Why Schemes

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1 Summary of affine varieties

Let k be an algebraic closed field. The main idea of classical algebraic geometry is that we have a correspondence

$$\{\text{subsets of } k^n \text{ cutout by polynomials}\} \leftrightarrow \{\text{finitely generated reduced } k- \text{ algebras }\}$$
$$\text{Geometry} \leftrightarrow \text{Algebra}$$

In particular, the above correspondence can be given as follows:

- $I \subset k[x_1, \ldots, x_n]$ ideal: Then we define

  $$X := Z(I) = \{a \in k^n | f(a) = 0 \quad \forall f \in I\}$$

  sometimes we use the notation $V(I)$ instead of $Z(I)$. This kind of set is an affine variety.

- $\mathbb{A}^n$: $n-$ dimensional affine space. As a set, it is just k^n, but we equip this set with *Zariski* topology - where the closed subsets are generated by $Z(I)$.

- $I(X) := \{f \in k[x_1, \ldots, x_n] | f(x) = 0 \quad \forall x \in X\}$. Then the quotient replacing $k[X] := \dfrac{k[x_1, \ldots, x_n]}{I(X)}$ is called *coordinate ring* of X.

- $k[X]$ parametrizes functions on X.

- Hilbert's weak Nullstellensatz:

  $$\{\text{ points of } X\} \leftrightarrow \{\text{ maximal ideals of } k[X]\}$$

- Hilbert Nullstellensatz: $I(Z(I)) = \sqrt{I} := \{f : f^n \in I \text{ for some } n\}$

- Morphisms: given X and $Y \in \mathbb{A}^n$ a morphism between two affine varieties is given by $\varphi = (f_1, \ldots, f_n)$. This morphism induces a $k-$ algebra homorphism

  $$\varphi^* k[Y] \to k[X], \quad \varphi^*(\psi) = \psi \circ \varphi$$

2 Why varieties are not good enough?

Some possible reasons are:

1. embedding into $\mathbb{A}^n$ shouldn't really be part of the data. It would be nice to have an intrinsic definition, since you can embed the same variety in different spaces.

2. for non-algebraic closed field, the Nullstellensatz doesn't work.

3. We can ask, on which topological space is $\mathbb{R}[x,y]/(x^2+y^2+1)$ naturally a functions space? Or $\mathbb{Z}[x]$? Or $\mathbb{Z}$? This leads to the ideal of considering all possible rings.

4. nilpotent arises naturally when deforming varieties, so ignoring them is not a good option.