SOLUTION FOR THE BONUS QUESTION

April 3, 2025

In this report, I will give a proof for the bonus question for the class MATH 538. First we recall the question

Problem 1

Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ viewed as a real Lie algebra. We will denote this Lie algebra by $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$. Show that

1. $\mathfrak{g}$ is simple as a real Lie algebra.
2. $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ is not a split real Lie algebra, i.e. it has no split Cartan subalgebra.

Proof.

1. First we will show that $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ is semisimple. Indeed, we have

$$\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}} \otimes \mathbb{C} = (\mathfrak{sl}(2, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}(2, \mathbb{R}) \otimes_{\mathbb{R}} (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$$

Now note that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$ as rings, so we clearly reduce to

$$\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}} \otimes \mathbb{C} \cong \mathfrak{sl}(2, \mathbb{R}) \otimes_{\mathbb{R}} (\mathbb{C} \oplus \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$$

This clearly shows that $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ is semi-simple. Now we show that it is in fact simple. Assume not, then $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ has a nontrivial simple ideal I. Then for any $\lambda \in \mathbb{C} \setminus \{0\}$, clearly $\lambda(I)$ is also an ideal as

$$I \supset [I, \lambda(I)] = \lambda[I, I] = \lambda(I) \neq 0$$

Clearly $\dim I = \dim \lambda(I)$, this implies that $I = \lambda(I)$. That means when we allow complex scalar multiplication the ideal I stays the same. Thus I is in fact an ideal of $\mathfrak{sl}(2, \mathbb{C})$. But since $\mathfrak{sl}(2, \mathbb{C})$ is simple as a complex Lie algebra, this implies $I = \mathfrak{sl}(2, \mathbb{C})$. The restriction of scalar doesn't change anything, which implies $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ is simple.

2. Assume not, then $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ is a split real Lie algebra. From the previous part, we regconized that $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ is a real form of the semisimple Lie algebra $\mathfrak{g}$. We first claim the following

   Claim: *If a complex Lie algebra $\mathfrak{g}$ has a real split form, then this split form must be unique, up to isomorphism.*

   Using the above claim, we can immediate point out that $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ is a split real form of the $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. Since $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ is semisimple but not simple, it is clearly not isomorphic to $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$. So we have proven that $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}|\mathbb{R}}$ is not split. The only thing left we need to prove is the above claim.

$\square$