

# **On Semi-stable lattices in higher rank**

by

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## Abstract

In this thesis, we study the notion of semi-stable lattices in two dimensional space and generalize it to finite dimensional space of dimension at least 3.

The semi-stable lattices can be defined using two different definition, as first defined using a plot as in [10] and later defined using Harish-Chandra map as in [7]. We showed that the two definition are equivalent and proved the equivalence between geometric properties using plot and algebraic properties using Harish-Chandra map. We achieve this by pointing out the volume of sublattices can be computed using pairing between Harish-Chandra map with the Weyl vectors. In this way, the geometric information and algebraic information are the same.

When the lattice is not semi-stable, which is called unstable, it can be attached a canonical plot in geometric sense and a canonical pair in algebraic sense. We also proved that these objects encode essentially the same data in finite dimensional case.

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## Preface

This thesis is an original work by Tri Nguyen. No part of this thesis has been previously published.

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# Chapter 1

## Introduction

Semi-stable lattices first appeared in the paper of Stuhler [22]. He was inspired by some work of Harder and Narasimhan on semi-stable vector bundles over curves, and studied the analogy in the context of lattice theory. The notion of semi-stability as well as canonical filtration were then introduced by him. Stuhler also discovered the connection between semi-stability and Minkowski's reduction theory. In particular, he showed that, if a lattice  $(L, q)$  has a Minkowski reduced basis  $\{e_1, \dots, e_n\}$  such that  $q(e_{i+1}) > \alpha \cdot q(e_i)$  for some suitable  $\alpha$  then the set

$$0 \subset \mathbb{Z}e_1 \subset \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \subset \dots \subset \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \dots \oplus \mathbb{Z}e_n$$

forms a canonical plot for  $L$ . We refer to the original paper [22] or the book [19] for a detailed treatment of Stuhler's result. In 1978, Grayson reproduced the results of Stuhler and used this to prove some theorems about arithmetic groups. The difference is that he associated each lattice to a graphical data called *canonical plot*, but "the effect on the clarity of arguments is dramatic." [6]. He also recovered some results in classical reduction theory without the need to use the notion of Siegel's fundamental domain. Along these lines, there are various result on semi-stability, such as

1. In [19], the author extended the original paper of Stuhler to study the shortest vector problem. In particular, he gave a relation between the first successive minima and the slope of the lattice.
2. In [6], Bill Casselman studied the partition of arithmetic groups using semi-stability. His paper hinted a connection between the partition using semistablity and Arthur's combinatorial lemma in the theory of automorphic forms.
3. In a series of 3 papers [21], [20] and [16], Uri Shapira et al. studied semi-stable lattices in the context of homogeneous dynamics. One of the main results involves the

relation between semi-stable lattices and well-rounded lattices, which then simplifies a conjecture of Minkowski on the geometry of number.

This thesis has two goals: The first is to give a survey of the geometric picture of semi-stable lattice, following [10] and [6]. The second is to compare the semi-stability in geometric sense, following Grayson, with semi-stability in Lie theoretic sense as introduced in the papers [7] and [18]. Ultimately, we will prove that these two notions of semi-stability are essentially the same.

## Outline of the thesis

- In chapter II, we go over the theory of two dimensional lattice. We also introduce the semistability for lattices, encoded by the upper half plane model. Using the graphical data, we show that semi-stability and  $\rho$ -semi-stability are the same.
- In Chapter III, we recalled the necessary Lie theory for  $\mathrm{SL}_n$  and  $\mathrm{GL}_n$ . They will be needed later in introducing the notion of  $\rho$ -semi-stability.
- In chapter IV, we defined lattices in higher rank and related properties. Then semi-stable lattice and canonical filtration are studied in detail, following the papers [10] and [6]. At the end of this chapter,  $\rho$ -semi-stability along with some of its basic properties are given. When the lattice is not semi-stable, each lattice is attached to a canonical filtration. Similarly, if a point  $x \in \mathrm{GL}_n(\mathbb{R})$  is not  $\rho$ -semi-stable, we associate  $x$  to a canonical pair  $\mathrm{cp}(x) = (P, \delta)$ .
- In the last chapter, we give two main theorems: The first theorem involves the equivalence between semi-stability in Grayson's sense and  $\rho$ -semi-stability. Furthermore, we showed that the canonical filtration and the canonical pair essentially encode the same data.

# Chapter 2

$$\mathrm{SO}_2(\mathbb{R}) \setminus \mathrm{SL}_2(\mathbb{R})$$

In this chapter, I will give an exposition on the structure of  $\mathrm{SL}_2(\mathbb{R})$  as the spaces of lattice, this space plays the role of a toy model before exploring the space of lattice in the higher rank. The exposition follows [9, Chapter I] closely.

## 2.1 $\mathrm{SL}_2(\mathbb{R})$ and its action on the upper half plane $\mathfrak{H}$

Consider the upper half plane

$$\mathfrak{H} = \{z : \Im z > 0\} \subset \mathbb{C}.$$

We define the action of  $G = \mathrm{SL}_2(\mathbb{R})$  on  $\mathfrak{H}$  as follows

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ (z) := \frac{dz - b}{-cz + a}$$

We need to check that this is indeed an action. It is routine to check that

$$\begin{bmatrix} \frac{1}{\sqrt{y}} & 0 \\ 0 & \sqrt{y} \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \circ (i) = x + iy \quad (\text{for } x \in \mathbb{R}, y > 0)$$

Moreover, we have

$$\begin{aligned} 2i \cdot \Im \left( \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \circ (z) \right) &= \frac{az + b}{cz + d} - \frac{d\bar{z} + b}{c\bar{z} + d} = \frac{(az + b)(c\bar{z} + d) - (a\bar{z} + b)(cz + d)}{|cz + d|^2} \\ &= \frac{adz - bc\bar{z} - bcz + ad\bar{z}}{|cz + d|^2} = \frac{z - \bar{z}}{|cz + d|^2} \end{aligned}$$

since  $ad - bc = 1$ . In particular, we have the following Proposition

**Proposition 2.1.0.1.** *The group  $\mathrm{SL}_2(\mathbb{R})$  acts transitively on  $\mathfrak{H}$ . In particular,*

$$\begin{bmatrix} \frac{1}{\sqrt{y}} & 0 \\ 0 & \sqrt{y} \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \circ (i) = x + iy \quad (\text{for } x \in \mathbb{R}, y > 0)$$

Further, for  $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$  and  $z \in \mathfrak{H}$ ,

$$\Im(g \circ z) = \frac{\Im z}{|cz + d|^2}.$$

**Remark.** *In many texts, the above action is defined as*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ (z) := \frac{az + b}{cz + d}$$

*The side of the action doesn't really matter here, as we can recover the whole theory by switching between the left/right action. However, the result of this thesis follows the recent paper [18], which use the inverse action. That's why we follow the same notations set up in that paper.*

The point  $z = i$  is special, in the sense that its stability group is the orthogonal group  $K = \mathrm{SO}_2(\mathbb{R})$ . Indeed, for any  $g = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \in SL_2(\mathbb{R})$  we have that

$$g \circ i = i \Leftrightarrow \frac{ai + b}{ci + d} = i \Leftrightarrow a = d \text{ and } b = -c$$

Combining with the fact that  $ad - bc = 1$ , we must have  $a^2 + b^2 = 1$ . This implies that there is a  $\theta$  such that  $a = \cos \theta$  and  $b = \sin \theta$ . Since  $G$  acts on  $\mathfrak{H}$  transitively, we know from group theory that there is a bijection between the collection of cosets of  $\mathrm{Stab}(i)$  in  $G$  and the orbits of  $i$ . It is also useful to consider the group that stabilizes the point at infinity, which we include in the following proposition

**Proposition 2.1.0.2.**

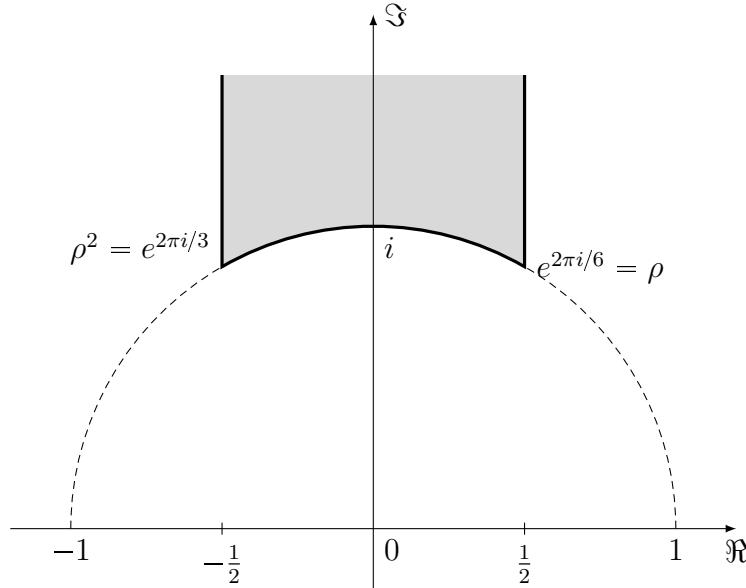
1.  $\mathrm{Stab}(i) = \mathrm{SO}_2(\mathbb{R})$ .
2. *We have a bijection between the left cosets spaces and the upper half plane given by*

$$\mathrm{SO}_2(\mathbb{R}) \backslash \mathrm{SL}_2(\mathbb{R}) \cong \mathfrak{H} \quad \text{via} \quad SO(2)g \rightarrow g^{-1}(i)$$

$$3. Stab(\infty) := \Gamma_\infty := \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\}$$

## 2.2 Fundamental domain

Let  $\Gamma = \text{SL}_2(\mathbb{Z})$  and consider its action on  $\mathfrak{H}$ .



The goal of this section is to prove that under the action of the  $\Gamma = \text{SL}_2(\mathbb{Z})$ , we can "move" every points on the upper half plane into a domain pictured above. This is similar to the fundamental domain given by the translation action of  $\mathbb{Z}$  to  $\mathbb{R}$  is the half-open unit interval  $[0, 1)$ . In general, this give a simpler description to the homogenous space of lattice.

Note that when we try to compute the fundamental domain of  $\mathbb{Z} \backslash \mathbb{R}$ , we have  $\mathbb{Z}$  plays a role of *discrete* subset of  $\mathbb{R}$ . We give a precise definition of discreteness as follows

**Definition 2.2.0.1.** Let a group  $G$  act continuously on a topological space  $X$ . A subset  $\Gamma \subset G$  is called **discrete** if for any two compact subse  $A, B$  in  $X$ , there are only finitely many  $g \in \Gamma$  such that  $g \circ A \cap B \neq \emptyset$ .

We will prove that the set

$$\Gamma = \text{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\}$$

is a discrete subgroup of  $G = \text{SL}_2(\mathbb{R})$ . To prove this, we first need the following lemma

**Lemma 2.2.0.2.** Fix a real number  $r > 0$  and  $0 < \delta < 1$ . We denote  $R_{r,\delta}$  the rectangle

$$R_{r,\delta} = \{z = x + iy : -r \leq x \leq r, 0 < \delta \leq y \leq \delta^{-1}\}$$

Then for any  $\epsilon > 0$  and any fixed set  $\mathbb{S}$  of coset representatives for  $\Gamma_\infty \backslash \Gamma$ , there are finitely many  $g \in \mathbb{S}$  such that  $\Im(g \circ z) > \epsilon$  for some  $z \in R_{r,\delta}$ .

*Proof.* Let  $g = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ . Then for  $z \in R_{r,\delta}$ ,

$$\operatorname{Im}(g \circ z) = \frac{y}{c^2 y^2 + (cx + d)^2} < \epsilon$$

if  $|c| > (y\epsilon)^{-\frac{1}{2}}$ . On the other hand, for  $|c| \leq (y\epsilon)^{-\frac{1}{2}} \leq (\delta\epsilon)^{-\frac{1}{2}}$ , we have

$$\frac{y}{(cx + d)^2} < \epsilon$$

if the following inequalities hold:

$$|d| > |c|r + (y\epsilon^{-1})^{\frac{1}{2}} \geq |c|r + (\epsilon\delta)^{-\frac{1}{2}}.$$

Consequently,  $\Im(g \circ z) > \epsilon$  only if

$$|c| \leq (\delta\epsilon)^{-\frac{1}{2}} \quad \text{and} \quad |d| \leq (\epsilon\delta)^{-\frac{1}{2}}(r + 1),$$

and the total number of such pairs (not counting  $(c, d) = (0, \pm 1), (\pm 1, 0)$ ) is at most  $\frac{4(r+1)}{(\epsilon\delta)}$ . This proves the lemma.  $\square$

It follows from Lemma 2.2.0.2 that  $\Gamma = \operatorname{SL}(2, \mathbb{Z})$  is a discrete subgroup of  $SL(2, \mathbb{R})$ . This is because:

1. It is enough to show that for any compact subset  $A \subset \mathfrak{H}$  there are only finitely many  $g \in SL(2, \mathbb{Z})$  such that  $(g \circ A) \cap A \neq \emptyset$ ;
2. Every compact subset of  $A \subset \mathfrak{H}$  is contained in a rectangle  $R_{r,\delta}$  for some  $r > 0$  and  $0 < \delta < \delta^{-1}$ ;
3.  $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} = \emptyset$ , except for finitely many  $\alpha \in \Gamma_\infty$ ,  $g \in \Gamma_\infty \backslash \Gamma$ .

To prove (3), note that Lemma 2.2.0.2 implies that  $(g \circ R_{r,\delta}) \cap R_{r,\delta} = \emptyset$  except for finitely many  $g \in \Gamma_\infty \backslash \Gamma$ . Let  $S \subset \Gamma_\infty \backslash \Gamma$  denote this finite set of such elements  $g$ . If  $g \notin S$ , then

Lemma 2.2.0.2 tells us that it is because  $\Im(g \circ z) < \delta$  for all  $z \in R_{r,\delta}$ . Since  $\Im(\alpha g \circ z) = \Im(g \circ z)$  for  $\alpha \in \Gamma_\infty$ , it is enough to show that for each  $g \in S$ , there are only finitely many  $\alpha \in \Gamma_\infty$  such that  $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} \neq \emptyset$ . This last statement follows from the fact that  $g \circ R_{r,\delta}$  itself lies in some other rectangle  $R_{r',\delta'}$ , and every  $\alpha \in \Gamma_\infty$  is of the form  $\alpha = \begin{bmatrix} 1 & -m \\ 0 & 1 \end{bmatrix}$  ( $m \in \mathbb{Z}$ ), so that

$$\alpha \circ R_{r',\delta'} = \{x + iy \mid -r' + m \leq x \leq r' + m, 0 < \delta' \leq \delta'^{-1}\},$$

which implies  $(\alpha \circ R_{r',\delta'}) \cap R_{r,\delta} = \emptyset$  for  $|m|$  sufficiently large. Now we are ready to describe the domain for  $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}$ . First we need to specify what we mean by the term fundamental domain

**Definition 2.2.0.3.** *Given a group  $G$  acting on the set  $X$ . A **fundamental domain** for this action is the set  $D \subset X$  such that*

1. *For any  $x \in X$ , there exists a  $d \in D$  such that  $xg = d$ .*
2. *if  $xg_1$  and  $xg_2$  are both in the domain  $D$ , then  $g_1 = g_2$ . In particular, the representative for the orbit of  $x$  under the action of  $G$  in  $D$  is unique.*

**Example 2.2.0.4.** *Let  $\mathbb{Z}$  acts on  $\mathbb{R}$  by translation, namely  $nx := n + x$ . Then it is easy to see that the fundamental domain for  $\mathbb{R}/\mathbb{Z}$  is the half interval  $[0, 1)$ .*

**Proposition 2.2.0.5.** *A fundamental domain for  $\mathfrak{H}/\mathrm{SL}_2(\mathbb{Z})$  can be given as the region*

$$\mathfrak{D} = \{z = x + iy \in \mathfrak{H} : |z| \geq 1, -1/2 \leq x < 1/2\} \cup \{|z| = 1, x \leq 0\}.$$

*Proof.* First we eliminated the repeated points on the boundary. Note that the line  $x = -1/2$  is the same as the line  $x = 1/2$  under the transformation  $z \mapsto z + 1$ . Similarly, given a point on the circle  $\{|z| = 1\}$ , the transformation  $z \mapsto -|z|^{-1}$  satisfies

$$\frac{-1}{x + iy} = \frac{-x + iy}{x^2 + y^2} = -x + iy,$$

which flips the sign of  $x$ . Thus it identifies the half circle on the right of the imaginary axis with that on the left.

Now we need to show two things:

1. For any  $z \in \mathfrak{H}$  we can find an element  $g \in \mathrm{SL}_2(\mathbb{Z})$  such that  $g \circ z \in \mathfrak{D}$ .
2. If  $z \equiv z' \in \mathfrak{D}$  modulor  $\mathrm{SL}_2(\mathbb{Z})$ , then either  $\Re(z) = \pm \frac{1}{2}$  and  $z' = z \mp 1$ , or  $|z| = 1$  and  $z' = \frac{-1}{z}$ .

First we prove for (1): Fix  $z \in \mathfrak{H}$ . It follows from Lemma 2.2.0.2 that for every  $\epsilon > 0$ , there are at most finitely many  $g \in \mathrm{SL}(2, \mathbb{Z})$  such that  $g \circ z$  lies in the strip

$$D_\epsilon := \left\{ w \mid -\frac{1}{2} \leq \mathrm{Re}(w) < \frac{1}{2}, \epsilon \leq \mathrm{Im}(w) \right\}.$$

Let  $B_\epsilon$  denote the finite set of such  $g \in \mathrm{SL}(2, \mathbb{Z})$ . Clearly, for sufficiently small  $\epsilon$ , the set  $B_\epsilon$  contains at least one element. We will show that there is at least one  $g \in B_\epsilon$  such that  $g \circ z \in D$ . Among these finitely many  $g \in B_\epsilon$ , choose one such that  $\Im(g \circ z)$  is maximal in  $D_\epsilon$ . If  $|g \circ z| < 1$ , then for  $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ,  $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  we have, for any  $m$ ,

$$\Im(T^m S g \circ z) = \Im\left(\frac{-1}{g \circ z}\right) = \frac{\Im(g \circ z)}{|g \circ z|^2} > \Im(g \circ z)$$

But we can choose  $m$  such that  $T^m S g \circ z \in D_\epsilon$ , which contradicts the maximality of  $\Im(g \circ z)$ .

Next we give a proof for (2): Let  $z \in D$ ,  $g = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ , and assume that  $g \circ z \in D$ .

Without loss of generality, we may assume that

$$\Im(g \circ z) = \frac{y}{|cz + d|^2} \geq \Im(z),$$

(otherwise just interchange  $z$  and  $g \circ z$  and use  $g^{-1}$ ). This implies that  $|cz + d| \leq 1$  which implies that  $1 \geq |cy| \geq \frac{1}{\sqrt{3}}|c|$ . This is clearly impossible if  $|c| \geq 2$ . So we only have to consider the cases  $c = 0, \pm 1$ . If  $c = 0$  then  $d = \pm 1$  and  $g$  is a translation by  $b$ . Since  $-\frac{1}{2} \leq \Re(z), \Re(g \circ z) \leq \frac{1}{2}$ , this implies that either  $b = 0$  and  $z = g \circ z$  or else  $b = \pm 1$  and  $\Re(z) = \pm \frac{1}{2}$  while  $\Re(g \circ z) = \mp \frac{1}{2}$ . If  $c = 1$ , then  $|z + d| \leq 1$  implies that  $d = 0$  unless  $z = e^{2\pi i/3}$  and  $d = 0, -1$ . The case  $d = 0$  implies that  $|z| \leq 1$  which implies  $|z| = 1$ . Also, in this case,  $c = 1, d = 0$ , we must have  $b = -1$  because  $ad - bc = 1$ . Then  $g \circ z = a - \frac{1}{z+1}$ . It follows that  $g \circ z = a - e^{2\pi i/3}$  and  $d = 1$ , then we must have  $a - b = 1$ . It follows that  $g \circ z = a - \frac{1}{z+1} = a + e^{2\pi i/3}$ , which implies that  $a = 0$  or  $1$ . A similar argument holds when  $z = e^{\pi i/3}$  and  $d = -1$ . Finally, the case  $c = -1$  can be reduced to the previous case  $c = 1$  by reversing the signs of  $a, b, c, d$ .  $\square$

## 2.3 Lattices, upper half plane and $\mathrm{SL}_2(\mathbb{R})$

In this section, we investigate the notion of semi-stable lattices and how the upper half plane  $\mathfrak{H}$  can be regarded as a space of two-dimensional lattices. First we need to define what a lattice

is.

### 2.3.1 Upper half plane and classification of similar lattices

**Definition 2.3.1.1.** A *lattice*  $L$  in  $\mathbb{C}$  is an additive group of the form

$$L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$$

where  $\omega_1, \omega_2$  are linearly independent over  $\mathbb{R}$ . We call the set  $\{\omega_1, \omega_2\}$  a **base** for  $L$ .

We want to classify 2-dimensional lattices, given they are "the same" under certain conditions. Note that, from the definition 2.3.1.1, a lattice is specified by giving a base. Let us denote  $\text{BAS}$  the set of of  $\mathbb{R}$ -bases for  $\mathbb{C}$  and  $\text{BAS}^+$  the set of all bases  $\{z, w\}$  such that  $\Im(z/w) > 0$ . We clearly have a map

$$\Phi: \text{BAS}^+ \rightarrow \{\text{lattices in } \mathbb{C}\}$$

where

$$\Phi(\{z, w\}) = \mathbb{Z}z \oplus \mathbb{Z}w$$

The maps is surjective, as we can always choose the sign of the elements in the base so that it is an element in  $\text{BAS}^+$ . Let  $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{R})$  acts on  $\text{BAS}^+$  in the following way

$$\gamma\{z, w\} = \{az + bw, cz + dw\}$$

It is clear that this is indeed an action. Modulo this action, we can define a bijective map as in the below lemma

**Lemma 2.3.1.2.** Two bases in  $\text{BAS}^+$  are mapped to the same lattice if they are in the same orbit under the action of  $\text{SL}_2(\mathbb{R})$ . In particular, the quotient set  $\text{BAS}^+ / \text{SL}_2(\mathbb{Z})$  are in bijection with the set of lattices in  $\mathbb{C}$ .

*Proof.* Let  $\{z, w\}$  and  $\{z', w'\}$  be two bases such that

$$\begin{cases} \Im(z/w) > 0 \\ \Im(z'/w') > 0 \\ \Phi(z, w) = \Phi(z', w') = L \end{cases}$$

Since  $z', w'$  are elements in the lattice  $L$  spanned by  $\{z, w\}$ , there exists  $a, b, c, d \in \mathbb{Z}$  such

that

$$\begin{bmatrix} z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}$$

By the same argument, there must be  $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$  such that

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} z' \\ w' \end{bmatrix}$$

Thus we must have

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  has integral entries, which implies that  $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm 1$ . Note that

$$\Im\left(\frac{z'}{w'}\right) = \Im\left(\frac{az + bw}{cz + dw}\right) = \Im\left(\frac{a(z/w) + b}{b(z/w) + d}\right) = (ad - bc)\Im\left(\frac{z}{w}\right)$$

Hence  $ad - bc = 1$  or  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ .  $\square$

Note that for each element  $(z, w) \in \mathrm{BAS}^+$ , we can rescaling by a factor  $w$  to get a base  $(z', 1)$  where  $\Im(z') = \Im(z/w) > 0$ . Since the rescaling action commutes with the action of  $\gamma$  on  $\mathrm{BAS}^+$ , it makes sense to define the space  $\mathbb{C}^\times \backslash \mathrm{BAS}^+ / \mathrm{SL}_2(\mathbb{Z})$ . By identifying the pair  $(z', 1)$  with  $z' \in \mathfrak{H}$  and using lemma 2.3.1.2, we have the following bijection

$$\mathfrak{H} / \mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{C}^\times \backslash \mathrm{BAS}^+ / \mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{C}^\times \backslash \{\text{lattices in } \mathbb{C}\}$$

**Proposition 2.3.1.3.** *The upper half plane  $\mathfrak{H}$  classifies similarity classes of two dimensional lattices.*

Now regard  $\mathbb{C} \cong \mathbb{R}^2$  via  $x + iy \mapsto (x, y)$ , and the inner product is defined to be

$$\langle z_1, z_2 \rangle = x_1 x_2 + y_1 y_2,$$

where  $z_i = x_i + iy_i$ . With this inner product, we can define the volume of the lattice

**Definition 2.3.1.4.** *Via the identification  $\mathbb{C} \cong \mathbb{R}^2$  and let  $e_1, e_2$  to be the usual standard*

basis for  $\mathbb{R}^2$ , the volume of a lattice  $L = \mathbb{Z}z \oplus \mathbb{Z}w$  is given by

$$\begin{bmatrix} \langle z, e_1 \rangle & \langle z, e_2 \rangle \\ \langle w, e_1 \rangle & \langle w, e_2 \rangle \end{bmatrix}$$

Now for any  $z \in \mathfrak{H}$ , the pair  $(1, z)$  can be identified with the lattice

$$L_z = \mathbb{Z}z \oplus \mathbb{Z}.$$

The lattice spanned by  $z$  and 1 has volume  $y$ , so the corresponding unit lattice is the one spanned by  $z/\sqrt{y}$  and  $1/\sqrt{y}$ .

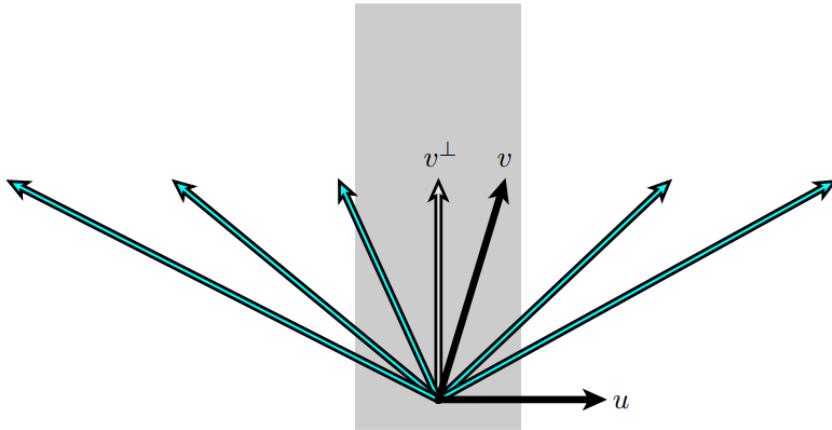
Using Proposition 2.2.0.5, it is immediate that every lattice spanned by 1 and  $z$  is similar to lattice generated by 1 and a point  $z'$  inside the region  $\mathfrak{D}$ .

**Definition 2.3.1.5.** A primitive vector is a vector such that it is not the multiple of any other vector in the lattice.

**Proposition 2.3.1.6.** If  $L$  is any lattice, and  $u$  is a primitive vector in  $L$ , and  $v'$  is a vector in the sublattice  $L' = L/\mathbb{Z}u$ , then there exists a unique representative  $v$  of  $v'$  such that its projection onto  $u$  lies in the interval  $(-u/2, u/2]$ . Moreover, the following inequality holds

$$\|v\|^2 \leq \frac{\|u\|^2}{4} + \|v'\|^2,$$

where we identify  $v'$  with a vector  $v^\perp$  in the orthogonal complement of  $u$ .



**Figure 2.1:** Reduction in two dimensional space

To see why every point  $z \in \mathfrak{H}$  can be transformed into a point inside  $\mathfrak{D}$ , we start with a lattice generated by the  $\mathbb{Z}$ -linear combination of 1,  $z$  and consider the shortest vector  $u$ . Applying

the above lemma, we can find a vector  $v$  with the length at least as large as that of  $u$ . So if we rotate and scale to get  $u = 1$ , the vector  $v$  will lie in the strip  $(-1/2, 1/2]$  and has the length at least 1. This clearly show that  $v$  is a point in the domain  $\mathfrak{D}$ .

### 2.3.2 Canonical plot and semistability in two dimensional space

Now Grayson - following a prior idea of Stuhler - associated every lattice to a sort of **Newton Polygon**. We will set up a graph coordinate in the following way:

1. First we construct a two dimensional coordinate, say  $Oxy$
2. We highlight the origin.
3. If we are dealing with the lattice  $L$ , compute the area of the fundamental domain of  $L$
4. Assign the point  $(2, \log(\text{vol}(L)))$  to the line  $x = 2$  in the coordinate.
5. If  $v$  is any primitive vector, we put the point  $(1, \log(\|v\|))$  in the set.

Note that the lattice is discrete, so we can find a shortest vector  $v$  of the lattice  $L$ . This will correspond to the lowest point on the axis  $x = 1$  in the diagram. Note the that  $x$ -coordinate of each of these point reflects its dimension.

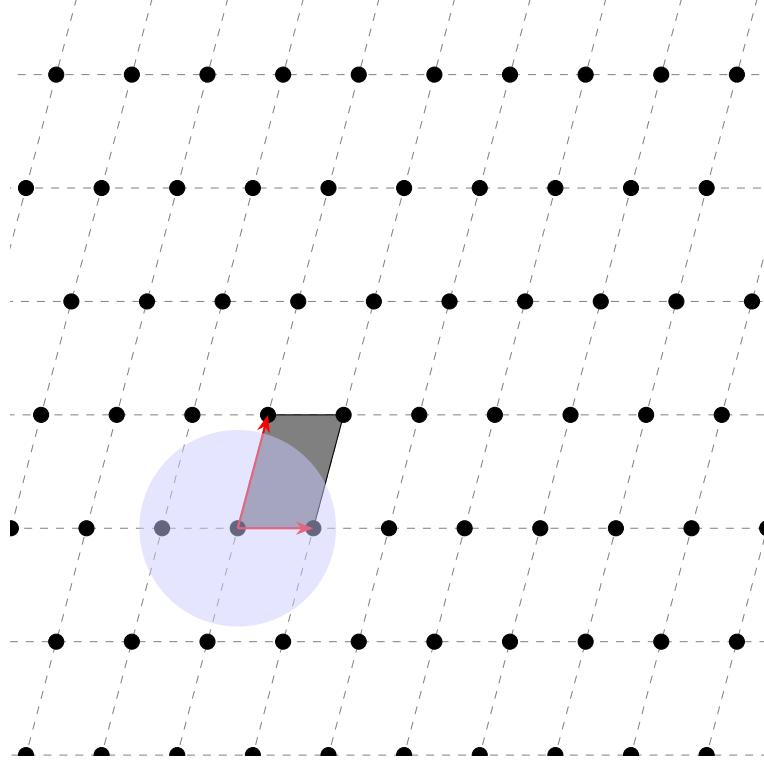
As an example, let's consider the lattice of the following shape - with the shortest vector  $u$  has the length  $\|u\| < 1$ . Now applying the above process, we get the figure on the left. If we further taking the convex hull of the diagram, we will get the figure on the right Clearly for each dimension, we have the corresponding lowest point, and so the convex hull of the plot is bounded from below. Grayson calls the plot on the left **canonical plot** of the lattice and the boundary of the convex hull of the canonical plot its **canonical polygon**. In the expository [6], Bill Casselman instead calls the canonical polygon as **profile**. We will use the terminology of Casselman.

Now we will try to understand the profile of a lattice associated to a point  $z \in \mathfrak{D}$ . First we prove a simple observation

**Lemma 2.3.2.1.** *If  $z \in \mathfrak{D}$  then the lattice  $L_z = \mathbb{Z}z \oplus \mathbb{Z}$  admits 1 as the shortest vector.*

*Proof.* We identify  $z = x + iy$  with  $(x, y) \in \mathbb{R}^2$  and 1 with  $(1, 0) \in \mathbb{R}^2$ . Assume that 1 is not the shortest vector, then there exists  $a, b \in \mathbb{Z}$  such that

$$|az + b|^2 < 1 \Leftrightarrow (ax + b)^2 + (ay)^2 < 1 \Leftrightarrow a^2|z|^2 + 2abx + b^2 < 1$$



**Figure 2.2:** Example of a lattice

Since  $z \in \mathfrak{D}$ , we clearly have  $|x| \leq \frac{1}{2}$  and  $|z| \geq 1$ , thus the integers  $a, b$  must satisfy

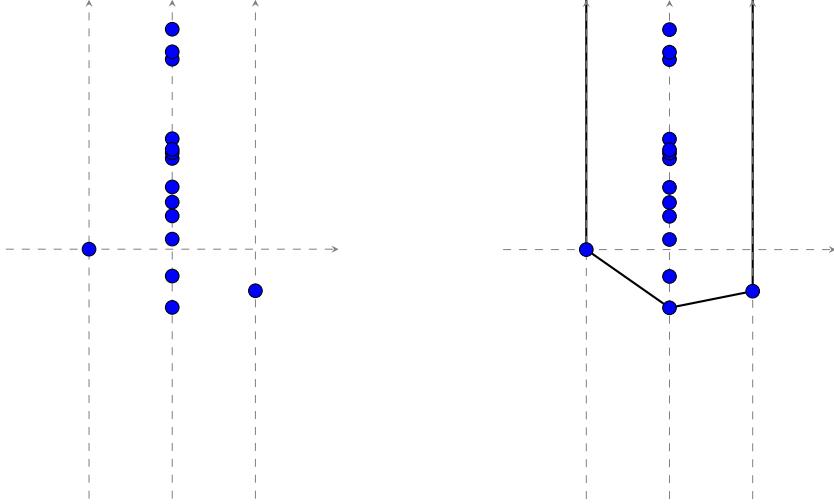
$$a^2 - |ab| + b^2 < 1$$

Since the above expression are symmetric, we can assume  $|a| \geq |b|$  and completing the square yields

$$\left(\frac{\sqrt{3}b}{2}\right)^2 \geq a^2 - ab + b^2 < 1 \Rightarrow b^2 < 4/3 \Rightarrow b \leq 1$$

Substituting  $|b| = 1$  yields  $|a|^2 - |a| < 0$ . There is no non-zero integer  $a$  satisfying this condition.  $\square$

The area of the lattice  $L_z$  is given by  $\det \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} = y$ . Note that we can scale the basis by a factor  $a = \sqrt{y}$  so that we get a lattice of volume 1. So the lowest points with respect to the axes  $x = 0, 1, 2$  are  $(0, 0)$ ,  $(1, -\log(a))$ , and  $(2, 0)$ . The interesting part of  $\mathfrak{D}$  is where  $y \leq 1$ . This corresponds to the lattices that have the canonical plots lying entirely on or above the  $x$ -axis. In particular, the profile of such a lattice only has the vertices at the origin and  $(2, 0)$ . Grayson and Stuhler call this kind of lattice **semi-stable**. If we don't normalize the area of such a lattice, then a semi-stable lattice has the bottom of the profile as a straight



line.

Conversely, the lattices assigned to the points  $z \in \mathfrak{H}$  with  $\Im(z) > 1$  correspond to lattices that have the canonical plot breaking at the lowest point on the axis  $x = 1$ . In the general case, this reflects the fact that a non semi-stable lattice has the shortest vector  $u$  satisfying  $\|u\| < \sqrt{\text{vol}(L_z)}$ . Following Casselman, we call such a lattice **unstable**. In some sense, we can see that **the degree of instability** is measured by the shortest vector compared to its volume. In the above lemma, we only find the semi-stable locus inside the fundamental domain. To find the semi-stable locus for the whole upper half plane  $\mathfrak{H}$ , we use the following lemma:

**Lemma 2.3.2.2.** *If  $L_z$  is semi-stable if and only if the lattice  $L_{g \circ z}$  is semi-stable, where  $g \in \text{SL}_2(\mathbb{R})$ .*

*Proof.* By definition, the lattice  $L_{g \circ z} := \mathbb{Z}(g \circ z) \oplus \mathbb{Z}$ . It is obvious that  $g \circ z$  and  $z$  are in the same coset of the quotient space  $\mathfrak{H} / \text{SL}_2(\mathbb{Z})$ . By Proposition 2.3.1.3, we have

$$\mathfrak{H} / \text{SL}_2(\mathbb{Z}) \cong \mathbb{C}^\times \setminus \{\text{lattices in } \mathbb{C}\}.$$

In particular, we must have that  $L_{g \circ z} = cL_z$  for some non-zero complex number  $c$ . In polar coordinate  $c = re^{it}$  for some real number  $t$ . Let's denote  $u$  the shortest vector of  $L_z$ . Then  $cu$  is also the shortest vector of  $cL_z$  and

$$|cu| = r|u|, \quad \text{and,} \quad \text{vol}(L_{g \circ z}) = \text{vol}(cL_z) = c^2 \text{vol}(L_z)$$

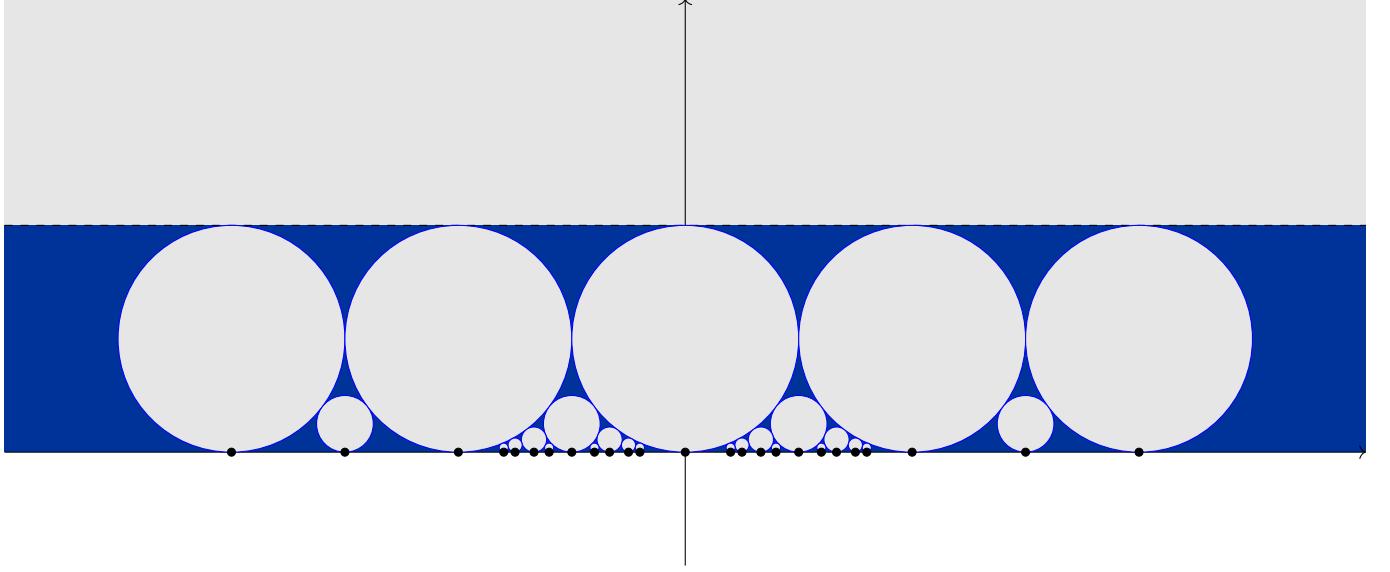
Using the assumption that  $L_z$  is semi-stable and the above equality, we get

$$L_z \text{ is semi-stable} \Leftrightarrow |u| \geq \sqrt{\text{vol}(L_z)}$$

$$\Leftrightarrow |cu| \geq c\sqrt{\text{vol}(L_z)} = \sqrt{\text{vol}(L_{goz})}$$

$$\Leftrightarrow L_{goz} \text{ is semi-stable} \quad \square$$

From this lemma, we can see that the semi-stable locus is the complement of the Farey balls in the upper half plane, as illustrated in the following figure, where the blue part is the semi-stable locus and the gray part is the unstable one.



**Figure 2.3:** Semi-stable locus in upper half plane

## 2.4 $\rho$ -semi-stability of lattices

### 2.4.1 Lie-theoretic definition of semi-stable lattices

The semi-stability can be defined in a more Lie-theoretic way. First we recall the Iwasawa decomposition for  $\text{SL}_2(\mathbb{R})$ .

**Proposition 2.4.1.1.** *We have*

$$\text{SL}_2(\mathbb{R}) \cong K \times A \times N$$

where

- $K = \text{SO}_2(\mathbb{R})$ : the special orthogonal group.

- $A = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a > 0 \right\}$ .

$$\bullet \quad N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right\}.$$

Combining with Proposition 2.1.0.1, we have the following identification

$$\mathfrak{H} \cong A \times N$$

via the map

$$x + iy \mapsto \begin{bmatrix} 1/\sqrt{y} & 0 \\ 0 & \sqrt{y} \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} = a(y) n(x)$$

Let's denote  $\mathfrak{sl}_2(\mathbb{R})$  the Lie algebra of the Lie group  $\mathrm{SL}_2(\mathbb{R})$  - the vector space of traceless matrices of size  $2 \times 2$ . We denote  $\mathfrak{h} = \mathbb{R}H$  where  $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  its standard Cartan subalgebra. We then have the map

$$H_B: \mathfrak{H} \rightarrow \mathfrak{h}, \quad z = x + iy \mapsto \log(a(y))H = \frac{-1}{2} \log(\Im(z))$$

Let  $\alpha: \mathfrak{h} \rightarrow \mathbb{R}$  be the unique linear function such that  $\alpha(H) = 2$ . If we let  $\rho = \frac{1}{2}\alpha$ , then we define

$$\deg_{\text{inst}}(z) := \min_{\gamma \in \Gamma / \Gamma \cap B} \langle \rho, H_B(z\gamma) \rangle$$

where  $B$  is the group of upper triangular matrices with invertible entries along the diagonal.

**Definition 2.4.1.2.** *The lattice  $L_z$  corresponds to the point  $z \in \mathfrak{H}$  is called  $\rho$ -semi-stable or just **semi-stable** if  $\deg_{\text{inst}}(z) \geq 0$ .*

## 2.4.2 Semi-stable locus in the upper half plane model

We shall use Definition 2.4.1.2 to find the semi-stable locus in the upper half plane  $\mathfrak{H}$ .

**Proposition 2.4.2.1.** *The locus of  $\rho$ -semi-stable points in the upper half plane  $\mathfrak{H}$  is the complement of the Farey balls.*

*Proof.* If  $\deg_{\text{inst}}(z)$  is achieved at some  $\gamma_0 \in \Gamma$  and  $z$  is  $\rho$ -semi-stable, then

$$\min_{\gamma \in \Gamma / \Gamma \cap B_{\mathbb{Q}}} \langle \rho, H_B(z\gamma) \rangle = \langle \rho, H_B(z\gamma_0) \rangle \geq 0$$

Hence  $\rho$ -semi-stable locus must be the set

$$\{z \in \mathfrak{H} : \langle \rho, H_B(z\gamma) \rangle \geq 0 \text{ for all } \gamma \in \Gamma\}$$

We identify  $z = x + iy$  with the product  $a(y)n(x)$  as above. Under the identification 2.1.0.2, we must have

$$\langle \rho, H_B(z\gamma) = -\log(a(\gamma^{-1} \circ z)) \rangle$$

Note that, for any pair of co-prime integers  $(c, d)$ , we can always find integers  $a, b$  such that  $ad - bc = 1$  by Bezout's theorem. Assume that  $\gamma^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , then  $\gamma_0^{-1} \circ z$  satisfies

$$\Im(\gamma^{-1} \circ z) = \frac{\Im(z)}{(cx + d)^2 + (cy)^2}$$

Thus,  $z$  is semi-stable if and only if it satisfies the inequalities

$$y \leq (cx + d)^2 + (cy)^2 \Leftrightarrow \left( x + \frac{c}{d} \right)^2 + \left( y - \frac{1}{2c^2} \right)^2 \geq \left( \frac{1}{2c^2} \right)^2.$$

Since the above equation is exactly the equation for Farey balls, we are done.  $\square$

### 2.4.3 Unstable lattices

Consider an unstable lattice  $L$  with a shortest vector  $u$ , then the bottom of the profile of  $L$  has a break at the point  $(1, \log(\|u\|))$ . Clearly  $u$  is a primitive vector, so  $L_1 = \mathbb{Z}u$  is a sublattice of  $L$ . This determines a **lattice flag**

$$\mathcal{F} : 0 \subset L_1 \subset L_2 = L$$

And this is called the **canonical flag** associated to  $L$ . By tensoring with  $\mathbb{R}$  we get a flag of rational subspaces

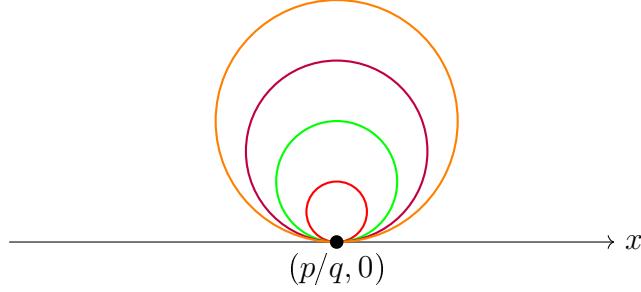
$$0 \subset V_1 = L_1 \otimes \mathbb{R} \subset V_2 = L_2 \otimes \mathbb{R}$$

Conversely, for any rational flag  $\mathcal{F}$  in  $\mathbb{R}^2$ , we can denote  $\mathcal{H}_{\mathcal{F}}$  the set of all unstable lattices that gives rise to the flag  $\mathcal{F}$ . What can we say about the set  $\mathcal{H}_{\mathcal{F}}$ ? It turns out that the can be parametrized by the Farey balls in the complex plane.

**Proposition 2.4.3.1.** *Given a flag of rational vector spaces*

$$\mathcal{F} : 0 \subset V_1 \subset V_2 = \mathbb{R}^2$$

*Assume that  $V_1 = \text{span}(pf_1 + qf_2)$  for some  $\mathbb{R}$ - linearly independent set  $\{f_1, f_2\}$ . Then the set  $\mathcal{H}_{\mathcal{F}}$  corresponds to the Farey balls that is tangent to the  $x$ -axis at the fraction  $p/q$ .*



**Figure 2.4:** The Farey Circles correspond to the same rational flag

*Proof.* Consider the lattice  $L = L_z \in \mathcal{H}_{\mathcal{F}}$  for  $z = x + iy$ . This lattice  $L$  has a basis

$$\begin{cases} f_1 = \frac{e_1}{\sqrt{y}} \\ f_2 = \frac{-xe_1}{\sqrt{y}} + e_2\sqrt{y} \end{cases}$$

Assume that  $u = pf_1 + qf_2$  is the shortest vector of  $L$ . Since  $u$  is primitive, we must have  $\gcd(p, q) = 1$ . Since the lattice  $L$  is unstable, it follows that  $\mu(L_1) \leq \mu(L/L_1)$  for  $L_1 = \mathbb{Z}u$ . In particular, we have

$$\frac{| -qz + p |}{\sqrt{y}} \leq 1 \Leftrightarrow (-qx + p)^2 + (qy)^2 \leq y$$

If  $q \neq 0$ , this is the equation for the Farey balls that tangent to the horizontal axis at  $(p/q, 0)$ .

If  $p = 0$  then  $q = 1$ , the equation  $\frac{| -qz + p |}{\sqrt{y}} \leq 1$  degenerates to  $y \geq 1$ . Hence we are done.  $\square$

**Remark.** Note that for every flag in  $\mathbb{R}^2$  corresponds to a parabolic subgroup. For example, the subgroup

$$B = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

stabilizes the flag  $0 \subset \mathbb{R}e_1 \subset \mathbb{R}^2$  for some choice of basis  $\{e_1, e_2\}$  of  $\mathbb{R}^2$ . Any parabolic subgroup of  $\mathrm{SL}_2(\mathbb{R})$  is conjugate with  $B$ . So there is a correspondence between rational parabolic subgroups of  $\mathrm{SL}_2(\mathbb{R})$ , that is, the parabolic subgroup that stabilizes a rational flag  $\mathcal{F}$  - and the collections  $\mathcal{H}_{\mathcal{F}}$  of lattices gives rise to the flag  $\mathcal{F}$ .

# Chapter 3

## BASIC LIE THEORY FOR $\text{SL}_n(\mathbb{R}) \& \text{GL}_n(\mathbb{R})$

In this chapter, we review the basic theory of roots and weights for the Lie groups  $\text{SL}_n(\mathbb{R})$  and  $\text{GL}_n(\mathbb{R})$ .

### 3.1 Lie algebras

Throughout this section, the ground field is  $k = \mathbb{R}$ .

#### 3.1.1 Definition

A Lie algebra  $\mathfrak{g}$  is a vector space over  $k$  such that the Lie bracket

$$\begin{aligned} [\cdot, \cdot] &: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \\ (x, y) &\mapsto [x, y] \end{aligned}$$

satisfies the following axioms

1.  $[x, y]$  is bilinear.
2.  $[x, x] = 0$  for all  $x \in \mathfrak{g}$ .
3.  $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$

The last property is called *Jacobi's identity*.

**Example 3.1.1.1.** The first example is the set of all matrices of size  $n \times n$ , denoted by  $\mathfrak{gl}_n(\mathbb{R})$ . We define the Lie bracket by

$$[A, B] := AB - BA,$$

for  $A, B \in \mathfrak{gl}_n(\mathbb{R})$ . Therefore  $\mathfrak{gl}_n(\mathbb{R})$  is a Lie algebra.

**Example 3.1.1.2.** Another example is the subspace  $\mathfrak{sl}_n(\mathbb{R})$  of  $\mathfrak{gl}_n(\mathbb{R})$ , defined by

$$\mathfrak{sl}_n(\mathbb{R}) = \{A \in \mathfrak{gl}_n(\mathbb{R}) : \text{tr}(A) = 0\}$$

The Lie bracket over  $\mathfrak{sl}_n(\mathbb{R})$  is just the restriction of the Lie bracket over  $\mathfrak{gl}_n(\mathbb{R})$ . Indeed, for any  $A, B \in \mathfrak{sl}_n(\mathbb{R})$  we have

$$\text{tr}[A, B] = \text{tr}(AB - BA) = \text{tr}(AB) - \text{tr}(BA) = 0$$

### 3.1.2 Cartan Subalgebras

First we need the notion of Cartan subalgebras

**Definition 3.1.2.1.** For any Lie algebra  $\mathfrak{g}$ , a subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  is said to be a Cartan algebra if it is

- $\mathfrak{h}$  is a nilpotent subalgebra, i.e.  $\mathfrak{h}^{(n)} = \underbrace{[\mathfrak{h}, [\mathfrak{h}, \dots, [\mathfrak{h}, \mathfrak{h}]]]}_{n\text{-times}} = 0$  for some integer  $n > 0$ .
- It is self normalizing. Namely, we have  $\mathfrak{h} = \{x \in \mathfrak{g} : [x, \mathfrak{g}] \subset \mathfrak{g}\}$ .

A priori, it is not clear whether such a subalgebra exists for any Lie algebras. However, we have the following theorem

**Theorem 3.1.2.2.** [2, Corollary 1.2] Let  $L$  be a Lie algebra of dimension  $n$  over a field  $F$  of at least  $n - 1$  elements. Then  $L$  has a Cartan subalgebra.

In particular, any finite dimensional Lie algebra over an infinite field has a Cartan subalgebra.

**Example 3.1.2.3.** For the simple Lie algebra  $\mathfrak{sl}_n(\mathbb{R})$ , one choice of Cartan subalgebra  $\mathfrak{h}$  is the set

$$\mathfrak{h} = \left\{ H = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}, a_1 + a_2 + \dots + a_n = 0 \right\} \quad (3.1)$$

*Proof.* Indeed, it is obvious that  $[\mathfrak{h}, \mathfrak{h}] = 0$ , thus the Lie algebra  $\mathfrak{h}$  is nilpotent. To show the self-normalizing property, we pick an any element  $\sum_{i,j} a_{ij} E_{ij}$  such that

$$\left[ \sum_{i,j} a_{ij} E_{ij}, \mathfrak{h} \right] \subset \mathfrak{h}$$

Here  $E_{ij}$  denotes the matrix that has 1 at the  $ij$  entry and 0 otherwise. Clearly the matrix  $E_{pp} - E_{qq} \in \mathfrak{h}$  for  $p \neq q$ . We then have

$$\left[ \sum_{i,j} a_{ij} E_{ij}, E_{pp} - E_{qq} \right] \in \mathfrak{h}$$

or equivalently

$$\sum_i a_{ip} E_{ip} - \sum_i a_{iq} E_{iq} - \sum_j a_{pj} E_{pj} + \sum_j a_{qj} E_{qj} \in \mathfrak{h}$$

The coefficients of  $E_{pq}$  in the above sum is  $-2a_{pq}$ . By our choice of  $\mathfrak{h}$  it must be the case that  $a_{pq} = 0$ . Thus, only the coefficients of  $E_{pq}$  for  $p = q$  survive. Hence  $\sum_{i,j} a_{ij} E_{ij} \in \mathfrak{h}$  and  $\mathfrak{h}$  is then a Cartan subalgebra of  $\mathfrak{sl}_n(\mathbb{R})$ .  $\square$

### 3.1.3 Root space decomposition

Consider a semisimple Lie algebra  $\mathfrak{g}$ . With respect to a choice of Cartan subalgebra  $\mathfrak{h}$ , we have a root space decomposition. In particular, there is a finite set  $\Phi \subset \mathfrak{h}^*$  of linear forms on  $\mathfrak{h}$ , whose elements are called **roots**, such that

$$\mathfrak{g} = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right),$$

where  $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$  for any  $\alpha \in \Phi$ . We refer to [13, Chapter 8] for a detail analysis of this decomposition in general case. For this thesis, most of the time, we stick with the following example

**Example 3.1.3.1.** Let  $I = \{1, 2, \dots, n-1\}$ . For the simple Lie algebra  $\mathfrak{sl}_n(\mathbb{R})$ , one choice of Cartan subalgebra  $\mathfrak{h}$  is as in (3.1). With respect to this Cartan subalgebra, we can define the linear function

$$L_i: \mathfrak{h} \rightarrow \mathbb{R}, \quad H \mapsto a_i$$

Then the roots are given by  $\alpha_{ij} := L_i - L_j$  for distinct  $i, j \in I$ . We have the root space

decomposition for  $\mathfrak{sl}_n(\mathbb{R})$  as follows

$$\mathfrak{g} = \mathfrak{h} \oplus \left( \bigoplus_{i \neq j} \mathfrak{g}_{\alpha_{ij}} \right) = \mathfrak{h} \oplus \left( \bigoplus_{i \neq j} \mathbb{R}E_{ij} \right)$$

For the sake of brevity, we will denote  $\alpha_{i,i+1}$  by  $\alpha_i$  - these are called **simple roots**. We can decompose the set of roots  $\Phi = \{\alpha_{ij}, i \neq j\}$  into disjoint subsets, namely

$$\Phi = \Phi_+ \coprod \Phi_-$$

where the positive set of roots  $\Phi_+$  comprises  $\alpha_{ij}$  for  $i < j$  and the remaining roots are in the negative set of roots  $\Phi_-$ .

**Lemma 3.1.3.2.** *Each  $\alpha \in \Phi$  can be written uniquely as a linear combination*

$$\alpha = m_1\alpha_1 + \dots + m_d\alpha_d$$

with all  $m_i \in \mathbb{Z}_{\geq 0}$  or  $m_i \in \mathbb{Z}_{\leq 0}$ . If  $\alpha \in \Phi_+$  then all  $m_i \geq 0$ , otherwise  $m_i \leq 0$  for all  $i$ .

### 3.1.4 Killing form

To work with the geometry of the root system, we need the notion of the Killing form.

**Definition 3.1.4.1.** *Let  $\mathfrak{g}$  be any Lie algebra and  $x, y \in \mathfrak{g}$ . We define the map*

$$\begin{aligned} \text{ad}(x) : \mathfrak{g} &\rightarrow \mathfrak{g} \\ y &\mapsto \text{ad}(x)(y) := [x, y] \end{aligned}$$

The bilinear map  $\kappa(x, y) := \text{Tr}(\text{ad}(x) \circ \text{ad}(y))$  is called **Killing form**. If there is no confusion, we write  $(x, y) := \kappa(x, y)$ . When  $\mathfrak{g}$  is semisimple, the Killing form has many nice properties, as recorded in the following theorem

**Theorem 3.1.4.2.** [13, Chapter 5] *A Lie algebra  $\mathfrak{g}$  is semi-simple if and only if the Killing form is non-degenerate.*

It can also be showed that, given the root space decomposition as in section 3.1.3, if  $\alpha, \beta \in \Delta$  satisfy  $\alpha + \beta \neq 0$ , then the space  $\mathfrak{g}_\alpha$  is orthogonal to  $\mathfrak{g}_\beta$  with respect to the Killing form. A corollary of this is

**Theorem 3.1.4.3.** [13, Chapter 8] *Given a semisimple Lie algebra  $\mathfrak{g}$  and a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$ . Then the restriction of the Killing form to  $\mathfrak{h}$  is non-degenerate.*

By Theorem 3.1.4.3, there exists a bijective linear map

$$\begin{aligned} f: \mathfrak{h} &\rightarrow \mathfrak{h}^* \\ x &\mapsto f_x \end{aligned}$$

where  $f_x(y) := \kappa(x, y)$  for all  $y \in \mathfrak{h}$ . We may thus define a bilinear map

$$\begin{aligned} (\cdot, \cdot): \mathfrak{h}^* \times \mathfrak{h}^* &\rightarrow \mathbb{R} \\ (f_x, f_y) &\mapsto \kappa(x, y) \end{aligned}$$

In this notation, it can be observed that  $(f_x, f_y) = (x, y)$ . A more thorough treatment of the Killing form can be found in [13, Chapter 5,8].

**Example 3.1.4.4.** Consider the lie algebra  $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$  and  $\mathfrak{h}$  as in (3.1). It can be shown that

$$\kappa(x, y) = 2n\text{tr}(xy)$$

where  $xy$  is just the usual matrix multiplication and  $\text{tr}$  is the trace of the matrix. We refer to [8, Section 4.3] for a detailed computation.

**Claim:**

$$2 \frac{E_{i,i} - E_{i+1,i+1}}{(E_{i,i} - E_{i+1,i+1}, E_{i,i} - E_{i+1,i+1})} = \frac{E_{i,i} - E_{i+1,i+1}}{2n}$$

gives rise to the root  $\alpha_i = L_i - L_{i+1}$  via the identification using Killing form.

Indeed, by direct computation, we have

$$\left( \frac{E_{i,i} - E_{i+1,i+1}}{2n}, E_{j,j} - E_{j+1,j+1} \right) = \begin{cases} 2, & \text{if } i = j \\ -1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise} \end{cases} \quad (3.2)$$

On the other hand we also have

$$\alpha_i (E_{j,j} - E_{j+1,j+1}) = (L_i - L_{i+1})(E_{j,j} - E_{j+1,j+1}) \begin{cases} 2, & \text{if } i = j \\ -1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Since  $\{E_{j,j} - E_{j+1,j+1}\}$  where  $j \in I$  forms a basis for  $\mathfrak{h}$ , we must have

$$\frac{(E_{i,i} - E_{i+1,i+1}, x)}{2n} = (L_i - L_{i+1})(x)$$

for all  $x \in \mathfrak{h}$ . This proves the desired claim.

### 3.1.5 Co-roots

In previous section, we showed that the Killing form gives rise to an inner product over the dual space  $\mathfrak{h}^* = \text{Hom}(\mathfrak{h}, \mathbb{R})$ . For each root  $\alpha \in \mathfrak{h}^*$ , we define the corresponding co-root  $\alpha^\vee \in \mathfrak{h}$  to be the element such that

$$\langle \beta, \alpha^\vee \rangle := \beta(\alpha^\vee) = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

for all  $\beta \in \mathfrak{h}^*$ . It is clear from this definition that  $\langle \alpha, \alpha^\vee \rangle = 2$ .

**Example 3.1.5.1.** *This continues the example 3.1.4.4. We want to determine the co-root  $\alpha_i^\vee$ . It is then sufficient to compute  $\langle \alpha_j, \alpha_i^\vee \rangle$  for  $j \in I$ . By definition*

$$\langle \alpha_j, \alpha_i^\vee \rangle = \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)}$$

By example 3.1.4.4, we have

$$(\alpha_j, \alpha_i) = \frac{1}{4n^2}(E_{j,j} - E_{j+1,j+1}, E_{i,i} - E_{i+1,i+1}) = \begin{cases} \frac{1}{n}, & \text{if } i = j \\ \frac{-1}{2n}, & \text{if } |i - j| = 1 \\ 0, & \text{otherwise} \end{cases}$$

In particular,

$$\alpha_j(\alpha_i^\vee) = \begin{cases} 2, & \text{if } i = j \\ -1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

This proves that  $\alpha_i^\vee = E_{i,i} - E_{i+1,i+1}$ .

### 3.1.6 Weights and co-weights

Fix  $i \in I$ , define  $\lambda_i \in \mathfrak{h}^*$  by specifying

$$\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

**Example 3.1.6.1.** Consider again the Lie algebra  $\mathfrak{sl}_n(\mathbb{R})$  with the co-root  $\alpha_{ij}^\vee = E_i - E_j$ .

Explicitly, it can be checked that  $\lambda_i = L_1 + L_2 + \dots + L_i$ . Indeed, we have

$$(L_1 + L_2 + \dots + L_i)(E_{j,j} - E_{j+1,j+1}) = 1 \Leftrightarrow i = j \quad \text{and } 0 \text{ otherwise}$$

which uniquely characterize  $\lambda_i$ .

In the same manner we define the co-weight  $\lambda_i^\vee$  such that

$$\langle \alpha_i, \lambda_j^\vee \rangle = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

**Example 3.1.6.2.** We work with  $\mathfrak{sl}_n(\mathbb{R})$ . Note that

$$(L_i - L_{i+1})(E_{1,1} + \dots + E_{j,j}) = 1 \Leftrightarrow i = j, \quad \text{and } 0 \text{ otherwise}$$

we can see that  $\lambda_i^\vee = E_{1,1} + \dots + E_{i,i}$

From these definitions, we can see that  $\hat{\Delta} = \{\lambda_i, i = 1, 2, \dots, n-1\}$  is a basis for the space  $\mathfrak{h}^*$ . We call  $\hat{\Delta}$  the set of **fundamental weights**. Similarly the set  $\hat{\Delta}^\vee = \{\lambda_i^\vee, i = 1, 2, \dots, n-1\}$  is a basis for the Cartan subalgebra  $\mathfrak{h}$ . We have the following

**Lemma 3.1.6.3.** We can write

$$\lambda_i := r_1\alpha_1 + r_2\alpha_2 + \dots + r_d\alpha_d$$

where  $r_i$ 's are rational number such that  $r_i \geq 0$ .

This coefficients  $r_i$ 's is determined by inverting the Cartan matrix of  $\mathfrak{sl}_n(\mathbb{R})$ . We refer to [23] for an explicit formula for  $r'_i$ 's.

### 3.1.7 Weyl group

We define the map

$$s_\alpha(\beta) := \beta - \langle \beta, \alpha^\vee \rangle \alpha = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha$$

where  $\alpha, \beta \in \mathfrak{h}^*$  are roots. The space  $\mathfrak{h}^*$  along with the bilinear map  $(\cdot, \cdot)$  have the structure of a Euclidean space, so we can think of  $s_\alpha$  as reflection transformation. When  $\alpha = \alpha_i$  - the simple root - we call  $s_\alpha$  the **simple reflection** and denote  $s_i := s_{\alpha_i}$ .

**Definition 3.1.7.1.** The Weyl group of  $\mathfrak{sl}_n(\mathbb{R})$  is defined to be the set generated by the simple reflections

$$W = \langle s_i, i = 1, 2, \dots, n-1 \rangle$$

**Example 3.1.7.2.** We continue the computation as in example 3.1.5.1. For any  $i \in \{1, 2, \dots, n-1\}$  we have

$$s_i(L_{i+1}) = L_{i+1} - \langle L_{i+1}, \alpha_i^\vee \rangle \alpha_i = L_{i+1} - (-1)(L_i - L_{i+1}) = L_i$$

and

$$s_i(L_i) = L_i - \langle L_i, \alpha_i^\vee \rangle \alpha_i = L_i - (L_i - L_{i+1}) = L_{i+1}$$

Thus, if we consider the  $l+1$  dimensional space generated by  $L_i, i = 1, 2, \dots, n$ , then the Weyl group is generated by the set of transposition  $(i, i+1)$ . Hence  $W \cong S_{l+1}$  - the symmetric group of  $l+1$  letters.

## 3.2 Lie groups

We only define linear Lie groups.

### 3.2.1 General linear group and special linear group

**Definition 3.2.1.1.** A general linear group over the real field, denoted  $GL_n(\mathbb{R})$ , is defined to be the set

$$GL_n(\mathbb{R}) = \{A \in \mathfrak{gl}_n(\mathbb{R}) : \det(A) \neq 0\}$$

An linear Lie group is a closed subgroup of  $GL_n(\mathbb{R})$ . The key example is the following group

$$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det(A) = 1\}$$

### 3.2.2 Relation with Lie algebras

We first need to define exponential map

**Definition 3.2.2.1.** Given a matrix  $X \in \mathfrak{gl}_n(\mathbb{R})$ , we define the exponential of  $X$  is series

$$e^X := I + X + \frac{X^2}{2!} + \dots = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

For each linear Lie group  $G \subset GL_n(\mathbb{R})$ , we associate the set

$$\text{Lie}(G) := \{X \in \mathfrak{gl}_n(\mathbb{R}) : e^{tX} \in G, \forall t \in \mathbb{R}\}$$

It can be proven, for example see [8, Theorem 3.2.1], that  $\text{Lie}(G)$  is a vector spaces. We will try to compute  $\text{Lie}(GL_n(\mathbb{R}))$  and  $\text{Lie}(SL_n(\mathbb{R}))$ .

**Example 3.2.2.2.** We need the following result

**Theorem 3.2.2.3.** [8, Section 2.1] For any  $X \in M_n(\mathbb{R})$ , we have

$$\det(e^X) = e^{\text{tr}(X)},$$

where  $\text{tr}(X)$  is just the sum of entries along the diagonal of  $X$ .

Using Theorem 3.2.2.3, it is clearly that for any  $X \in M_n(\mathbb{R})$ ,  $e^X$  is an invertible matrix, thus  $\text{Lie}(GL_n(\mathbb{R})) = \mathfrak{gl}_n(\mathbb{R})$ . Let  $X \in \mathfrak{sl}_n(\mathbb{R})$  so that  $\text{tr}(X) = 0$ , and apply Theorem 3.2.2.3 again, we have

$$\det(e^{tX}) = e^{t\text{tr}(X)} = 1,$$

, which implies  $\text{Lie}(SL_n(\mathbb{R})) \supset \mathfrak{sl}_n(\mathbb{R})$ . Conversely, if  $X \in SL_n(\mathbb{R})$ , then

$$e^{tX} \in SL_n(\mathbb{R}), \forall t \in \mathbb{R} \Leftrightarrow e^{t\text{tr}(X)} = \det(e^{tX}) = 1, \forall t \in \mathbb{R}$$

Hence

$$\text{tr}(X) = \left( \frac{d}{dt} e^{t\text{tr}(X)} \right)_0 = 0$$

which implies  $\text{Lie}(SL_n(\mathbb{R})) \subset \mathfrak{sl}_n(\mathbb{R})$ . We then proved  $\text{Lie}(SL_n(\mathbb{R})) = \mathfrak{sl}_n(\mathbb{R})$ .

### 3.2.3 Roots at group level

We want to understand how the roots behave at Lie group level. The analog for a Cartan subalgebra is a maximal torus given by the set

$$T = \left\{ t = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix} : \prod_{i=1}^n a_i = 1 \right\},$$

Then  $T$  acts on  $\mathfrak{g}$  by conjugation:  $\text{Ad}(t)(X) := tXt^{-1}$  for any  $t \in T$  and  $X \in \mathfrak{g}$ . Explicitly, we can check that

$$\text{Ad}(t)(E_{ij}) = a_i a_j^{-1} E_{ij}$$

Since every element of  $T$  are diagonal and commutative, they are simultaneously diagonal. If we let  $\mathfrak{g}_\chi := \{X \in G : \text{Ad}(t)(X) = \chi(t)X, \forall t \in T\}$ , then we have the decomposition

$$\mathfrak{g} = \bigoplus_{\chi \in X(T)} \mathfrak{g}_\chi,$$

where  $X(T)$  is just the set of multiplicative character from the torus  $T$  above to  $\mathbb{R}^*$ . This is the **root space decomposition** at Lie group level. Therefore, at the group level, the character  $\chi_{ij}(\text{diag}(t_1, \dots, t_n)) = t_i t_j^{-1}$  is a root whenever  $i \neq j$ . We define the set of **simple roots** as

$$\Delta = \{\chi_i \mid i = 1, \dots, n\}$$

where

$$\chi_i: T \mapsto \mathbb{R}, t \mapsto \frac{t_i}{t_{i+1}}.$$

### 3.2.4 Weights

Similarly, at Lie group level, we also have the notion of **fundamental weights**. For each  $i \in I$ , we define a character  $\lambda_i \in \text{Hom}(T, \mathbb{R}^*)$  where  $T$  is the torus defined in 3.2.3

$$\omega_i: T \rightarrow \mathbb{R}^*, \quad \omega_i(t) = a_1 \dots a_i$$

**Remark.** If we define the logarithm map over the torus  $T$  by

$$t = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix} \mapsto \log(t) := \begin{bmatrix} \log(a_1) & 0 & \dots & 0 \\ 0 & \log(a_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \log(a_n) \end{bmatrix}$$

Then it is clear that  $\log$  is a map from  $T$  to  $\mathfrak{h}^*$ . In this way, we can think  $\chi_{ij}$  as  $\alpha_{ij} \circ \log$ . When there are no confusion, we use the symbol  $\alpha_{ij}$  to indicate the root at both Lie algebra level and Lie group level. Similarly we will also use  $\lambda_i$  for the notation of simple weight in both Lie algebra and Lie group level as in the case of root.

**Example 3.2.4.1.** When  $n = 3$ , we have the following relations

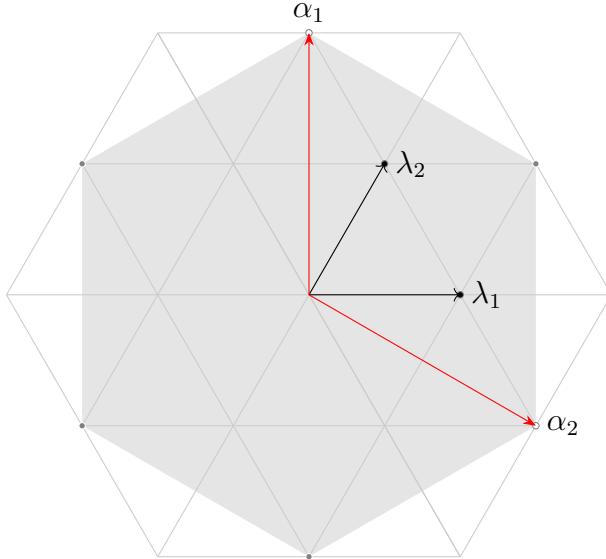
$$\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \quad \lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$$

We refer to the figure 3.1 for an illustration.

**Definition 3.2.4.2.** A weight  $\lambda$  is called **dominant** if it satisfies  $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$  for all  $\alpha$ .

Clearly by lemma 3.1.3.2, the weight  $\lambda$  is dominant if and only if  $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$  for all fundamental root  $\alpha_i$ . It is also clear that the set of dominant weights is given by

$$\Lambda^+ := \{c_1\lambda_1 + \dots + c_d\lambda_d \mid c_i \in \mathbb{Z}_{\geq 0}\}$$



**Figure 3.1:** Roots and weights for the Lie group  $\mathrm{SL}_3(\mathbb{R})$

The set of dominant weights is denoted  $\Lambda^+$ . A weight  $\lambda = \sum n_i \lambda_i$  is called strongly dominant if  $n_i > 0$  for all  $i$ . One important example is the minimal strongly dominant weight given by

$$\rho = \sum \lambda_i$$

This is called **Weyl vector** and is characterized in several ways:

1.  $\langle \rho, \alpha_i^\vee \rangle = 1$  for all  $i$ .

2.

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

To prove the last equation we use the action of the Weyl group  $W$ . Let  $\mu = \frac{1}{2} \sum \alpha$ . Apply the simple reflection  $s_i$  given by

$$s_i(x) = x - \langle x, \alpha_i^\vee \rangle \alpha_i$$

We know that  $s_i$  sends  $\alpha_i$  to  $-\alpha_i$  and permutes the other positive roots. So:

$$s_i(\mu) = \mu - \langle x, \alpha_i^\vee \rangle \alpha_i$$

Therefore,  $(\mu, \alpha_i) = \mu(h_i) = 1$  for all  $i$ . So,  $\mu = \rho$ .

We of course also have an action of  $W$  on the weights. For example, one can verify that

$$s_i(\lambda_i) = \lambda_i - \alpha_i \quad \text{and} \quad s_j(\lambda_i) = \lambda_i \text{ for } i \neq j.$$

Unlike Lemma 3.1.6.3, if we try to express the fundamental weights in terms of the fundamental roots, we do not always get positive coefficients. However, it is true that all the coefficients must be integer. In particular, we have

$$\alpha_j = \sum_{n_j} n_j \lambda_j, \quad n_j \in \mathbb{Z}.$$

To put it another way, the root lattice

$$\mathbb{Z}\Delta = \{m_1\alpha_1 + \dots + m_d\alpha_d, m_i \in \mathbb{Z}\}$$

is contained inside the weight lattice  $\mathbb{Z}\Pi = \{n_1\lambda_1 + \dots + n_d\lambda_d, n_i \in \mathbb{Z}\}$ .

### 3.2.5 Cartan matrix

We fix a set of simple roots  $\Delta = \{\alpha_1, \dots, \alpha_d\}$  is defined to be the matrix

$$A = [\langle \alpha_i, \alpha_j^\vee \rangle]$$

If we let  $a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$  then the Cartan matrix has the following simple properties:

**Lemma 3.2.5.1.**

- For any  $i$ , we have  $a_{ii} = 2$ .
- For any  $i \neq j$ ,  $a_{ij}$  is a non-positive integer, i.e.  $a_{ij} \in \mathbb{Z}_{\leq 0}$ .

We give an explicit example for  $\mathfrak{sl}_n(\mathbb{R})$ , which has the root system  $A_n$ . The corresponding Cartan matrix is

$$A_n = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix}$$

The Cartan matrix provides a linear combination presentation of the fundamental roots  $\alpha_j$ 's in terms of fundamental weights  $\lambda_i$ , as stated in the following lemma

**Lemma 3.2.5.2.** Fix a number  $n$  and consider the set of simple roots  $\alpha_j$  as well as the set of weights  $\lambda_i$  of  $SL_n(\mathbb{R})$ , we have the following relations

$$\alpha_i = \begin{cases} 2\lambda_1 - \lambda_2, & \text{if } i = 1 \\ -\lambda_{i-1} + 2\lambda_i - \lambda_{i+1}, & \text{if } 1 < i < n \\ -\lambda_{n-1} + 2\lambda_n, & \text{if } i = n \end{cases}$$

*Proof.* Note that we have

$$\langle \alpha_i, \alpha_j^\vee \rangle = \begin{cases} 2, & \text{if } i = j \\ -1, & \text{if } |i - j| = 1 \\ 0, & \text{otherwise} \end{cases}.$$

So if we let  $\alpha_j = \sum_{i=1}^n a_{ij} \lambda_i$  and compute  $\langle \alpha_i, \alpha_j^\vee \rangle$ , the result follows immediately.  $\square$

### 3.3 Parabolic subgroups and parabolic subalgebras

We shall provide two equivalent viewpoints on parabolic subgroups. They will play different roles in defining different notions of semi-stability in the next chapter.

#### 3.3.1 Parabolic subgroups I: An explicit description

For our purposes, it is enough to define the standard parabolic subgroups. There exists a bijection between standard parabolic subgroups of  $SL_n(\mathbb{R})$  and partitions of  $n$ , and partition  $n = n_1 + \dots + n_k$  defines the standard parabolic subgroup corresponding to the partition as follows

**Definition 3.3.1.1.** The standard parabolic subgroup  $P$  of  $GL_n(\mathbb{R})$  associated to the partition  $n = n_1 + n_2 + \dots + n_r$  is denoted  $P_{n_1, \dots, n_r}$  and is defined to be the group of all matrices of the form

$$\begin{bmatrix} M_{n_1} & 0 & \dots & 0 \\ 0 & M_{n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_{n_k} \end{bmatrix},$$

where  $M_{n_i} \in GL(n_i, \mathbb{R})$  for  $1 \leq i \leq r$ . The integer  $r - 1$  is called the rank of the parabolic subgroup  $P_{n_1, \dots, n_r}$ . We denote the collection of such subgroups  $\mathbf{ParSt}$ .

We denote  $\{e_i\}$  the standard basis of  $\mathbb{R}^n$ . Let

$$d_i = n_1 + n_2 + \dots + n_i$$

Then any element in  $P_{n_1, \dots, n_r}$  stabilizes the chain of vector spaces

$$0 \subset \mathbb{R}^{d_1} \subset \mathbb{R}^{d_2} \subset \dots \subset \mathbb{R}^{d_k} = \mathbb{R}^n$$

Here  $\mathbb{R}^k$  is generated by the linearly independent subset  $\{e_1, \dots, e_k\}$  of the standard basis. We call such chain of vector spaces a **flag** of type  $(n_1, \dots, n_k)$ .

**Definition 3.3.1.2.** *The maximal standard parabolic subgroups in  $GL_n(\mathbb{R})$  corresponds to the stabilizer of the flag of type  $\rho_i = (i, n-i)$ , where  $i = 1, \dots, n-1$  of  $n$ . We will further denote  $Q_i = P_{\rho_i}$  and **MaxParSt** the collection of such maximal parabolic subgroups.*

**Example 3.3.1.3.** *Below we list all the standard parabolic subgroup in  $GL_3(\mathbb{R})$ . For  $GL_3(\mathbb{R})$ , there are three standard parabolic subgroups corresponding to three partitions of 3, namely*

$$3 = 1 + 1 + 1, \quad 3 = 1 + 2, \quad 3 = 2 + 1$$

For a partition  $(r_1, \dots, r_{s+1})$ , we denote  $P_{(r_1, \dots, r_{s+1})}$  the corresponding parabolic subgroups. Thus, we have

$$P_{1,1,1} = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\}, \quad P_{1,2} = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\}$$

$$P_{2,1} = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$$

Clearly **MaxParSt** =  $\{P_{2,1}, P_{1,2}\}$ .

### 3.3.2 Parabolic subgroups II: Using BN-pairs

There is another way to look at the parabolic subgroups

**Definition 3.3.2.1** (BN-pairs). *A **BN-pairs** is a 4-tuple  $(G, B, N, S)$  where  $G$  is a group generated by subgroups  $B$  and  $N$ . The subgroup  $H = B \cap N$ ,  $S$  is a finite set of involutions which generate the Weyl group  $W = N/H$ . Moreover, the following axioms holds*

- If  $r \in R$  and  $w \in W$ , then  $rBw \subset BwB \cup BrwB$ .

- If  $r \in R$ ,  $rBr \neq B$ .

Let  $J \subset R$ , we define  $W_J$  to be the subgroup of  $W$  generated by the involutions  $r \in J$ . We call it **standard parabolic subgroup** of  $W$ . Set  $P_J = BW_JB$  as in the notation of BN-pairs. We have the following theorem

**Theorem 3.3.2.2.**

- $P_J$  is a subgroup of  $G$ . In particular, we have

$$G = BWB,$$

which is called **Bruhat decomposition** of  $G$ .

- If  $P_I = P_J$  then we have  $I = J$ .
- All subgroups of  $G$  containing  $B$  arises in this way.

For our purpose, it is enough to concentrate on the following example

**Example 3.3.2.3.** Let  $G = GL_n(\mathbb{R})$ , then the sets  $B, N, R$  are given explicitly as follows

- $B =$  upper triangular matrices
- $N =$  monomial matrices, namely, matrices that have exactly one non-zero entry in each row and column.
- From the above, it is clear that  $H = B \cap N$  is the diagonal group, and this group is normal in  $N$ .
- It can be shown that  $W = N/H \cong S_n$ , and we can choose  $R = \{(i, i-1)\}$  - the set of transpositions.

The above theorem leads to the following definition of parabolic subgroups

**Definition 3.3.2.4** (Parabolic subgroups). Using the same notation in the previous theorem, we call the subgroups  $P_J$  with  $J \subset R$  standard parabolic subgroups of group  $G$ .

We would like to explicitly describe the parabolic subgroup for  $SL_n(\mathbb{R})$ .

**Example 3.3.2.5.** As introduced in previous section, the set  $\Phi = \{\alpha_{ij}\}$  forms a root system for  $SL_n(\mathbb{R})$ . The Borel subgroup is just  $B \cap SL_n(\mathbb{R})$ , the group of upper triangular matrices with determinant 1. We also consider the set  $N$  of monomial matrices with determinant 1.

Then it is easy to check that  $N/H$  is the set of all permutation matrices. That is,  $N/H$  is generated by the matrices of the form

$$r_i := \begin{bmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{n-i-1} \end{bmatrix}$$

Via the identification  $s_i \mapsto (i, i+1)$ , we identify the Weyl group  $W = N/H \cong S_n$ . So  $R = \{(i, i+1) | i = 1, 2, \dots, n\}$ . We consider the set  $I = R \setminus \{r_i\}$  for some  $i < n$ . The associated parabolic subgroup of  $W$  is

$$W_I = \langle r_1, r_2, \dots, r_{i-1}, r_{i+1}, \dots, r_{n-1} \rangle \cong S_i \times S_{n-i}$$

The corresponding parabolic subgroup is

$$P_i := P_I = \left\{ \begin{bmatrix} A & * \\ & B \end{bmatrix} \in SL_n(\mathbb{R}) : A \in GL_i, B \in GL_{n-i} \right\}$$

### 3.3.3 Langlands decomposition

We fix a partition of  $n$  as

$$n = n_1 + n_2 + \dots + n_k$$

and consider the parabolic subgroup of this type, i.e. the subgroup

$$P_{n_1, \dots, n_k} = \left\{ \begin{bmatrix} M_1 & * & \dots & * \\ 0 & M_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_k \end{bmatrix} \right\}$$

where  $M_i \in GL_{n_i}(\mathbb{R})$  is invertible of size  $n_i \times n_i$ .

This group can be factored as

$$P_{n_1, \dots, n_k} = M_{n_1, \dots, n_k} \ltimes N_{n_1, \dots, n_k}$$

where

$$N_{n_1, \dots, n_k} = \left\{ \begin{bmatrix} I_1 & * & \dots & * \\ 0 & I_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_k \end{bmatrix} \right\} \quad (I_k \text{ is the } n_k \times n_k \text{ identity matrix})$$

and

$$M_{n_1, \dots, n_k} = \left\{ \begin{bmatrix} M_1 & 0 & \dots & 0 \\ 0 & M_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_k \end{bmatrix} \right\}$$

The subgroup  $M_{n_1, \dots, n_k}$  is called **Levi component**. We can further factor this subgroup as

$$M_{n_1, \dots, n_k} = M_{n_1, \dots, n_k}^1 \cdot A_{n_1, \dots, n_k}$$

with  $A_{n_1, \dots, n_k}$  plays the role of the connected center of  $M_{n_1, \dots, n_k}$ :

$$A_{n_1, \dots, n_k} = \left\{ \begin{bmatrix} t_1 I_1 & 0 & \dots & 0 \\ 0 & t_2 I_k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_k I_k \end{bmatrix} : t_i > 0 \right\}$$

and

$$M_{n_1, \dots, n_k}^1 = \left\{ \begin{bmatrix} M'_1 & 0 & \dots & 0 \\ 0 & M'_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M'_k \end{bmatrix} \right\},$$

where  $\det(M'_i) = \pm 1$ .

**Definition 3.3.3.1.** For a given parabolic subgroup  $P$ , the factorization

$$P = M_P^1 \times A_P \times N_P$$

as above is called **Langlands decomposition**.

### 3.3.4 Iwasawa decomposition and $P$ -horospherical decomposition

We introduce two important ways to decompose a matrix in  $\mathrm{GL}_n(\mathbb{R})$  into simpler parts. This will be frequently used in the remaining of this thesis.

**Proposition 3.3.4.1.** *Let*

$$K = \mathrm{O}_n(\mathbb{R}), \quad A = \left\{ \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix} : a_i > 0 \right\}, \quad U = \left\{ \begin{bmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \right\}$$

*Then the natural product map*

$$p: K \times A \times U \rightarrow GL_n(\mathbb{R}), \quad p(k, a, n) = kan$$

*is a bijection.*

*Proof.* The Iwasawa decomposition is essentially the Gram-Schmidt orthogonalization process. For more detail, see [14, Subsection 10.1]  $\square$

Thus, the following decomposition makes sense

**Definition 3.3.4.2.** *The decomposition of  $g = kan$  for  $k \in K, a \in A$  and  $n \in N$  is called **Iwasawa decomposition**.*

Using the Langlands decomposition as well as the Iwasawa decomposition  $G = KAN = KB$ , we obtain the decomposition

$$G = KP = KM_P A_P N_P \cong KM_P \times A_P \times N_P$$

As a consequence, we get

$$X_G = K \backslash G = X_{M_P} \times A_P \times N_P$$

for  $X_{M_P} := (K \cap M_P) \backslash M_P$ .

This is called the  **$P$ -horospherical** decomposition of the lattice space  $X_G$ . In this way, we can identify  $x \in X_G$  with the triple  $(m_P(x), a_P(x), n_P(x))$ .

**Example 3.3.4.3.** *When  $n = 2$ , the only nontrivial standard parabolic subgroup is the Borel subgroup  $P = B$ , so the  $P$ -horospherical decomposition is*

$$X_G = K \backslash G = A_B \times N_B$$

*which is in fact just the Iwasawa decomposition.*

Furthermore, if  $Q \supset P$  is also a parabolic subgroup, it gives rise to a parabolic subgroup  $*P := P \cap M_Q$  of  $M_Q$ . In particular, we have the following result

**Theorem 3.3.4.4.** [12, Lemma 2] Let  $P, Q$  be two standard parabolic subgroup such that  $Q \supset P$ . Then the map

$$P \mapsto \star P := P \cap M_Q$$

is a bijection.

The space  $X_{M_Q}$  itself has the decomposition

$$X_{M_Q} = X_{*P} \times A(Q)_{*P} \times N(Q)_{*P}$$

where

$$X_{*P} = X_{M_P}, \quad A(Q)_{*P} A_Q = A_P, \quad N(Q)_{*P} N_Q = N_P.$$

This is called **relative  $P$ -horospherical decomposition** with respect to  $Q$ .

**Example 3.3.4.5.** Consider the group  $G = \mathrm{SL}_3(\mathbb{R})$ . We consider the rational standard parabolic subgroup

$$P_1 = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\} \subset \mathrm{SL}_3(\mathbb{R}) \Rightarrow M_{P_1} = \left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\}$$

and  $B$  the Borel subgroup. Then

$$*B = B \cap M_{P_1} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & y \\ 0 & 0 & z \end{bmatrix} : xz = 1 \right\}$$

Then it is clear that  $X_{M_{P_1}}$  can be identified with the space of lattices of rank 2. The  $B$ -horospherical decomposition with respect to  $P_1$  is just Iwasawa decomposition in disguise.

### 3.3.5 Parabolic sets and parabolic subalgebras

**Definition 3.3.5.1.** Given a root system  $\Phi$ , a **parabolic subset**  $\Phi(P)$  is a subset of  $\Phi$  that satisfies the following conditions:

1.  $\Phi(P) \cup -\Phi(P) = \Phi$ .
2. It is closed, in the sense that, for any two root  $\alpha, \beta \in \Phi(P)$  such that  $\alpha + \beta$  is a root, then  $\alpha + \beta \in \Phi(P)$ .

The parabolic set parametrizes the corresponding parabolic subalgebra with a choice of base  $\Delta$  of the root system  $\Phi$  as given in the following theorem

**Theorem 3.3.5.2.** Consider a semisimple Lie algebra  $\mathfrak{g}$  with the root system  $\Phi$  and a base  $\Delta$ . We denote  $\Phi^+$  the set of positive roots with respect to this base. There exists a correspondence between parabolic subset  $\Phi(P)$  of  $\Phi$  and subalgebras of  $\mathfrak{g}$  containing the Borel subalgebra  $\mathfrak{b}$ . The correspondence is given by

$$\Phi(P) \longleftrightarrow \mathfrak{p} := \mathfrak{b} \oplus \bigoplus_{\alpha \in \Phi(P) \setminus \Phi^+} \mathfrak{g}_\alpha$$

*Proof.* We refer to [5, Chapter VI, §1.7, Proposition 20] for a proof of this fact.  $\square$

**Example 3.3.5.3.** We consider the case  $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$ . Denote  $\Delta = \{\alpha, \beta\}$  base for the root system of  $\mathfrak{g}$ , so that

$$\Phi^+ = \{\alpha, \beta, \alpha + \beta\}$$

There are 4 parabolic sets, corresponding to 4 parabolic subalgebras given as follows

$$\begin{aligned}\Phi(P) &= \Phi^+ \longleftrightarrow \mathfrak{p} = \mathfrak{b} \\ \Phi(P) &= \Phi^+ \cup \{-\beta\} \longleftrightarrow \mathfrak{p} = \mathfrak{b} \oplus \mathfrak{g}_{-\beta} \\ \Phi(P) &= \Phi^+ \cup \{-\alpha\} \longleftrightarrow \mathfrak{p} = \mathfrak{b} \oplus \mathfrak{g}_{-\alpha} \\ \Phi(P) &= \Phi^+ \cup -\Phi^+ \longleftrightarrow \mathfrak{p} = \mathfrak{g}\end{aligned}$$

### 3.3.6 On the function $H_P$

Recall that for the Lie group  $\mathrm{SL}_n(\mathbb{R})$ , we attach to it a root system  $\Phi = \{\alpha_{i,j}\}$  with

$$\Delta = \{\alpha_i \mid i \in I\}, \quad I = \{1, 2, \dots, n-1\}$$

as the set of fundamental roots. It is a fact that the Weyl group  $W$  satisfies

$$W = \langle r_{\alpha_i} : i \in I \rangle$$

For the sake of brevity, we denote  $r_{\alpha_i} := r_i$ . For a standard parabolic subgroup  $P = P_J$  where  $J \subset I$ , we define the following subset of a fixed Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{sl}_n(\mathbb{R})$

$$\begin{aligned}\mathfrak{a}(P) &:= \mathrm{span}\{\alpha_i^\vee : i \in J\} \\ \mathfrak{a}_P &:= \{H \in \mathfrak{h} : \langle \alpha_i, H \rangle = 0 \text{ for } i \in J\}\end{aligned}$$

By the definition of weights, i.e.  $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$ , we can see that the subspace  $\mathfrak{a}_P$  of  $\mathfrak{h}$  contains the subspace

$$\text{span} \{ \lambda_i^\vee : i \notin J \}$$

It is also clear that the dimension of  $\mathfrak{a}(P)$  is  $|J|$  as the set  $\{\alpha_i^\vee : i \in J\}$  is a linearly independent set. Assume that

$$\sum_{i \notin J} c_i \lambda_i^\vee = 0$$

Pairing both sides with  $\alpha_i$ , we deduce that  $c_i = 0$  for all  $i \notin J$ . In particular, the set  $\{\lambda_i^\vee : i \notin J\}$  contains linearly independent vectors and thus spans a subspace of  $\mathfrak{a}_P$  of dimension  $|I| - |J|$ . Therefore

$$\mathfrak{a}_P = \text{span} \{ \lambda_i^\vee : i \notin J \}$$

To put it another way, we have a direct sum decomposition

$$\mathfrak{h} = \mathfrak{a}(P) \oplus \mathfrak{a}_P$$

We also write  $\lambda_{j,J}^\vee$  for the basis of  $\mathfrak{a}(P)$  containing the fundamental weight, namely  $\langle \alpha_k, \lambda_{j,J}^\vee \rangle = \delta_{kj}$  for  $j, k \in J$ . We have the following lemma

**Lemma 3.3.6.1.** [18, Subsection 2.3.5] *Let  $H \in \mathfrak{g}$  be arbitrary. For  $J \subsetneq I$  we have the decomposition*

$$H = H(J) + H_J,$$

where  $H(J) \in \mathfrak{a}(P)$  and  $H_J \in \mathfrak{a}_P$ . If  $H = \sum_{i=1}^n p_i \lambda_i^\vee$  then

$$H(J) = \sum_{i \in J} p_i \lambda_{i,J}^\vee$$

*Proof.* Let  $H(J) = \sum_{i \in J} c_i \lambda_{i,J}^\vee$ , then for  $i \in J$

$$p_j = \langle \alpha_i, H \rangle = \langle \alpha_i, H_J + H(J) \rangle = \sum_{i \in J} \langle \alpha_i, c_i \lambda_{i,J}^\vee \rangle = c_i$$

Hence  $H(J) = \sum_{i \in J} p_i \lambda_{i,J}^\vee$ . □

We now define the function  $H_P$  as follows:

**Definition 3.3.6.2.** *Assume that for  $x \in X$ , we have a  $P$ -horospherical decomposition*

$$x = (m_P(x), a_P(x), n_P(x))$$

Then we define

$$H_P: X \rightarrow \mathfrak{a}_P$$

$$x \mapsto H_P(x) = \log(a_P(x)),$$

where  $\log$  is defined as in subsection 3.2.4.

This definition is well-defined since  $\log(a_P(x))$  is an element of  $\mathfrak{a}_P$  by definition. We prove the following lemma

**Lemma 3.3.6.3.** *Let  $P \subset Q$  be standard parabolic subgroups of  $G$ , and  $x \in K \backslash G$ . Then we have*

$$H_P(x) = H_Q(x) + H_{*P}(pr_{M_Q}(x))$$

where  $pr_{M_Q}(x)$  stands for the natural projection of  $x \in X$  on  $M_Q/K \cap M_Q$ .

*Proof.* This follows immediately from the observation that

$$A_P = A_Q A(Q)_{*P}$$

and the definition 3.3.6.2 of the function  $H_P$  for parabolic subgroup  $P$ . □

### 3.3.7 Some additional notations

We define some important set that will be used in the later part of this thesis. Given a standard parabolic subgroup  $P = P_J$ , let

$$\Delta(P) := \{\alpha_i : i \in J\} \quad \text{and} \quad \Delta_P := \{\alpha_{i|\mathfrak{a}_P} : i \notin J\}$$

Similarly, we define

$$\hat{\Delta}(P) := \{\lambda_{i|\mathfrak{a}(P)} : i \in J\} \quad \text{and} \quad \hat{\Delta}_P := \{\lambda_i : i \notin J\}$$

**Example 3.3.7.1.** *Let  $G = \mathrm{SL}_3(\mathbb{R})$ . We can choose a base  $\Delta = \{\alpha_1, \alpha_2\}$  as before, thus  $I = \{1, 2\}$ . Assume that  $J = \{2\}$ . Then  $P_1 := P_{I \setminus J}$ . We have the following sets*

$$\Delta_{P_1} = \{\alpha_{1|\mathfrak{a}_{P_1}}\}, \quad \hat{\Delta}_{P_1} = \{\lambda_1\}$$

**Remark.** Over  $\mathfrak{a}(P)$  we can also define the notion of weight  $\lambda_{i,J}$  by the relations

$$\langle \omega_j^J, \alpha_i^\vee \rangle = \delta_{ij}$$

for  $j \in J$ . It is known that we have the relation

$$\omega_i^J = \sum_{i \in J} c_i \alpha_j,$$

where  $c_j \geq 0$ . Thus we can naturally define  $\lambda_{i,J}$  on the Cartan subalgebra  $\mathfrak{h}$  by letting  $\lambda_{i,J}$  to be identically zero on  $\mathfrak{a}_P$ . For example, the weights in  $\mathfrak{h}(J)$  are different from that of  $\mathfrak{h}$ . In example 3.3.7.1, we have the identification

$$\lambda_1|_{\mathfrak{a}(P)} = \frac{1}{2}\alpha_1$$

But they extend differently over  $\mathfrak{h}$ .

### 3.3.8 The linear functionals $\rho_P$ and $\rho_Q^P$

We define an analog for Weyl vector  $\rho$ , which is essential in studying the semi-stability in the later of this thesis.

**Definition 3.3.8.1.** Given a standard parabolic subgroup  $P = P_J$  of  $G = SL_n(\mathbb{R})$  or  $G = GL_n(\mathbb{R})$ , we define

$$\rho(P) := \sum_{\omega \in \widehat{\Delta}(P)} \omega \quad \text{and} \quad \rho_P := \rho - \rho(P),$$

where  $\rho$  is the Weyl vector in section 3.2.4.

If we are given another standard parabolic subgroup  $Q \subset P$  then we also set

$$\rho_Q^P := \rho(P) - \rho(Q) = \rho_Q - \rho_P$$

We have a more concrete description for  $\rho_P$  as follows

**Lemma 3.3.8.2.** [18, Subsection 4.1.3] Fix a standard parabolic subgroup  $P$  and define  $d_P(\alpha) := \langle \rho(P), \alpha^\vee \rangle$  for each  $\alpha \in \Delta_P$ . Then  $d_P(\alpha)$  is non-positive and

$$\rho_P = \sum_{\alpha \in \Delta_P} (1 - d_P(\alpha)) \lambda_\alpha$$

*Proof.* We write  $\rho_P = \sum_{i \in I} c_i \lambda_i$  and evaluate at  $\alpha_i$ . Then

$$c_i = \langle \rho_P, \alpha_i^\vee \rangle = \langle \rho, \alpha_i^\vee \rangle - \langle \rho(P), \alpha_i^\vee \rangle = \begin{cases} 0, & \text{if } i \in J \\ 1 - d_P(\alpha), & \text{otherwise} \end{cases}$$

Since  $\rho(P)$  is a non-negative linear combination of roots in  $\Delta(P)$ , says

$$\rho(P) = \sum_{i \in J} m_i \alpha_i$$

we have

$$d_P(\alpha) = \sum_{i \in J} m_i \langle \alpha_i, \alpha^\vee \rangle$$

where  $\alpha \in \Delta_P$ . It is known in Lie theory that the angle between two simple roots are obtuse, hence  $\langle \alpha_i, \alpha^\vee \rangle \leq 0$ , which implies  $d_P(\alpha)$  is non-positive.  $\square$

# Chapter 4

## SEMI-STABLE LATTICE IN HIGHER RANK

In this chapter, we will establish the notion of semi-stable lattice. Heuristically, this is a lattice that achieve all its successive minima at the same time, see [3].

We will provide two different definitions of semi -stable lattices; one is geometric - which follows Grayson's notion of a canonical plot in [10], and one is purely algebraic, which makes use of the maximal standard parabolic subgroups. We will show that the two definitions coincide in the next chapter.

### 4.1 Lattices in higher rank

Since lattices is the central object for studying in this thesis, we will define the notion of lattice of rank at least 3 in the last section in chapter I. The definitions and properties of a general lattice will be used extensively in the later part of this thesis. Note that, in many reference, such as [10], the lattice is defined to be a module over a ring of algebraic integers, but we can always restrict to a  $\mathbb{Z}$ -module.

#### 4.1.1 Abstract lattices

The following definition is due to [15, section 4]

**Definition 4.1.1.1** ( Abstract  $\mathbb{Z}$ -lattices). *Let  $L$  be a finitely generated  $\mathbb{Z}$ -module. In particular, it is a free  $\mathbb{Z}$ -module of finite rank. Suppose that  $L$  is endowed with a map  $Q: L \rightarrow \mathbb{R}$ , such that the following conditions hold*

1. *The set  $\{x \in L : Q(x) \leq r\}$  is finite for any real number  $r$  - the parallelogram law.*

2.  $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$  for all  $x, y \in L$ .
3.  $Q(x) \neq 0$  for all  $x \in L \setminus \{0\}$ .

We will call the pair  $(L, Q)$  a **abstract  $\mathbb{Z}$ -lattice**.

**Example 4.1.1.2.** Take  $L = \mathbb{Z}^n$  and choose our quadratic form to be the standard one. In particular, for  $x = (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$ ,

$$Q(x) := x_1^2 + x_2^2 + \dots + x_n^2$$

Here the multiplication is just the usual dot product between 2 vectors. In term of matrix, this quadratic form is assigned to the identity matrix  $I_n$ .

If there is no further confusion, we can just denote a Euclidean lattice by  $L$ , without specifying the bilinear form  $Q$ . The lattice  $L$  determines a full-rank lattice inside  $L_{\mathbb{R}}$ , namely, the rank of the lattice  $L$  is equal to the dimension of  $L_{\mathbb{R}}$ .

### 4.1.2 An alternative definition of lattices

For the sake of computation, we also usually adopt another definition of the lattice. In particular, we view lattice as a free  $\mathbb{Z}$ -module of rank  $n$  that is isomorphic to  $\mathbb{R}^n$  via base changing.

**Definition 4.1.2.1.** A lattice in  $\mathbb{R}^n$  is a subset  $L \subset \mathbb{R}^n$  such that there exists a basis  $b_1, \dots, b_n$  of  $\mathbb{R}^n$  such that

$$L = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \dots \oplus \mathbb{Z}b_n$$

If we put the vector  $b_1, b_2, \dots, b_n$  in columns, with respect to the standard basis, namely

$$g = [b_1 | b_2 | \dots | b_n],$$

then  $L = g\mathbb{Z}^n$ .

It can be shown that  $L$  is a discrete subset of  $\mathbb{R}^n$ . The discrete property turns out to be the sufficient condition for a subset of  $\mathbb{R}^n$  to be a lattice, as stated in the following proposition

**Theorem 4.1.2.2.** [17, Proposition 4.2] Given a vector space  $V$  and a subset  $L \subset V$ . Then  $L$  is a lattice if and only if  $L$  is discrete.

### 4.1.3 Equivalence between two definitions of lattices

In this subsection, we will show that every abstract  $\mathbb{Z}$ -lattice is isomorphic to an Euclidean  $\mathbb{Z}$ -lattice. This will be helpful in visualizing the abstract lattices, as we are just looking at concrete lattices with deformation by a linear transformation. First we need to specify the notion of isomorphic lattices - in the first definition

**Definition 4.1.3.1.** A map  $f: (L, Q) \rightarrow (L', Q')$  is an **isomorphism** between lattices if it is a group isomorphism and for all  $x \in L$ , we have

$$Q(x) = Q'(f(x))$$

**Proposition 4.1.3.2.** Any abstract lattice is isomorphic to a Euclidean  $\mathbb{Z}$ -lattice.

*Proof.* Assume that  $L = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \dots \mathbb{Z}b_n$ , and choose the associated quadratic form as in example 4.1.1.2, we can directly verify that the 3 conditions defining an abstract lattice hold. Conversely, we define

$$\langle x, y \rangle := \frac{Q(x+y) - Q(x-y)}{4}$$

We will show that this can be extended to an inner product on  $\mathbb{R}^n = L \otimes \mathbb{R}$ . By the parallelogram law 1, we have  $Q(y-x) = 2Q(x) + 2Q(y) - Q(x+y) = Q(x-y)$ , thus

$$\langle x, y \rangle = \frac{Q(x+y) - Q(x-y)}{4} = \frac{Q(x+y) - Q(y-x)}{4} = \langle y, x \rangle$$

We will prove by induction that  $Q(mx) = m^2Q(x)$  for all positive integer  $m$ . The statement is trivial for  $m = 1$ . By parallelogram law

$$Q(mx) = 2Q[(m-1)x] + 2Q(x) - Q[(m-2)x] = [2(m-1)^2 + 2 - (m-2)^2]Q(x) = m^2Q(x)$$

As a consequence,  $Q(x) \geq 0$  for all  $x \in L$ . Indeed, if  $Q(x) < 0$  for some  $x \in L$ , then the set  $\{v \in L : Q(v) < 0\} \supset \mathbb{Z}x$ , hence infinite. But this contradicts the condition 3 of an abstract lattice. Finally, consider the relations

$$\begin{aligned} Q(a+b+c) + Q(a-b+c) &= 2Q(a+c) + 2Q(b) \\ Q(a+b-c) + Q(a-b-c) &= 2Q(a-c) + 2Q(b) \\ Q(a+b+c) + Q(a-b-c) &= 2Q(b+c) + 2Q(a) \\ Q(a+b-c) + Q(a-b+c) &= 2Q(b-c) + 2Q(a) \end{aligned}$$

Taking the alternative sum yields

$$2Q(a + b + c) - 2Q(a + b - c) = 2Q(a + c) - 2Q(a - c) + 2Q(b + c) - 2Q(b - c)$$

or equivalently

$$\langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle$$

Thus we can extend the function  $\langle \cdot, \cdot \rangle : L \times L \rightarrow \mathbb{R}$  to a  $\mathbb{R}$ -bilinear function over  $L_{\mathbb{R}} \times L_{\mathbb{R}}$  by taking the tensor product. Clearly  $Q(x/m) = \frac{1}{m^2}Q(x)$  for any  $m \in \mathbb{Z} \setminus \{0\}$  and  $x \in L$ . Moreover the set  $\{x/m : x \in L, m \in \mathbb{Z} \setminus \{0\}\}$  is dense in  $L_{\mathbb{R}}$ , hence  $Q$  define a norm on  $L_{\mathbb{R}}$ . In particular,  $\langle \cdot, \cdot \rangle$  extends to an inner product on  $L_{\mathbb{R}}$ . The discrete condition of the abstract lattice ensure that  $L$  is a discrete subset of  $L_{\mathbb{R}}$  hence an Euclidean lattice by theorem 4.1.2.2.  $\square$

#### 4.1.4 Space of lattices

Let  $K = O_n(\mathbb{R})$  be the orthogonal subgroup of  $G = \mathrm{GL}_n(\mathbb{R})$ . We can identify the space  $X_n = K \backslash G$  with the space of lattices up to rotation. Indeed, assume that we have a lattice  $(L, Q)$  where

$$L = g\mathbb{Z}^n, \quad Q = BB^t$$

for some  $g, B \in \mathrm{GL}_n(\mathbb{R})$ . Define the norm  $\|\cdot\|_{gB}$  on  $\mathbb{Z}^n$  by setting

$$\|v\|_{gB} = \|Bgv\|, \quad \forall v \in \mathbb{Z}^n$$

where the norm on the right hand side of the above equation is just the standard norm. Then the lattice  $(L, Q)$  and  $(\mathbb{Z}^n, \|\cdot\|_{gB})$  are isomorphic. Hence each isomorphic class of lattices has a representative of the form  $(\mathbb{Z}^n, \|\cdot\|_g)$  for some  $g \in \mathrm{GL}_n(\mathbb{R})$ . It can be observed that the  $\|\cdot\|_g = \|\cdot\|_{kg}$  for any  $k \in O(n)$ . This proves that  $X_n$  parametric isomorphic classes of lattice bases up to rotation.

#### 4.1.5 Covolume of a lattice

Now that for every abstract lattice  $L$  we can find an invertible matrix  $g$  such that

$$L \cong g\mathbb{Z}^n$$

The number  $n$  is called the **rank** of the lattice  $L$ .

Let  $\{e_1, e_2, \dots, e_n\}$  be an orthonormal basis of  $L_{\mathbb{R}} \cong \mathbb{R}^n$  and

$$g = [b_1 | b_2 | \dots | b_n].$$

The covolume of the lattice  $L$  is defined as

**Definition 4.1.5.1.** *The covolume of  $L$  is given by the formula*

$$\text{vol}(L) = |\det(\langle b_i, e_j \rangle)|$$

The rank and covolume are invariant numerical values of  $L$ , as they don't depend on the choice of basis. Indeed, two bases of a rank  $n$  lattice  $L$  are related by a transformation  $g \in \text{GL}_n(\mathbb{Z})$ . Clearly this preserves the volume and the rank as a  $\mathbb{Z}$ -module.

### 4.1.6 Sublattices

To work with semi-stable lattice  $L$ , we need to consider all the sublattices contained inside  $L$ .

**Definition 4.1.6.1** (sublattice). *Let  $(L, Q)$  be a Euclidean  $\mathbb{Z}$ -lattice. We say that a  $\mathbb{Z}$ -submodule  $M$  of  $L$  a **sublattice** if and only if  $L/M$  is torsion free.*

From this definition, we can prove that  $M$  is a sublattice of  $L$  if it satisfies one of the following equivalent properties:

1.  $M$  is a summand of  $L$ .
2. every basis of  $M$  can be extended to a basis of  $L$ .
3. The group  $M$  is an intersection of  $L$  with a rational subspace of  $L_{\mathbb{R}}$ .

We refer to the [19, Section 1.4] for a proof of these equivalences.

**Example 4.1.6.2.** *If  $L = \mathbb{Z}^2$ , then any sublattice of  $L$  is a primitive vector  $u = (a, b)$ , i.e  $\gcd(a, b) = 1$ . Indeed,  $u = (a, b)$  is a sublattice of  $\mathbb{Z}^2$  if and only if there exists a vector  $v \in \mathbb{Z}^2$  such that  $L = \mathbb{Z}u \oplus \mathbb{Z}v$ . With respect to the usual inner product on  $\mathbb{R}^2$ , we have*

$$1 = \text{vol}(\mathbb{Z}^2) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

*This happens if and only if  $\gcd(a, b) = 1$ .*

**Definition 4.1.6.3.** Given a lattice  $L$  and a subgroup  $M \subset L$ , then we call the  $M$  **primitive** if  $L/M$  is torsion free. Equivalently,  $M \subset L$  is a primitive sublattice if no non-zero multiples of non-members of  $M$  are in  $M$ . In formula

$$(M \otimes \mathbb{Q}) \cap L = M$$

**Example 4.1.6.4.** We consider the subgroup  $M$  generated by a single element  $v \in L$ . Then  $\mathbb{Z}v$  is primitive if and only if there is no  $w \in L \setminus \{v\}$  such that

$$w = c \cdot v,$$

for some integer  $c$ .

### 4.1.7 Exterior product and covolume of lattice

In certain situation, it is more convenient to work with exterior product of lattices. We briefly recap the related definitions and results here. Further information can be found in [11, Chapter 5].

**Definition 4.1.7.1.** Given a vector space  $V$ . Then the exterior power  $\bigwedge^r V$  is defined to be the vector space spanned by element of the form

$$v_1 \wedge v_2 \wedge \dots \wedge v_r, \quad v_i \in V$$

subjected to the following relations

1.  $v \wedge v = 0$  for all  $v \in V$ .
2.  $v \wedge w = -w \wedge v$  for any  $v, w \in V$ .
3.  $(av_1 + bv_2) \wedge w = a(v_1 \wedge w) + b(v_2 \wedge w)$  for any  $v_1, v_2, w \in V$ .

If we identify  $V \cong \mathbb{R}^n$ , then the action of  $g \in \mathrm{GL}_n(\mathbb{R})$  on  $\mathbb{R}^n$  via matrix multiplication  $gv$  can be extended to  $\bigwedge^r V$  by defining

$$g(v_1 \wedge \dots \wedge v_r) := (gv_1 \wedge \dots \wedge gv_r), \quad \forall v_i \in V$$

Given an inner product  $\langle \cdot, \cdot \rangle$  on  $V$ , the vector space  $\bigwedge^r V$  also has an inner product as stated in the following theorem

**Theorem 4.1.7.2.** [11, Section 5.6] There is an inner product on  $\bigwedge^r V$  induced by the inner product  $\langle \cdot, \cdot \rangle$  on  $V$  given by

$$\langle v_1 \wedge \dots \wedge v_r, w_1 \wedge \dots \wedge w_r \rangle = \det \begin{bmatrix} \langle v_1, w_1 \rangle & \langle v_2, w_1 \rangle & \dots & \langle v_r, w_1 \rangle \\ \langle v_1, w_2 \rangle & \langle v_2, w_2 \rangle & \dots & \langle v_r, w_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_1, w_r \rangle & \langle v_2, w_r \rangle & \dots & \langle v_r, w_r \rangle \end{bmatrix}$$

From the above theorem we can define the norm on  $\bigwedge^r V$  to be

$$|v_1 \wedge \dots \wedge v_r| := \sqrt{\langle v_1 \wedge \dots \wedge v_r, v_1 \wedge \dots \wedge v_r \rangle} \quad (4.1)$$

From the equation (4.1), we can derive the following result

**Lemma 4.1.7.3.** Let  $L$  be a lattice of rank  $n$  and  $\{v_1, \dots, v_n\}$  be a basis for  $L$ . Then

$$|v_1 \wedge \dots \wedge v_n| = \text{vol}(L)$$

*Proof.* Let  $\{e_1, \dots, e_n\}$  be an orthonormal basis of the vector space  $L_{\mathbb{R}} = L \otimes \mathbb{R}$ . By definition 4.2.2.2, we have

$$\text{vol}(L) = \langle v_1 \wedge \dots \wedge v_n, e_1 \wedge \dots \wedge e_n \rangle$$

On the other hand, the space  $\bigwedge^n L_{\mathbb{R}}$  has dimension one, so there exists  $c \in \mathbb{R}$  such that

$$v_1 \wedge v_2 \wedge \dots \wedge v_n = c e_1 \wedge e_2 \wedge \dots \wedge e_n$$

Let  $v = v_1 \wedge v_2 \wedge \dots \wedge v_n$  and  $e = e_1 \wedge e_2 \wedge \dots \wedge e_n$ , then we have

$$|c| \text{vol}(L) = |c| \langle v, e \rangle = c^2 \langle e, e \rangle = c^2$$

which implies that  $|c| = \text{vol}(L)$ . On the other hand  $\langle v, v \rangle = c^2 \langle e, e \rangle = c^2$ , thus we are done proving  $|v_1 \wedge \dots \wedge v_n| = \text{vol}(L)$ .  $\square$

## 4.1.8

## 4.2 Grayson's definition of semi-stability

Recall that a lattice is a pair  $(L, \langle \cdot, \cdot \rangle)$  where  $L$  is a  $\mathbb{Z}$ -module and  $\langle \cdot, \cdot \rangle$  is the associated inner product. If there is no confusion, we can will just denote this lattice by  $L$  - the underlying  $\mathbb{Z}$ -module, and  $\langle \cdot, \cdot \rangle$  for the associated inner product.

### 4.2.1 Grayson's definition

In this subsection, we introduce Grayson's definition of *semi-stable* lattices, follows [10]. Grayson associates to every lattice a convex hull of certain points in  $\mathbb{R}^2$  - called its *profile*. To define the profile, observe first that if  $M \subset L$  is a sublattice, then the space  $M_{\mathbb{R}} = M \otimes \mathbb{R}$  is a subspace of  $L_{\mathbb{R}}$ , equipped with the restriction of the positive definite symmetric form  $Q$  of  $L$ , then  $M$  is also a lattice of rank not exceeding rank of  $L$ .

**Definition 4.2.1.1** (slope). *The slope of a non-zero lattice  $L$  is the number*

$$\mu(L) = \frac{\log \text{vol}(L)}{\dim L}$$

**Definition 4.2.1.2.** *Consider a lattice  $L$ . For any sublattice  $M \subset L$ , define*

$$\ell(M) = (\dim M, \log \text{vol}(M)) \in \mathbb{R}^2.$$

*The collection of all points  $\ell(M)$  where  $M$  ranges over all sublattices of  $L$  is called **the canonical plot** of the lattice  $L$ . By convention, we assign the lattice of zero rank to the origin of the plane.*

The following lemma asserts that, on each vertical line  $x = n$  where ( $n = 0, 1, \dots, \text{rk}(L)$ ), there is a lowest point and so the canonical plot is bounded below.

**Lemma 4.2.1.3.** *Given a lattice  $L$  and a number  $c$ , there exists only a finite number of sublattices  $M \subset L$  such that  $\text{vol}(M) < c$ .*

*Proof.* We will prove this by induction on the rank of the sublattices.

- For  $r = 1$ , the collection of all rank 1 sublattices of  $L$  is just the set of all vectors in  $L$ . So we reduce to show that for any  $c > 0$ , the set  $B(0, c) \cap L$  has finitely many elements. But the lattice  $L$  is a discrete subset of  $L_{\mathbb{R}}$ , thus the intersection of  $L$  with a bounded set has only finitely many elements.
- Let  $r > 1$ . Consider

$$M = \mathbb{Z}m_1 \oplus \dots \oplus \mathbb{Z}m_r$$

The wedge product  $\bigwedge^r L$ , then clearly  $m_1 \wedge m_2 \dots \wedge m_r$  is a vector in the lattice  $\bigwedge^r L$ . By the previous case, there are finitely many vectors with bounded length inside lattice. So we only need to show that the map

$$M \mapsto \bigwedge^r M$$

is finite to one, then we are done. Indeed, let  $v = \bigwedge^r M$  be a vector in  $\bigwedge^r L$ . We want to show that there are only finitely many sublattices  $N \subset L$  of rank  $r$  such that  $\bigwedge^r N = v$ . Indeed, let  $\{n_1, n_2, \dots, n_r\}$  be a basis for  $N$  then

$$v = m_1 \wedge m_2 \dots \wedge m_r = n_1 \wedge n_2 \dots \wedge n_r \Rightarrow M \otimes \mathbb{Q} = N \otimes \mathbb{Q}$$

If this is not true, we would find a factor  $u \in \text{span}_{\mathbb{Q}}(n_1, \dots, n_r) - \text{span}_{\mathbb{Q}}(m_1, \dots, m_r)$ . But this implies

$$0 = u \wedge n_1 \wedge \dots \wedge n_r = u \wedge m_1 \wedge \dots \wedge m_r \neq 0$$

which is a contradiction. Furthermore, we note that  $N$  and  $M'$  are lattice of  $L$  with the same rank, so  $N$  must have finite index in  $M'$ . This index is the same as index of  $M$  in  $M'$ . In particular, the preimage of the map  $M \mapsto \bigwedge^r M$  must be a abelian subgroup of  $M'$  with fixed index. There are finitely many such group, by [1, Proposition A.1]. Hence we are done.  $\square$

**Definition 4.2.1.4.** *The boundary polygon of the convex hull of the canonical plot is called **profile** of the lattice  $L$ .*

In theory, we can compute the profile by searching for the shortest vector in each of its exterior product, but this computation is infeasible when the dimension of the lattice grows. Since there are lattices with arbitrarily large volume of any rank smaller than that of  $L$ , we add to the side the point  $(0, \infty)$  and  $(n, \infty)$ . The sides of the profile are therefore two vertical lines. The bottom is a convex polygonal consisting of consecutive lines connecting the origin with the point  $\ell(L)$ .

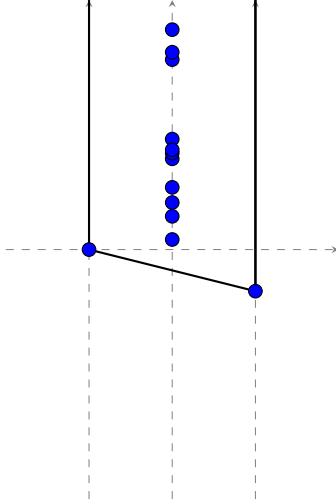
**Definition 4.2.1.5.** *If the bottom of the profile contains only two points  $(0, 0)$  and  $(n, \log \text{vol } L)$ , then the lattice  $L$  is said to be **semi-stable**. Otherwise  $L$  is said to be **unstable**.*

Below are the picture of two lattices. Visually, a lattice is called **semi-stable** if it satisfies the other equivalent conditions: If  $M$  is an arbitrary sublattice of  $L$  then  $\mu(M) \geq \mu(L)$ .

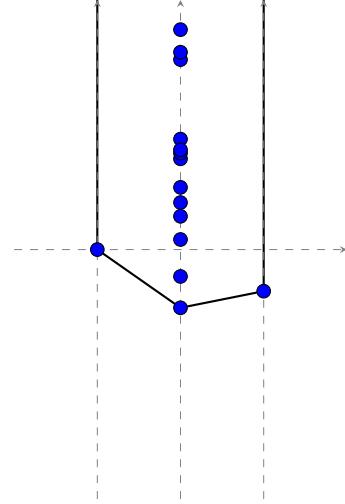
## 4.2.2 Canonical filtration

Given a lattice  $L$  and a sublattice  $M \subset L$ , the quotient group  $L/M$  has the structure of a lattice. Indeed, consider the exact sequence of lattices

$$0 \rightarrow M \rightarrow L \rightarrow L/M \rightarrow 0$$



**Figure 4.1:** A semi-stable lattice



**Figure 4.2:** An unstable lattice

By tensoring with  $\mathbb{R}$  we get a short exact sequence of  $\mathbb{R}$ -vector subspaces

$$0 \rightarrow M_{\mathbb{R}} \rightarrow L_{\mathbb{R}} \rightarrow (L/M)_{\mathbb{R}} \rightarrow 0,$$

which is split. Thus we have the isomorphisms

$$(L/M)_{\mathbb{R}} \cong L_{\mathbb{R}}/M_{\mathbb{R}} \cong M_{\mathbb{R}}^{\perp}$$

Therefore, by restricting of the inner product over  $L_{\mathbb{R}}$  to  $M_{\mathbb{R}}^{\perp}$ , we clearly see that  $L/M$  also inherits an inner product. Explicitly, the norm over  $L/M$  can be defined by restricting the norm  $\|\cdot\|$  on  $L_{\mathbb{R}}/M_{\mathbb{R}}$ , which is given by

$$\|x\| := \inf\{\|x + m\| : m \in M\}$$

To verify that  $L/M$  along with the given norm on the quotient group forms a lattice, we refer to [15].

**Definition 4.2.2.1.** *Given a lattice  $L$  containing a sublattice  $M$ , then  $L/M$  is a lattice. We call this lattice **quotient lattice**.*

**Lemma 4.2.2.2.** [6, Proposition 4.1] *If  $L$  is a lattice and  $M \subset L$  is a sublattice, we have*

$$\text{vol}(L) = \text{vol}(M) \cdot \text{vol}(L/M)$$

and if  $N$  is any sublattice of  $L$  that satisfies  $N + M = L$  then

$$\text{vol}(N) \geq \text{vol}(L/M)$$

*Proof.* Assume that  $\{m_i\}$  is a basis for the lattice  $M$  and  $\{e_i\}$  be an orthonormal basis for the vector space  $M_{\mathbb{R}}$ . Since  $M$  is a sublattice of  $L$ , we can extend the basis  $\{m_i\}$  to get a basis  $\{m_i\} \cup \{n_j\}$  for the lattice  $L$ . Similarly, we can extend  $\{e_i\}$  to get an orthonormal basis  $\{e_i\} \cup \{f_j\}$  for the vector space  $L_{\mathbb{R}}$ . In particular, we would have  $\langle m_i, f_j \rangle = 0$  for all  $i, j$ . By definition 4.1.5.1, we have

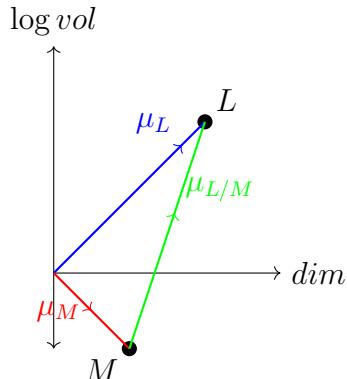
$$\begin{aligned} \text{vol}(L) &= \det \begin{bmatrix} \langle m_i, e_i \rangle & \langle n_j, e_i \rangle \\ \langle m_i, f_j \rangle & \langle n_j, f_j \rangle \end{bmatrix} \\ &= \det \begin{bmatrix} \langle m_i, e_i \rangle & \langle n_j, e_i \rangle \\ 0 & \langle n_j, f_j \rangle \end{bmatrix} \\ &= \text{vol}(M) \cdot \text{vol}(L/M) \end{aligned}$$

Hence we are done. The last inequality follows from the fact that the volume decrease under the orthogonal projection and the observation that  $N_{\mathbb{R}} \supset (L/M)_{\mathbb{R}}$ .  $\square$

In the canonical plot, the import of lemma 4.2.2.2 is that the slope of the quotient lattice  $L/M$  appears as the slope of the line segment connecting the points corresponding to the sublattice  $M$  and the lattice  $L$ . This is due to lemma 4.2.2.2

$$(\text{rank}(M), \log(\text{vol}(M))) + (\text{rank}(L/M), \log \text{vol}(L/M)) = (\text{rank}(L), \log \text{vol}(L))$$

Given two sublattices  $M_1, M_2 \subset L$ , if we apply lemma 4.2.2.2 to the sublattices  $M_1/M_1 \cap M_2$



**Figure 4.3:** Geometric meaning of slope in canonical plot

and  $M_2/M_1 \cap M_2$  in  $M_1 + M_2/M_1 \cap M_2$ , we get

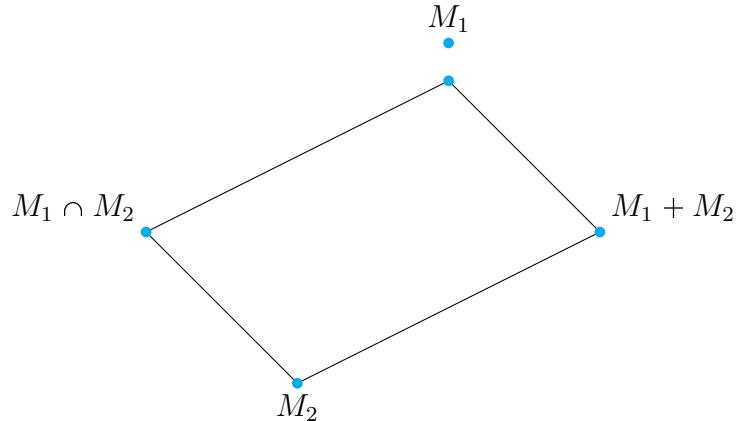
$$\text{vol}(M_1/M_1 \cap M_2) \text{vol}(M_2/M_1 \cap M_2) \geq \text{vol}(M_1 + M_2/M_1 \cap M_2)$$

or equivalently

**Proposition 4.2.2.3.** [6, Proposition 4.3] Suppose  $M_1, M_2$  are sublattices of  $L$  then

$$\log \text{vol}(M_1) + \log \text{vol}(M_2) \geq \log \text{vol}(M_1 + M_2) + \log \text{vol}(M_1 \cap M_2)$$

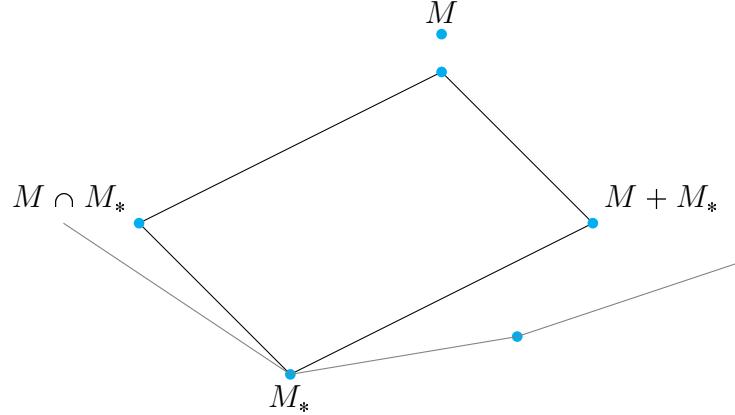
Grayson called this the parallelogram rule, as the geometric meaning of proposition 4.2.2.3 is illustrated in the figure 4.4. The **vertices** of the profile are the extremal/lowest points



**Figure 4.4:** Grason's parallelogram rule

of the plot. Clearly the two points  $(0, 0)$  and  $(n, \log \text{vol}(L))$  are vertices of the plot of the lattices  $L$  with  $\text{rank}(L) = n$ . The following lemma states the situation for other vertices of the profile

**Lemma 4.2.2.4.** [6, Lemma 4.4] Suppose that  $M_*$  is a lattice with the point  $\ell(M_*)$  as a vertex of the profile. If  $M$  is any other sublattice such that  $\ell(M)$  is also a vertex of the profile, then we must have either  $M \subset M_*$  or  $M_* \subset M$ .



**Figure 4.5:** Illustration for lemma 4.2.2.4

*Proof.* We refer to figure 4.5. Clearly we can see that the point  $\ell(M \cap M_*)$  lies somewhere on the left of both  $\ell(M)$  and  $\ell(M_*)$ . Similarly, the point  $\ell(M + M_*)$  lies somewhere inclusively to the right of  $\ell(M)$  and  $\ell(M_*)$ . By the parallelogram rule, the point  $\ell(M)$  will therefore be above the boundary points  $\ell(M \cap M_*)$ ,  $\ell(M_*)$  and  $\ell(M + M_*)$ . If  $M$  is also a vertex of the profile, the parallelogram must degenerate. In particular we either have  $M \cap M_* = M$ , in which  $M \subset M_*$ , or  $M + M_* = M$ , in which  $M_* \subset M$ .  $\square$

An immediate consequence is that

**Theorem 4.2.2.5.** [10, Theorem 1.18] *The vertices of the profile of a lattice  $L$  are represented by unique sublattices, and they form a chain.*

For a given lattice  $L$ , we call the chain of sublattices in theorem 4.2.2.5 the **canonical filtration** of  $L$ . Assume that the canonical filtration for  $L$  is

$$\mathcal{F} : 0 = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_{k-1} \subset L_k = L$$

then it can be seen from the diagram that

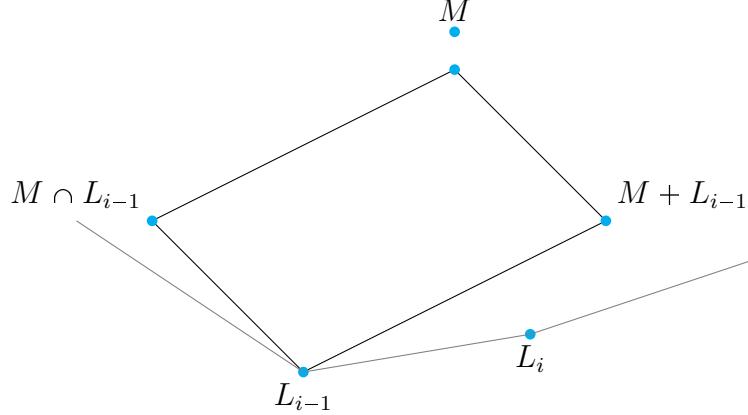
1.  $L_i/L_{i-1}$  is semi-stable for all  $1 \leq i \leq k$ , as the profile of  $L_i/L_{i-1}$  is just a straight line segment connecting  $\ell(L_{i-1})$  to  $\ell(L_i)$ .
2.  $\mu(L_i/L_{i-1}) \leq \mu(L_{i+1}/L_i)$  for all  $1 \leq i \leq k-1$ .

These two conditions is also sufficient for a chain to be a canonical filtration

**Theorem 4.2.2.6.** [10, Corollary 1.30] *Suppose*

$$\mathcal{F} : 0 = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_{k-1} \subset L_k = L$$

is a chain of lattices such that  $L_i/L_{i-1}$  is semi-stable and the slope  $L_i/L_{i-1}$  is not larger than the slope of  $L_{i+1}/L_i$ . Then this chain is the canonical filtration.



**Figure 4.6:** Canonical filtration

*Proof.* Suppose  $M$  to be any other sublattice of  $L$ . We want to know that  $\ell(M)$  lies above the plot  $P$  of the  $\ell(L_i)$ . We prove this by induction on the index  $i$  such that if  $M \subset L_i$

- If  $i = 1$  then  $M \subset L_1$ . Since  $L_1$  is semi-stable, we must have  $\ell(M)$  lies above the line connecting  $(0, 0)$  and  $\ell(L_1)$ . Hence  $\ell(M)$  lies above the plot  $P$ .
- Suppose that  $M \subset L_i$  for  $i > 1$ . Then  $M + L_{i-1} \subset L_i$  contains  $L_{i-1}$ . Therefore, it corresponds to the point lies above the line connecting  $\ell(L_{i-1})$  and  $\ell(L_i)$ , thus lies above  $P$ . By induction, the fact that  $(M \cap L_{i-1}) \subset L_{i-1}$  implies  $\ell(M \cap L_{i-1})$  lies above the plot  $P$ . Using the parallelogram rule, the point  $\ell(M)$  must then also lie above the plot  $P$ .

Thus any chain satisfies the conditions given in the theorem is a canonical filtration.  $\square$

## 4.3 $\rho$ semi-stability

### 4.3.1 $\rho$ -definition

We are now ready to define the  $\rho$ -definition of semi-stable lattice. Recall that we define the space of lattices of rank  $n$  by  $X_n := K \backslash \mathrm{GL}_n(\mathbb{R})$ , where  $K$  is the orthogonal subgroup

**Definition 4.3.1.1** ( $\rho$ -definition). *Let  $x \in X_n$  be an arbitrary lattice, then the lattice  $x$  is called  **$\rho$ -semi-stable** if and only if its degree of instability  $\deg_{inst}(x) \geq 0$ , where*

$$\deg_{inst}(x) := \min_{Q \in ParSt, \gamma \in GL(\mathbb{Q})/Q_i(\mathbb{Q})} \langle \rho_Q, H_Q(x\gamma) \rangle$$

and  $\text{ParSt}$  stands for the set of standard parabolic subgroup of  $GL_n(\mathbb{R})$  as defined in definition 3.3.1.1.

### 4.3.2 Some basic properties

We first have an elementary lemma and refer to the references for the proof.

**Lemma 4.3.2.1.** [7, Lemma 2.2.1; 18, Proposition 4.2.3]

Given  $x \in X_n$ , then the following are equivalent

1.  $\deg_{inst}(x) \geq 0$ .
2. For every standard parabolic subgroup  $P \subset G$  and  $\omega \in \hat{\Delta}_P$  we have

$$\langle \omega, H_P(x\delta) \rangle \geq 0$$

for each  $\delta \in G(\mathbb{Q})/P(\mathbb{Q})$ .

3. For every maximal parabolic subgroup  $Q \subset G$  and  $\omega \in \hat{\Delta}_Q$  we have

$$\langle \omega, H_Q(x\delta) \rangle \geq 0$$

for each  $\delta \in G(\mathbb{Q})/Q(\mathbb{Q})$ .

### 4.3.3 Check for semi-stability

From lemma 4.3.2.1, to check whether  $x \in G$  is semi-stable, we just need to verify whether

$$\langle \omega, H_Q(x\delta) \rangle \geq 0.$$

for  $\omega \in \hat{\Delta}_Q$  and  $Q \subset G$  a standard maximal parabolic subgroup. From lemma 3.3.6.1

$$H_B = H_Q + H(Q)_{*B}$$

where  $H_Q$  is a scalar multiple of  $\lambda_i^\vee$  for  $Q = Q_i$  and  $H(Q)_{*B}$  is a linear combination of  $\alpha_j^\vee$  for  $j \neq i$ . On the other hand, since  $Q$  is a maximal standard parabolic subgroup of  $G$ ,  $\omega$  is proportional to  $\lambda_i$ . Thus

$$\langle \omega H(Q)_{*B}(x\gamma) \rangle = 0.$$

In particular, we can replace  $H_Q$  by  $H_B$  in verifying semi-stability. Write

$$x\gamma = kan, \quad k \in K, a \in A, n \in N,$$

as in Iwasawa decomposition, and set  $H_B(x) = H$  where  $H = \exp(a)$ . In particular, if

$$a = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}$$

then

$$\langle \rho_{Q_i}, H_B(x\gamma) \rangle = \frac{n}{2} \log(a_1 a_2 \dots a_i) \quad (4.2)$$

Thus, to check for the semi-stability of a lattice  $x$ , we just need to look at the  $A$ -coordinate of  $x\gamma$  for every  $\gamma \in G(\mathbb{Q})/Q_i(\mathbb{Q})$ , and verify whether the following system holds

$$\left\{ \begin{array}{l} a_1 \geq 1 \\ a_1 a_2 \geq 1 \\ \dots \\ a_1 a_2 \dots a_n \geq 1 \end{array} \right.$$

## 4.4 Canonical pair

Consider a pair  $(P, \delta)$  of a standard parabolic subgroup  $P \subsetneq G$  and  $\delta \in G(\mathbb{Q})/P(\mathbb{Q})$ . Such a pair is called **destabilizing** for  $x$  if

$$\langle \rho_P, H_P(x\delta) \rangle = \deg_{\text{inst}}(x)$$

Such a pair is called **extremal** for  $x$  if for any standard parabolic  $Q \subset P$  such that

$$\langle \rho_Q, H_Q(x\delta) \rangle = \langle \rho_P, H_P(x\delta) \rangle$$

then  $Q = P$ .

**Definition 4.4.0.1.** A pair  $(P, \delta)$  that is both destabilizing and extremal for  $x$  is called a **canonical pair** for  $x$ .

A canonical pair for  $x$ , if exists, will be denoted by  $\mathbf{cp}(x) := (P, \delta)$ . A priori, it is not clear whether  $\mathbf{cp}(x)$  exists or not. We will show that it is in fact equivalent to the notion of canonical filtration introduced in the previous sections, and deduce that  $\mathbf{cp}(x)$  must exist and it is unique.

We obtain the following lemma as a consequence of the definition of canonical pair

**Lemma 4.4.0.2.** [7, Lemma 2.3.2; 18, Lemma 4.2.5] Let  $x \in X_n$  and  $(P, \delta)$  be a pair with  $P \subsetneq G$  be standard parabolic subgroup and  $\delta \in G(\mathbb{Q})/P(\mathbb{Q})$ . Then

1. If  $(P, \delta)$  is destabilizing for  $x$ , then  $\deg_{inst}^P(x\delta) \geq 0$ .
2. If  $(P, \delta)$  is extremal for  $x$ , then  $\langle \alpha, H_P(x\delta) \rangle < 0$  for any  $\alpha \in \Delta_P$ .

*Proof.* For the first part, let  $Q \subset P$  be any standard parabolic, then

$$\rho_Q = \rho_P + \rho_Q^P$$

Thus, for any  $\eta \in P(\mathbb{Q})/Q(\mathbb{Q})$

$$\begin{aligned} \langle \rho_Q^P, H_Q(x\gamma\eta) \rangle &= \langle \rho_Q, H_Q(x\gamma\eta) \rangle - \langle \rho_P, H_P(x\gamma\eta) \rangle \\ &= \langle \rho_Q, H_Q(x\gamma\eta) \rangle - \deg_{inst}(x) \geq 0 \end{aligned}$$

For the second part, we can pick a standard parabolic  $Q \subset G$  containing  $P$  such that  $P$  is maximal in  $Q$ . In particular, we have  $\Delta_P^Q = \{\alpha\}$ . Then

$$\begin{aligned} \langle \rho_P^Q, H_P(x\delta) \rangle &= \langle \rho_P, H_P(x\gamma) \rangle - \langle \rho_Q, H_Q(x\gamma\eta) \rangle \\ &= \deg_{inst}(x) - \langle \rho_Q, H_Q(x\gamma\eta) \rangle < 0 \end{aligned}$$

Since  $\rho_P^Q$  and  $\alpha$  are proportional by a positive number, the result follows immediately.  $\square$

# Chapter 5

## Equivalence between Grayson's semistability and $\rho$ -semistability

### 5.1 The equivalence between definitions of semi-stable lattices

Recall that we have two distinct definitions of semi-stability, as defined in definition 4.2.1.5 and definition 4.3.1.1. The following theorem asserts their equivalence:

**Theorem 5.1.0.1.** *Let  $x \in X_n = K \setminus GL_n(\mathbb{R})$  and  $L_x$  be the corresponding Euclidean lattice. Then the following conditions on  $x$  are equivalent*

1. *The bottom of the profile of  $L_x$  is a line connecting the origin and  $(n, 0)$ .*
2. *The degree of instability of  $x$  is nonnegative, i.e.,  $\deg_{inst}(x) \geq 0$ .*

#### 5.1.1 A useful lemma

**Lemma 5.1.1.1.** *There exists a bijection*

$$GL_n(\mathbb{Z}) / (Q_i(\mathbb{Q}) \cap GL_n(\mathbb{Z})) \longleftrightarrow \{ \text{sublattices of rank } i \text{ of } \mathbb{Z}^n \}$$

*Proof.* We define the map from the collection of sublattices of rank  $i$  to the cosets space as follows: for any sublattice  $M \subset \mathbb{Z}^n$ , there exists a basis of  $M$ , denoted by

$$\mathfrak{C} = \{v_1, v_2, \dots, v_i\}$$

Extend this basis to get a basis of  $\mathbb{Z}^n$

$$\mathfrak{C}_1 = \{v_1, v_2, \dots, v_n\}$$

In  $\mathbb{Z}^n$  we have the standard basis  $\mathfrak{B} = \{e_1, e_2, \dots, e_n\}$ , and there exists  $\gamma \in \mathrm{GL}_n(\mathbb{Z})$  such that

$$\gamma \cdot e_k = v_k \quad \forall k = 1, 2, \dots, n$$

So we define the map

$$\begin{aligned} \varphi: \{\text{sublattices of rank } i \text{ of } \mathbb{Z}^n\} &\rightarrow \mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})) \\ M &\mapsto [\gamma] \end{aligned}$$

where  $[\gamma]$  denotes the equivalence class of  $\gamma$  in the quotient space. This is a well-defined map. Indeed, assume that we have a different basis

$$\mathfrak{C}_2 = \{w_1, \dots, w_n\}$$

where

$$\mathfrak{C}' = \{w_1, \dots, w_i\}$$

is a basis for sublattice  $M$ . By definition 4.1.2.1, we can regard  $\mathfrak{C}, \mathfrak{C}'$  as bases of the real vector space  $\mathbb{M} \otimes \mathbb{R} = M_{\mathbb{R}}$ . Similarly,  $\mathfrak{C}_1, \mathfrak{C}_2$  are basis of  $\mathbb{R}^n$ . It is known from linear algebra that there exists a linear  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $T(v_k) = w_k$  for  $k = 1, 2, \dots, n$ . Moreover,  $T(M_{\mathbb{R}}) = M_{\mathbb{R}}$ . Thus  $T$  must have matrix representation as

$$[T]_{\mathbb{C}} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

where  $A$  is the matrix representation for  $T|_{M_{\mathbb{R}}}$  with respect to the bases  $\mathfrak{C}$ , hence is of size  $i \times i$ . Note that we also have  $\gamma' \cdot (\gamma)^{-1} v_k = \gamma' e_k = w_k \quad \forall k$ . Thus

$$\gamma' \cdot (\gamma)^{-1} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \in Q_i(\mathbb{R}) \cap \mathrm{GL}_n(\mathbb{Z}) = Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$$

This is the desired result. □

### 5.1.2 Proof of the main theorem

Now we give a proof for theorem 5.1.0.1

*Proof.* We first need a slight reduction - we identify the quotient  $\mathrm{GL}_n(\mathbb{Q})/Q_i(\mathbb{Q})$  with the quotient  $\mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z}))$ . This is due to the fact that  $\mathrm{GL}_n$  is split and hence  $\mathrm{GL}_n(\mathbb{Q}) = \mathrm{GL}_n(\mathbb{Z})P(\mathbb{Q})$  for any parabolic subgroup  $P$ , by [4, Lemma 1]. Now let  $x$  be an arbitrary lattice of rank  $n$ . Lemma 5.1.1.1 generalizes in the obvious way for lattice  $x = g\mathbb{Z}^n$  for some  $g \in \mathrm{GL}_n(\mathbb{R})$ . Indeed, we just define the map

$$\begin{aligned}\phi_g: \{\text{sublattices of rank } i \text{ of } x\mathbb{Z}^n\} &\rightarrow \mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})) \\ gM = g \bigoplus_{k=1}^i \mathbb{Z}v_i &\mapsto [\gamma]\end{aligned}$$

where  $\gamma e_k = v_k$  in  $\mathbb{Z}^n$  for all  $k \leq i$  and  $\phi_g^{-1}([\gamma]) = g \bigoplus_{k=1}^i \mathbb{Z}(\gamma \cdot e_i)$ .

Now apply lemma 4.3.2.1, we have that  $\deg_{\text{inst}}(x) \geq 0$  if and only if, for every maximal parabolic subgroup  $Q \subset G$  and  $\omega \in \hat{\Delta}_Q$  we have

$$\langle \omega, H_Q(x\delta) \rangle \geq 0$$

for each  $\delta \in G(\mathbb{Q})/Q_i(\mathbb{Q})$ . Let  $M = \phi_g([\delta])$ . Note that for  $\omega \in \hat{\Delta}_{Q_i}$ , we have

$$\langle \omega, H_Q(x\delta) \rangle = \langle \omega, H_B(x\delta) \rangle = \frac{n}{2} \log(a_1 \cdots a_i)$$

by equation 4.2 and the observation that  $\hat{\Delta}_{Q_i}$  has only one element as defined in section 3.3.7. Hence

$$\langle \omega, H_B(x\delta) \rangle = c \langle \rho_{Q_i}, H_B(x\gamma) \rangle = \frac{n}{2} \log(a_1 a_2 \dots a_i) = c \mathrm{vol}(M)$$

for some positive integer  $c$ . It is clear that the bottom of the profile of  $L_x$  is a straight line if and only if  $\mathrm{vol}(M) \geq 0$  for any sublattice  $M \subset L_x$ . But this condition is equivalent to  $\langle \omega, H_B(x\delta) \rangle \geq 0$  for  $\omega \in \hat{\Delta}_Q$  and  $Q \subset G$  a standard maximal parabolic subgroup. Using the remark in section 4.3.3, we can conclude that the profile of  $L_x$  has bottom as a line if and only if the degree of instability of  $x$  is nonnegative.  $\square$

## 5.2 Canonical pair and Canonical filtration

### 5.2.1 Equivalence between Canonical pair and canonical filtration

We can further prove that, the equivalence between different notions of semistability comes from the equivalence between the notion of canonical pair and the bottom of the profile of a lattice - the canonical filtration. The main result is the following proposition

**Theorem 5.2.1.1.** *Given an  $x \in X_n = K \backslash G$ . Then  $(P, \delta)$  is the canonical pair for  $x$  if and only if  $P$  is the stabilizer of the canonical filtration for the lattice  $L_x = x\mathbb{Z}^n = x\delta\mathbb{Z}^n$ .*

Recall that, from theorem 4.2.2.6 and lemma 4.4.0.2, both canonical pair and canonical filtration are characterized by two conditions. We will show that these two conditions are equivalent.

## 5.2.2 Equivalence between chain condition and condition 2

Throughout this section, we fix  $x \in X_n$  and the corresponding lattice  $L_x = x\mathbb{Z}^n$ .

**Lemma 5.2.2.1.** *The following conditions are equivalent for  $x \in X_n$ .*

1.  $\langle \alpha, H_P(x\delta) \rangle < 0$  for any  $\alpha \in \Delta_P$  and for some standard parabolic subgroup  $P$  and  $\delta \in G_{\mathbb{Q}}/P_{\mathbb{Q}}$
2.  $L_x$  has a chain of lattices

$$\mathcal{F} : 0 = L_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{k-1} \subset M_k = L_x$$

such that  $\mu(M_i/M_{i-1}) < \mu(M_{i+1}/M_i)$  for all  $i$  and

$$\mathcal{F} \otimes \mathbb{Q} : L_0 \subset M_1 \otimes \mathbb{Q} \subset M_2 \otimes \mathbb{Q} \subset \cdots \subset M_{k-1} \otimes \mathbb{Q} \subset M_k \otimes \mathbb{Q} = \mathbb{Q}^n$$

is stabilized by a rational standard parabolic subgroup  $P_{\mathbb{Q}}$ .

**Remark.** We are not assuming  $M_i/M_{i-1}$  and  $M_{i+1}/M_i$  are semi-stable, so the chain  $\mathcal{F}$  is not necessarily a canonical flag.

*Proof.* Assume that the standard parabolic  $P$  in the canonical pair is of type  $(n_1, \dots, n_k)$  as defined in subsection 3.3.1, and set

$$d_i = n_1 + n_2 + \dots + n_i$$

In particular, the  $A_P$ -coordinate of  $x\delta$  in the  $P$ -horospherical decomposition is

$$a_P(x\delta) = \begin{bmatrix} a_1 I_{n_1} & 0 & \dots & 0 \\ 0 & a_2 I_{n_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_k I_{n_k} \end{bmatrix} \quad (5.1)$$

Then from the chain of standard lattice

$$0 \subset \mathbb{Z}^{d_1} \subset \mathbb{Z}^{d_2} \subset \cdots \subset \mathbb{Z}^{d_k}$$

we obtain a chain of sublattices of  $L_x$  as follows

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_k$$

where

$$M_i := \bigoplus_{m=1}^{d_i} \mathbb{Z} x \delta \cdot e_m$$

Note that for  $M_k = x \delta \mathbb{Z}^{d_k} = x \mathbb{Z}^n = L_x$ .

Using the  $P$ -horospherical coordinate of section 3.3.4, we may write

$$x \delta = k m_P(x \delta) a_P(x \delta) n_P(x \delta),$$

where  $k \in \mathrm{SO}_n(\mathbb{R})$ ,  $n_P(x \delta) \in N_P$ ,  $m_p(x \delta) \in M_P$  are as in section 3.3.4 and  $a_P(x)$  is defined as in equation (5.1). Using equation (4.1), we compute

$$\mathrm{vol}(M_i/M_{i-1}) = \left| k m_P(x) a_P(x) n_P(x) \bigwedge_{j=d_i+1}^{d_{i+1}} e_j \right| = \left| a_P(x) \bigwedge_{j=d_i+1}^{d_{i+1}} e_j \right| = a_i^{n_i}.$$

The second equality is because  $k \in \mathrm{SO}_n(\mathbb{R})$  preserves the length,  $n_p(x \delta)$  stabilizes the flag  $0 \subset \mathbb{Z}^{d_1} \subset \mathbb{Z}^{d_2} \subset \cdots \subset \mathbb{Z}^{d_k}$  and the fact the  $m_p$  contains sub-block of determinant  $\pm 1$ . Thus,

$$\begin{aligned} \mu(M_i/M_{i-1}) - \mu(M_{i+1}/M_i) &= \frac{\log \mathrm{vol}(M_i/M_{i-1})}{n_i} - \frac{\log \mathrm{vol}(M_{i+1}/M_i)}{n_{i+1}} \\ &= \log(a_i) - \log(a_{i+1}) \end{aligned}$$

**Claim:**

$$H_P(x \delta) = m_1 \lambda_{d_1}^\vee + m_2 \lambda_{d_2}^\vee + \cdots + m_k \lambda_{d_k}^\vee \text{ where} \quad (5.2)$$

$$m_i = \log(a_i) - \log(a_{i+1})$$

If we can prove this then

$$m_i = \langle \alpha_{d_i}, H_P(x \delta) \rangle < 0 \Leftrightarrow \mu(M_i/M_{i-1}) - \mu(M_{i+1}/M_i) < 0$$

By the definition of  $(H_P)$ , we know that

$$H_P(x\delta) = \log a_P(x\delta) = \begin{bmatrix} \log(a_1)I_{n_1} & 0 & \dots & 0 \\ 0 & \log(a_2)I_{n_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \log(a_k)I_{n_k} \end{bmatrix}$$

The set  $(\hat{\Delta}_P)^\vee = \{\lambda_{d_i}\}$  is a basis for  $\mathfrak{a}_P$ , which clearly yields the expression (5.2). Thus we just need to evaluate the value for  $m_i$ , but

$$m_i = \langle \alpha_{d_i}, H_P(x\delta) \rangle = \log(a_i) - \log(a_{i+1})$$

This is the desired result.  $\square$

**Remark.** From the proof the lemma 5.2.2.1, we explicitly construct an equivalence

$$xG_{\mathbb{Q}}/P_{\mathbb{Q}} \longleftrightarrow \{ \text{flag of sublattices of } x\mathbb{Z}^n \text{ of the same type of partition type of } P \}$$

### 5.2.3 $P$ -semistability and $M_P$ -semistability

Before proving the remaining equivalence, we need the following lemma

**Lemma 5.2.3.1.** Let  $P$  be a standard parabolic subgroup of  $GL_n(\mathbb{R})$ . For  $x \in X = K \backslash GL_n(\mathbb{R})$ , we let  $m := pr_{M_P}(x)$ , the projection of  $x$  on the space  $X_{M_P} := M_P/M_P \cap K$ . Then

$$\deg_{inst}^P(x) = \deg_{inst}^{M_P}(m).$$

In particular,  $x$  is  $P$ -semi-stable if and only if its projection  $m$  is  $M_P$ -semi-stable.

*Proof.* The identity follows essentially from the definition:

$$\deg_{inst}^P(x) = \min_{R \subset P, \delta \in P_{\mathbb{Q}}/R_{\mathbb{Q}}} \langle \rho_R^P, H_R(x\delta) \rangle \quad \text{and} \quad \deg_{inst}^{M_P}(m) = \min_{*R \subset M_P, \bar{\delta} \in M_{P,\mathbb{Q}}/R_{\mathbb{Q}}} \langle \rho_R^P, H_R(m\bar{\delta}) \rangle$$

where  $R$  ranges over all standard parabolic subgroup of  $P$ . Note that  $*R = R \cap M_P$  is also a standard parabolic subgroup of  $M_P$ . Moreover, the map

$$\begin{aligned} \text{pr}_{M_P}: P_{\mathbb{Q}}/R_{\mathbb{Q}} &\rightarrow M_{P,\mathbb{Q}}/*R_{\mathbb{Q}} \\ \delta &\mapsto \bar{\delta} \end{aligned}$$

is a bijection by theorem 3.3.4.4. This implies that one takes the minimum over the same set in evaluating the degree of  $P$ -instability and  $M_P$ -instability. By 3.3.6.3, For any  $R \subset P$  a standard parabolic subgroup, we have

$$H_R(x\delta) = H_P(x\delta) + H_{*R}(\text{pr}_{M_P}(x\delta)) = H_P(x\delta) + H_{*R}(m\bar{\delta})$$

Note that, by definition, we have  $\rho_R^P = \rho(P) - \rho(R)$  which vanishes identically on  $\mathfrak{a}_P$ . Thus

$$\begin{aligned} \deg_{inst}^P(x) &= \min_{R \subset P, \delta \in P_{\mathbb{Q}}/R_{\mathbb{Q}}} \langle \rho_R^P, H_R(x\delta) \rangle \\ &= \min_{R \subset P, \delta \in P_{\mathbb{Q}}/R_{\mathbb{Q}}} \langle \rho_R^P, H_P(x) + H_{*R}(m\bar{\delta}) \rangle \\ &= \min_{*R \subset M_P, \bar{\delta} \in M_P, \mathbb{Q}/R_{\mathbb{Q}}} \langle \rho_R^P, H_P(x) + H_{*R}(m\bar{\delta}) \rangle \\ &= \deg_{inst}^{M_P}(m) \end{aligned}$$

This is what we want.  $\square$

#### 5.2.4 Equivalence between increasing chain condition and condition 1

**Lemma 5.2.4.1.** *Keeping the notations  $(P, \delta)$  of lemma 5.2.2.1, the following conditions are equivalent*

1.  $M_i/M_{i-1}$  is semi-stable for all  $i$ , where the flag

$$\mathcal{F} : 0 \subset M_1 \subset M_2 \subset \cdots \subset M_{k-1} \subset M_k = L_x$$

corresponds to  $x\delta$ .

2.  $x\delta$  is  $P$ -semi-stable.

*Proof.* By lemma 5.2.3.1, the condition  $x\delta$  is  $P$ -semi-stable if and only if  $m\bar{\delta}$  is  $M_P$ -semi-stable. Observe that, if  $P$  is of type  $(n_1, \dots, n_k)$ , then

$$m\bar{\delta} = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_k \end{bmatrix}$$

where  $A_i$  is a matrix of size  $n_i \times n_i$  with determinant  $\pm 1$ . We can then construct a flag of lattices

$$\mathcal{F}' : 0 \subset M'_1 \subset M'_2 \dots \subset M'_k = L'_x$$

where  $L'_x$  is just the normalization of lattice  $L_x$ , i.e. we rescale  $L_x$  to get a similar lattice  $L'_x$  of volume 1. The sublattice  $M'_i$  is constructed by setting

$$M'_i := \bigoplus_{m=1}^{d_i} \mathbb{Z} m \bar{\delta} \cdot e_m$$

It is clear that  $M'_i/M'_{i-1} \cong A_i \mathbb{Z}^{n_i}$ . It is also clear that,  $m\bar{\delta}$  is semi-stable if and only if each quotient  $M'_i/M'_{i-1}$  is semi-stable. Rescaling does not change the semistability, so we can rescale the flag  $\mathcal{F}'$  by a suitable factor to get the desired flag  $\mathcal{F}$

$$\mathcal{F} : 0 \subset M_1 \subset M_2 \subset \dots \subset M_{k-1} \subset M_k = L_x$$

where  $M_i/M_{i-1}$  is semi-stable for all  $i$ .

### 5.2.5 Proof of theorem 5.2.1.1

By [18, Subsection 4.3.4], the two conditions in lemma 4.4.0.2 are sufficient to guarantee that  $\text{cp}(x) = (P, \delta)$ . Thus, by lemma 5.2.4.1 and lemma 5.2.2.1,  $\text{cp}(x) = (P, \delta)$  if and only if there exists a flag of sublattices

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset \dots \subset M_k = L_x$$

stabilized by  $P$ , such that

- $\mu(M_i/M_{i-1}) < \mu(M_{i+1}/M_i)$ .
- $M_i/M_{i-1}$  is semi-stable for all  $i$ .

By theorem 4.2.2.6,  $\mathcal{F}$  satisfies the above conditions if and only if  $\mathcal{F}$  is the canonical filtration for  $L_x$ .  $\square$

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