Compactification in low dimension

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In this expository note, I will try to explain explicitly how to compactify $\Gamma\backslash\mathbb{H}$ by adding points in two ways.

1 Some preparations

We will always denote Γ a subgroup of the group $SL_2(\mathbb{Z})$ of finite index, and this group acts on the upper half complex plane $\mathbb{H}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ z := \frac{az+b}{cz+d}$$

When z tends to infinity, we have

$$\lim_{z\to\infty} \frac{az+b}{cz+d} = \frac{a}{c},$$

so we add the rational line to define the action of this group at ∞. In particular, we consider the set

$$\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$$

Note that on the projective rational line, we define the action to be the multiplication of a 2×2 matrix with a 2×1 vector. Then under this action, we have the following lemma

Lemma 1. *$SL_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$.*

Proof. For each point in $\mathbb{P}^1(\mathbb{Q})$, we can choose the representative to be of the form $[a : b]$, where $\gcd(a, b) = 1$. Then there exists $x, y \in \mathbb{Z}$ such that

$$ax - by = 1$$

Thus we get the following equality

$$\begin{bmatrix} b & a \\ -x & y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This implies any points in $\mathbb{P}^1(\mathbb{Q})$ can be moved to $[0 : 1]$, and thus the action is transitive. $\square$

Corollary 2. *If Γ is a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$ then $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$ has only finite orbits.*

2 Compactification of $\Gamma\backslash\mathbb{H}$ by adding points.

We introduce a topology on $\overline{\mathbb{H}}$. For the usual upper half plane, the topology is the usual metric topology on $\mathbb{C}$, and we only define the system of the neighborhood of $r \in \mathbb{P}^1(\mathbb{Q})$.

Let $S(c,\omega)$ be the circle that touches the real line at $\omega = p/q$ and has the radius $\frac{c}{2q^2}$. Then the collection of circles $D(c,\omega) = \bigcup_{0<c'\leqslant c} S(c',\omega)$ is called *Farey disk*. Let $c \to 0$, these disks define a neighborhood of ω. The Farey disks at ∞ are defined to be the region

$$D(T,\infty) = \{z : \Im z \geqslant T\}$$

It can be checked easily that the matrix under inversion, the Farey disk at ∞ is mapped to $D(1/T,0)$. In general, if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \circ \infty = \omega$ then $D(T,\infty)$ is mapped to $D(1/T,\omega)$. With the above topology on the extended upper half plane, we could show that

Lemma 3. $\Gamma\backslash\overline{\mathbb{H}}$ *is a compact set.*

Proof. We first prove for the case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. It is well known that the quotient space $\Gamma\backslash\mathbb{H}$ is identical to the set

$$\mathcal{F} = \{z \in \mathbb{H} : \Re z \in [-1/2, 1/2), |z| \geqslant 1 \text{ and } |z| > 1 \text{ if } \Re z > 0\}$$

By lemma 1, the projective rational line "shrinks" to a point under the action of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, Thus we can identify $\Gamma\backslash\overline{\mathbb{H}}$ with the set $\tilde{\mathcal{F}} = \mathcal{F} \cup \{\infty\}$. Consider an open cover $\{U_i\}_{i\in I}$ of $\tilde{\mathcal{F}}$ and the natural projection $\pi \colon \overline{\mathbb{H}} \to \tilde{\mathbb{F}}$. Then the set $\{\pi^{-1}(U_i)\}_{i\in I}$ forms an open cover of $\overline{\mathbb{H}}$. There must be an index i_0 such that $\pi^{-1}(U_{i_0})$ contains a neighborhood of ∞, namely contains a Farey disk $D(T,\infty)$ for some $T > 0$. Since $\overline{\mathcal{F}} - D(T,\infty)$ is a compact set, its image under π is compact, hence it can be covered by $U_{i_1}, \ldots, U_{i_m}$. Altogether, $\tilde{\mathcal{F}}$ admits a finite subcover $U_{i_0}, \ldots, U_{i_m}$.
Now we proceed to the general case. Note that

$$\overline{\mathbb{H}} = \mathrm{SL}_2(\mathbb{Z}) \circ \tilde{\mathcal{F}} = \bigcup \Gamma a_i \circ \tilde{\mathcal{F}}$$

by corollary 2. Then under the surjective map $\pi \colon \overline{\mathbb{H}} \to \Gamma\backslash\overline{\mathbb{H}}$, we have

$$\Gamma\backslash\overline{\mathbb{H}} = \bigcup \pi\left(\Gamma a_i \circ \tilde{\mathcal{F}}\right),$$

which shows that the set $Y(\Gamma) = \Gamma\backslash\overline{\mathbb{H}}$ is compact as it is the union of compact sets. □

The orbit of $\mathbb{P}^1(\mathbb{Q})$ under the action of Γ is called *cusps*. We have the obvious equality that

$$\Gamma\backslash\overline{\mathbb{H}} = \Gamma\backslash\mathbb{H} \cup \underbrace{\Gamma\backslash\,\mathbb{P}^1(\mathbb{Q})}_{\text{cusps}}$$

So in fact lemma 3 tells us that we only need to add a finite cusp to get a compact domain. That means we only need to consider the actions of Γ on the projective rational line. By the orbit-stabilizer theorem, we get the decomposition

$$\Gamma\backslash\bigcup_{\omega} D(c,\omega) = \bigcup \Gamma_{\omega_i}\backslash D(c,\omega_i)$$

where ω_i is the set of representative for the action of Γ on $\mathbb{P}^1(\mathbb{Q})$ and Γ_{ω_i} are the stabilizer of $\omega_i \in \Gamma$.
Again, since the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$ is transitive, for each $r \in \mathbb{P}^1(\mathbb{Q})$, there exists an element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \circ \infty = r$. So we have $\Gamma_r = \gamma\Gamma_\infty\gamma^{-1}$. Hence we only need to know the "shape" of the domain $\Gamma_\infty\backslash D(T,\infty)$. WLOG, we could assume $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and hence

$$\Gamma_\infty = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}$$

Geometrically, $\Gamma_\infty\backslash D(T,\infty)$ is the strip $\{\Re z \in [-1/2, 1/2), \Im z \geqslant T\}$. But this is biholomorphic to a closed disk that misses a point on the boundary. So compactification is obtained by filling in the missing points to get finitely many compact disks.

3 Borel - Serre compactification of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$

We consider another compactification, by looking at the Farey disk $D(c,\omega)$ for fixed parameters c,ω. Then for any points $y \neq \omega$ on the Farey circle $S(c,\omega)$, we could connect y with ω by a unique geodesic in the upper half-plane.
These geodesics are either upper half circles that are orthogonal to the real line or the vertical line passing through ω. We thus can identify the Farey disks as follows

$$D(c,\omega) - \{\omega\} = X_{\infty,\omega} \times (0,c],$$

since a point θ on the Farey circles $S(c,\omega)$ is defined by its radius, up to a scaling of c, and the intersection of the geodesic $\overline{\theta\omega}$ with the real line. The uniqueness of the geodesics gives us a bijection between two sets. Here we let $X_{\infty,\omega} = \mathbb{P}^1(\mathbb{R}) - \{\omega\}$
How does the group Γ act on the set on the RHS set in the above identification? First, we look at the special case where $\omega = \infty$. In this case, the identification is

$$D(T,\infty) - \infty = X_{\infty,\infty} \times [T,\infty)$$

On the left, stabilizer subgroup Γ_∞ can be thought of as a subgroup of the group of translation, which leaves all the Farey circles $S(t,\infty)$ - which are the line $\{\Im z = t \geqslant T\}$ in this case - intact. Thus on the right-hand side, the action of Γ_∞ only affects the first coordinate. In general case, we need a lemma

Lemma 4. *If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \circ \infty = \omega$ then $\gamma \circ D(T,\infty) = D(1/T,\omega)$.*

Assume lemma 4 with the note that $\Gamma_\omega = \gamma\Gamma_\infty\gamma^{-1}$, we conclude that the action of Γ_ω only affects $X_{\infty,\omega}$ for all $\omega \in \mathbb{P}^1(\mathbb{Q})$. Since $\Gamma_\omega\backslash X_{\infty,\omega}$ is a circle, it is compact. Hence we can compactify the quotient space $\Gamma_\omega\backslash D(c,\omega) - \{\omega\}$ as

$$\Gamma_\omega\backslash D(c,\omega) - \{\omega\} \hookrightarrow \Gamma_\omega\backslash X_{\infty,\omega} \times [0,c]$$

As in section 1, we only need to compactify finitely many such quotient spaces and get the compactification of $\Gamma\backslash\mathbb{H}$.
Now we give a proof of lemma 4

Proof. Assume $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is an element that sends ∞ to $\omega = \frac{p}{q}$. Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}.$$

Thus we must have $a = p, c = q$ and b, c are integers such that $aq - cp = 1$. A Farey circle in the neighborhood of ∞ is, in fact, a line $S(T,\infty) = \{\Im z = T\}$, and this line is mapped to a circle tangent to the real line. Direct calculation shows that, for $z = x + iT$

$$\Im(\gamma \circ z) = \frac{\Im z}{|cz+d|^2} = \frac{T}{(cx+d)^2 + c^2T^2} \leqslant \frac{1}{q^2 T}$$

The equality happens if $x = -d/c$. Since this $\gamma \circ z$ is a point on the circle tangent to the real line at p/q and has the largest distance to the real line, the segment connect p/q and $\gamma \circ z$ must be the diameter of the image circle. In particular, the radius of the image circle is $\frac{1}{2Tq^2}$. Lemma 4 follows immediately. $\square$