Lie theory - homework 1

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Problem 1

1. By definition, we only need to check that

$$[[a,b],c] \in \mathfrak{g} \quad \forall a,b \in \mathfrak{h}, c \in \mathfrak{g},$$

but this is clear as $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, we could use Jacobi's identity to get

$$[[a,b],c] = [b,[c,a]] + [a,[b,c]] \in [\mathfrak{h},\mathfrak{h}].$$

2. Recall that $\mathcal{D}^{k+1}\,\mathfrak{g} = [\mathfrak{g}^k, \mathfrak{g}^k]$. Clearly $\mathfrak{g}$ is itself an ideal, so the fact that $\mathcal{D}^{k+1}\,\mathfrak{g}$ follows immediately from part a and induction on k.

3. In class, we called $\mathfrak{g}$ semisimple iff it has no nontrivial solvable ideal. Note that abelian ideals are solvable, hence all abelian ideals are zero if $\mathfrak{g}$ is semisimple. Conversely, assume that $\mathfrak{g}$ is not semisimple, then it has a non trivial solvable ideal $\mathfrak{h}$. In particular, we have a strictly decreasing chain of ideals as follows:

$$\mathfrak{h} = \mathfrak{h}^{(0)} \supset \mathfrak{h}^{(1)} \supset \ldots \supset \mathfrak{h}^{(n)} \supset \mathfrak{h}^{(n+1)} = (0)$$

But this implies that $\mathfrak{h}^{(n)}$ is a non trivial abelian ideal of $\mathfrak{g}$ by part a.

Problem 2 We compute $\operatorname{ad} x$ with respect to this basis. The other two are computed similarly.

$$\operatorname{ad} x(x) = [x,x] = 0 \tag{1}$$

$$\operatorname{ad} x(y) = [x,y] = xy - yx = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = h \tag{2}$$

$$\operatorname{ad} x(h) = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} = -2x \tag{3}$$

So with respecto to the basis $\{x,h,y\}$, the linear map $\operatorname{ad} x$ correspond to the matrix

$$\operatorname{ad} x = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

similarly, we could find that

$$\operatorname{ad} y = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \operatorname{ad} h = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Problem 4

1. The given matrix algebra L is generated by 3 following linearly independent elements

$$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   Moreover, it is easy to check that, with respect to the usual lie bracket over matrix algebra, we have

$$[a, c] = b, \quad [a, b] = [b, c] = 0$$

   Since L and V has the same dimension, they are isomorphic as vector spaces. Thus, there exists a linear map $f \colon V \to L$ such that $f(x) = a$, $f(y) = c$ and $f(z) = b$. Hence

$$f([x, y]) = b = [a, c] = [f(x), f(y)]$$

   Thus f is a Lie algebra isomorphism.

2. From the definition, we could see that the derived algebra of V is generated by 1 element z, hence abelian, and hence the second term in lower central series vanishes.

Problem 3

Since the matrix x has n distinct eigenvalues and has size $n \times n$, x has n linearly independent eigenvectors.

We denote v_i be the vector such that $x \cdot v_i = a_i v_i$. Then $\{v_i\}_i^n$ can be chosen as a basis of $\mathbb{R}^n$. With respect to this basis, x can be associated with the diagonal vector, with a_i's on the diagonal. Let $\{e_{ij}\}_{i=1}^n$ be the standard basis of $\mathfrak{gl}(\mathrm{n}, \mathrm{F})$. It can be verified that

$$\operatorname{ad} x(e_{ij}) = xe_{ij} - e_{ij}x = a_i - a_j, \quad \forall 1 \leqslant i, j \leqslant n$$

This implies that the standard basis $\{e_{ij}\}_{i=1}^n$ is the full set of eigenvectors of $\operatorname{ad} x$, and the corresponding eigenvalues are scalars $a_i - a_j$.