

Compactification in low dimension

Tri Nguyen - University of Alberta

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In this expository note, I will try to explain explicitly how to compactify $\Gamma \backslash \mathbb{H}$ by adding points in two ways.

1 Some preparations

We will always denote Γ a subgroup of the group $SL_2(\mathbb{Z})$ of finite index, and this group acts on the upper half complex plane \mathbb{H} by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ z := \frac{az + b}{cz + d}$$

When z tends to infinity, we have

$$\lim_{z \rightarrow \infty} \frac{az + b}{cz + d} = \frac{a}{c},$$

so we add the rational line to define the action of this group at ∞ . In particular, we consider the set

$$\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$$

Note that on the projective rational line, we define the action to be the multiplication of a 2×2 matrix with a 2×1 vector. Then under this action, we have the following lemma

Lemma 1. $SL_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$.

Proof. For each point in $\mathbb{P}^1(\mathbb{Q})$, we can choose the representative to be of the form $[a : b]$, where $\gcd(a, b) = 1$. Then there exists $x, y \in \mathbb{Z}$ such that

$$ax - by = 1$$

Thus we get the following equality

$$\begin{bmatrix} b & a \\ -x & y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This implies any points in $\mathbb{P}^1(\mathbb{Q})$ can be moved to $[0 : 1]$, and thus the action is transitive. \square

Corollary 2. If Γ is a subgroup of finite index in $SL_2(\mathbb{Z})$ then $\Gamma \backslash \mathbb{P}^1(\mathbb{Q})$ has only finite orbits.

2 Compactification of $\Gamma \backslash \mathbb{H}$ by adding points.

We introduce a topology on $\overline{\mathbb{H}}$. For the usual upper half plane, the topology is the usual metric topology on \mathbb{C} , and we only define the system of the neighborhood of $r \in \mathbb{P}^1(\mathbb{Q})$.

Let $S(c, \omega)$ be the circle that touches the real line at $\omega = p/q$ and has the radius $\frac{c}{2q^2}$. Then the collection of circle $D(c, \omega) = \bigcup_{0 < c' \leq c} S(c', \omega)$ is called *Farey disk*. Let $c \rightarrow 0$, these disks define a neighborhood of ω . The Farey disks at ∞ are defined to be the region

$$D(T, \infty) = \{z : \Im z \geq T\}$$

It can be checked easily that the matrix under inversion, the Farey disk at ∞ is mapped to $D(1/T, 0)$. In general, if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \circ \infty = \omega$ then $D(T, \infty)$ is mapped to $D(1/T, \omega)$.

With the above topology on the extended upper half plane, we could show that

Lemma 3. $\Gamma \backslash \overline{\mathbb{H}}$ is a compact set.

Proof. We first prove for the case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. It is well known that the quotient space $\Gamma \backslash \mathbb{H}$ is identical to the set

$$\mathcal{F} = \{z \in \mathbb{H} : \Re z \in [-1/2, 1/2), |z| \geq 1 \text{ and } |z| > 1 \text{ if } \Re z > 0\}$$

By lemma 1, the projective rational line "shrinks" to a point under the action of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, Thus we can identify $\Gamma \backslash \overline{\mathbb{H}}$ with the set $\tilde{\mathcal{F}} = \mathcal{F} \cup \{\infty\}$. Consider an open cover $\{U_i\}_{i \in I}$ of $\tilde{\mathcal{F}}$ and the natural projection $\pi: \overline{\mathbb{H}} \rightarrow \tilde{\mathcal{F}}$. Then the set $\{\pi^{-1}(U_i)\}_{i \in I}$ forms an open cover of $\overline{\mathbb{H}}$. There must be an index i_0 such that $\pi^{-1}(U_{i_0})$ contains a neighborhood of ∞ , namely contains a Farey disk $D(T, \infty)$ for some $T > 0$. Since $\overline{\mathcal{F}} - D(T, \infty)$ is a compact set, its image under π is compact, hence it can be covered by U_{i_1}, \dots, U_{i_m} . Altogether, $\tilde{\mathcal{F}}$ admits a finite subcover U_{i_0}, \dots, U_{i_m} .

Now we proceed to the general case. Note that

$$\overline{\mathbb{H}} = \mathrm{SL}_2(\mathbb{Z}) \circ \tilde{\mathcal{F}} = \bigcup \Gamma a_i \circ \tilde{\mathcal{F}}$$

by corollary 2. Then under the surjective map $\pi: \overline{\mathbb{H}} \rightarrow \Gamma \backslash \overline{\mathbb{H}}$, we have

$$\Gamma \backslash \overline{\mathbb{H}} = \bigcup \pi \left(\Gamma a_i \circ \tilde{\mathcal{F}} \right),$$

which shows that the set $Y(\Gamma) = \Gamma \backslash \overline{\mathbb{H}}$ is compact as it is the union of compact sets. \square

The orbit of $\mathbb{P}^1(\mathbb{Q})$ under the action of Γ is called *cusps*. We have the obvious equality that

$$\Gamma \backslash \overline{\mathbb{H}} = \Gamma \backslash \mathbb{H} \cup \underbrace{\Gamma \backslash \mathbb{P}^1(\mathbb{Q})}_{\text{cusps}}$$

So in fact lemma 3 tells us that we only need to add finite cups.