CHAPTER II :ROOTS AND WEIGHTS FOR $\mathrm{SL_n}(\mathbb{R})/\mathrm{GL_n}(\mathbb{R})$

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In this chapter, we review some basis theory of roots and weight. We will first recall the general theory and compute explicitly the examples for $\mathrm{SL_n}(\mathbb{R})/\mathrm{GL_n}(\mathbb{R})$.

1 Structure theory

1.1 The Cartan subalgebra

First we need the notion of Cartan subalgebra

Definition 1.1. *For any Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be Cartan algebra if it is*

- *$\mathfrak{h}$ is a nilpotent subalgebra.*
- *It is self normalizing. In particular, we have $\mathfrak{h} = \{x \in \mathfrak{g} : [x, \mathfrak{g}] \subset \mathfrak{g}\}$.*

When $\mathfrak{g}$ is a semisimple Lie algebra, we have the following theorem

Theorem 1.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field k of charateristic 0 with a subalgebra $\mathfrak{h}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ if and only if it is a maximal toral subalgebra, i.e. is is maximal among all subalgebras contanining only semisimple elements.*

add citation.

1.2 Root space decomposition

With respect to some choice of Cartan subalgebra, we have a root space decomposition. In particular, there is a finite set $\Phi \subset \mathfrak{h}^*$ of linear forms on H, whose elements are called **roots**, such that

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right),$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \forall h \in \mathfrak{h}\}$ for any $\alpha \in \Phi$.

1.3 A specific example: root space decomposition for $\mathfrak{sl}_n(\mathbb{R})$

For the semisimple Lie algebra $\mathfrak{sl}_\mathrm{n}(\mathbb{R})$, a typical choice of the Cartan subalgebra is the set

$$\mathfrak{h} = \left\{ H = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}, a_1 + a_2 + \ldots + a_n = 0 \right\}$$

With respect to this Cartan subalgebra, we can define the linear function

$$L_i \colon \mathfrak{h} \to \mathbb{R}, \quad H \mapsto L_i(H) = a_i$$

Then the roots are given by $\alpha_{ij} := L_i - L_j$ for distinct i, j. We have the root space decomposition for $\mathfrak{sl}_n(\mathbb{R})$ as follows

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus \mathfrak{g}_{\alpha_{ij}} \right).$$

For the sake of brevity, we will denote $\alpha_{i,i+1}$ by α_i - these are called **simple roots**.

1.4 Roots at group level

Since the main object in this thesis is the Lie groups, we want to understand how the roots behave at group level. The analog for the Cartan subalgebra is the maximal torus

$$T = \left\{ t = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix} : a_i \neq 0 \right\},$$

Then T acts on $\mathfrak{g}$ by conjugation. Explicitly, we can check that

$$\mathrm{Ad}(t)(E_{ij}) = t_i t_j^{-1} E_{ij}$$

Therefore, at the group level, the character $\alpha_{ij}(\mathrm{diag}(t_1, \dots, t_n)) = t_i t_j^{-1}$ is a root whenever $i \neq j$. The set

$$\Delta = \left\{ \alpha_i \mid i = \overline{1, n} \right\}$$

where

$$\alpha_i \colon T \mapsto \mathbb{R}, t \mapsto \frac{t_i}{t_{i+1}}$$

is the set of **simple roots**. We can decompose the set of root into to disjoint subsets, namely

$$\Phi = \{\alpha_{ij}, i \neq j\} = \Phi_+ \coprod \Phi_-$$

where the set Φ_+ comprises of α_{ij} for $i < j$ and the remaining roots are in Φ_-. The former comprises of **positive roots** while the latter contains **negative roots**. We have the following lemma

Lemma 1.3. *Each $\alpha \in \Phi$ can be written uniquely as a linear combination*

$$\alpha = m_1 \alpha_1 + \ldots + m_d \alpha_d$$

with all $m_i \in \mathbb{Z}_{\geqslant 0}$ or $m_i \in \mathbb{Z}_{\leqslant 0}$. If $\alpha \in \Phi_+$ then all $m_i \geqslant 0$, otherwise $m_i \leqslant 0$ for all i.

1.5 Weights

define in term of Lie algebra

Another class of linear forms that we are interested in are the **fundamental weights**. For each fundamental weights λ_i, we define

$$\lambda_i \colon T \to \mathbb{R}, \quad \lambda_i(t) = a_1 \ldots a_i$$

We have the following

Lemma 1.4. *We can write*

$$\lambda_i := r_1 \alpha_1 + r_2 \alpha_2 + \ldots + r_d \alpha_d$$

where r_i's are rational number such that $r_i \geqslant 0$.add proof

Example 1.5. *When $n = 3$, we have the following relations add picture*

$$\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \quad \lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$$

Definition 1.6. *A weight λ is called* ***dominant*** *if it satisfies $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geqslant 0}$ for all α*

Clearly by lemma 1.3, the weight λ is dominant if and only if $\langle \lambda, \alpha_i^\vee \rangle$ for all fundamental root α_i. It is also clearly that the set of fundamental weight is given by addition of the fundamental weights, namely

$$\Lambda^+ := \{c_1\lambda_1 + \ldots + c_d\lambda_d \mid c_i \in \mathbb{Z}_{\geqslant 0}\}$$

The set of dominant weights is denoted Λ^+. A weight $\lambda = \sum n_i\lambda_i$ is called strongly dominant if $n_i > 0$ for all i. One important example is the minimal strongly dominant weight given by

$$\rho = \sum \lambda_i$$

This is called **Weyl vector** and is characterized in several ways:

1. $\langle \rho, \alpha_i^\vee \rangle = 1$ for all i.
2. $$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

To prove the last equation we use the action of the Weyl group W. Let $\mu = \frac{1}{2}\sum \alpha$. Apply the simple reflection s_i given by

$$s_i(x) = x - \langle x, \alpha_i^\vee \rangle \alpha_i$$

We know that s_i sends α_i to $-\alpha_i$ and permutes the other positive roots. So:

$$s_i(\mu) = \mu - \langle x, \alpha_i^\vee \rangle \alpha_i$$

Therefore, $(\mu, \alpha_i) = \mu(h_i) = 1$ for all i. So, $\mu = \rho$.
Unlike lemma 1.4, if we try to express the fundamental weights in term of the fundamental roots, we don't always get positive coefficients. However, it is true that all the coefficients must be integer. In particular, we have

$$\alpha_j = \sum_{n_j} \lambda_j, \quad n_j \in \mathbb{Z}.$$

To put it another way, the root lattice $\mathbb{Z}\Delta$ is contained inside the weight lattice.

1.6 Weyl group

We only define the Weyl group explicitly for the group $\mathrm{SL_n}(\mathbb{R})$ or $\mathrm{GL_n}(\mathbb{R})$. It is a fact that the Weyl groups for these two Lie groups are the same and equal to $W = S_n$ - the permutation group of n letters. We recall some basis observation about this group

1. Every $\sigma \in W$ can be written (non-uniquely) as a product of $w_{i_1} \cdots w_{i_k}$ for some integer k. Such a sequence is said to have length k. If k is the minimum, over all such writings, it is called the length of σ and written $\ell(\sigma)$. Any expression of length $\ell(\sigma)$ for σ is called a reduced expression.
2. The group S_n is generated by S subject to the following two types of relations:
   - (Reflection) $w_i^2 = 1$ for $i \in I$.
   - (Braid relations) $w_i\, w_{i+1}\, w_i = w_{i+1}w_iw_{i+1}$ for $i = 1, \ldots, n-2$ and $w_iw_j = w_jw_i$ for $|j - i| \geqslant 2$.

Note that W acts on $\mathrm{Hom}(H, \mathbb{R}^*)$ in the natural way: $w.\varphi(h) = \varphi(w^{-1}h)$. More explicitly, σ sens $\alpha := \alpha_{ij}$ to $\alpha_{\sigma(i),\sigma(j)}$. Hence we find that

$$w_i\alpha_j = \begin{cases} -\alpha_i & \text{if } i = j \\ \alpha_j & \text{if } |j-i| > 1 \\ \alpha_i + \alpha_j & \text{if } |j-i| = 1 \end{cases}$$

We of course also have an action of W on the weights. For example, one can verify that

$$s_i(\lambda_i) = \lambda_i - \alpha_i \quad \text{and} \quad s_j(\lambda_i) = \lambda_i \text{ for } i \neq j \ .$$

Recall the definition of Weyl vector ρ, we have the following generalized action of Weyl group of ρ:

$$w\rho = \rho - \sum_{\alpha \in \Delta_{w^{-1}}} \alpha,$$

where check this explicitly

$$\Delta_\sigma := \{\alpha \in \Phi_+ \mid \sigma(\alpha) \in \Phi_-\}$$

1.7 Cartan matrix

We fix a set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_d\}$ is define to be the matrix

$$A = \left[\left\langle \alpha_i, \alpha_j^\vee \right\rangle\right]$$

If we let $a_{ij} = \left\langle \alpha_i, \alpha_j^\vee \right\rangle$ then the Cartan matrix has the following simple properties:

Lemma 1.7.

- *For any i, we have $a_{ii} = 2$.*
- *For any $i \neq j$, a_{ij} is a non-positive integer, i.e. $a_{ij} \in \mathbb{Z}_{\leqslant 0}$.*

We give an explicit example for $\mathfrak{sl}_n(\mathbb{R})$, which has the root system A_n. The corresponding Cartan matrix is

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix}$$

2 Parabolic subgroups

2.1 Parabolic sets and parabolic subalgebras

Definition 2.1. *Given a root system Δ. A **parabolic subset** Δ_P is a subset of Δ such that it satisfies the following conditions:*

1. *For any $\alpha \in \Delta$, at most one of the two elements $\alpha, -\alpha$ is contained in Δ_P.*
2. *It is closed, in the sense that, for any two root $\alpha, \beta \in \Delta_P$ such that $\alpha + \beta$ is a root, then $\alpha + \beta \in \Delta_P$.*

The parabolic set parametrizes the parabolic subalgebra with the root system Δ, as given in the following theorem

Theorem 2.2. *Given a semisimple Lie algebra $\mathfrak{g}$ with the root system Δ. There exists a correspondence between parabolic subset of Δ_P of Δ and the subalgebra of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{b}$. The correspondence is given by*

$$\Delta_P \longleftrightarrow \mathfrak{p} := \mathfrak{b} \oplus \bigoplus_{\alpha \in \Delta \backslash \Delta_P} g_\alpha$$

Proof. We refer to [?] for a proof of this fact. □

Example 2.3. *We consider the case* $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$*. Let's denote* $\Pi = \{\alpha, \beta\}$ *a base for the root system of* $\mathfrak{g}$*. It is clear that the set of positive root is* $\Delta_+ \{\alpha, \beta, \gamma\}$*. There are 4 parabolic sets, corresponding to 4 parabolic subalgebra given as follows*

$$\begin{gathered}
\Delta_P = \Delta_+ \longleftrightarrow \mathfrak{p} = \mathfrak{b} \\
\Delta_P = \Delta \cup \{-\alpha\} \longleftrightarrow \mathfrak{p} = \mathfrak{b} \oplus \mathfrak{g}_{-\alpha} \\
\Delta_P = \Delta \cup \{-\beta\} \longleftrightarrow \mathfrak{p} = \mathfrak{b} \oplus \mathfrak{g}_{-\beta} \\
\Delta_P = \Delta \longleftrightarrow \mathfrak{p} = \mathfrak{g}
\end{gathered}$$

2.2 Parabolic subgroups

For our purpose, it is enough to define the standard parabolic subgroups. Similar to the parabolic subalgebras, which is formed by adding some "negative" roots, we have the same situation for parabolic subgroups. It can be seen that the parabolic subalgebra contains elements in the form of block-upper triangular.
Therefore, each parabolic subalgebra of $\mathfrak{g} = \mathfrak{sl}_{\mathrm{n}}(\mathbb{R})$ corresponds to a partition of n. In the same manner, there exists a bijection between each parabolic subgroup of $\mathrm{SL}_{\mathrm{n}}(\mathbb{R})$ and each partition of n. We can therefore define the parabolic subgroup explicitly as follows:

Definition 2.4. *The standard parabolic subgroup associated to the partition* $n = n_1 + n_2 + \cdots + n_r$ *is denoted* $P_{n_1,\ldots,n_r}$ *and is defined to be the group of all matrices of the form*

$$\begin{pmatrix} m_{n_1} & * & \cdots & * \\ 0 & m_{n_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_{n_r} \end{pmatrix},$$

where $m_{n_i} \in \mathrm{GL}(n_i, \mathbb{R})$ *for* $1 \leqslant i \leqslant r$*. The integer* r *is termed the rank of the parabolic subgroup* $P_{n_1,\ldots,n_r}$*.*

Definition 2.5. *The maximal standard parabolic subgroups in* $GL_n(k)$ *corresponds to the stabilizer of the flag of type* $\rho_i = (i, n-i)$*, where* $i = 1, \ldots, n-1$ *of* n*. We will further denote* $Q_i = P_{\rho_i}$ *and* ***MaxParSt*** *the collection of such maximal parabolic subgroups.*

Example 2.6. *Below we list all the standard parabolic subgroup in* $GL_3(\mathbb{R})$ *and* $GL_4(\mathbb{R})$*.*

- *For* $GL_3(\mathbb{R})$*, there are three standard parabolic subgroups corresponding to three partitions of* 3*, namely*

  $$3 = 1 + 1 + 1, \quad 3 = 1 + 2, \quad 3 = 2 + 1$$

  For a partition $(r_1, \ldots, r_{s+1})$*, we denote* $P_{(r_1,\ldots,r_{s+1})}$ *the corresponding parabolic subgroups. Thus we have*

  $$P_{1,1,1} = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\}, \quad P_{1,2} = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\}$$

  $$P_{2,1} = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$$

  Clearly ***MaxParSt*** $= \{P_{2,1}, P_{1,2}\}$.

- *For* $GL_4(\mathbb{R})$*, there are seven standard parabolic subgroups for seven partitions*

  $$4 = 1 + 1 + 1 + 1, \quad 4 = 1 + 1 + 2, \quad 4 = 1 + 2 + 1$$

$$4 = 2 + 1 + 1, \quad 4 = 1 + 3, \quad 4 = 2 + 1, \quad 4 = 3 + 1$$

Explicitly, we have the following subgroups

$$P_{1,1,1,1} = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\}, \qquad P_{1,1,2} = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \right\}$$

$$P_{1,2,1} = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\}, \qquad P_{2,1,1} = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\}$$

$$P_{1,2,1} = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\}, \qquad P_{2,1,1} = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\}$$

$$P_{1,3} = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \right\}, \qquad P_{3,1} = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\}$$

$$P_{2,2} = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \right\}$$

Clearly $\boldsymbol{MaxParSt} = \{P_{1,3}, P_{3,1}, P_{2,2}\}$.

2.3 Langlands decomposition

We fix a partition of n as

$$n = n_1 + n_2 + \ldots + n_k$$

and consider the parabolic subgroup of this type, i.e. the subgroup

$$P_{n_1,\ldots,n_k} = \left\{ \begin{bmatrix} \mathfrak{m}_1 & * & \ldots & * \\ 0 & \mathfrak{m}_2 & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathfrak{m}_k \end{bmatrix} \right\}$$

where $\mathfrak{m}_\mathfrak{i}$ is invertible of size $n_i \times n_i$
This group can be factored as

$$P_{n_1,\ldots,n_k} = M_{n_1,\ldots,n_k} N_{n_1,\ldots,n_k}$$

where

$$N_{n_1,\ldots,n_k} = \left\{ \begin{bmatrix} I_1 & * & \ldots & * \\ 0 & I_2 & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_k \end{bmatrix} \right\} \quad (I_k \text{ is the } n_k \times n_k \text{ identity matrix})$$

and

$$M_{n_1,\ldots,n_k} = \left\{ \begin{bmatrix} \mathfrak{m}_1 & 0 & \ldots & 0 \\ 0 & \mathfrak{m}_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathfrak{m}_k \end{bmatrix} \right\}$$

The subgroup $M_{n_1,\dots,n_k}$ is called **Levi component**. We can further factor this subgroup as

$$M_{n_1,\dots,n_k} = M'_{n_1,\dots,n_k} \cdot A_{n_1,\dots,n_k}$$

with $A_{n_1,\dots,n_k}$ plays the role of the connected center of $M_{n_1,\dots,n_k}$:

$$A_{n_1,\dots,n_k} = \left\{ \begin{bmatrix} t_1 I_1 & 0 & \dots & 0 \\ 0 & t_2 I_k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_k I_k \end{bmatrix} : t_i \neq 0 \right\}$$

and

$$M'_{n_1,\dots,n_k} = \left\{ \begin{bmatrix} \mathfrak{m}'_1 & 0 & \dots & 0 \\ 0 & \mathfrak{m}'_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathfrak{m}'_k \end{bmatrix} \right\},$$

where $\det(\mathfrak{m}'_i) = \pm 1$.

Definition 2.7. *For a given parabolic subgroup P, the factorization*

$$P = M_P \times A_P \times N_P$$

*as above is called **Langlands decomposition**.*