

# Compactification in low dimension

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In this expository note, I will try to explain explicitly how to compactify  $\Gamma \backslash \mathbb{H}$  by adding points in two ways.

## 1 Some preparations

We will always denote  $\Gamma$  a subgroup of the group  $SL_2(\mathbb{Z})$  of finite index, and this group acts on the upper half complex plane  $\mathbb{H}$  by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ z := \frac{az + b}{cz + d}$$

When  $z$  tends to infinity, we have

$$\lim_{z \rightarrow \infty} \frac{az + b}{cz + d} = \frac{a}{c},$$

so we add the rational line to define the action of this group at  $\infty$ . In particular, we consider the set

$$\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$$

Note that on the projective rational line, we define the action to be the multiplication of a  $2 \times 2$  matrix with a  $2 \times 1$  vector. Then under this action, we have the following lemma

**Lemma 1.**  $SL_2(\mathbb{Z})$  acts transitively on  $\mathbb{P}^1(\mathbb{Q})$ .

*Proof.* For each point in  $\mathbb{P}^1(\mathbb{Q})$ , we can choose the representative to be of the form  $[a : b]$ , where  $\gcd(a, b) = 1$ . Then there exists  $x, y \in \mathbb{Z}$  such that

$$ax - by = 1$$

Thus we get the following equality

$$\begin{bmatrix} b & a \\ -x & y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This implies any points in  $\mathbb{P}^1(\mathbb{Q})$  can be moved to  $[0 : 1]$ , and thus the action is transitive.  $\square$

**Corollary 2.** If  $\Gamma$  is a subgroup of finite index in  $SL_2(\mathbb{Z})$  then  $\Gamma \backslash \mathbb{P}^1(\mathbb{Q})$  has only finite orbits.

## 2 Compactification of $\Gamma \backslash \mathbb{H}$ by adding points.

We introduce a topology on  $\overline{\mathbb{H}}$ . For the usual upper half plane, the topology is the usual metric topology on  $\mathbb{C}$ , and we only define the system of the neighborhood of  $r \in \mathbb{P}^1(\mathbb{Q})$ .

Let  $S(c, \omega)$  be the circle that touches the real line at  $\omega = p/q$  and has the radius  $\frac{c}{2q^2}$ . Then the collection of circles  $D(c, \omega) = \bigcup_{0 < c' \leq c} S(c', \omega)$  is called *Farey disk*. Let  $c \rightarrow 0$ , these disks define a neighborhood of  $\omega$ . The Farey disks at  $\infty$  are defined to be the region

$$D(T, \infty) = \{z : \Im z \geq T\}$$

It can be checked easily that the matrix under inversion, the Farey disk at  $\infty$  is mapped to  $D(1/T, 0)$ . In general, if  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  such that  $\gamma \circ \infty = \omega$  then  $D(T, \infty)$  is mapped to  $D(1/T, \omega)$ . With the above topology on the extended upper half plane, we could show that

**Lemma 3.**  $\Gamma \backslash \overline{\mathbb{H}}$  is a compact set.

The proof is taken from [1] and I rewrite it here for completeness.

*Proof.* We first prove for the case  $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ . It is well known that the quotient space  $\Gamma \backslash \mathbb{H}$  is identical to the set

$$\mathcal{F} = \{z \in \mathbb{H} : \Re z \in [-1/2, 1/2), |z| \geq 1 \text{ and } |z| > 1 \text{ if } \Re z > 0\}$$

By lemma 1, the projective rational line "shrinks" to a point under the action of  $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ . Thus we can identify  $\Gamma \backslash \overline{\mathbb{H}}$  with the set  $\tilde{\mathcal{F}} = \mathcal{F} \cup \{\infty\}$ . Consider an open cover  $\{U_i\}_{i \in I}$  of  $\tilde{\mathcal{F}}$  and the natural projection  $\pi: \overline{\mathbb{H}} \rightarrow \tilde{\mathcal{F}}$ . Then the set  $\{\pi^{-1}(U_i)\}_{i \in I}$  forms an open cover of  $\overline{\mathbb{H}}$ . There must be an index  $i_0$  such that  $\pi^{-1}(U_{i_0})$  contains a neighborhood of  $\infty$ , namely contains a Farey disk  $D(T, \infty)$  for some  $T > 0$ . Since  $\tilde{\mathcal{F}} - D(T, \infty)$  is a compact set, its image under  $\pi$  is compact, hence it can be covered by  $U_{i_1}, \dots, U_{i_m}$ . Altogether,  $\tilde{\mathcal{F}}$  admits a finite subcover  $U_{i_0}, \dots, U_{i_m}$ . Now we proceed to the general case. Note that

$$\overline{\mathbb{H}} = \mathrm{SL}_2(\mathbb{Z}) \circ \tilde{\mathcal{F}} = \bigcup \Gamma a_i \circ \tilde{\mathcal{F}}$$

by corollary 2. Then under the surjective map  $\pi: \overline{\mathbb{H}} \rightarrow \Gamma \backslash \overline{\mathbb{H}}$ , we have

$$\Gamma \backslash \overline{\mathbb{H}} = \bigcup \pi \left( \Gamma a_i \circ \tilde{\mathcal{F}} \right),$$

which shows that the set  $Y(\Gamma) = \Gamma \backslash \overline{\mathbb{H}}$  is compact as it is the union of compact sets.  $\square$

The orbit of  $\mathbb{P}^1(\mathbb{Q})$  under the action of  $\Gamma$  is called *cusps*. We have the obvious equality that

$$\Gamma \backslash \overline{\mathbb{H}} = \Gamma \backslash \mathbb{H} \cup \underbrace{\Gamma \backslash \mathbb{P}^1(\mathbb{Q})}_{\text{cusps}}$$

So in fact lemma 3 tells us that we only need to add a finite cusp to get a compact domain. That means we only need to consider the actions of  $\Gamma$  on the projective rational line. By the orbit-stabilizer theorem, we get the decomposition

$$\Gamma \backslash \bigcup_{\omega} D(c, \omega) = \bigcup \Gamma_{\omega_i} \backslash D(c, \omega_i)$$

where  $\omega_i$  is the set of representative for the action of  $\Gamma$  on  $\mathbb{P}^1(\mathbb{Q})$  and  $\Gamma_{\omega_i}$  are the stabilizer of  $\omega_i \in \Gamma$ .

Again, since the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathbb{P}^1(\mathbb{Q})$  is transitive, for each  $r \in \mathbb{P}^1(\mathbb{Q})$ , there exists an element  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  such that  $\gamma \circ \infty = r$ . So we have  $\Gamma_r = \gamma \Gamma_{\infty} \gamma^{-1}$ . Hence we only need to know the "shape" of the domain  $\Gamma_{\infty} \backslash D(T, \infty)$ . WLOG, we could assume  $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ , and hence

$$\Gamma_{\infty} = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}$$

Geometrically,  $\Gamma_{\infty} \backslash D(T, \infty)$  is the strip  $\{\Re z \in [-1/2, 1/2), \Im z \geq T\}$ . But this is biholomorphic to a closed disk that misses a point on the boundary. So compactification is obtained by filling in the missing points to get finitely many compact disks.

### 3 Borel - Serre compactification of $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$

We consider another compactification, by looking at the Farey disk  $D(c, \omega)$  for fixed parameters  $c, \omega$ . Then for any points  $y \neq \omega$  on the Farey circle  $S(c, \omega)$ , we could connect  $y$  with  $\omega$  by a unique geodesic in the upper half-plane.

These geodesics are either upper half circles that are orthogonal to the real line or the vertical line passing through  $\omega$ . We thus can identify the Farey disks as follows

$$D(c, \omega) - \{\omega\} = X_{\infty, \omega} \times (0, c],$$

since a point  $\theta$  on the Farey circles  $S(c, \omega)$  is defined by its radius, up to a scaling of  $c$ , and the intersection of the geodesic  $\overline{\theta\omega}$  with the real line. The uniqueness of the geodesics gives us a bijection between two sets. Here we let  $X_{\infty, \omega} = \mathbb{P}^1(\mathbb{R}) - \{\omega\}$

How does the group  $\Gamma$  act on the set on the RHS set in the above identification? First, we look at the special case where  $\omega = \infty$ . In this case, the identification is

$$D(T, \infty) - \infty = X_{\infty, \infty} \times [T, \infty)$$

On the left, stabilizer subgroup  $\Gamma_\infty$  can be thought of as a subgroup of the group of translation, which leaves all the Farey circles  $S(t, \infty)$  - which are the line  $\{\Im z = t \geq T\}$  in this case - intact. Thus on the right-hand side, the action of  $\Gamma_\infty$  only affects the first coordinate. In general case, we need a lemma

**Lemma 4.** *If  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  such that  $\gamma \circ \infty = \omega$  then  $\gamma \circ D(T, \infty) = D(1/T, \omega)$ .*

Assume lemma 4 with the note that  $\Gamma_\omega = \gamma \Gamma_\infty \gamma^{-1}$ , we conclude that the action of  $\Gamma_\omega$  only affects  $X_{\infty, \omega}$  for all  $\omega \in \mathbb{P}^1(\mathbb{Q})$ . Since  $\Gamma_\omega \backslash X_{\infty, \omega}$  is a circle, it is compact. Hence we can compactify the quotient space  $\Gamma_\omega \backslash D(c, \omega) - \{\omega\}$  as

$$\Gamma_\omega \backslash D(c, \omega) - \{\omega\} \hookrightarrow \Gamma_\omega \backslash X_{\infty, \omega} \times [0, c]$$

As in section 1, we only need to compactify finitely many such quotient spaces and get the compactification of  $\Gamma \backslash \mathbb{H}$ .

Now we give a proof of lemma 4

*Proof.* Assume  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  is an element that sends  $\infty$  to  $\omega = \frac{p}{q}$ . Let  $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}.$$

Thus we must have  $a = p, c = q$  and  $b, d$  are integers such that  $aq - cp = 1$ . A Farey circle in the neighborhood of  $\infty$  is, in fact, a line  $S(T, \infty) = \{\Im z = T\}$ , and this line is mapped to a circle tangent to the real line. Direct calculation shows that, for  $z = x + iT$

$$\Im(\gamma \circ z) = \frac{\Im z}{|cz + d|^2} = \frac{T}{(cx + d)^2 + c^2 T^2} \leq \frac{1}{q^2 T}$$

The equality happens if  $x = -d/c$ . Since this  $\gamma \circ z$  is a point on the circle tangent to the real line at  $p/q$  and has the largest distance to the real line, the segment connect  $p/q$  and  $\gamma \circ z$  must be the diameter of the image circle. In particular, the radius of the image circle is  $\frac{1}{2Tq^2}$ . Lemma 4 follows immediately.  $\square$

The above process can be applied to finitely many Farey disks as in section 2 to get a compactification of  $\Gamma \backslash \mathbb{H}$ .

## 4 Equivalent definitions of semi-stability

In this section, we give two definitions of semi-stability in  $\mathbb{H}$ , and show that they are essentially the same. We will also compute the semi-stable locus in  $\mathbb{H}$ .

We first introduce some terminology: for each complex number  $z \in \mathbb{H}$ , we assign to it a lattice in  $\mathbb{C}$ , where the lattice is spanned by two vectors  $\{z, 1\}$ . By identifying  $\mathbb{C} \cong \mathbb{R}^2$ , we can compute the volume of the fundamental domain given by this lattice is  $y = \Im(z)$ . Then we scale two vectors in the basis to get a unit lattice.

Following this process, each  $z \in \mathbb{H}$  is assigned with a unique unit lattice, namely  $\Gamma_z = \text{span}_{\mathbb{Z}} \{a, z/a\}$ , where  $a = \Im(z)$ . Now we are ready to define semi-stability. Furthermore, to each lattice, we assigned to it a plot in the following way:

- We start with the point  $(0, 0)$  in the plane.
- Let  $u$  be the shortest vector in the lattice  $\Gamma_z$ , we highlight the point  $(1, \log |u|)$  in the plane.
- Finally, we attached the point  $(2, \log(\text{vol}(A)))$ , where  $A$  is the fundamental domain of  $\Gamma_z$ .
- We connect consecutive points by line segments. The union of these line segments is called *profile* of the lattice.

In our setting, since we already normalize all lattices to unit lattice, the final point is in fact  $(2, 0)$ .

**Definition 1.** *The lattice assigned to the number  $z \in \mathbb{H}$  is call semi-stable if and only if the point  $(1, \log |u|)$  lies above the  $x$ -axis.*

Before giving the second definition of semi-stability, we will try to find the semi-stable locus using this definition. First, we restrict our attention to the fundamental domain

$$\mathcal{F} = \{z \in \mathbb{H} : \Re z \in [-1/2, 1/2), |z| \geq 1 \text{ and } |z| > 1 \text{ if } \Re z > 0\}$$

For each point  $z \in \mathcal{F}$ , it is easily to see that the shortest vector in the corresponding lattice is  $a$ . Thus the 1-dim point is  $(1, \log(a)) = (1, -\log(y)/2)$ . By definition,  $\Gamma_z$  is semi-stable iff  $\log(y) \leq 0$ , i.e.  $y \geq 1$ . To find the semi-stable locus in the whole upper half plane, we need the following result

**Lemma 5.** *If  $\Gamma_z$  is semi-stable, then so is the lattice  $\Gamma_{g \circ z}$ , where  $g \in \text{SL}_2(\mathbb{Z})$ .*

*Proof.* If we denote  $L_z = \text{span}_{\mathbb{Z}} \{1, z\}$ , then  $L_{\gamma \circ z} = cL_z$  for some complex number  $c$ . Indeed, we just need to check for  $\gamma$  be an inversion or translation, since these two transformations generate  $\text{SL}_2(\mathbb{Z})$ , but this is easy. Now let  $c = re^{it}$ . Multiplying by  $e^{it}$  doesn't change the length, hence doesn't change the semi-stability. Multiplying by a positive number  $r$  will shift  $(1, \log |u|)$  to  $(1, \log |u| + \log r)$  and  $(2, \log(\text{vol}(A)))$  to  $(2, \log(\text{vol}(A)) + 2 \log r)$ .

The line segment  $d$  connecting origin with the final point intersect the line  $x = 1$  at  $(1, \log(\text{vol}(A)) + \log r)$ . By the semi-stability of the original lattice the point  $(1, \log |u| + \log r)$  is above the line segment  $d$ .  $\square$

From this lemma, we could see that the semi-stable locus is the complement of the Farey balls in the upper half plane.

Now we give the second definition of semi-stability. First we note that  $\mathbb{H}$  can be identified with the set  $\text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R})$ . Using Iwasawa's decomposition, we could identify  $z = x + iy$  with the pair  $(a(z), n(z))$  where

$$a(z) = \begin{bmatrix} y^{-1/2} & 0 \\ 0 & y^{1/2} \end{bmatrix} \quad n(z) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

Then we can define a map

$$H_B: \mathbb{H} \rightarrow \mathfrak{sl}_2, \quad z \mapsto \log(a(z))H,$$

where  $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ . Then we define a unique linear map  $\alpha: \mathbb{R}H \rightarrow \mathbb{C}$  such that  $\alpha(H) = 2$ . Set  $\rho = \alpha/2$ . For each  $T = -kH$  where  $k > 0$ , we define the degree of instability of  $x \in \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$  as follows

$$\deg_{\mathrm{inst}}^T(x) := \min_{\gamma \in \Gamma/\Gamma \cap B} \langle \rho, H_B(x\gamma) - T \rangle$$

In this particular case, the RHS of the above definition can be simplified as

$$\deg_{\mathrm{inst}}^T(x) = \min_{\gamma \in \Gamma} \log(a(x\gamma) + k)$$

For the case  $k = 0$ , the point  $x$  is call semi-stable iff  $\deg_{\mathrm{inst}}^T(x) \geq 0$ . Since the minimum is always achieved, this implies  $\frac{-1}{2} \log(\Im(x\gamma)) \geq 0$ , i.e.  $\Im(x\gamma) \leq 1$  for all  $\gamma$ .

A key observation is that the minimum of  $a(z)$  is achieved inside the Siegel's set, which is exactly the fundamental domain  $\mathcal{F}$  in this case. This implies the semi-stable locus inside the fundamental domain is the intersection of  $\mathcal{F}$  with  $\{\Im(z) < 1\}$ . So the semi-stable locus is the union of orbits who has a representative in this part.

## References

- [1] Diamond, F., & Shurman, J. M. (2005). A first course in modular forms (Vol. 228, pp. xvi-436). New York: Springer.