# CHAPTER I : $\mathrm{SL}_2(\mathbb{R})$

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In this chapter, I will give an exposition on the structure of $\mathrm{SL}_2(\mathbb{R})$ as the spaces of lattice, this space plays the role of a toy model before exploring the space of lattice in the higher rank. The exposition follows the paper [**?**] and [**?**] closely.

## 1 $\mathbf{SL}_2(\mathbb{R})$ and its action on the upper half plane $\mathfrak{H}$

A priori, the upper half plane

$$\mathfrak{H} = \{z : \Im z > 0\} \subset \mathbb{C}$$

has no group structure on its. However, we will show below that it can identify topologically with the space with the space of cosets $\mathrm{SO}_2(\mathbb{R}) \setminus \mathrm{SL}_2(\mathbb{R})$, and thus we can study the spaces $\mathfrak{H}$ via the spac of lattices $\mathrm{SO}_2(\mathbb{R}) \setminus \mathrm{SL}_2(\mathbb{R})$. We define the action of $G = \mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{H}$ as follows

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ (z) = \frac{az+b}{cz+d}$$

**Proposition 1.1.** *The group $SL_2(\mathbb{R})$ stabilizes $\mathfrak{H}$ and acts transitively on it. In particular,*

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} (i) = x + iy \quad (\text{for } x \in \mathbb{R},\ y > 0)$$

*Further, for* $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ *and* $z \in \mathfrak{H}$,

$$\Im\, g(z) = \frac{\Im\, z}{|cz+d|^2}.$$

*Proof.* The first formula is clear. The second formula would imply that the upper half-plane is stabilized. Compute directly:

$$2i \cdot \Im \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) \right) = \frac{az+b}{cz+d} - \frac{d\overline{z}+b}{c\overline{z}+d} = \frac{(az+b)(c\overline{z}+d) - (a\overline{z}+b)(cz+d)}{|cz+d|^2}$$

$$= \frac{adz - bc\overline{z} - bcz + ad\overline{z}}{|cz+d|^2} = \frac{z - \overline{z}}{|cz+d|^2}$$

since $ad - bc = 1$. □

The point $z = i$ is special, in the sense that its stability group is the orthogonal group $K = \mathrm{SO}_2(\mathbb{R})$. Indeed, for any $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ we have that

$$g \circ i = i \Leftrightarrow \frac{ai+b}{ci+d} = i \Leftrightarrow a = d \text{ and } b = -c$$

Combinining with the fact that $ad - bc = 1$, we must have $a^2 + b^2 = 1$. This implies that there is a $\theta$ such that $a = \cos\theta$ and $b = \sin\theta$. Since $G$ acts on $\mathfrak{H}$ transitively, we know from group theory that there is a bijection between the collection of cosets of $\mathrm{Stab}(i)$ in $G$ and the orbits of $i$. In particular

**Proposition 1.2.** *We have an isomorphism of $SL_2(\mathbb{R})$-spaces*

$$\mathrm{SO}_2(\mathbb{R}) \setminus SL_2(\mathbb{R}) \approx \mathfrak{H} \quad via \quad SO(2)g \to g^{-1}(i)$$

*That is, the map respects the action of $SL_2(\mathbb{R})$, in the sense that*

$$(\mathrm{SO}_2(\mathbb{R})\, g) \cdot h \;\longrightarrow\; h^{-1}(g^{-1}i)$$

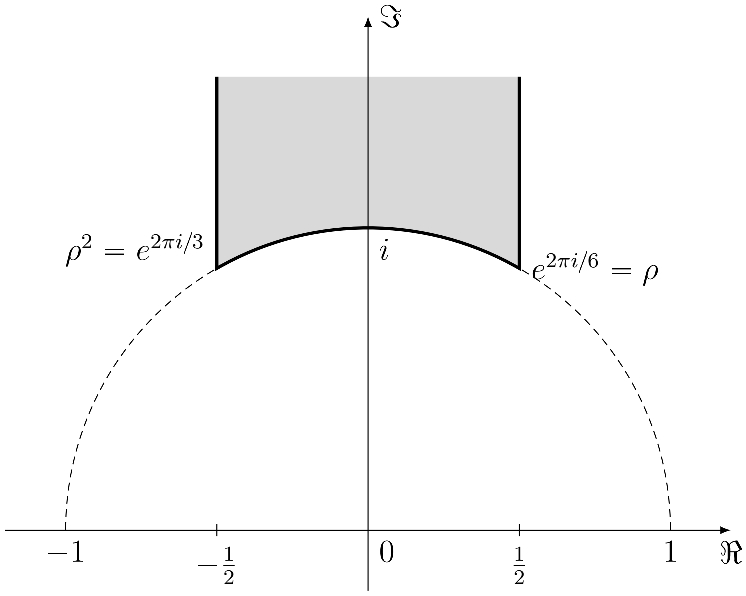
*Proof.* This is because of *associativity*:

$$(\mathrm{SO}_2(\mathbb{R})\, g) \cdot h = (\mathrm{SO}_2(\mathbb{R})) \cdot (gh) \;\longrightarrow\; (gh)^{-1}(i) = h^{-1}(g^{-1}(i))$$

giving the result. □

## 2 Fundamental domain for $\Gamma = SL_2(\mathbb{Z})$ on $\mathfrak{H}$

Here is a picture of the fundamental domain $\mathfrak{H}/\Gamma$.




The goal of this section is to prove that under the action of the $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, we can "move" every points on the upper half plane to a domain, under an equivalence given by a specific action. This is similar to the fundamental domain given by the translation action of $\mathbb{Z}$ to $\mathbb{R}$ is the half-open unit interval $[0, 1)$. In general, this give a simpler description to the homogenous space of lattice. Note that when we try to compute the fundamental domain of $\mathbb{Z}\backslash\mathbb{R}$, we have $\mathbb{Z}$ plays a role of *discrete* subset of $\mathbb{R}$. We give a precise definition of discreteness as follows

**Definition 2.1.** *Let a group $G$ act continuously on a topological space $X$. A subset $\Gamma \subset G$ is called* ***discrete*** *if for any two compact subse $A, B$ in $X$, there are only finitely many $g \in \Gamma$ such that $g \circ A \cap B \neq \varnothing$.*

We will prove that the set

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\}$$

is a discrete subgroup of $G = \mathrm{SL}_2(\mathbb{R})$. To prove this, we first need the following lemma

**Lemma 2.1.** *Fix a real number $r > 0$ and $0 < \delta < 1$. We denote $R_{r,\delta}$ the rectangle*

$$R_{r,\delta} = \left\{ z = x + iy : -r \leqslant x \leqslant r, 0 < \delta \leqslant y \leqslant \delta^{-1} \right\}$$

*Then for any $\epsilon > 0$ and any fixed set $\mathbb{S}$ of coset representatives for $\Gamma_\infty\backslash\Gamma$, there are finitely many $g \in \mathbb{S}$ such that $\Im(g \circ z) > \epsilon$ for some $z \in R_{r,\delta}$.*

In the above lemma, the notation $\Gamma_\infty$ is defined to be the set

$$\Gamma_\infty = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}.$$

It can be seen easily that this is the stability group of $\infty$ in $\mathfrak{H}$.

*Proof.* Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then for $z \in R_{r,\delta}$,

$$\operatorname{Im}(g \circ z) = \frac{y}{c^2y^2 + (cx+d)^2} < \epsilon$$

if $|c| > (y\epsilon)^{-\frac{1}{2}}$. On the other hand, for $|c| \leqslant (y\epsilon)^{-\frac{1}{2}} \leqslant (\delta\epsilon)^{-\frac{1}{2}}$, we have

$$\frac{y}{(cx+d)^2} < \epsilon$$

if the following inequalities hold:

$$|d| > |c|r + (y\epsilon^{-1})^{\frac{1}{2}} \geqslant |c|r + (\epsilon\delta)^{-\frac{1}{2}}.$$

Consequently, $\Im(g \circ z) > \epsilon$ only if

$$|c| \leqslant (\delta\epsilon)^{-\frac{1}{2}} \quad \text{and} \quad |d| \leqslant (\epsilon\delta)^{-\frac{1}{2}}(r+1),$$

and the total number of such pairs (not counting $(c,d) = (0, \pm 1), (\pm 1, 0)$) is at most $\frac{4(r+1)}{(\epsilon\delta)}$. This proves the lemma. □

It follows from Lemma 2.1 that $\Gamma = SL(2,\mathbb{Z})$ is a discrete subgroup of $SL(2,\mathbb{R})$. This is because:

1. It is enough to show that for any compact subset $A \subset \mathfrak{H}$ there are only finitely many $g \in SL(2,\mathbb{Z})$ such that $(g \circ A) \cap A \neq \phi$;
2. Every compact subset of $A \subset \mathfrak{H}$ is contained in a rectangle $R_{r,\delta}$ for some $r > 0$ and $0 < \delta < \delta^{-1}$;
3. $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$, except for finitely many $\alpha \in \Gamma_\infty$, $g \in \Gamma_\infty \backslash \Gamma$.

To prove (3), note that Lemma 2.1 implies that $(g \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$ except for finitely many $g \in \Gamma_\infty \backslash \Gamma$. Let $S \subset \Gamma_\infty \backslash \Gamma$ denote this finite set of such elements $g$. If $g \notin S$, then Lemma 2.1 tells us that it is because $\Im(g \circ z) < \delta$ for all $z \in R_{r,\delta}$. Since $\Im(\alpha g \circ z) = \Im(g \circ z)$ for $\alpha \in \Gamma_\infty$, it is enough to show that for each $g \in S$, there are only finitely many $\alpha \in \Gamma_\infty$ such that $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} \neq \phi$. This last statement follows from the fact that $g \circ R_{r,\delta}$ itself lies in some other rectangle $R_{r',\delta'}$, and every $\alpha \in \Gamma_\infty$ is of the form $\alpha = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$ $(m \in \mathbb{Z})$, so that

$$\alpha \circ R_{r',\delta'} = \{x + iy \mid -r' + m \leqslant x \leqslant r' + m,\ 0 < \delta' \leqslant \delta''^{-1}\},$$

which implies $(\alpha \circ R_{r',\delta'}) \cap R_{r,\delta} = \phi$ for $|m|$ sufficiently large. Now we are ready to describe the fundamental domain for $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}$.

**Proposition 2.1.** *A fundamental domain for* $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}$ *can be given as the region*

$$\mathfrak{D} = \{z = x + iy \in \mathfrak{H} : |z| \geqslant 1, -1/2 \leqslant x \leqslant 1/2\}$$

*, modulo the congruent boundary points symmetric with respect to the imaginary axis.*

*Proof.* First we eliminated the repeated points on the boundary. Note that the line $x = -1/2$ is the same as the line $x = 1/2$ under the transformation $z \mapsto z+1$. Similarly, given a point on the circle $\{|z| = 1\}$, the transformation $z \mapsto -|z|^{-1}$ satisfies

$$\frac{-1}{x+iy} = \frac{-x+iy}{x^2+y^2} = -x+iy,$$

which flips the sign of $x$. Thus it identifies the half circle on the right of the imaginary axis with that on the left.
Now we need to show two things:

1. For any $z \in \mathfrak{H}$ we can find an element $g \in \mathrm{SL}_2(\mathbb{Z})$ such that $g \circ z \in \mathfrak{D}$.
2. If $z \equiv z' \in \mathfrak{D}$ modulor $\mathrm{SL}_2(\mathbb{Z})$, then either $\Re(z) = \pm\frac{1}{2}$ and $z' = z \mp 1$, or $|z| = 1$ and $z' = \frac{-1}{z}$.

First we prove for (1): Fix $z \in \mathfrak{H}$. It follows from Lemma 2.1 that for every $\epsilon > 0$, there are at most finitely many $g \in \mathrm{SL}(2,\mathbb{Z})$ such that $g \circ z$ lies in the strip

$$D_\epsilon := \left\{ w \;\middle|\; -\frac{1}{2} \leqslant \mathrm{Re}(w) < \frac{1}{2},\ \epsilon \leqslant \mathrm{Im}(w) \right\}.$$

Let $B_\epsilon$ denote the finite set of such $g \in \mathrm{SL}(2,\mathbb{Z})$. Clearly, for sufficiently small $\epsilon$, the set $B_\epsilon$ contains at least one element. We will show that there is at least one $g \in B_\epsilon$ such that $g \circ z \in D$. Among these finitely many $g \in B_\epsilon$, choose one such that $\Im(g \circ z)$ is maximal in $D_\epsilon$. If $|g \circ z| < 1$, then for $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ we have, for any $m$,

$$\Im\left(T^m S g \circ z\right) = \Im\left(\frac{-1}{g \circ z}\right) = \frac{\Im(g \circ z)}{|g \circ z|^2} > \Im(g \circ z)$$

But we can choose $m$ such that $T^m S g \circ z \in D_\epsilon$, which contradicts the maximality of $\Im(g \circ z)$.
Next we give a proof for (2):Let $z \in D$, $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, and assume that $g \circ z \in D$. Without loss of generality, we may assume that

$$\Im(g \circ z) = \frac{y}{|cz+d|^2} \geqslant \Im(z),$$

(otherwise just interchange $z$ and $g \circ z$ and use $g^{-1}$). This implies that $|cz+d| \leqslant 1$ which implies that $1 \geqslant |cy| \geqslant \frac{1}{\sqrt{3}}|c|$. This is clearly impossible if $|c| \geqslant 2$. So we only have to consider the cases $c = 0, \pm 1$. If $c = 0$ then $d = \pm 1$ and $g$ is a translation by $b$. Since $-\frac{1}{2} \leqslant \Re(z), \Re(g \circ z) \leqslant \frac{1}{2}$, this implies that either $b = 0$ and $z = g \circ z$ or else $b = \pm 1$ and $\Re(z) = \pm\frac{1}{2}$ while $\Re(g \circ z) = \mp\frac{1}{2}$. If $c = 1$, then $|z+d| \leqslant 1$ implies that $d = 0$ unless $z = e^{2\pi i/3}$ and $d = 0, -1$. The case $d = 0$ implies that $|z| \leqslant 1$ which implies $|z| = 1$. Also, in this case, $c = 1$, $d = 0$, we must have $b = -1$ because $ad - bc = 1$. Then $g \circ z = a - \frac{1}{z+1}$. It follows that $g \circ z = a - e^{2\pi i/3}$ and $d = 1$, then we must have $a - b = 1$. It follows that $g \circ z = a - \frac{1}{z+1} = a + e^{2\pi i/3}$, which implies that $a = 0$ or $1$. A similar argument holds when $z = e^{\pi i/3}$ and $d = -1$. Finally, the case $c = -1$ can be reduced to the previous case $c = 1$ by reversing the signs of $a, b, c, d$. $\square$