Lie theory - homework 1

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Problem 1

1. By definition, we only need to check that

$$[[a,b],c] \in \mathfrak{g} \quad \forall a,b \in \mathfrak{h}, c \in \mathfrak{g},$$

but this is clear as $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, we could use Jacobi's identity to get

$$[[a,b],c] = [b,[c,a]] + [a,[b,c]] \in [\mathfrak{h},\mathfrak{h}].$$

2. Recall that $\mathcal{D}^{k+1}\,\mathfrak{g} = [\mathfrak{g}^k, \mathfrak{g}^k]$. Clearly $\mathfrak{g}$ is itself an ideal, so the fact that $\mathcal{D}^{k+1}\,\mathfrak{g}$ follows immediately from part a and induction on k.

3. In class, we called $\mathfrak{g}$ semisimple iff it has no nontrivial solvable ideal. Note that abelian ideals are solvable, hence all abelian ideals are zero if $\mathfrak{g}$ is semisimple. Conversely, assume that $\mathfrak{g}$ is not semisimple, then it has a non trivial solvable ideal $\mathfrak{h}$. In particular, we have a strictly decreasing chain of ideals as follows:

$$\mathfrak{h} = \mathfrak{h}^{(0)} \supset \mathfrak{h}^{(1)} \supset \ldots \supset \mathfrak{h}^{(n)} \supset \mathfrak{h}^{(n+1)} = (0)$$

But this implies that $\mathfrak{h}^{(n)}$ is a non trivial abelian ideal of $\mathfrak{g}$ by part a.

Problem 2 We compute $\operatorname{ad} x$ with respect to this basis. The other two are computed similarly.