

# Montel theorem and some related results

Tri Nguyen - University of Alberta

July 15, 2024

In this expository note, I will try to explain explicitly how to compactify  $\Gamma \backslash \mathbb{H}$  by adding points in two ways.

## 1 Some preparations

We will always denote  $\Gamma$  a subgroup of the group  $SL_2(\mathbb{Z})$  of finite index, and this group acts on the upper half complex plane  $\mathbb{H}$  by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ z := \frac{az + b}{cz + d}$$

When  $z$  tends to infinity, we have

$$\lim_{z \rightarrow \infty} \frac{az + b}{cz + d} = \frac{a}{c},$$

so we add the rational line to define the action of this group at  $\infty$ . In particular, we consider the set

$$\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$$

Note that on the projective rational line, we define the action to be the multiplication of a  $2 \times 2$  matrix with a  $2 \times 1$  vector. Then under this action, we have the following lemma

**Lemma 1.**  $SL_2(\mathbb{Z})$  acts transitively on  $\mathbb{P}^1(\mathbb{Q})$ .

*Proof.* For each point in  $\mathbb{P}^1(\mathbb{Q})$ , we can choose the representative to be of the form  $[a : b]$ , where  $\gcd(a, b) = 1$ . Then there exists  $x, y \in \mathbb{Z}$  such that

$$ax - by = 1$$

Thus we get the following equality

$$\begin{bmatrix} b & a \\ -x & y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This implies any points in  $\mathbb{P}^1(\mathbb{Q})$  can be moved to  $[0 : 1]$ , and thus the action is transitive.  $\square$

**Corollary 2.** If  $\Gamma$  is a subgroup of finite index in  $SL_2(\mathbb{Z})$  then  $\Gamma \backslash \mathbb{P}^1(\mathbb{Q})$  has only finite orbits.

## 2 Compactification of $\Gamma \backslash \mathbb{H}$ by adding points.

We introduce a topology on  $\overline{\mathbb{H}}$ . For the usual upper half plane, the topology is the usual metric topology on  $\mathbb{C}$ , and we only define the system of the neighborhood of  $r \in \mathbb{P}^1(\mathbb{Q})$ .

Let  $S(c, \omega)$  be the circle that touches the real line at  $\omega = p/q$  and has the radius  $\frac{c}{2q^2}$ . Then the collection of circle  $D(c, \omega) = \bigcup_{0 < c' \leq c} S(c', \omega)$  is called *Farey disk*. Let  $c \rightarrow 0$ , these disks define a neighborhood of  $\omega$ . The Farey disks at  $\infty$  are defined to be the region

$$D(T, \infty) = \{z : \Im z \geq T\}$$

It can be checked easily that the matrix under inversion, the Farey disk at  $\infty$  is mapped to  $D(1/T, 0)$ . In general, if  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  such that  $\gamma \circ \infty = \omega$  then  $D(T, \infty)$  is mapped to  $D(1/T, \omega)$ .

With the above topology on the extended upper half plane, we could show that

**Lemma 3.**  $\Gamma \backslash \overline{\mathbb{H}}$  is a compact set.

*Proof.* We first prove for the case  $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ . It is well known that the quotient space  $\Gamma \backslash \mathbb{H}$  is identical to the set

$$\mathcal{F} = \{z \in \mathbb{H} : \Re z \in [-1/2, 1/2), |z| \geq 1 \text{ and } |z| > 1 \text{ if } \Re z > 0\}$$

By lemma 1, the projective rational line "shrinks" to a point under the action of  $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ , Thus we can identify  $\Gamma \backslash \overline{\mathbb{H}}$  with the set  $\tilde{\mathcal{F}} = \mathcal{F} \cup \{\infty\}$ . Consider an open cover  $\{U_i\}_{i \in I}$  of  $\tilde{\mathcal{F}}$  and the natural projection  $\pi: \overline{\mathbb{H}} \rightarrow \tilde{\mathcal{F}}$ . Then the set  $\{\pi^{-1}(U_i)\}_{i \in I}$  forms an open cover of  $\overline{\mathbb{H}}$ . There must be an index  $i_0$  such that  $\pi^{-1}(U_{i_0})$  contains a neighborhood of  $\infty$ , namely contains a Farey disk  $D(T, \infty)$  for some  $T > 0$ . Since  $\overline{\mathcal{F}} - D(T, \infty)$  is a compact set, its image under  $\pi$  is compact, hence it can be covered by  $U_{i_1}, \dots, U_{i_m}$ . Altogether,  $\tilde{\mathcal{F}}$  admits a finite subcover  $U_{i_0}, \dots, U_{i_m}$ .

Now we proceed to the general case. Note that

$$\overline{\mathbb{H}} = \mathrm{SL}_2(\mathbb{Z}) \circ \tilde{\mathcal{F}} = \bigcup \Gamma a_i \circ \tilde{\mathcal{F}}$$

by corollary 2. Then under the surjective map  $\pi: \overline{\mathbb{H}} \rightarrow \Gamma \backslash \overline{\mathbb{H}}$ , we have

$$\Gamma \backslash \overline{\mathbb{H}} = \bigcup \pi \left( \Gamma a_i \circ \tilde{\mathcal{F}} \right),$$

which shows that the set  $Y(\Gamma) = \Gamma \backslash \overline{\mathbb{H}}$  is compact as it is the union of compact sets.  $\square$

The orbit of  $\mathbb{P}^1(\mathbb{Q})$  under the action of  $\Gamma$  is called *cusps*. We have the obvious equality that

$$\Gamma \backslash \overline{\mathbb{H}} = \Gamma \backslash \mathbb{H} \cup \underbrace{\Gamma \backslash \mathbb{P}^1(\mathbb{Q})}_{\text{cusps}}$$

So in fact lemma 3 tells us that we only need to add finite cups.