MATH 506 HOMEWORK 1-3 SOLUTIONS

1. Homework 1

Problem 1.1. Use the definition of H_1 to prove the following: Let D_1 and D_2 be two connected open sets in $\mathbb{C}$. If $H_1(D_1) = H_1(D_2) = 0$ and $D_1 \cap D_2$ is connected, then $H_1(D_1 \cup D_2) = 0$. Hint: Show that every closed curve γ in $D_1 \cup D_2$ is homologous to the sum $\sum \gamma_\alpha$, where each γ_α is either a closed curve in D_1 or a closed curve in D_2.

Proof. Let γ be parameterized by $\gamma : [0,1] \to D = D_1 \cup D_2$. Let us consider $\gamma^{-1}(D_1)$ and $\gamma^{-1}(D_2)$. Each is a disjoint union of intervals open in $[0,1]$ and their union covers $[0,1]$. By compactness of $[0,1]$, we can find a finite open cover of $[0,1]$ in the form of

$$[0,1] = [a_0, b_0) \cup (a_1, b_1) \cup ... \cup (a_{2n}, b_{2n}]$$

with $0 = a_0 < a_1 < b_0 < a_2 < b_1 < a_3 < b_2 < ... < b_{2n} = 1$ such that $(a_k, b_k) \subset \gamma^{-1}(D_1)$ if k is even and $(a_k, b_k) \subset \gamma^{-1}(D_2)$ if k is odd. Here we assume that $\gamma(0) = \gamma(1) \in D_1$ WLOG. So

$$[0,1] = [0, c_1] \cup [c_1, c_2] \cup ... \cup [c_{2n}, 1] \text{ for } c_m = \frac{1}{2}(b_{m-1} + a_m)$$

with $\gamma(c_m) \in D_1 \cap D_2$ for $1 \le m \le 2n$ and $\gamma([c_{m-1}, c_m]) \subset D_k$ if $2 \mid (m-k)$, where we let $c_0 = 0$ and $c_{2n+1} = 1$.

We let γ_m be the curve $\gamma : [c_{m-1}, c_m] \to D$. Then $\gamma_m \subset D_k$ if $2 \mid (m-k)$. For each pair $\{c_i, c_{2n+1-i}\}$, we choose a continuous curve $\sigma_i : [0,1] \to D_1 \cap D_2$ such that $\sigma_i(0) = \gamma(c_i)$ and $\sigma_i(1) = \gamma(c_{2n+1-i})$. This is possible since $D_1 \cap D_2$ is connected. Then

$$\gamma = (\gamma_1 + \sigma_1 + \gamma_{2n+1}) + \sum_{i=2}^{n} (\gamma_i + \sigma_i + \gamma_{2n+2-i} - \sigma_{i-1}) + (\gamma_{n+1} - \sigma_n)$$

in $H_1(D)$. Clearly, each of

$$\gamma_1 + \sigma_1 + \gamma_{2n+1},\ \gamma_i + \sigma_i + \gamma_{2n+2-i} - \sigma_{i-1},\ \gamma_{n+1} - \sigma_n$$

lies entirely in one of D_1 and D_2. So they are homologous to 0 since $H_1(D_1) = H_1(D_2) = 0$. So $\gamma = 0$ in $H_1(D)$. □

Problem 1.2. Find all entire functions $f(z)$ satisfying

$$f(z_1 + z_2) = f(z_1) f(z_2) \text{ for all } z_1, z_2 \in \mathbb{C}.$$

Do there exist nonconstant entire functions $f(z)$ satisfying

$$f(z_1 z_2) = f(z_1) + f(z_2) \text{ for all } z_1, z_2 \in \mathbb{C}?$$

Justify your answer.

Proof. If $f(z)$ has a zero at z_0, then $f(z) = f(z_0)f(z - z_0) = 0$ for all z.

Suppose that $f(z)$ is nowhere vanishing. Then $f'(z)/f(z)$ has a complex anti-derivative $g(z)$ on $\mathbb{C}$. Then

$$\frac{(e^{g(z)})'}{e^{g(z)}} = g'(z) = \frac{f'(z)}{f(z)} \Rightarrow \left(\frac{f(z)}{e^{g(z)}}\right)' = 0$$

for all $z \in \mathbb{C}$. Therefore, $f(z) \equiv ce^{g(z)}$ for some constant $c \neq 0$. We may choose $g(z)$ such that $c = 1$. So $f(z) \equiv e^{g(z)}$ for some entire function $g(z)$. Then

$$1 = \frac{f(z_1)f(z_1)}{f(z_1 + z_2)} = \exp(g(z_1) + g(z_2) - g(z_1 + z_2))$$
$$\Rightarrow g(z_1) + g(z_2) - g(z_1 + z_2) \in \{2n\pi i : n \in \mathbb{Z}\}$$

for all $z_1, z_2 \in \mathbb{C}$. And since $g(z_1) + g(z_2) - g(z_1 + z_2) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is continuous, we must have

$$g(z_1) + g(z_2) - g(z_1 + z_2) = 2n\pi i$$

for all $z_1, z_2 \in \mathbb{C}$ and some $n \in \mathbb{Z}$. Differentiating it with respect to z_1, we obtain

$$g'(z_1) = g'(z_1 + z_2)$$

for all $z_1, z_2 \in \mathbb{C}$. Hence $g'(z) \equiv a$ and $g(z) \equiv az + 2n\pi i$. So $f(z) \equiv \exp(az)$.

In conclusion, either $f(z) \equiv 0$ or $\exp(az)$ for some constant $a \in \mathbb{C}$. □

Problem 1.3. Show that if f and g are analytic functions on a region G (i.e. a connected open set in $\mathbb{C}$) such that $\overline{f}g$ is analytic on G, then either f is constant or $g \equiv 0$.

Proof. If $g \equiv 0$, we are done. Otherwise, $D = \{g(z) \neq 0\}$ is a dense open subset of G. Then f and $\overline{f} = (\overline{f}g)/g$ are analytic on D. By Cauchy-Riemann equations,

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial \overline{f}}{\partial \overline{z}} = 0 \Rightarrow \frac{\partial f}{\partial \overline{z}} = \overline{\frac{\partial \overline{f}}{\partial \overline{z}}} = 0 \Rightarrow \frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

on D and hence $f(z) \equiv c$ is constant on D. By continuity, $f(z) \equiv c$ on G since D is dense in G. □

Problem 1.4. Let $D \subset \mathbb{C}$ be a region and let $f(z)$ be a meromorphic functions on D (i.e. the quotient of two analytic functions on D). Show that if

$$a_0(z) + a_1(z)f(z) + a_2(z)(f(z))^2 + ... + a_{n-1}(z)(f(z))^{n-1} + (f(z))^n \equiv 0$$

for some analytic functions $a_0(z), a_1(z), ..., a_{n-1}(z)$ on D, then $f(z)$ is analytic on D.

Proof. Otherwise, $f(z)$ has a pole at $z_0 \in D$. Then $f(z) = (z - z_0)^{-m}g(z)$ in a disk $\Delta = \{|z - z_0| < r\}$ for some $m \in \mathbb{Z}^+$ and some analytic function $g(z)$ in Δ satisfying $g(z_0) \neq 0$. Then

$$\sum_{r=0}^{n-1} a_r(z)(z - z_0)^{m(n-r)}(g(z))^r + (g(z))^n = 0$$

in Δ. Setting $z = z_0$, we obtain $g(z_0) = 0$. Contradiction. So $f(z)$ is analytic on D. □

Problem 1.5. Suppose that the power series $\sum a_n z^n$ has radius of convergence 1. If $\sum a_n$ converges to A, show that

$$\lim_{r\to 1^-} \sum a_n r^n = A.$$

Use this to show that

$$\log 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

Proof. Let

$$A_n = \sum_{m=0}^{n} a_m.$$

Then

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (A_n - A_{n-1}) z^n = \sum_{n=0}^{\infty} A_n z^n (1-z)$$

for $|z| < 1$. For $0 < r < 1$,

$$\begin{aligned}\left|\sum a_n r^n - A\right| &= \left|\sum_{n=0}^{\infty} A_n r^n (1-r) - \sum_{n=0}^{\infty} A r^n (1-r)\right| \\ &\le \sum_{n=0}^{\infty} |A_n - A| r^n (1-r) \\ &= \sum_{n=0}^{N-1} |A_n - A| r^n (1-r) + \sum_{n=N}^{\infty} |A_n - A| r^n (1-r) \\ &\le 2M(1-r^N) + \varepsilon_N r^N\end{aligned}$$

for all $N \in \mathbb{Z}^+$, $\varepsilon_N = \sup\{|A_n - A| : n \ge N\}$ and $M = \sup |A_n|$. Therefore,

$$\limsup_{r\to 1^-} \left|\sum a_n r^n - A\right| \le \varepsilon_N$$

for all $N \in \mathbb{Z}^+$. And since $\varepsilon_N \to 0$ as $N \to \infty$,

$$\limsup_{r\to 1^-} \left|\sum a_n r^n - A\right| = 0 \Rightarrow \lim_{r\to 1^-} \sum a_n r^n = A.$$

Let $\mathrm{Log}(z)$ be the principal branch of $\log z$. Then

$$\mathrm{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$$

for $|z| < 1$. Since $\{1/n\}$ is decreasing and $\lim_{n\to\infty} 1/n = 0$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges. Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \lim_{r \to 1^-} \operatorname{Log}(1+r) = \ln 2.$$

□

Problem 1.6. Show that if $f : \mathbb{C} \to \mathbb{C}$ is a continuous function and $f(z)$ is analytic on $\mathbb{C} \backslash \{\operatorname{Re}(z) = 0\}$, then $f(z)$ is entire.

Proof. By Morera's Theorem, it suffices to show that $\int_\gamma f(z)dz = 0$ for all triangles γ.

Since $f(z)$ is analytic on $\{\operatorname{Re}(z) > 0\}$, $\int_\gamma f(z)dz = 0$ for all continuous closed curves γ contained in $\{\operatorname{Re}(z) > 0\}$. For every continuous closed curve $\gamma \in \{\operatorname{Re}(z) \geq 0\}$,

$$\int_\gamma f(z)dz = \lim_{\varepsilon \to 0^+} \int_{\gamma_\varepsilon} f(z)dz = 0$$

where $\gamma_\varepsilon(t) = \gamma(t) + \varepsilon \in \{\operatorname{Re}(z) > 0\}$ for $\varepsilon > 0$. In conclusion,

$$\int_\gamma f(z)dz = 0$$

for all continuous closed curves $\gamma \subset \{\operatorname{Re}(z) \geq 0\}$. Similarly,

$$\int_\gamma f(z)dz = 0$$

for all continuous closed curves $\gamma \subset \{\operatorname{Re}(z) \leq 0\}$.

For every triangle γ, it is easy to see that

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$$

for some closed polygons γ_1 and γ_2 satisfying that $\gamma_1 \subset \{\operatorname{Re}(z) \leq 0\}$ and $\gamma_2 \subset \{\operatorname{Re}(z) \geq 0\}$. Therefore, $\int_\gamma f(z)dz = 0$ and $f(z)$ is entire. □

Problem 1.7. Let $f(z)$ and $g(z)$ be two analytic functions on an open set D. Show that if $f(z)$ and $g(z)$ have finitely many zeros in D and they do not have common zeros, then there exist analytic functions $a(z)$ and $b(z)$ on D such that $a(z)f(z) + b(z)g(z) \equiv 1$ on D.

Proof. Let $p_1, p_2, ..., p_n$ be the zeros of $f(z)$ with multiplicities $m_1, m_2, ..., m_n$, respectively. We claim that there exists a polynomial $b(z)$ in z of degree $\deg b(z) < m_1 + m_2 + ... + m_n$ such that $1 - b(z)g(z)$ has zeros at $p_1, p_2, ..., p_n$ of multiplicities at least $m_1, m_2, ..., m_n$.

Let $h(z) = 1/g(z)$. Since $g(z)$ does not vanish at p_j, $h(z)$ is analytic at p_j for $j = 1, 2, ..., n$. By Chinese Remainder Theorem, there exists $b(z) \in \mathbb{C}[z]$ of $\deg b(z) < m_1 + m_2 + ... + m_n$ such that

$$b(z) \equiv \sum_{l=0}^{m_j - 1} \frac{h^{(l)}(p_j)}{l!}(z - p_j)^l \pmod{(z - p_j)^{m_j}}$$

for $j = 1, 2, ..., n$. Therefore, $h(z) - b(z)$ has zeros at p_j of multiplicities at least m_j. The same holds for $1 - b(z)g(z) = g(z)(h(z) - b(z))$. So

$$a(z) = \frac{1 - b(z)g(z)}{f(z)}$$

is analytic on D. We are done. □

2. Homework 2

Problem 2.1. Let $f(z)$ be an entire function with two periods λ_1 and λ_2, i.e.,

$$f(z) = f(z + \lambda_1) = f(z + \lambda_2)$$

for all $z \in \mathbb{C}$. If λ_1 and λ_2 are linearly independent over $\mathbb{Q}$, then $f(z)$ must be constant.

Proof. Suppose that λ_1 and λ_2 are linearly independent over $\mathbb{R}$. Then every complex number z is a linear combination of λ_1 and λ_2 over $\mathbb{R}$. That is,

$$z = c_1\lambda_1 + c_2\lambda_2$$

for some real numbers c_1 and c_2. Let

$$m_1 = \lfloor c_1 \rfloor \text{ and } m_2 = \lfloor c_2 \rfloor$$

be the largest integers less than or equal to c_1 and c_2, respectively. Then

$$f(z) = f(z - m_1\lambda_1 - m_2\lambda_2) = f((c_1 - m_1)\lambda_1 + (c_2 - m_2)\lambda_2).$$

Since $0 < c_1 - m_1 \le 1$ and $0 < c_2 - m_2 \le 1$,

$$|(c_1 - m_1)\lambda_1 + (c_2 - m_2)\lambda_2| \le (c_1 - m_1)|\lambda_1| + (c_2 - m_2)|\lambda_2| \le |\lambda_1| + |\lambda_2|.$$

Let M be the maximum of $|f(z)|$ on $\{|z| \le |\lambda_1| + |\lambda_2|\}$. Then

$$|f(z)| = |f((c_1 - m_1)\lambda_1 + (c_2 - m_2)\lambda_2)| \le M$$

for all $z \in \mathbb{C}$. So $f(z)$ is constant by Louville.

Suppose that λ_1 and λ_2 are linearly independent over $\mathbb{Q}$. If they are linearly independent over $\mathbb{R}$, then we are done. Otherwise, λ_1 and λ_2 are linearly independent over $\mathbb{Q}$ and dependent over $\mathbb{R}$. That is, $\lambda = \lambda_1/\lambda_2 \in \mathbb{R}\backslash\mathbb{Q}$. Namely, it is an irrational real number.

We claim that for every $\varepsilon > 0$, there exist integers m_1 and m_2 such that

$$0 < |m_1\lambda - m_2| < \varepsilon$$

Fixing a positive integer n, let us consider

$$a_k = k\lambda - \lfloor k\lambda \rfloor$$

for $k = 0, 1, 2, ..., n$. These are $n+1$ numbers in the interval $[0, 1]$. By Pigeon Hole principle, there exist a_k and a_l such that $0 \le k \ne l \le n$ and

$$|a_k - a_l| = |(k - l)\lambda - (\lfloor k\lambda \rfloor - \lfloor l\lambda \rfloor)| \le \frac{1}{n}$$

Let $m_1 = k - l$ and $m_2 = \lfloor k\lambda \rfloor - \lfloor l\lambda \rfloor$. Then

$$|m_1\lambda - m_2| \le \frac{1}{n}.$$

This proves our claim.

For every positive integer n, there exist integers m_1 and m_2 such that

$$0 < |m_1\lambda - m_2| \le \frac{1}{n}$$

So

$$0 < |m_1\lambda_1 - m_2\lambda_2| = |\lambda_2(m_1\lambda - m_2)| \le \frac{|\lambda_2|}{n}.$$

Let $z_n = m_1\lambda_1 - m_2\lambda_2$. Since $f(z_n) = f(m_1\lambda_1 - m_2\lambda_2) = f(0)$, we conclude that there exists a sequence $\{z_n\}$ such that

$$0 < |z_n| \le \frac{|\lambda_2|}{n} \text{ and } f(z_n) = f(0).$$

This means that the set $\{z : f(z) = f(0)\}$ has a cluster point at 0. So $f(z) \equiv f(0)$. $\square$

Problem 2.2. Let $f(z)$ be an entire function. Show that $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$ if and only if $f^{(n)}(0) \in \mathbb{R}$ for all $n = 0, 1, 2, ...$

Proof. If $f^{(n)}(0)$ is real, then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

for all $z \in \mathbb{R}$ and hence $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$.

Suppose that $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. We can prove by induction that $f^{(n)}(z) \in \mathbb{R}$ for $z \in \mathbb{R}$. By Cauchy-Riemann equations,

$$f'(z) = \frac{\partial f}{\partial x} = f_x(z).$$

For $z \in \mathbb{R}$, since $f(z) \in \mathbb{R}$, $f_x(z) \in \mathbb{R}$. Therefore, $f'(z) \in \mathbb{R}$ for all $z \in \mathbb{R}$. Then, inductively, we have $f''(z), ..., f^{(n)}(z), ... \in \mathbb{R}$ for all $z \in \mathbb{R}$. $\square$

Problem 2.3. Let $f_1(z)$ and $f_2(z)$ be two analytic functions on $D = \{|z| < 1\}$. Suppose that $f_1(0) = f_2(0)$, f_2 is biholomorphic and $f_1(D) \subset f_2(D)$. Show that

$$|f_1'(0)| \le |f_2'(0)|.$$

Find a necessary and sufficient condition for the equality to hold.

Proof. Let us consider $g(z) = f_2^{-1} \circ f_1(z) : D \to D$, which is well defined since $f_1(D) \subset f_2(D)$.

Since $f_1(0) = f_2(0)$, $g(0) = 0$. Applying Schwartz Lemma to g, we obtain $|g'(0)| \le 1$ and hence

$$|g'(0)| = \frac{|f_1'(0)|}{|f_2'(0)|} \le 1 \Rightarrow |f_1'(0)| \le |f_2'(0)|.$$

By Schwartz Lemma, the equality holds if and only if $g(z) = cz$ for some $|c| = 1$, i.e., $f_1(z) = f_2(cz)$ for all z. $\square$

Problem 2.4. Let $f(z)$ be a holomorphic function on $D = \{|z| < 1\}$. If $f(0) = 0$, show that the series

$$\sum_{n=1}^{\infty} f(z^n)$$

uniformly converges on every compact subset of D.

Proof. It suffices to show that the series converges uniformly on the closed disk $\{|z| \le r\}$ for all $r < 1$. Let

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

on D. Since $\sum a_n z^n$ has radius of convergence at least 1, for every $0 < R < 1$, there exists a constant M such that $|a_n| \le MR^{-n}$ for all n. We choose some $r < R < 1$. Then

$$\begin{aligned} |f(z^n)| &= \left| \sum_{m=1}^{\infty} a_m z^{mn} \right| \le \sum_{m=1}^{\infty} |a_m||z|^{mn} \\ &\le MR^{-n} r^{mn} = \frac{Mr^n}{R - r^n} \le \frac{Mr^n}{R - r} \end{aligned}$$

for all $|z| \le r$. Clearly,

$$\sum_{n=1}^{\infty} \frac{Mr^n}{R - r} = \frac{M}{R - r} \sum_{n=1}^{\infty} r^n$$

converges and hence $\sum f(z^n)$ converges uniformly on $\{|z| \le r\}$. $\square$

Problem 2.5. Show that for a complex polynomial $f(z)$ of degree n, the function $M(r)/r^n$ is nonincreasing for $r \in (0, \infty)$, where

$$M(r) = \max_{|z| \le r} |f(z)|.$$

Proof. Let $g(z) = z^n f(z^{-1})$. Then by Maximum Modulus,

$$\max_{|z| \le 1/r} |g(z)| = \max_{|z| = 1/r} |g(z)| = \frac{1}{r^n} \max_{|z| = r} |f(z)| = \frac{M(r)}{r^n}.$$

Hence

$$\max_{|z| \le 1/r_1} |g(z)| \ge \max_{|z| \le 1/r_2} |g(z)| \Rightarrow \frac{M(r_1)}{r_1^n} \ge \frac{M(r_2)}{r_2^n}$$

for all $0 < r_1 < r_2$. $\square$

Problem 2.6. Let $D = \{r \le |z| \le R\}$ for some $0 < r < R$. Show that there exists a positive constant ε, depending on r and R, such that

$$\left\| f(z) - \frac{1}{z} \right\|_D = \max_{z \in D} \left| f(z) - \frac{1}{z} \right| \ge \varepsilon$$

for all entire functions $f(z)$.

Proof. By Maximum Modulus,

$$\left|\left|f(z)-\frac{1}{z}\right|\right|_D \geq \max_{|z|=r}\left|f(z)-\frac{1}{z}\right| = \frac{1}{r}\max_{|z|=r}|zf(z)-1|$$
$$\geq \frac{1}{r}|0f(0)-1| = \frac{1}{r}.$$

□

Problem 2.7. Let a be a complex number satisfying $|a| > 5/2$. Show that the power series

$$F(z) = \sum_{n=0}^{\infty} \frac{z^n}{a^{n^2}}$$

defines an entire function which does not vanish on the boundary of the annulus

$$|a^{2n-2}| < |z| < |a^{2n}|$$

and has exactly one zero inside the annulus for $n = 1, 2, ...$

Proof. We apply Rouché's theorem to $f(z)$ and $f_n(z) = -a^{-n^2}z^n$ in $|z| < |a^{2n}|$.

For $|z| = |a^{2n}|$,

$$\left|\frac{z^{m-1}a^{-(m-1)^2}}{z^m a^{-m^2}}\right| = a^{2m-2n-1} \leq |a|^{-3} \text{ if } m \leq n-1 \text{ and}$$
$$\left|\frac{z^{m+1}a^{-(m+1)^2}}{z^m a^{-m^2}}\right| = a^{2n-2m-1} \leq |a|^{-3} \text{ if } m \leq n+1$$

Therefore,

$$
\begin{aligned}
|f(z)+f_n(z)| &= \left|\sum_{m=0}^{n-1} a^{-m^2}z^m + \sum_{m=n+1}^{\infty} a^{-m^2}z^m\right| \\
&\le \sum_{m=0}^{n-1} |a^{-m^2}z^m| + \sum_{m=n+1}^{\infty} |a^{-m^2}z^m| \\
&= |a^{-(n-1)^2}z^{n-1}| \sum_{m=0}^{n-1} \left|\frac{z^m a^{-m^2}}{z^{n-1}a^{-(n-1)^2}}\right| \\
&\quad + |a^{-(n+1)^2}z^{n+1}| \sum_{m=n+1}^{\infty} \left|\frac{z^m a^{-m^2}}{z^{n+1}a^{-(n+1)^2}}\right| \qquad (2.1)\\
&< |a^{n^2-1}| \sum_{m=0}^{\infty} |a|^{-3m} + |a^{n^2-1}| \sum_{m=0}^{\infty} |a|^{-3m} \\
&= \frac{2|a|^{n^2-1}}{1-|a|^{-3}} = |f_n(z)|\frac{2|a|^{-1}}{1-|a|^{-3}} \\
&= |f_n(z)|\frac{2}{|a|-|a|^{-2}} < |f_n(z)|\frac{2}{(5/2)-(5/2)^{-2}} \\
&= \frac{100}{117}|f_n(z)| < |f_n(z)|
\end{aligned}
$$

for $|z| = |a^{2n}|$ and $|a| > 5/2$. In conclusion, we have

$$|f(z)+f_n(z)| < |f(z)| + |f_n(z)| \qquad (2.2)$$

for $|z| = |a^{2n}|$ and all $n = 0, 1, 2, ...$ By Rouché's Theorem, $f(z)$ and $f_n(z)$ have the same number of zeros in $|z| < |a^{2n}|$, counted with multiplicity. Therefore, $f(z)$ has exactly n zeros in $|z| < |a^{2n}|$, counted with multiplicity. This holds for all $n \in \mathbb{N}$.

Finally, since $f(z)$ has n zeros in $|z| < |a^{2n}|$ and $n-1$ zeros in $|z| < |a^{2n-2}|$, it has exactly one zero in $|a^{2n-2}| \le |z| < |a^{2n}|$. By (2.2), $f(z) \neq 0$ for $|z| = |a^{2n}|$ and all $n \in \mathbb{N}$. Therefore, $f(z)$ has exactly one zero in $|a^{2n-2}| < |z| < |a^{2n}|$. □

Problem 2.8. For an entire function $f(z)$, we let

$$M(r) = \max_{|z|\le r} |f(z)|.$$

Let $f(z)$ be an entire function with

$$\limsup_{r\to\infty} \frac{\log M(r)}{r} = l.$$

Show that the infinite series

$$F(z) = \sum_{n=0}^{\infty} f^{(n)}(z)$$

converges if $l < 1$ and diverges if $l > 1$.

Proof. Suppose that $l < 1$. So there exists $\lambda < 1$ such that $|f(z)| \le ce^{\lambda|z|}$ for some constant $c > 0$ and all z. By Cauchy Integral Formula,

$$|f^{(n)}(z)| = \left|\frac{n!}{2\pi i}\int_{|w|=R}\frac{f(w)}{(w-z)^{n+1}}dw\right| \le \frac{c(n!)R}{(R-|z|)^{n+1}}e^{\lambda R} \tag{2.3}$$

for all $n \in \mathbb{N}$.

We fix $r > 0$ and want to show that

$$\sum_{n=0}^{\infty}|f^{(n)}(z)| < \infty \tag{2.4}$$

in $\{|z| \le r\}$. We choose $R = r + \lambda^{-1}n$. Then

$$|f^{(n)}(z)| \le ce^{\lambda r}\left(1 + \frac{\lambda r}{n}\right)\lambda^n e^n \frac{n!}{n^n} \tag{2.5}$$

for all $n \ge 1$ and $|z| \le r$ by (2.3). So it suffices to show the convergence of the series

$$\sum_{n=1}^{\infty} ce^{\lambda r}\left(1 + \frac{\lambda r}{n}\right)\lambda^n e^n \frac{n!}{n^n} = \sum_{n=1}^{\infty} a_n \tag{2.6}$$

which follows from the ratio test:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\lambda e\left(\frac{n^n}{(n+1)^n}\right) = \lambda < 1. \tag{2.7}$$

When $l > 1$, if $F(z)$ converges for $z = 0$, then

$$\lim_{n\to\infty} f^{(n)}(0) = 0 \Rightarrow |f^{(n)}(0)| \le c \tag{2.8}$$

for a constant c and all n. Then

$$|f(z)| = \left|\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n\right| \le \sum_{n=0}^{\infty}\frac{c|z|^n}{n!} = ce^{|z|} \tag{2.9}$$

which contradicts

$$\limsup_{r\to\infty}\frac{\log M(r)}{r} = l > 1. \tag{2.10}$$

□

Problem 2.9. Let $f(z)$ be an entire function with $M(r)$ defined in the previous problem. Show that if there is a constant $0 < \alpha < 1$ such that

$$\lim_{r\to\infty}\frac{M(\alpha r)}{M(r)} > 0,$$

then $f(z)$ is a polynomial and the above limit is α^n with $n = \deg f$.

Proof. Since the limit

$$\lim_{r\to\infty} \frac{M(\alpha r)}{M(r)} > 0, \tag{2.11}$$

exists, there exists a constant $c > 0$ such that $M(\alpha r) \ge cM(r)$ for all $r \ge 1$. Therefore,

$$M(\alpha^n r) \ge c^n M(r) \Rightarrow c^{-n} M(1) \ge M(\alpha^{-n}). \tag{2.12}$$

By Cauchy Integral Formula, we have

$$\begin{aligned} |f^{(m)}(0)| &= \left| \frac{m!}{2\pi i} \int_{|z|=\alpha^{-n}} \frac{f(z)}{z^{m+1}} dz \right| \le (m!) M(\alpha^{-n}) \alpha^{mn} \\ &\le (m!) M(1) \left(\frac{\alpha^m}{c} \right)^n \end{aligned} \tag{2.13}$$

for all m and n. Then for all m satisfying $\alpha^m < c$, $f^{(m)}(0) = 0$ by taking $n \to \infty$ in (2.13). Therefore, $f(z)$ is a polynomial.

If $f(z)$ is a polynomial of degree n, then

$$\lim_{z\to\infty} \left| \frac{f(z)}{z^n} \right| = c \Rightarrow \lim_{r\to\infty} \frac{M(r)}{r^n} = c \Rightarrow \lim_{r\to\infty} \frac{M(\alpha r)}{M(r)} = \alpha^n. \tag{2.14}$$

□

Problem 2.10. Let $f(z)$ be an analytic function on $\{|z| < 1\}$. If $f(0) = 0$ and $|f(z)| < 1$ for all $z \in D$, show that

$$|f''(0)| \le 2 - 2|f'(0)|^2.$$

Hint: Apply Schwartz's Lemma to the function

$$\frac{g(z) - g(0)}{1 - \overline{g(0)} g(z)}$$

for $g(z) = z^{-1} f(z)$.

Proof. By Schwartz's Lemma, $|g(z)| < 1$ for $|z| < 1$ and $g(z) = z^{-1} f(z)$ unless $f(z) = cz$ for $|c| = 1$, where the inequality is obvious.

Therefore, $|h(z)| < 1$ for $|z| < 1$ and

$$h(z) = \frac{g(z) - g(0)}{1 - \overline{g(0)} g(z)}.$$

And since $h(0) = 0$, we conclude that

$$|h'(0)| = \frac{|g'(0)|}{1 - |g(0)|^2} \le 1 \Rightarrow |g'(0)| \le 1 - |g(0)|^2$$

by Schwartz's Lemma.

Suppose that

$$f(z) = \sum_{n=1}^{\infty} a_n z^n.$$

Then

$$g(z) = \sum_{n=0} a_{n+1} z^n$$

and hence

$$g^{(n)}(0) = (n!) a_{n+1} = \frac{(n!) f^{(n+1)}(0)}{(n+1)!} = \frac{f^{(n+1)}(0)}{n+1}.$$

Therefore,

$$|g'(0)| \le 1 - |g(0)|^2 \Rightarrow |f''(0)| \le 2 - 2|f'(0)|^2.$$

□

3. Homework 3

Problem 3.1. We call a map $f : X \to Y$ *proper* if $f^{-1}(K)$ is compact for all compact sets $K \subset Y$. Then an entire function $f : \mathbb{C} \to \mathbb{C}$ is proper if and only if $f(z)$ is a nonconstant polynomial in z.

Proof. Suppose that f is proper. Let $K = \{|w| \le 1\}$. Since f is proper, $f^{-1}(K)$ is compact. Therefore, $f^{-1}(K) \subset \{|z| \le R\}$ and hence

$$f(\{|z| > R\}) \cap K = \emptyset.$$

So $f(\{|z| > R\})$ cannot be dense in $\mathbb{C}$. By Casorati-Weierstrass, $f(z)$ has at worst a pole at ∞ and hence $f(z)$ is a polynomial. Clearly, $f(z)$ cannot be constant; otherwise, $f^{-1}(c) = \mathbb{C}$ is not compact for some c.

Suppose that $f(z) = a_0 + a_1 z + ... + a_n z^n$ is a nonconstant polynomial. To show that f is proper, it suffices to show that $f^{-1}(K_r)$ is bounded for all $K_r = \{|w| \le r\}$. Since

$$\lim_{z\to\infty} f(z) = \infty,$$

there exists $R > 0$ such that $|f(z)| > r$ for all $|z| > R$. It follows that

$$f^{-1}(K_r) \subset \{|z| \le R\}.$$

□

Problem 3.2. Prove the following variation of Rouché's Theorem: Let γ be a continuous closed curve homologous to 0 in an open set $D \subset \mathbb{C}$ and let $f(z)$ and $g(z)$ be two analytic functions on D satisfying

$$|f(z) + c_1 g(z)| > |f(z) + c_2 g(z)|$$

for some constants $c_1, c_2 \in \mathbb{C}$ satisfying $|c_1| \le |c_2|$ and all z on γ. Then $f(z)$ and $g(z)$ have the same number of zeros in the interior of γ, counted with multiplicities, i.e.,

$$\sum_{f(p)=0} \nu(\gamma, p) \operatorname{mult}_p f = \sum_{g(q)=0} \nu(\gamma, q) \operatorname{mult}_q g$$

where $\nu(\gamma, z_0)$ is the winding number of γ at z_0.

Proof. Let $h(z) = f(z)/g(z)$. Applying Argument Principle to $h(z)$ on γ,

$$\begin{aligned}\nu(h\circ\gamma, 0) &= \frac{1}{2\pi}\int_\gamma \frac{h'(z)}{h(z)}dz = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz\frac{1}{2\pi} - \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}dz \\ &= \sum_{f(p)=0} \nu(\gamma, p)\operatorname{mult}_p f - \sum_{g(q)=0}\nu(\gamma, q)\operatorname{mult}_q g.\end{aligned}$$

It suffices to show that $\nu(h\circ\gamma, 0) = 0$. By our hypothesis,

$$|h(z) + c_1| > |h(z) + c_2|$$

for all $z \in \gamma$ and hence

$$h\circ\gamma \subset G = \big\{|w + c_1| > |w + c_2|\big\}.$$

Note that $0 \notin G$ since $|c_1| \le |c_2|$. And G is a half plane and hence simply connected. Therefore, $\nu(h\circ\gamma, 0) = 0$. □

Problem 3.3. Compute the integral

$$\int_0^\infty \frac{dx}{1+x^r}$$

for $r > 1$.

Solution. Let us first assume that $r = p/q$ is rational for some positive integer p and q such that $\gcd(p, q) = 1$. Since $r > 1$, $p > q$. Then

$$\int_0^\infty \frac{dx}{1+x^r} = \int_0^\infty \frac{dx}{1+t^{p/q}} = \int_0^\infty \frac{qt^{q-1}}{1+t^p}dt$$

after the substitution $x = t^q$.

Let $\alpha = \exp(2\pi i/p)$ and let us consider the complex integral

$$\int_\gamma \frac{qz^{q-1}}{1+z^p}dz = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}\right)\frac{qz^{q-1}}{1+z^p}dz \tag{3.1}$$

along the curve $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ given by

$$\begin{cases}\gamma_1(t) = t \text{ for } 0 \le t \le R \\ \gamma_2(t) = Re^{it} \text{ for } 0 \le t \le 2\pi/p \\ \gamma_3(t) = (R-t)\alpha \text{ for } 0 \le t \le R\end{cases}$$

for some large R.

For γ_2, we have

$$\begin{aligned}&\left|\int_{\gamma_2}\frac{qz^{q-1}}{1+z^p}dz\right| \le \left(\frac{2\pi R}{p}\right)\frac{qR^{q-1}}{R^p - 1} = \frac{2\pi R^q}{r(R^p-1)} \\ &\Rightarrow \lim_{R\to\infty}\int_{\gamma_2}\frac{qz^{q-1}}{1+z^p}dz = 0\end{aligned} \tag{3.2}$$

since $p > q$.

For γ_1, we have

$$\int_{\gamma_1} \frac{qz^{q-1}}{1+z^p} dz = \int_0^R \frac{qt^{q-1}}{1+t^p} dt. \tag{3.3}$$

For γ_3, we have

$$\begin{aligned}\int_{\gamma_3} \frac{qz^{q-1}}{1+z^p} dz &= -\int_0^R \frac{q\alpha^q(R-t)^{q-1}}{1+\alpha^p(R-t)^p} dt \\ &= -\alpha^q \int_0^R \frac{q(R-t)^{q-1}}{1+(R-t)^p} dt \\ &= -\alpha^q \int_0^R \frac{qt^{q-1}}{1+t^p} dt\end{aligned} \tag{3.4}$$

since $\alpha^p = 1$.

Combining (3.1)-(3.4), we obtain

$$\lim_{R\to\infty} \int_{\gamma} \frac{qz^{q-1}}{1+z^p} dz = (1-\alpha^q) \int_0^\infty \frac{qt^{q-1}}{1+t^p} dt \tag{3.5}$$

The roots of $1 + z^p$ are $\exp((2n+1)\pi i/p)$ for $0 \le n < p$; among them, only $\beta = \exp(\pi i/p)$ lies inside the curve γ. Therefore, by Residue Theorem,

$$\begin{aligned}\int_{\gamma} \frac{qz^{q-1}}{1+z^p} dz &= 2\pi i \operatorname{Res}\left(\frac{qz^{q-1}}{1+z^p}, \beta\right) \\ &= 2\pi i \left.\frac{qz^{q-1}}{(1+z^p)'}\right|_{\beta} = 2\pi i \left(\frac{q}{p}\right) \beta^{q-p} = -\frac{2\pi i \beta^q}{r}\end{aligned} \tag{3.6}$$

where $qz^{q-1}(1+z^p)^{-1}$ has a simple pole at β since $1+z^p$ has a zero at β of multiplicity one.

Combining (3.5) and (3.6), we obtain

$$\begin{aligned}\int_0^\infty \frac{qt^{q-1}}{1+t^p} dt &= -\left(\frac{2\pi i}{r}\right) \frac{\beta^q}{1-\alpha^q} = -\left(\frac{2\pi i}{r}\right) \frac{\beta^q}{1-\beta^{2q}} \\ &= -\left(\frac{2\pi i}{r}\right) \frac{1}{\beta^{-q}-\beta^q} \\ &= -\left(\frac{2\pi i}{r}\right) \frac{1}{\exp(-q\pi i/p) - \exp(q\pi i/p)} \\ &= -\left(\frac{2\pi i}{r}\right) \frac{1}{(-2i)\sin(q\pi/p)} = \frac{\pi}{r\sin(\pi/r)}\end{aligned}$$

where we notice that $\alpha = \beta^2$. Therefore,

$$\int_0^\infty \frac{dx}{1+x^r} = \frac{\pi}{r\sin(\pi/r)} \tag{3.7}$$

for all rational numbers $r > 1$. It is not hard to prove that the function

$$F(r) = \int_0^\infty \frac{dx}{1+x^r}$$

is continuous for $r > 1$. Therefore,

$$F(r) \equiv \frac{\pi}{r\sin(\pi/r)}$$

(3.7) holds for all real numbers $r > 1$. □

Problem 3.4. Find $\mathrm{Aut}(\mathbb{C}^*) = \mathrm{Aut}(\mathbb{C} - \{0\})$ and $\mathrm{Aut}(\mathbb{C} - \{0, 1\})$.

Proof. Let us prove the following lemma:

Lemma 3.1. *For a finite set S of points on $\mathbb{C}$, every univalent function $f(z)$ on $\mathbb{C}\backslash S$ is a linear fractional transformation with singularity in S.*

By the above lemma, if $f \in \mathrm{Aut}(\mathbb{C}^*)$, then $f(z) = az + b$ or $a + bz^{-1}$ for some constants $a, b \in \mathbb{C}$. And since $0 \notin f(\mathbb{C}^*)$, it is easy to see

$$\mathrm{Aut}(\mathbb{C}^*) = \{bz : b \neq 0\} \cup \left\{\frac{b}{z} : b \neq 0\right\}.$$

Similar, $f \in \mathrm{Aut}(\mathbb{C} - \{0, 1\})$ must be one of the following:

$$az + b,\ a + \frac{b}{z},\ a + \frac{b}{z-1}$$

And since $0, 1 \notin f(\mathbb{C} - \{0, 1\})$, it is easy to see

$$\mathrm{Aut}(\mathbb{C} - \{0, 1\}) = \left\{z, 1 - z, \frac{1}{z}, 1 - \frac{1}{z}, \frac{1}{1-z}, \frac{z}{z-1}\right\}.$$

It remains to prove the lemma.

First, we show that f has at worst poles at $S \cup \{\infty\}$. We choose a closed disk $D \subset \mathbb{C}\backslash S$ of positive radius. By Open Mapping, $f(D)$ contains a nonempty open set G. Since f is 1-1,

$$f(\mathbb{C}\backslash D) \cap G = \emptyset.$$

Therefore, $f(\{0 < |z - p| < \varepsilon\}) \cap G = \emptyset$ for some $\varepsilon > 0$ and $p \in S$ as long as

$$\{|z - p| < \varepsilon\} \cap D = \emptyset.$$

By Casorati-Weierstrass, $f(z)$ has at worst poles at S.

Similarly, $f(\{|z| > R\}) \cap G = \emptyset$ as long as $D \subset \{|z| \leq R\}$. So $f(z)$ has at worst poles at ∞. In conclusion, $f(z)$ is an analytic function on $\mathbb{C}\backslash S$ with at worst poles at $S \cup \{\infty\}$. So $f(z)$ has to be a rational function $f(z)$ with poles in $S \cup \{\infty\}$.

Second, we prove that $f(z)$ has simple poles at every singularity among $S\cup\{\infty\}$. Otherwise, suppose that $f(z)$ has a pole of order $m \geq 2$ at p. Then there exists an open neighborhood U of p such that $f(z) \neq 0$ in $U^* = U\backslash\{p\}$. So $f : U^* \to \mathbb{C}^*$ is analytic and 1-1. Consequently, $g(z) = 1/f(z)$ is also 1-1 on U^*. Since $g(z)$ has a removable singularity at p and $g(p) = 0$, $g(z)$ extends to a univalent function on U. So $g'(p) \neq 0$. But $g(z)$ has a zero of multiplicity $m \geq 2$ at p. Contradiction.

Finally, we prove that $f(z)$ has at most one pole among $S \cup \{\infty\}$. Otherwise, suppose that $f(z)$ has two poles $p \neq q$. We choose U_p and U_q to be open neighborhoods of p and q, respectively, such that $U_p \cap U_q = \emptyset$ and $f(z) \neq 0$ on $U_p^* \cup U_q^*$ for $U_p^* = U_p \backslash \{p\}$ and $U_q^* = U_q \backslash \{q\}$. As before, $g(z) = 1/f(z)$ is 1-1 on $U_p^* \sqcup U_q^*$ and extends to an analytic function on $U_p \sqcup U_q$ with $g(p) = g(q) = 0$. By Open Mapping, $g(U_p) \cap g(U_q)$ contains an open disk D with $0 \in D$. Thus, for every $w \in D \backslash \{0\}$, there exist $z_p \in U_p^*$ and $z_q \in U_q^*$ such that $g(z_p) = g(z_q) = w$. This contradicts the fact that g is 1-1 on $U_p^* \sqcup U_q^*$.

In conclusion, $f(z)$ has at most one simple pole and hence a linear fractional transformation. □

Problem 3.5. Let $\lambda_1, \lambda_2 \neq 0, 1$ be two complex numbers. Show that $\mathbb{C} - \{0, 1, \lambda_1\}$ and $\mathbb{C} - \{0, 1, \lambda_2\}$ are biholomorphic if and only if

$$\lambda_1 \in \left\{ \lambda_2, \frac{1}{\lambda_2}, 1 - \lambda_2, 1 - \frac{1}{\lambda_2}, \frac{1}{1 - \lambda_2}, \frac{\lambda_2}{\lambda_2 - 1} \right\}.$$

In other words, they are biholomorphic if and only if there exists $f \in \mathrm{Aut}(\mathbb{C} - \{0, 1\})$ such that $\lambda_1 = f(\lambda_2)$.

Proof. By Lemma 3.1, f must be one of the following:

$$az + b, \ a + \frac{b}{z}, \ a + \frac{b}{z - 1}, \ a + \frac{b}{z - \lambda_1}$$

And since $0, 1, \lambda_2 \notin f(\mathbb{C} - \{0, 1, \lambda_1\})$, we conclude

$$\begin{aligned} f(z) = z, \ 1 - z, \ \frac{z}{\lambda_1}, \ 1 - \frac{z}{\lambda_1}, \ \frac{1 - z}{1 - \lambda_1}, \ \frac{z - \lambda_1}{1 - \lambda_1}, \\ \frac{1}{z}, \ 1 - \frac{1}{z}, \ \frac{\lambda_1}{z}, \ 1 - \frac{\lambda_1}{z}, \ \frac{\lambda_1(z - 1)}{(\lambda_1 - 1)z}, \ \frac{z - \lambda_1}{z(1 - \lambda_1)}, \\ \frac{z(1 - \lambda_1)}{z - \lambda_1}, \ \frac{\lambda_1(z - 1)}{z - \lambda_1}, \ \frac{z}{z - \lambda_1}, \ \frac{\lambda_1}{\lambda_1 - z}, \ \frac{z - 1}{z - \lambda_1} \text{ or } \frac{1 - \lambda_1}{z - \lambda_1}. \end{aligned}$$

It is easy to see that λ_2 is one of the limits of $f(z)$ as $z \to 0, 1, \lambda_1, \infty$. Then it follows

$$\lambda_2 \in \left\{ \lambda_1, \frac{1}{\lambda_1}, 1 - \lambda_1, 1 - \frac{1}{\lambda_1}, \frac{1}{1 - \lambda_1}, \frac{\lambda_1}{\lambda_1 - 1} \right\}$$

which is equivalent to

$$\lambda_1 \in \left\{ \lambda_2, \frac{1}{\lambda_2}, 1 - \lambda_2, 1 - \frac{1}{\lambda_2}, \frac{1}{1 - \lambda_2}, \frac{\lambda_2}{\lambda_2 - 1} \right\}.$$

□

Problem 3.6. Let $D = \{|z| < 1\}$ and $H(D)$ be the space of holomorphic functions on D. Show that $F \subset H(D)$ is normal if and only if there is a sequence $\{M_n\}$ of positive constants such that $\limsup \sqrt[n]{M_n} \leq 1$ and $|a_n| \leq M_n$ for all n and all $f(z) = \sum_{n=0}^{\infty} a_n z^n \in F$.

Proof. Suppose that F is normal. Let

$$M_n = \sup_{f \in F} \frac{|f^{(n)}(0)|}{n!} = \sup \left\{ |a_n| : \sum a_m z^m \in F \right\}.$$

Since F is normal, $\{f^{(n)}(z) : f \in F\}$ is normal for all $n \in \mathbb{N}$. So the set $\{|f^{(n)}(0)| : f \in F\}$ is uniformly bounded. Consequently, $M_n < \infty$ for all n. By the definition of M_n, $|a_n| \le M_n$ for all n and $\sum a_m z^m \in F$.

For all $0 < r < 1$, F is uniformly bounded on $\{|z| \le r\}$. Hence there exists $C > 0$ such that $|f(z)| \le C$ for all $f \in F$ and $|z| \le r$. Then

$$\frac{|f^{(n)}(0)|}{n!} = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz \right| \le \frac{C}{r^n}$$

for all n and all $f \in F$. Then $M_n \le C/r^n$ and

$$\limsup \sqrt[n]{M_n} \le \frac{1}{r} \limsup \sqrt[n]{C} = \frac{1}{r}$$

for all $0 < r < 1$. Therefore, $\limsup \sqrt[n]{M_n} \le 1$.

On the other hand, suppose that there exists such a sequence $\{M_n\}$. Since $\limsup \sqrt[n]{M_n} \le 1$, for every $0 < R < 1$, there exists C such that $M_n \le CR^{-n}$ for all n. Then

$$|f(z)| \le \sum_{n=0}^{\infty} M_n r^n \le \sum_{n=0}^{\infty} CR^{-n} r^n = \frac{CR}{R - r}$$

for all $f \in F$ and $|z| \le r < R$. So F is uniformly bounded on $\{|z| \le r\}$ for all $r < 1$. Consequently, F is normal. □

Problem 3.7. Let G be a connected open set in $\mathbb{C}$ and $H(G)$ be the space of holomorphic functions on G. For a sequence $\{f_n\} \subset H(G)$ of one-to-one functions which converge to some $f \in H(G)$ locally uniformly, show that f is either one-to-one or a constant function.

Proof. It suffices to show that for every $c \in \mathbb{C}$, either $f(z) \equiv c$ or $f(z) - c$ has at most one zero in G. Otherwise, suppose that $f(z) \not\equiv c$ and $f(z) - c$ has two zeros $z_1 \ne z_2$ in G.

We choose $r > 0$ such that $K = \{|z - z_1| \le r\} \sqcup \{|z - z_2| \le r\} \subset G$ and $f(z) \ne c$ for all $z \in K \backslash \{z_1, z_2\}$. Let

$$M = \min_{z \in \partial K} |f(z) - c| = \min \left(\min_{|z - z_1| = r} |f(z) - c|, \min_{|z - z_2| = r} |f(z) - c| \right).$$

Since f_n converges to f uniformly on K, there exists N such that

$$||f - f_n||_K < M$$

for all $n > N$. By Rouché's Theorem, since

$$|(f(z) - c) - (f_n(z) - c)| \le ||f - f_n||_K < M \le |f(z) - c|$$

for $n > N$, $|z - z_j| = r$ and $j = 1, 2$, $f(z) - c$ and $f_n(z) - c$ has the same number of zeros in $|z - z_j| < r$. Therefore, $f_n(z) - c$ has at least two zeros for $n > N$, which contradicts the hypothesis that f_n is 1-1. □

Problem 3.8. Let $G_1, G_2 \subsetneq \mathbb{C}$ be simply connected open sets and $f : G_1 \to G_2$ be a biholomorphic map from G_1 to G_2. Suppose that $f(z_1) = z_2$. Show that for every one-to-one holomorphic map $g : G_1 \to G_2$ satisfying $g(z_1) = z_2$, $|g'(z_1)| \le |f'(z_1)|$.

Proof. By Riemann Mapping Theorem, there exist biholomorphic maps $s_j : G_j \to D$ for $D = \{|z| < 1\}$ and $j = 1, 2$. We can choose s_j such that $s_j(z_j) = 0$ for $j = 1, 2$.

By Problem 2.3,

$$|(s_2 \circ g \circ s_1^{-1})'(0)| \le |(s_2 \circ f \circ s_1^{-1})'(0)| \Rightarrow \left|\frac{s_2'(z_2)g'(z_1)}{s_1'(z_1)}\right| \le \left|\frac{s_2'(z_2)f'(z_1)}{s_1'(z_1)}\right|$$
$$\Rightarrow |g'(z_1)| \le |f'(z_1)|.$$

□

Problem 3.9. Let $f(z)$ and $g(z)$ be entire functions such that $e^{f(z)}, e^{g(z)}$ and 1 are linearly dependant over $\mathbb{C}$, i.e., there exist $c_1, c_2, c_3 \in \mathbb{C}$, not all zero, such that $c_1 e^{f(z)} + c_2 e^{g(z)} + c_3 = 0$ for all z. Then $f(z), g(z)$ and 1 are linearly dependent over $\mathbb{C}$.

Proof. If one of c_1, c_2, c_3 vanishes, then it is obvious that $f(z), g(z)$ and 1 are linearly dependent over $\mathbb{C}$. Otherwise, suppose that $c_1, c_2, c_3 \ne 0$. Then

$$e^{f(z)} = -\frac{c_2}{c_1} e^{g(z)} - \frac{c_3}{c_1} \notin \left\{0, -\frac{c_3}{c_1}\right\}$$

for all $z \in \mathbb{C}$. By Picard's Little Theorem, $e^{f(z)}$ is constant and hence $f(z)$ is constant. So $f(z)$ and 1 are linearly dependent over $\mathbb{C}$. □

Problem 3.10. Let $f(x, y)$ and $g(x, y)$ be real-valued harmonic functions on $\mathbb{R}^2$ such that $e^{f(x,y)}$, $e^{g(x,y)}$ and 1 are linearly dependant over $\mathbb{R}$. Then $f(x, y)$, $g(x, y)$ and 1 are linearly dependent over $\mathbb{R}$.

Proof. Suppose that $c_1 e^{f(x,y)} + c_2 e^{g(x,y)} + c_3 = 0$ for some $c_1, c_2, c_3 \in \mathbb{R}$, not all zero, and all $(x, y) \in \mathbb{R}^2$.

If one of c_1, c_2, c_3 vanishes, it is obvious that $f(x, y), g(x, y)$ and 1 are linearly dependent over $\mathbb{R}$. Otherwise, suppose that $c_1, c_2, c_3 \ne 0$.

WLOG, suppose that $c_1 > 0$. If $c_2 > 0$, then

$$c_1 e^{f(x,y)} = c_3 - c_2 e^{g(x,y)} < c_3 \Rightarrow f(x, y) < \ln c_3 - \ln c_1$$

and hence $f(x, y)$ is constant by Louiville's Theorem on harmonic functions over $\mathbb{R}^2$. Suppose that $c_2 < 0$. If $c_3 > 0$, then

$$-c_2 e^{g(x,y)} = c_1 e^{f(x,y)} + c_3 > c_3 \Rightarrow g(x, y) > \ln c_3 - \ln(-c_2)$$

and hence $g(x, y)$ is constant. If $c_3 < 0$, then

$$c_1 e^{f(x,y)} = -c_2 e^{g(x,y)} - c_3 > -c_3 \Rightarrow f(x, y) > \ln(-c_3) - \ln c_1$$

and hence $f(x, y)$ is constant. In conclusion, $f(x, y), g(x, y)$ and 1 are linearly dependent over $\mathbb{R}$. □