

CHAPTER I : $SL_2(\mathbb{R})$

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In this chapter, I will give an exposition on the structure of $SL_2(\mathbb{R})$ as the spaces of lattice, this space plays the role of a toy model before exploring the space of lattice in the higher rank. The exposition follows the paper [?] and [?] closely.

1 $SL_2(\mathbb{R})$ and its action on the upper half plane \mathfrak{H}

A priori, the upper half plane

$$\mathfrak{H} = \{z : \Im z > 0\} \subset \mathbb{C}$$

has no group structure on its. However, we will show below that it can identify topologically with the space with the space of cosets $SO_2(\mathbb{R}) \backslash SL_2(\mathbb{R})$, and thus we can study the spaces \mathfrak{H} via the space of lattices $SO_2(\mathbb{R}) \backslash SL_2(\mathbb{R})$. We define the action of $G = SL_2(\mathbb{R})$ on \mathfrak{H} as follows

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ (z) = \frac{az + b}{cz + d}$$

Proposition 1.1. *The group $SL_2(\mathbb{R})$ stabilizes \mathfrak{H} and acts transitively on it. In particular,*

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} (i) = x + iy \quad (\text{for } x \in \mathbb{R}, y > 0)$$

Further, for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathfrak{H}$,

$$\Im g(z) = \frac{\Im z}{|cz + d|^2}.$$

Proof. The first formula is clear. The second formula would imply that the upper half-plane is stabilized. Compute directly:

$$\begin{aligned} 2i \cdot \Im \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) \right) &= \frac{az + b}{cz + d} - \frac{d\bar{z} + b}{c\bar{z} + d} = \frac{(az + b)(c\bar{z} + d) - (a\bar{z} + b)(cz + d)}{|cz + d|^2} \\ &= \frac{adz - bc\bar{z} - bcz + ad\bar{z}}{|cz + d|^2} = \frac{z - \bar{z}}{|cz + d|^2} \end{aligned}$$

since $ad - bc = 1$. □

The point $z = i$ is special, in the sense that its stability group is the orthogonal group $K = SO_2(\mathbb{R})$.

Indeed, for any $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ we have that

$$g \circ i = i \Leftrightarrow \frac{ai + b}{ci + d} = i \Leftrightarrow a = d \text{ and } b = -c$$

Combining with the fact that $ad - bc = 1$, we must have $a^2 + b^2 = 1$. This implies that there is a θ such that $a = \cos \theta$ and $b = \sin \theta$. Since G acts on \mathfrak{H} transitively, we know from group theory that there is a bijection between the collection of cosets of $\text{Stab}(i)$ in G and the orbits of i . In particular

Proposition 1.2. *We have an isomorphism of $SL_2(\mathbb{R})$ -spaces*

$$SO_2(\mathbb{R}) \backslash SL_2(\mathbb{R}) \approx \mathfrak{H} \quad \text{via} \quad SO(2)g \rightarrow g^{-1}(i)$$

That is, the map respects the action of $SL_2(\mathbb{R})$, in the sense that

$$(SO_2(\mathbb{R})g) \cdot h \longrightarrow h^{-1}(g^{-1}i)$$

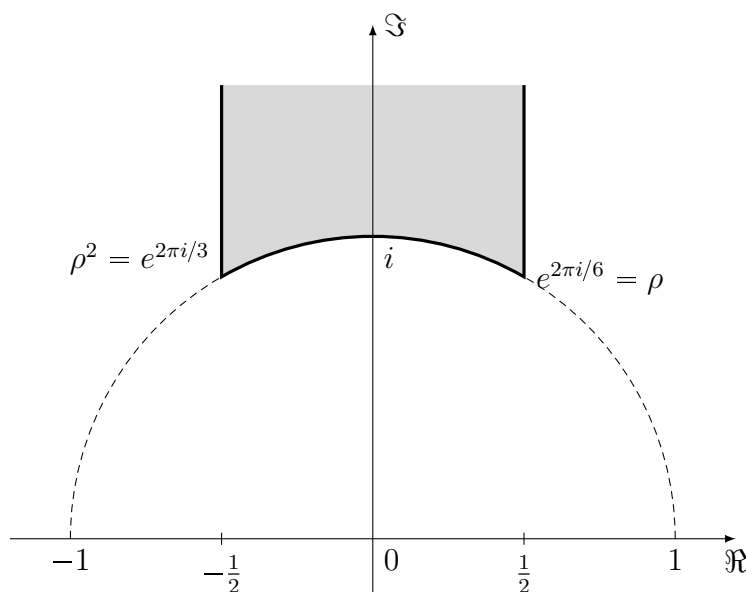
Proof. This is because of *associativity*:

$$(SO_2(\mathbb{R})g) \cdot h = (SO_2(\mathbb{R})) \cdot (gh) \longrightarrow (gh)^{-1}(i) = h^{-1}(g^{-1}(i))$$

giving the result. □

2 Fundamental domain for $\Gamma = SL_2(\mathbb{Z})$ on \mathfrak{H}

Here is a picture of the fundamental domain \mathfrak{H}/Γ .



The goal of this section is to prove that under the action of the $\Gamma = SL_2(\mathbb{Z})$, we can "move" every points on the upper half plane to a domain, under an equivalence given by a specific action. This is similar to the fundamental domain given by the translation action of \mathbb{Z} to \mathbb{R} is the half-open unit interval $[0, 1)$. In general, this give a simpler description to the homogenous space of lattice. Note that when we try to compute the fundamental domain of $\mathbb{Z} \backslash \mathbb{R}$, we have \mathbb{Z} plays a role of *discrete* subset of \mathbb{R} . We give a precise definition of discreteness as follows

Definition 2.1. *Let a group G act continuously on a topological space X . A subset $\Gamma \subset G$ is called **discrete** if for any two compact subse A, B in X , there are only finitely many $g \in \Gamma$ such that $g \circ A \cap B \neq \emptyset$.*

We will prove that the set

$$\Gamma = SL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\}$$

is a discrete subgroup of $G = SL_2(\mathbb{R})$. To prove this, we first need the following lemma

Lemma 2.1. *Fix a real number $r > 0$ and $0 < \delta < 1$. We denote $R_{r,\delta}$ the rectangle*

$$R_{r,\delta} = \{z = x + iy : -r \leq x \leq r, 0 < \delta \leq y \leq \delta^{-1}\}$$

Then for any $\epsilon > 0$ and any fixed set \mathbb{S} of coset representatives for $\Gamma_\infty \backslash \Gamma$, there are finitely many $g \in \mathbb{S}$ such that $\Im(g \circ z) > \epsilon$ for some $z \in R_{r,\delta}$.

In the above lemma, the notation Γ_∞ is defined to be the set

$$\Gamma_\infty = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}.$$

It can be seen easily that this is the stability group of ∞ in \mathfrak{H} .

Proof. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then for $z \in R_{r,\delta}$,

$$\text{Im}(g \circ z) = \frac{y}{c^2 y^2 + (cx + d)^2} < \epsilon$$

if $|c| > (y\epsilon)^{-\frac{1}{2}}$. On the other hand, for $|c| \leq (y\epsilon)^{-\frac{1}{2}} \leq (\delta\epsilon)^{-\frac{1}{2}}$, we have

$$\frac{y}{(cx + d)^2} < \epsilon$$

if the following inequalities hold:

$$|d| > |c|r + (y\epsilon^{-1})^{\frac{1}{2}} \geq |c|r + (\epsilon\delta)^{-\frac{1}{2}}.$$

Consequently, $\Im(g \circ z) > \epsilon$ only if

$$|c| \leq (\delta\epsilon)^{-\frac{1}{2}} \quad \text{and} \quad |d| \leq (\epsilon\delta)^{-\frac{1}{2}}(r + 1),$$

and the total number of such pairs (not counting $(c, d) = (0, \pm 1), (\pm 1, 0)$) is at most $\frac{4(r+1)}{(\epsilon\delta)}$. This proves the lemma. \square

It follows from Lemma 2.1 that $\Gamma = SL(2, \mathbb{Z})$ is a discrete subgroup of $SL(2, \mathbb{R})$. This is because:

1. It is enough to show that for any compact subset $A \subset \mathfrak{H}$ there are only finitely many $g \in SL(2, \mathbb{Z})$ such that $(g \circ A) \cap A \neq \emptyset$;
2. Every compact subset of $A \subset \mathfrak{H}$ is contained in a rectangle $R_{r,\delta}$ for some $r > 0$ and $0 < \delta < \delta^{-1}$;
3. $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} = \emptyset$, except for finitely many $\alpha \in \Gamma_\infty$, $g \in \Gamma_\infty \setminus \Gamma$.

To prove (3), note that Lemma 2.1 implies that $(g \circ R_{r,\delta}) \cap R_{r,\delta} = \emptyset$ except for finitely many $g \in \Gamma_\infty \setminus \Gamma$. Let $S \subset \Gamma_\infty \setminus \Gamma$ denote this finite set of such elements g . If $g \notin S$, then Lemma 2.1 tells us that it is because $\Im(g \circ z) < \delta$ for all $z \in R_{r,\delta}$. Since $\Im(\alpha g \circ z) = \Im(g \circ z)$ for $\alpha \in \Gamma_\infty$, it is enough to show that for each $g \in S$, there are only finitely many $\alpha \in \Gamma_\infty$ such that $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} \neq \emptyset$. This last statement follows from the fact that $g \circ R_{r,\delta}$ itself lies in some other rectangle $R_{r',\delta'}$, and every $\alpha \in \Gamma_\infty$ is of the form $\alpha = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$ ($m \in \mathbb{Z}$), so that

$$\alpha \circ R_{r',\delta'} = \{x + iy \mid -r' + m \leq x \leq r' + m, 0 < \delta' \leq \delta'^{-1}\},$$

which implies $(\alpha \circ R_{r',\delta'}) \cap R_{r,\delta} = \emptyset$ for $|m|$ sufficiently large. Now we are ready to describe the fundamental domain for $SL_2(\mathbb{Z}) \setminus \mathfrak{H}$.

Proposition 2.1. *A fundamental domain for $SL_2(\mathbb{Z}) \setminus \mathfrak{H}$ can be given as the region*

$$\mathfrak{D} = \{z = x + iy \in \mathfrak{H} : |z| \geq 1, -1/2 \leq x \leq 1/2\}$$

, modulo the congruent boundary points symmetric with respect to the imaginary axis.

Proof. First we eliminated the repeated points on the boundary. Note that the line $x = -1/2$ is the same as the line $x = 1/2$ under the transformation $z \mapsto z + 1$. Similarly, given a point on the circle $\{|z| = 1\}$, the transformation $z \mapsto -|z|^{-1}$ satisfies

$$\frac{-1}{x + iy} = \frac{-x + iy}{x^2 + y^2} = -x + iy,$$

which flips the sign of x . Thus it identifies the half circle on the right of the imaginary axis with that on the left.

Now we need to show two things:

1. For any $z \in \mathfrak{H}$ we can find an element $g \in SL_2(\mathbb{Z})$ such that $g \circ z \in \mathfrak{D}$.
2. If $z \equiv z' \in \mathfrak{D}$ modulator $SL_2(\mathbb{Z})$, then either $\Re(z) = \pm \frac{1}{2}$ and $z' = z \mp 1$, or $|z| = 1$ and $z' = \frac{-1}{\bar{z}}$.

First we prove for (1): Fix $z \in \mathfrak{H}$. It follows from Lemma 2.1 that for every $\epsilon > 0$, there are at most finitely many $g \in SL(2, \mathbb{Z})$ such that $g \circ z$ lies in the strip

$$D_\epsilon := \left\{ w \mid -\frac{1}{2} \leq \Re(w) < \frac{1}{2}, \epsilon \leq \Im(w) \right\}.$$

Let B_ϵ denote the finite set of such $g \in SL(2, \mathbb{Z})$. Clearly, for sufficiently small ϵ , the set B_ϵ contains at least one element. We will show that there is at least one $g \in B_\epsilon$ such that $g \circ z \in D$. Among these finitely many $g \in B_\epsilon$, choose one such that $\Im(g \circ z)$ is maximal in D_ϵ . If $|g \circ z| < 1$, then for $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ we have, for any m ,

$$\Im(T^m S g \circ z) = \Im\left(\frac{-1}{g \circ z}\right) = \frac{\Im(g \circ z)}{|g \circ z|^2} > \Im(g \circ z)$$

But we can choose m such that $T^m S g \circ z \in D_\epsilon$, which contradicts the maximality of $\Im(g \circ z)$.

Next we give a proof for (2): Let $z \in D$, $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z})$, and assume that $g \circ z \in D$. Without loss of generality, we may assume that

$$\Im(g \circ z) = \frac{y}{|cz + d|^2} \geq \Im(z),$$

(otherwise just interchange z and $g \circ z$ and use g^{-1}). This implies that $|cz + d| \leq 1$ which implies that $1 \geq |cy| \geq \frac{1}{\sqrt{3}}|c|$. This is clearly impossible if $|c| \geq 2$. So we only have to consider the cases $c = 0, \pm 1$. If $c = 0$ then $d = \pm 1$ and g is a translation by b . Since $-\frac{1}{2} \leq \Re(z), \Re(g \circ z) \leq \frac{1}{2}$, this implies that either $b = 0$ and $z = g \circ z$ or else $b = \pm 1$ and $\Re(z) = \pm \frac{1}{2}$ while $\Re(g \circ z) = \mp \frac{1}{2}$. If $c = 1$, then $|z + d| \leq 1$ implies that $d = 0$ unless $z = e^{2\pi i/3}$ and $d = 0, -1$. The case $d = 0$ implies that $|z| \leq 1$ which implies $|z| = 1$. Also, in this case, $c = 1, d = 0$, we must have $b = -1$ because $ad - bc = 1$. Then $g \circ z = a - \frac{1}{z+1}$. It follows that $g \circ z = a - e^{2\pi i/3}$ and $d = 1$, then we must have $a - b = 1$. It follows that $g \circ z = a - \frac{1}{z+1} = a + e^{2\pi i/3}$, which implies that $a = 0$ or 1 . A similar argument holds when $z = e^{\pi i/3}$ and $d = -1$. Finally, the case $c = -1$ can be reduced to the previous case $c = 1$ by reversing the signs of a, b, c, d . \square