = c_1 f_1 + \dots + c_n f(n)$$

Applying both sides to  $a_i$  yields

$$0 = c_i f_i(a_i) = c_i$$

which means  $\mathfrak{B}$  is a set of linearly independent vectors. Thus  $\dim S = \#\mathfrak{B} = n$ . In the note, we proved that the character  $\chi_1, \chi_2, \dots, \chi_m$  are mutually orthogonal. In particular, they are a subset of  $S$  comprising of linearly independent vectors. Thus

$$m = \#\hat{G} \leq n = \#G.$$

**Remark:** In fact, if we assume the theorem about the structure of finite abelian group, we can prove that the equality always happens, namely  $\#\hat{G} = \#G$  □

**Problem 3**

Consider the finite abelian group  $G_9$  consisting of the invertible elements of  $\mathbb{Z}/9\mathbb{Z}$ . Find all the characters of  $G_9$ . Make sure you prove that the characters of  $G_9$  you find are all distinct and there are no others.

*Proof.* Using the Remark in the previous exercise, we predict that there are 6 characters in total. It can be checked easily that 2 generates  $G_9$ , so it is a cyclic group of order 6. Indeed, we have

$$2^1 \equiv 2 \pmod{9}$$

$$2^2 \equiv 4 \pmod{9}$$

$$2^3 \equiv 8 \pmod{9}$$

$$2^4 \equiv 7 \pmod{9}$$

$$2^5 \equiv 5 \pmod{9}$$

$$2^6 \equiv 1 \pmod{9}$$

So the character is defined solely by determining where it sends [2] in  $\mathbb{C}$ . Let  $\chi \in \hat{G}_9$  be any character, we then have

$$\chi(2)^6 = \chi(2^6) = \chi(1) = 1 \in \mathbb{C}$$

thus implies  $\chi(2)$  must be a 6-th root of unity. There are 6 choices in total, namely

$$\chi(2) = e^{\frac{2i\pi k}{6}}, 0 \leq k \leq 6$$

Clearly these are 6 distinct characters, and they are all possible characters of  $G_9$ .

We can further compute a character table as follows:  $\chi(n) \pmod{9}$

$n$	1	2	4	5	7	8
$\chi_1(n)$	1	1	1	1	1	1
$\chi_2(n)$	1	$\omega$	$\omega^2$	$-\omega^2$	$-\omega$	-1
$\chi_3(n)$	1	$\omega^2$	$-\omega$	$-\omega$	$\omega^2$	1
$\chi_4(n)$	1	-1	1	-1	1	-1
$\chi_5(n)$	1	$-\omega$	$\omega^2$	$\omega^2$	$-\omega$	1
$\chi_6(n)$	1	$-\omega^2$	$-\omega$	$\omega$	$\omega^2$	-1

where  $\omega = e^{i\pi/3}$ . □

**Problem 4**

Show that there is exactly one real, non-principal Dirichlet character  $\chi \pmod{9}$ . Find  $\chi(916)$ .

*Proof.* Continuing from the previous exercise, with a note that a character is called real character if its image lies entirely in  $\mathbb{R}$ , namely we have a group homomorphism  $\chi: G_9 \rightarrow \mathbb{R}$ . Since we know that all character  $\chi \in \hat{G}$  satisfy

$$|\chi(a)| = 1, \text{ for all } a \in G_9,$$

we can deduce that a real character  $\chi$  must satisfy  $\chi(G_9) \subset \{\pm 1\}$ . Since we want to find a non-principal character, the only choices is that  $\chi(G_9) = \{\pm 1\}$ . As shown above,  $\chi$  is defined solely by its value at 2, so we must choose  $\chi(2) = -1$ . This implies that there is only one non-principal, real Dirichlet character over  $G_9$ .

Note that  $916 \equiv 16 = 2^4 \pmod{9}$ . Thus

$$\chi(916) = \chi(2^4) = (\chi(2)^4) = (-1)^4 = 1.$$

And we are done. □