Chapter 1

$\mathrm{SL}_2(\mathbb{R})$

In this chapter, I will give an exposition on the structure of $\mathrm{SL}_2(\mathbb{R})$ as the spaces of lattice, this space plays the role of a toy model before exploring the space of lattice in the higher rank. The exposition follows the paper [?] and [?] closely.

1.1 $\mathrm{SL}_2(\mathbb{R})$ and its action on the upper half plane $\mathfrak{H}$

A priori, the upper half plane

$$\mathfrak{H} = \{z : \Im z > 0\} \subset \mathbb{C}$$

has no group structure on its. However, we will show below that it can identify topologically with the space with the space of cosets $\mathrm{SO}_2(\mathbb{R})\,2\backslash \mathrm{SL}_2(\mathbb{R})$, and thus we can study the spaces $\mathfrak{H}$ via the spac of lattices $\mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})$. We define the action of $G = \mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{H}$ as follows

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ (z) = \frac{dz - b}{-cz + a}$$

Proposition 1.1.1. *The group* $\mathrm{SL}_2(\mathbb{R})$ *stabilizes* $\mathfrak{H}$ *and acts transitively on it. In particular,*

$$\begin{bmatrix} \frac{1}{\sqrt{y}} & 0 \\ 0 & \sqrt{y} \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} (i) = x + iy \quad (\textit{for } x \in \mathbb{R},\ y > 0)$$

Further, for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ *and* $z \in \mathfrak{H}$,

$$\Im\, g(z) = \frac{\Im\, z}{|cz + d|^2}.$$

Proof. The first formula is clear. The second formula would imply that the upper half-plane is stabilized. Compute directly:

$$2i \cdot \Im\left(\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \circ (z)\right) = \frac{az+b}{cz+d} - \frac{d\overline{z}+b}{c\overline{z}+d} = \frac{(az+b)(c\overline{z}+d) - (a\overline{z}+b)(cz+d)}{|cz+d|^2}$$

$$= \frac{adz - bc\overline{z} - bcz + ad\overline{z}}{|cz+d|^2} = \frac{z - \overline{z}}{|cz+d|^2}$$

since $ad - bc = 1$. □

The point $z = i$ is special, in the sense that its stability group is the orthogonal group $K = \mathrm{SO}_2(\mathbb{R})$. Indeed, for any $g = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \in SL_2(\mathbb{R})$ we have that

$$g \circ i = i \Leftrightarrow \frac{ai+b}{ci+d} = i \Leftrightarrow a = d \text{ and } b = -c$$

Combinining with the fact that $ad - bc = 1$, we must have $a^2 + b^2 = 1$. This implies that there is a θ such that $a = \cos\theta$ and $b = \sin\theta$. Since G acts on $\mathfrak{H}$ transitively, we know from group theory that there is a bijection between the collection of cosets of $\mathrm{Stab}(i)$ in G and the orbits of i. In particular

Proposition 1.1.2. *We have an isomorphism of $SL_2(\mathbb{R})$-spaces*

$$\mathrm{SO}_2(\mathbb{R})\setminus\mathrm{SL}_2(\mathbb{R}) \cong \mathfrak{H} \quad \textit{via} \quad SO(2)g \to g^{-1}(i)$$

That is, the map respects the action of $SL_2(\mathbb{R})$, in the sense that

$$(\mathrm{SO}_2(\mathbb{R})\, g)\cdot h \longrightarrow h^{-1}(g^{-1}i)$$

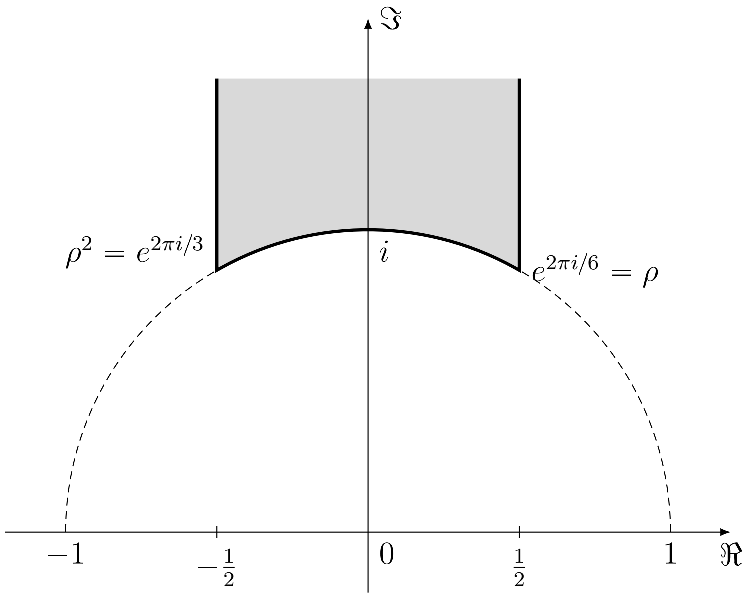
Proof. This is because of *associativity*:

$$(\mathrm{SO}_2(\mathbb{R})\, g)\cdot h = (\mathrm{SO}_2(\mathbb{R}))\cdot(gh) \longrightarrow (gh)^{-1}(i) = h^{-1}(g^{-1}(i))$$

giving the result. □

1.2 Fundamental domain for $\Gamma = SL_2(\mathbb{Z})$ on $\mathfrak{H}$

Here is a picture of the fundamental domain $\mathfrak{H}/\Gamma$.



The goal of this section is to prove that under the action of the $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, we can "move" every points on the upper half plane to a domain, under an equivalence given by a specific action. This is similar to the fundamental domain given by the translation action of $\mathbb{Z}$ to $\mathbb{R}$ is the half-open unit interval $[0, 1)$. In general, this give a simpler description to the homogenous space of lattice. Note that when we try to compute the fundamental domain of $\mathbb{Z}\backslash\mathbb{R}$, we have $\mathbb{Z}$ plays a role of *discrete* subset of $\mathbb{R}$. We give a precise definition of discreteness as follows

Definition 1.2.1. *Let a group G act continuously on a topological space X. A subset $\Gamma \subset G$ is called* ***discrete*** *if for any two compact subse A, B in X, there are only finitely many $g \in \Gamma$ such that $g \circ A \cap B \neq \varnothing$.*

We will prove that the set

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\}$$

is a discrete subgroup of $G = \mathrm{SL}_2(\mathbb{R})$. To prove this, we first need the following lemma

Lemma 1.2.2. *Fix a real number* $r > 0$ *and* $0 < \delta < 1$. *We denote* $R_{r,\delta}$ *the rectangle*

$$R_{r,\delta} = \left\{z = x + iy : -r \leqslant x \leqslant r, 0 < \delta \leqslant y \leqslant \delta^{-1}\right\}$$

Then for any $\epsilon > 0$ *and any fixed set* $\mathbb{S}$ *of coset representatives for* $\Gamma_\infty \backslash \Gamma$, *there are finitely many* $g \in \mathbb{S}$ *such that* $\Im(g \circ z) > \epsilon$ *for some* $z \in R_{r,\delta}$.

In the above lemma, the notation Γ_∞ is defined to be the set

$$\Gamma_\infty = \left\{\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z}\right\}.$$

It can be seen easily that this is the stability group of ∞ in $\mathfrak{H}$.

Proof. Let $g = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Then for $z \in R_{r,\delta}$,

$$\mathrm{Im}(g \circ z) = \frac{y}{c^2y^2 + (cx+d)^2} < \epsilon$$

if $|c| > (y\epsilon)^{-\frac{1}{2}}$. On the other hand, for $|c| \leqslant (y\epsilon)^{-\frac{1}{2}} \leqslant (\delta\epsilon)^{-\frac{1}{2}}$, we have

$$\frac{y}{(cx+d)^2} < \epsilon$$

if the following inequalities hold:

$$|d| > |c|r + (y\epsilon^{-1})^{\frac{1}{2}} \geqslant |c|r + (\epsilon\delta)^{-\frac{1}{2}}.$$

Consequently, $\Im(g \circ z) > \epsilon$ only if

$$|c| \leqslant (\delta\epsilon)^{-\frac{1}{2}} \quad \text{and} \quad |d| \leqslant (\epsilon\delta)^{-\frac{1}{2}}(r+1),$$

and the total number of such pairs (not counting $(c,d) = (0, \pm 1), (\pm 1, 0)$) is at most $\frac{4(r+1)}{(\epsilon\delta)}$. This proves the lemma. $\square$

It follows from Lemma 1.2.2 that $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ is a discrete subgroup of $SL(2,\mathbb{R})$. This is because:

1. It is enough to show that for any compact subset $A \subset \mathfrak{H}$ there are only finitely many $g \in SL(2,\mathbb{Z})$ such that $(g \circ A) \cap A \neq \phi$;
2. Every compact subset of $A \subset \mathfrak{H}$ is contained in a rectangle $R_{r,\delta}$ for some $r > 0$ and $0 < \delta < \delta^{-1}$;
3. $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$, except for finitely many $\alpha \in \Gamma_\infty$, $g \in \Gamma_\infty \backslash \Gamma$.

To prove (3), note that Lemma 1.2.2 implies that $(g \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$ except for finitely many $g \in \Gamma_\infty \backslash \Gamma$. Let $S \subset \Gamma_\infty \backslash \Gamma$ denote this finite set of such elements g. If $g \notin S$, then Lemma 1.2.2 tells us that it is because $\Im(g \circ z) < \delta$ for all $z \in R_{r,\delta}$. Since $\Im(\alpha g \circ z) = \Im(g \circ z)$ for $\alpha \in \Gamma_\infty$, it is enough to show that for each $g \in S$, there are only finitely many $\alpha \in \Gamma_\infty$ such that $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} \neq \phi$. This last statement follows from the fact that $g \circ R_{r,\delta}$ itself lies in some other rectangle $R_{r',\delta'}$, and every $\alpha \in \Gamma_\infty$ is of the form $\alpha = \begin{bmatrix} 1 & -m \\ 0 & 1 \end{bmatrix}$ $(m \in \mathbb{Z})$, so that

$$\alpha \circ R_{r',\delta'} = \{x + iy \mid -r' + m \leqslant x \leqslant r' + m,\ 0 < \delta' \leqslant \delta''^{-1}\},$$

which implies $(\alpha \circ R_{r',\delta'}) \cap R_{r,\delta} = \phi$ for $|m|$ sufficiently large. Now we are ready to describe the fundamental domain for $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}$.

Proposition 1.2.3. *A fundamental domain for $\mathfrak{H}/\mathrm{SL}_2(\mathbb{Z})$ can be given as the region*

$$\mathfrak{D} = \{z = x + iy \in \mathfrak{H} : |z| \geqslant 1, -1/2 \leqslant x \leqslant 1/2\},$$

modulo the congruent boundary points symmetric with respect to the imaginary axis.

Proof. First we eliminated the repeated points on the boundary. Note that the line $x = -1/2$ is the same as the line $x = 1/2$ under the transformation $z \mapsto z + 1$. Similarly, given a point on the circle $\{|z| = 1\}$, the transformation $z \mapsto -|z|^{-1}$ satisfies

$$\frac{-1}{x+iy} = \frac{-x+iy}{x^2+y^2} = -x + iy,$$

which flips the sign of x. Thus it identifies the half circle on the right of the imaginary axis with that on the left.

Now we need to show two things:

1. For any $z \in \mathfrak{H}$ we can find an element $g \in \mathrm{SL}_2(\mathbb{Z})$ such that $g \circ z \in \mathfrak{D}$.
2. If $z \equiv z' \in \mathfrak{D}$ modulor $\mathrm{SL}_2(\mathbb{Z})$, then either $\Re(z) = \pm\frac{1}{2}$ and $z' = z \mp 1$, or $|z| = 1$ and $z' = \frac{-1}{z}$.

First we prove for (1): Fix $z \in \mathfrak{H}$. It follows from Lemma 1.2.2 that for every $\epsilon > 0$, there are at most finitely many $g \in \mathrm{SL}(2,\mathbb{Z})$ such that $g \circ z$ lies in the strip

$$D_\epsilon := \left\{ w \;\middle|\; -\frac{1}{2} \leqslant \mathrm{Re}(w) < \frac{1}{2},\ \epsilon \leqslant \mathrm{Im}(w) \right\}.$$

Let B_ϵ denote the finite set of such $g \in \mathrm{SL}(2,\mathbb{Z})$. Clearly, for sufficiently small ϵ, the set B_ϵ contains at least one element. We will show that there is at least one $g \in B_\epsilon$ such that $g \circ z \in D$. Among these finitely many $g \in B_\epsilon$, choose one such that $\Im(g \circ z)$ is maximal in D_ϵ. If $|g \circ z| < 1$, then for $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ we have, for any m,

$$\Im\left(T^m S g \circ z\right) = \Im\left(\frac{-1}{g \circ z}\right) = \frac{\Im(g \circ z)}{|g \circ z|^2} > \Im(g \circ z)$$

But we can choose m such that $T^m S g \circ z \in D_\epsilon$, which contradicts the maximality of $\Im(g \circ z)$.

Next we give a proof for (2):Let $z \in D$, $g = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, and assume that $g \circ z \in D$. Without loss of generality, we may assume that

$$\Im(g \circ z) = \frac{y}{|cz+d|^2} \geqslant \Im(z),$$

(otherwise just interchange z and $g \circ z$ and use g^{-1}). This implies that $|cz + d| \leqslant 1$ which implies that $1 \geqslant |cy| \geqslant \frac{1}{\sqrt{3}}|c|$. This is clearly impossible if $|c| \geqslant 2$. So we only have to consider the cases $c = 0, \pm 1$. If $c = 0$ then $d = \pm 1$ and g is a translation by b. Since $-\frac{1}{2} \leqslant \Re(z), \Re(g \circ z) \leqslant \frac{1}{2}$, this implies that either $b = 0$ and $z = g \circ z$ or else $b = \pm 1$ and $\Re(z) = \pm\frac{1}{2}$ while $\Re(g \circ z) = \mp\frac{1}{2}$. If $c = 1$, then $|z + d| \leqslant 1$ implies that $d = 0$ unless $z = e^{2\pi i/3}$ and $d = 0, -1$. The case $d = 0$ implies that $|z| \leqslant 1$ which implies $|z| = 1$. Also, in this case, $c = 1$, $d = 0$, we must have $b = -1$ because $ad - bc = 1$. Then $g \circ z = a - \frac{1}{z+1}$. It follows that $g \circ z = a - e^{2\pi i/3}$ and $d = 1$, then we must have $a - b = 1$. It follows that $g \circ z = a - \frac{1}{z+1} = a + e^{2\pi i/3}$, which implies that $a = 0$ or 1. A similar argument holds when $z = e^{\pi i/3}$ and $d = -1$. Finally, the case $c = -1$ can be reduced to the previous case $c = 1$ by reversing the signs of a, b, c, d. $\square$

1.3 Lattices and semi-stability in dimension 2

In this section, we investigate the notion of semi-stable lattices and how the upper half plane $\mathfrak{H}$ can be regard as a spaces of two dimensional lattices.
Now regard $\mathbb{C} \cong \mathbb{R}^2$ via $x + iy \mapsto (x, y)$, and the inner product is defined to be

$$\langle z_1, z_2 \rangle = x_1x_2 + y_1y_2,$$

where $z_i = x_i + iy_i$. Now for any $z \in \mathfrak{H}$, the pair $(1, z)$ can be identified with the lattice

$$L_z = \mathbb{Z}z \oplus \mathbb{Z}$$

First we prove the following statement

Proposition 1.3.1. *The upper half plane $\mathfrak{H}$ classifies similarity classes of two dimensional lattice.*

Proof. Let $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ be any lattice in $\mathbb{R}_2$. Then using the above identification, we can find two complex number z_1, z_2 such that $|z_1| = ||e_1||$ and $|z_2| = |e_2|$ □

add this later, refer to the book by Anton Deitmar

Clearly in each classes of similar lattice, there is a unique one that has unit covolume. The lattice spanned by z and 1 has volume y, so the corresponding unit lattice is the one spanned by $z/\sqrt{y}$ and $1/\sqrt{y}$.
Using Proposition 1.2.3, it is immediate that every lattice spanned by 1 and z is similar to lattice generated by 1 and a point z' inside ther region $\mathfrak{D}$.
Historically, in two dimension, Proposition 1.2.3 is first discovered by Lagrange, with the distribution of Gauss to solve for the shortest vector problem in two dimensional space. In the language of modern mathematics, it can be phrased as follows:

Proposition 1.3.2. *If L is any lattice, and u is a primitive vector in L, and v' is a vector in the sublattice $L' = L/\mathbb{Z}u$, then there exists a unique representative v of v' such that its projection onto u lies in the interval $(-u/2, u/2]$. Moreover, the following inequality holds*

$$||v||^2 \leqslant \frac{||u||^2}{4} + ||v'||^2,$$

where we identify v' with a vector $v^\perp$ in the orthogonal complement of u.

Remark. *Here the primitive vector is the vector such that it is not the multiple of any other vector in the lattice.*

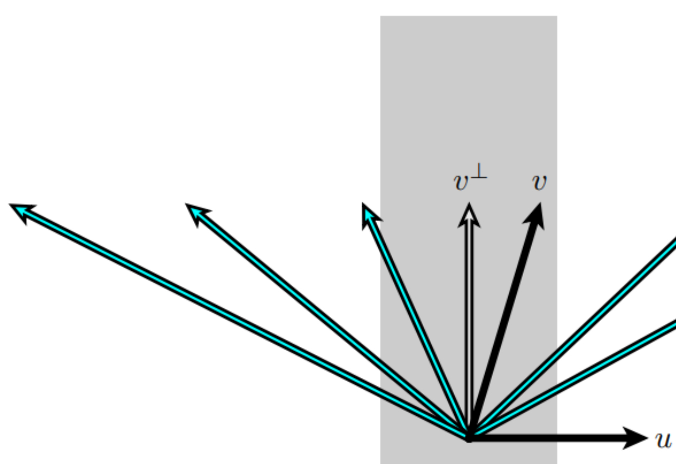


Figure 1.1

To see why every point $z \in \mathfrak{H}$ can be transformed into a point inside $\mathfrak{D}$, we start with a lattice generated by the $\mathbb{Z}$-linear combination of $1, z$ and consider the shortest vector u. Applying the above lemma, we can find a vector v with the length as least as large as that of u. So if we rotate and scale to get $u = 1$, the vector v will lie in the strip $(-1/2, 1/2]$ and has the length at least 1. This clearly show that v is a point in the domain $\mathfrak{D}$.

Now Grayson - following a prior idea of Stuhler - associated every lattice to a sort of **Newton Polygon**. We will set up a graph coordinate in the following way:

1. First we construct a two dimensional coordinate, say Oxy
2. We highlight the origin.
3. If we are dealing with the lattice L, compute the area of the fundamental domain of L
4. Assign the point $(2, \log(\mathrm{vol}(L)))$ to the line $x = 2$ in the coordinate.
5. If v is any primitive vector, we put the point $(1, \log(||v||))$ in the set.

Note that the lattice is discrete, so we can find a shortest vector v of the lattice L. This will correspond to the lowest point on the axis $x = 1$ in the diagram. Note the that x-coordinate of each of these point reflects its dimension.
As an example, let's consider the lattice of the following shape - with the shortest vector u has the length $||u|| < 1$. Now applying the above process, we get the figure on the left. If we further taking

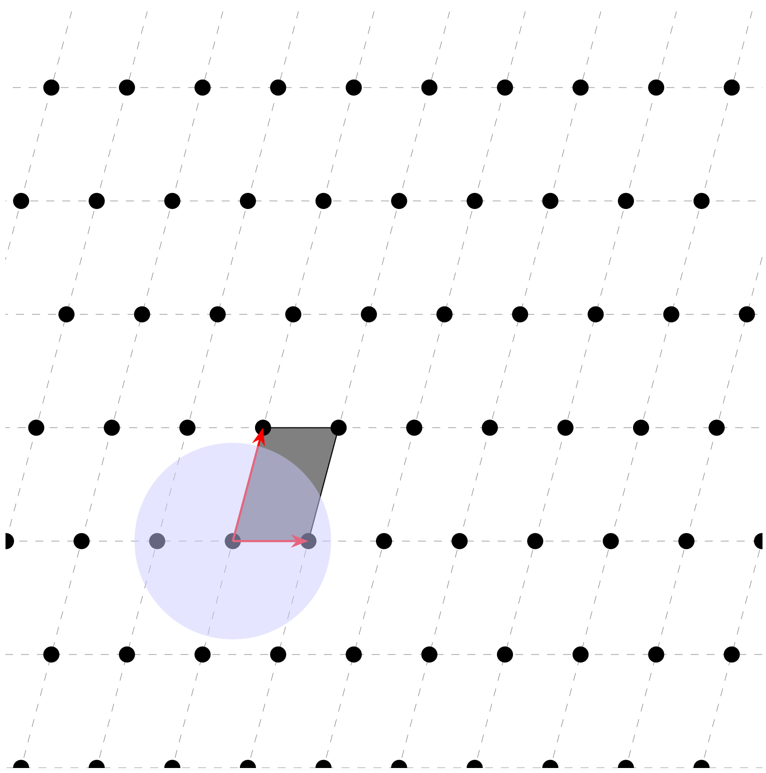
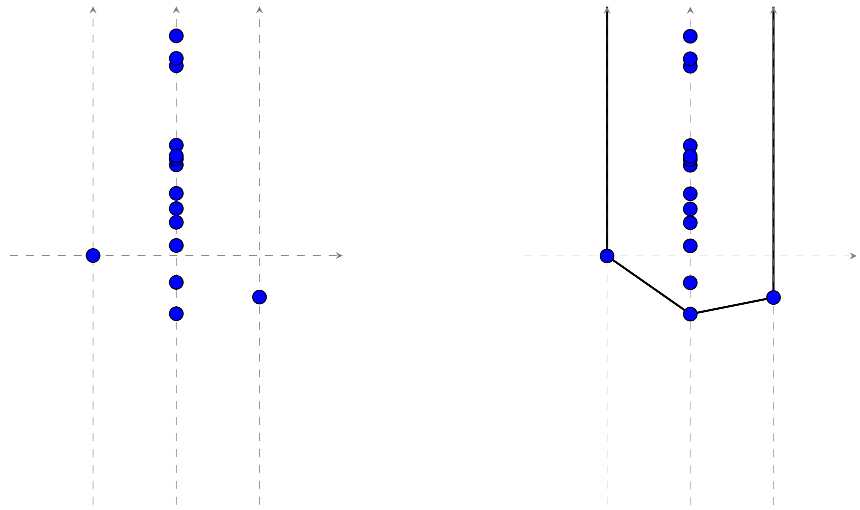


Figure 1.2: Example of a lattice

the convex hull of the diagram, we will get the figure on the right Clearly for each dimension, we have the corresponding lowest point, and so the convex hull of the plot is bounded from below. Grayson calls the plot on the left **canonical plot** of the lattice and the boundary of the convex hull of the canonical plot its **canonical polygon**. In the expository [?] of Bill Casselman, he instead calls the canonical polygon as **profile**. We will use the terminology of Casselman.
Now we will try to understand the profile of a lattice associated to a point $z \in \mathfrak{D}$. First we prove a simple observation



Lemma 1.3.3. *If $z \in \mathfrak{D}$ then the lattice $L_z = \mathbb{Z}z \bigoplus \mathbb{Z}$ admits 1 as the shortest vector.*

Proof. We identify $z = x + iy$ with $(x, y) \in \mathbb{R}^2$ and 1 with $(1, 0) \in \mathbb{R}^2$. Assume that 1 is not the shortest vector, then there exists $a, b \in \mathbb{Z}$ such that

$$|az + b|^2 < 1 \Leftrightarrow (ax + b)^2 + (ay)^2 < 1 \Leftrightarrow a^2|z|^2 + 2abx + b^2 < 1$$

Since $z \in \mathfrak{D}$, we clearly have $|x| \leqslant \frac{1}{2}$ and $|z| \geqslant 1$, thus the integers a, b must satisfy

$$a^2 - |ab| + b^2 < 1$$

Since the above expression are symmetric, we can assume $|a| \geqslant |b|$ and completing the square yields

$$\left(\frac{\sqrt{3}b}{2}\right)^2 \geqslant a^2 - ab + b^2 < 1 \Rightarrow b^2 < 4/3 \Rightarrow b \leqslant 1$$

Substituting $|b| = 1$ yields $|a|^2 - |a| < 0$. There is no non-zero integer a satisfying this condition. □

The area of the lattice L_z is given by $\det \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} = y$. Note that we can scale the basis by a factor $a = \sqrt{y}$ so that we get a lattice of volume 1. So the lowest points with respect to the axes $x = 0, 1, 2$ are $(0, 0), (1, -\log(a))$, and $(2, 0)$. The interesting part of $\mathfrak{D}$ is where $y \leqslant 1$. This corresponds to the lattices that have the canonical plots lying entirely on or above the x-axis. In particular, the profile of such a lattice only has the vertices at the origin and $(2, 0)$. Grayson and Stuhler call this kind of lattice **semi-stable**. If we don't normalize the area of such a lattice, then a semi-stable lattice has the bottom of the profile as a straight line.
Conversely, the lattices assigned to the points $z \in \mathfrak{H}$ with $\Im(z) > 1$ correspond to lattices that have the canonical plot breaking at the lowest point on the axis $x = 1$. In the general case, this reflects the fact that a non semi-stable lattice has the shortest vector u satisfying $||u|| < \sqrt{\mathrm{vol}(L_z)}$. Following Casselman, we call such a lattice **unstable**. In some sense, we can see that *the degree of instability* is measured by the shortest vector compared to its volume. In the above lemma, we only find the semi-stable locus inside the fundamental domain. To find the semi-stable locus for the whole upper half plane $\mathfrak{H}$, we use the following lemma:

Lemma 1.3.4. *If L_z is semi-stable, then so is the lattice $L_{g \circ z}$, where $g \in \mathrm{SL}_2(\mathbb{R})$.*

Proof. If we denote $L_z = \mathrm{span}_{\mathbb{Z}}\{1, z\}$, then $L_{\gamma \circ z} = cL_z$ for some complex number c. Indeed, we just need to check for γ being an inversion or translation, since these two transformations generate $\mathrm{SL}_2(\mathbb{R})(\mathbb{Z})$, but this is easy. Now let $c = re^{it}$. Multiplying by e^{it} doesn't change the length, hence doesn't change the semi-stability. Multiplying by a positive number r will shift $(1, \log|u|)$ to $(1, \log|u| + \log r)$ and $(2, \log(\mathrm{vol}(A)))$ to $(2, \log(\mathrm{vol}(A)) + 2\log r)$.

I think in 2-dimensio case, c

The line segment d connecting the origin with the final point intersects the line $x = 1$ at $(1, \log(\mathrm{vol}(A)) + \log r)$. By the semi-stability of the original lattice, the point $(1, \log|u| + \log r)$ is above the line segment d. $\square$

From this lemma, we can see that the semi-stable locus is the complement of the Farey balls in the upper half plane, as illustrated in the following figure, where the blue part is the semi-stable locus and the gray part is the unstable one.

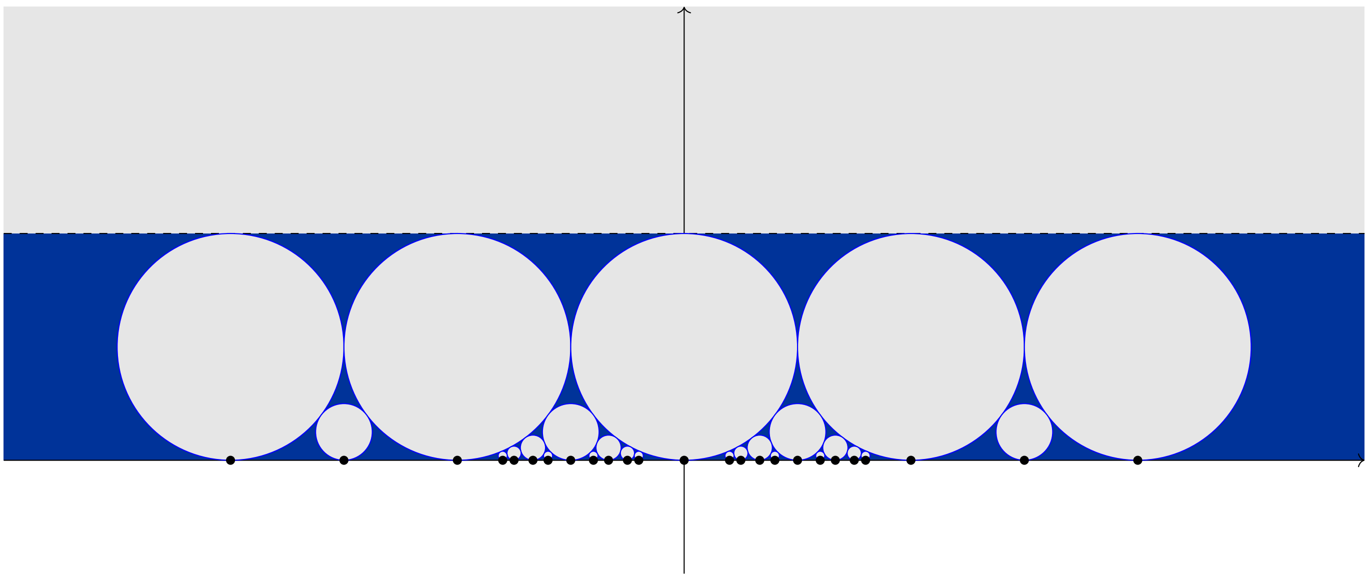


Figure 1.3: Semistable locus in upper half plane

1.4 ρ-semi-stability of lattices

The semi-stability can be defined in a more Lie-theoretic way. First we recall the Iwasawa decomposition for $\mathrm{SL}_2(\mathbb{R})$.

Proposition 1.4.1. *We have*

$$\mathrm{SL}_2(\mathbb{R}) \cong K \times A \times N$$

where

- $K = \mathrm{SO}_2(\mathbb{R})$*: the special orthogonal group.*
- $A = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a > 0 \right\}$.
- $N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right\}$.

Combinining with Proposition 1.1.1, we have the following identification

$$\mathfrak{H} \cong A \times N$$

via the map

$$x + iy \mapsto \begin{bmatrix} 1/\sqrt{y} & 0 \\ 0 & \sqrt{y} \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} = a\,(y)\,n(x)$$

Let's denote $\mathfrak{sl}_2(\mathbb{R})$ the Lie algebra of the Lie group $\mathrm{SL}_2(\mathbb{R})$ - the vector space of traceless matrices of size 2×2. We denote $\mathfrak{h} = \mathbb{R}H$ where $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ its standard Cartan subalgebra. We then have the map

$$H_B\colon \mathfrak{H} \to \mathfrak{h}, \quad z = x + iy \mapsto \log(a(y))H = \frac{-1}{2}\log(\Im(z))$$

Let $\alpha\colon \mathfrak{h} \to \mathbb{R}$ be the unique linear function such that $\alpha(H) = 2$. If we let $\rho = \frac{1}{2}\alpha$, then we define

$$\deg_{\text{inst}}(z) := \min_{\gamma \in \Gamma/\Gamma\cap B} \langle \rho, H_B(z\gamma)\rangle$$

where B is the group of upper triangular matrices with invertible entries along the diagonal.

Definition 1.4.2. *The lattice L_z corresponds to the point $z \in \mathfrak{H}$ is called ρ-**semistable** or just **semi-stable** if $\deg_{inst}(z) \geqslant 0$.*

We shall use this definition to find the semi-stable locus in the upper half plane $\mathfrak{H}$.

Proposition 1.4.3. *The locus of ρ-semistable points in the upper half plane $\mathfrak{H}$ is the complement of the Farey balls.*

Proof. We first make simple observation: If $\deg_{\text{inst}}(z)$ is achieved at some $\gamma_0 \in \Gamma$ and z is ρ-semistable, then

$$\langle \rho, H_B(z\gamma)\rangle \geqslant 0 \text{ for all } \gamma \in \Gamma$$

From this observation, the ρ-semistable locus must be the set

$$\{z \in \mathfrak{H} : \langle \rho, H_B(z\gamma)\rangle \geqslant 0 \text{ for all } \gamma \in \Gamma\}$$

We identify $z = x + iy$ with the product $a(y)n(x)$ as above. Under the identification 1.1.2, we must have

$$\langle \rho, H_B(z\gamma) = -\log(a(\gamma^{-1} \circ z))\rangle$$

Assume that $\gamma^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\gamma^{-1} \circ z$ satisfies

$$\Im(\gamma^{-1} \circ z) = \frac{\Im(z)}{(cx+d)^2 + (cy)^2}$$

Thus, z is semistable if and only if it satifies the inequalities

$$y \leqslant (cx+d)^2 + (cy)^2 \Leftrightarrow \left(x + \frac{c}{d}\right)^2 + \left(y - \frac{1}{2c^2}\right)^2 \geqslant \left(\frac{1}{2c^2}\right)^2$$

for any c, d coprime. Since the above equation is exactly the equation for Farey balls, we are done. ☐

In particular, we just proved that the ρ-semistablility and the semistablility in Grayson's sense are equivalent.

1.4.1 Unstable lattices

Consider an unstable lattice L with a shortest vectur u, then the bottom of the profile of L has a break at the point $(1, \log(||u||))$. Clearly u is a primitive vector, so $L_1 = \mathbb{Z}u$ is a sublattice of L. This determines a **lattice flag**

$$\mathcal{F} : 0 \subset L_1 \subset L_2 = L$$

And this is called the **canonical flag** associated to L. By tensoring with $\mathbb{R}$ we get a flag of rational subspaces

$$0 \subset V_1 = L_1 \otimes \mathbb{R} \subset V_2 = L_2 \otimes \mathbb{R}$$

Conversely, for any rational flag $\mathcal{F}$ in $\mathbb{R}^2$, we can denote $\mathcal{H}_{\mathcal{F}}$ the set of all unstable lattices that gives rise to the flag $\mathcal{F}$. What can we say about the set $\mathcal{H}_{\mathcal{F}}$? It turns out that the can be parametrized by the Farey balls in the complex plane.

Proposition 1.4.4. *Given a flag of rational vector spaces*

$$\mathcal{F} : 0 \subset V_1 \subset V_2 = \mathbb{R}^2$$

Assume that $V_1 = span(pf_1 + qf_2)$ *for some* $\mathbb{R}$*- linearly independent set* $\{f_1, f_2\}$ *. Then the set* $\mathcal{H}_{\mathcal{F}}$ *corresponds to the Farey balls that is tangent to the* $x-axis$ *at the fraction* p/q*.*

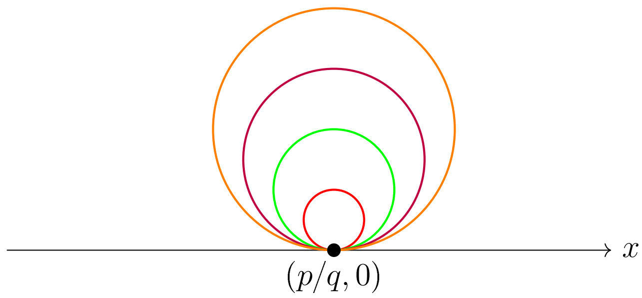


Figure 1.4: The Farey Circles correspond to the same rational flag

Proof. Consider the lattice $L = L_z \in \mathcal{H}_{\mathcal{F}}$ for $z = x + iy$. This lattice L has a basis

$$\begin{cases} f_1 = \dfrac{e_1}{\sqrt{y}} \\ f_2 = \dfrac{-xe_1}{\sqrt{y}} + e_2\sqrt{y} \end{cases}$$

Assume that $u = pf_1 + qf_2$ is the shortest vector of L. Since u is primitive, we must have $\gcd(p, q) = 1$. Since the lattice L is unstable, it follows that $\mu(L_1) \leqslant \mu(L/L_1)$ for $L_1 = \mathbb{Z}u$. In particular, we have

$$\frac{|-qz+p|}{\sqrt{y}} \leqslant 1 \Leftrightarrow (-qx+p)^2 + (qy)^2 \leqslant y$$

If $q \neq 0$, this is the equation for the Farey balls that tangent to the horizonal axis at $(p/q, 0)$. If $p = 0$ then $q = 1$, the equation $\dfrac{|-qz+p|}{\sqrt{y}} \leqslant 1$ degenerates to $y \geqslant 1$. Hence we are done. ☐

Remark. *Note that for every flag in* $\mathbb{R}^2$ *corresponds to a parabolic subgroup. For example, the subgroup*

$$B = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

stabilizes the flag $0 \subset \mathbb{R}e_1 \subset \mathbb{R}^2$ *for some choice of basis* $\{e_1, e_2\}$ *of* $\mathbb{R}^2$*. Any parabolic subgroup of* $\mathrm{SL}_2(\mathbb{R})$ *is conjugate with* B*. So there is a correspondence between rational parabolic subgroups of* $\mathrm{SL}_2(\mathbb{R})$*, that is, the parabolic subgroup that stabilizes a rational flag* $\mathcal{F}$ *- and the collections* $\mathcal{H}_{\mathcal{F}}$ *of lattices gives rise to the flag* $\mathcal{F}$*.*

Chapter 2

BASIS LIE THEORY FOR $\mathbf{SL}_{\mathrm{n}}(\mathbb{R})$ & $\mathbf{GL}_{\mathrm{n}}(\mathbb{R})$

In this chapter, we review the basic theory of roots and weights for Lie groups. We will first recall the general theory and then compute explicitly the examples for $\mathrm{SL}_{\mathrm{n}}(\mathbb{R})$ & $\mathrm{GL}_{\mathrm{n}}(\mathbb{R})$.

2.1 Structure theory

Throughout this section, the ground field is $k = \mathbb{R}$.

2.1.1 Lie algebras

A Lie algebra $\mathfrak{g}$ is a vector space over k such that the Lie bracket

$$\begin{aligned} [\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g} &\to \mathfrak{g} \\ (x,y) &\mapsto [x,y] \end{aligned}$$

satisfies the following axioms

1. $[x,y]$ is bilinear.
2. $[x,x] = 0$ for all $x \in \mathfrak{g}$.
3. $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$

The last property is called *Jacobi's identity.*

Example 2.1.1. *The first example is the set of all matrices of size $n \times n$, denoted by $\mathfrak{gl}_{\mathrm{n}}(\mathbb{R})$. We define the Lie bracket by*

$$[A,B] := AB - BA,$$

for arbitrary matrices $A, B \in \mathfrak{gl}_{\mathrm{n}}(\mathbb{R})$. It is clearly that $[A,B]$ is bilinear and $[A,A] = 0$ for all matrices A. We need to verify the Jacobi's identity. Note that

$$\begin{aligned} [A,[B,C]] &= A(BC - CB) - (BC - CB)A \\ &= ABC - ACB - BCA + CBA \end{aligned}$$

It follows that

$$\begin{aligned}&[A,[B,C]]+[C,[A,B]]+[B,[C,A]]\\=&ABC-ACB-BCA+CBA\\+&CAB-CBA-ABC+BAC\\+&BCA-BAC-CAB+ABC\\=&0\end{aligned}$$

Therefore this is a Lie algebra.

Example 2.1.2. *A less trivial example is the subspace $\mathfrak{sl}_n(\mathbb{R})$ of $\mathfrak{gl}_n(\mathbb{R})$, defined by*

$$\mathfrak{sl}_n(\mathbb{R}) = \{A \in \mathfrak{gl}_n(\mathbb{R}) : tr(A) = 0\}$$

The Lie bracket over $\mathfrak{sl}_n(\mathbb{R})$ is just the restriction of the Lie bracket over $\mathfrak{gl}_n(\mathbb{R})$. Indeed, for any $A, B \in slnr$ we have

$$tr[A,B] = tr(AB - BA) = tr(AB) - tr(BA) = 0$$

Hence the restriction of the Lie bracket over $\mathfrak{gl}_n(\mathbb{R})$ is a Lie bracket over $\mathfrak{sl}_n(\mathbb{R})$. The 3 axioms for the Lie bracket is automatically satisfied.

Definition 2.1.3. *An ideal $I \subset \mathfrak{g}$ is a subspace of $\mathfrak{g}$ such that $[I, \mathfrak{g}] \subset \mathfrak{g}$.*

Definition 2.1.4. *A Lie algebra $\mathfrak{g}$ is called **simple** if it has no nontrivial ideals.*

From the computation in 2.1.2, it is immediate that $\mathfrak{sl}_n(\mathbb{R})$ is an ideal of $\mathfrak{gl}_n(\mathbb{R})$. This implies that $\mathfrak{gl}_n(\mathbb{R})$ is not a simple Lie algebra. However, the Lie algebra $\mathfrak{sl}_n(\mathbb{R})$ is simple.

2.1.2 Cartan Subalgebras

First we need the notion of Cartan subalgebras

Definition 2.1.5. *For any Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be a Cartan algebra if it is*

- *$\mathfrak{h}$ is a nilpotent subalgebra.*
- *It is self normalizing. In particular, we have $\mathfrak{h} = \{x \in \mathfrak{g} : [x, \mathfrak{g}] \subset \mathfrak{g}\}$.*

Example 2.1.6. *For the simple Lie algebra $\mathfrak{sl}_n(\mathbb{R})$, one choice of Cartan subalgebra $\mathfrak{h}$ is the set*

$$\mathfrak{h} = \left\{ H = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}, a_1 + a_2 + \ldots + a_n = 0 \right\}$$

Proof. Indeed, it is obvious that $[\mathfrak{h}, \mathfrak{h}] = 0$, thus the Lie algebra $\mathfrak{h}$ is nilpotent. To show the self-normalizing property, we pick an any element $\sum_{i,j} a_{ij} E_{ij}$ such that

$$\left[\sum_{i,j} a_{ij} E_{ij}, \mathfrak{h}\right] \subset \mathfrak{h}$$

Here E_{ij} denotes the matrix that has 1 at the ij entry and 0 otherwise. Clearly the matrix $E_{pp} - E_{qq} \in \mathfrak{h}$ for $p \neq q$. We then have

$$\left[\sum_{i,j} a_{ij} E_{ij}, E_{pp} - E_{qq}\right] \in \mathfrak{h}$$

or equivalently

$$\sum_i a_{ip}E_{ip} - \sum_i a_{iq}E_{iq} - \sum_j a_{pj}E_{pj} - \sum_j a_{qj}E_{qj} \in \mathfrak{h}$$

The coefficients of E_{pq} in the above sum is $-2a_{pq}$. By our choice of $\mathfrak{h}$ it must be the case that $a_{pq} = 0$. Thus, only the coefficients of E_{pq} for $p = q$ survive. Hence $\sum_{i,j} a_{ij}E_{ij} \in \mathfrak{h}$ and $\mathfrak{h}$ is then a Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{R})$. □

2.1.3 Root space decomposition

With respect to a choice of Cartan subalgebra $\mathfrak{h}$, we have a root space decomposition. In particular, there is a finite set $\Phi \subset \mathfrak{h}^*$ of linear forms on $\mathfrak{h}$, whose elements are called **roots**, such that

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right),$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$ for any $\alpha \in \Phi$.

2.1.4 A specific example: root space decomposition for $\mathfrak{sl}_n(\mathbb{R})$

For the simple Lie algebra $\mathfrak{sl}_n(\mathbb{R})$, one choice of Cartan subalgebra $\mathfrak{h}$ is the set

$$\mathfrak{h} = \left\{ H = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}, a_1 + a_2 + \dots + a_n = 0 \right\}$$

With respect to this Cartan subalgebra, we can define the linear function

$$L_i \colon \mathfrak{h} \to \mathbb{R}, \quad H \mapsto a_i$$

Then the roots are given by $\alpha_{ij} := L_i - L_j$ for distinct i, j. We have the root space decomposition for $\mathfrak{sl}_n(\mathbb{R})$ as follows

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus \mathfrak{g}_{\alpha_{ij}} \right) = \mathfrak{h} \oplus \left(\bigoplus \mathbb{R}E_{ij} \right)$$

For the sake of brevity, we will denote $\alpha_{i,i+1}$ by α_i - these are called **simple roots**.

2.1.5 Killing form

to be added.

2.1.6 Co-roots

In previous section, we showed that the Killing form gives rise to an inner product over the dual space $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{R})$. For each root $\alpha \in \mathfrak{h}^*$, we define the corresponding co-root $\alpha^\vee$ to be the linear functional over $\mathfrak{h}^*$ such that

$$\langle \beta, \alpha^\vee \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

On the other hand, since $\alpha^\vee$ is the dual space of $\mathfrak{h}^*$, it can be regarded as an element h_α of $\mathfrak{h}$. So we have

$$\beta(h_\alpha) = \langle \beta, \alpha^\vee \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

Example 2.1.7. *We compute explicitly the fundamental coroot $\alpha_i\vee$ of the fundamental root α_i for the Lie algebra $\mathfrak{sl}_{\mathrm{n}}(\mathbb{R})$. In previous section, we defined the fundamental roots to be $\alpha_{ij} = L_i - L_j$. It can be verified directly that $\alpha_{ij}^\vee = E_i - E_j$ as*

$$\left\langle \alpha_{ij}, \alpha_{ij}^\vee \right\rangle = (L_i - L_J)(E_i - E_j) = 2$$

Similarly, we can also define the λ_i such that

$$\left\langle \lambda_i, \alpha_j^\vee \right\rangle = \delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

Explicitly, it can be checked that $\lambda_i = L_1 + L_2 + \ldots + L_i$.

2.1.7 Roots at group level

We want to understand how the roots behave at group level. The analog for the Cartan subalgebra is the maximal torus

$$T = \left\{ t = \begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} : \prod_{i=1}^n a_i = 1 \right\},$$

Then T acts on $\mathfrak{g}$ by conjugation. Explicitly, we can check that

$$\mathrm{Ad}(t)(E_{ij}) = t_i t_j^{-1} E_{ij}$$

Therefore, at the group level, the character $\alpha_{ij}(\mathrm{diag}(t_1, \ldots, t_n)) = t_i t_j^{-1}$ is a root whenever $i \neq j$. We define the set of **simple roots** as

$$\Delta = \{\alpha_i \mid i = 1, \ldots, n\}$$

where

$$\alpha_i \colon T \mapsto \mathbb{R}, t \mapsto \frac{t_i}{t_{i+1}}.$$

We can decompose the set of roots $\Phi = \{\alpha_{ij}, i \neq j\}$ into disjoint subsets, namely

$$\Phi = \Phi_+ \coprod \Phi_-$$

where the positive set of roots Φ_+ comprises α_{ij} for $i < j$ and the remaining roots are in the negative set of roots Φ_-. We have the following lemma

Lemma 2.1.8. *Each $\alpha \in \Phi$ can be written uniquely as a linear combination*

$$\alpha = m_1\alpha_1 + \ldots + m_d\alpha_d$$

with all $m_i \in \mathbb{Z}_{\geqslant 0}$ or $m_i \in \mathbb{Z}_{\leqslant 0}$. If $\alpha \in \Phi_+$ then all $m_i \geqslant 0$, otherwise $m_i \leqslant 0$ for all i.

2.1.8 Weyl group

We only define the Weyl group explicitly for the group $\mathrm{SL}_{\mathrm{n}}(\mathbb{R})$ or $\mathrm{GL}_{\mathrm{n}}(\mathbb{R})$. It is a fact that the Weyl groups for these two Lie groups are the same and equal to $W = S_n$ - the permutation group of n letters. We recall some basis observation about this group

1. Every $\sigma \in W$ can be written (non-uniquely) as a product of $s_{i_1} \cdots s_{i_k}$ for some integer k. Such a sequence is said to have length k. If k is the minimum, over all such writings, it is called the length of σ and written $\ell(\sigma)$. Any expression of length $\ell(\sigma)$ for σ is called a reduced expression.

2. The group S_n is generated by S subject to the following two types of relations:
   - (Reflection) $s_i^2 = 1$ for $i \in I$.
   - (Braid relations) $s_i\, s_{i+1}\, s_i = s_{i+1}s_is_{i+1}$ for $i = 1, \dots, n-2$ and $s_is_j = s_js_i$ for $|j-i| \geqslant 2$.

Note that W acts on $\mathrm{Hom}(T, \mathbb{R}^*)$ in the natural way: $w.\varphi(h) = \varphi(w^{-1}h)$. More explicitly, σ sends $\alpha := \alpha_{ij}$ to $\alpha_{\sigma(i),\sigma(j)}$. Hence we find that

$$w_i\alpha_j = \begin{cases} -\alpha_i & \text{if } i = j \\ \alpha_j & \text{if } |j-i| > 1 \\ \alpha_i + \alpha_j & \text{if } |j-i| = 1 \end{cases}$$

define in terms of Lie algebra

2.1.9 Weights

Another class of linear forms that we are interested in are the **fundamental weights**. For each $i \in I$, we define a character $\lambda_i \in \mathrm{Hom}(T, \mathbb{R}^*)$ where T is the torus defined in 2.1.7

$$\lambda_i \colon T \to \mathbb{R}^*, \quad \lambda_i(t) = a_1 \dots a_i$$

We have the following

Lemma 2.1.9. *We can write*

$$\lambda_i := r_1\alpha_1 + r_2\alpha_2 + \ldots + r_d\alpha_d$$

where r_i's are rational number such that $r_i \geqslant 0$.

This coefficients r_i's is determined by inverting the Cartan matrix of $\mathfrak{sl}_n(\mathbb{R})$. Hence we postpone a proof of this until reviewing the notion of Cartan matrices.

Example 2.1.10. *When $n = 3$, we have the following relations*

$$\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \quad \lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$$

We refer to the figure 2.1 for an illustration.

Definition 2.1.11. *A weight λ is called **dominant** if it satisfies $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geqslant 0}$ for all α.*

Clearly by lemma 2.1.8, the weight λ is dominant if and only if $\langle \lambda, \alpha_i^\vee \rangle$ for all fundamental root α_i. It is also clear that the set of dominant weights is given by

$$\Lambda^+ := \{c_1\lambda_1 + \ldots + c_d\lambda_d \mid c_i \in \mathbb{Z}_{\geqslant 0}\}$$

The set of dominant weights is denoted Λ^+. A weight $\lambda = \sum n_i\lambda_i$ is called strongly dominant if $n_i > 0$ for all i. One important example is the minimal strongly dominant weight given by

$$\rho = \sum \lambda_i$$

This is called **Weyl vector** and is characterized in several ways:

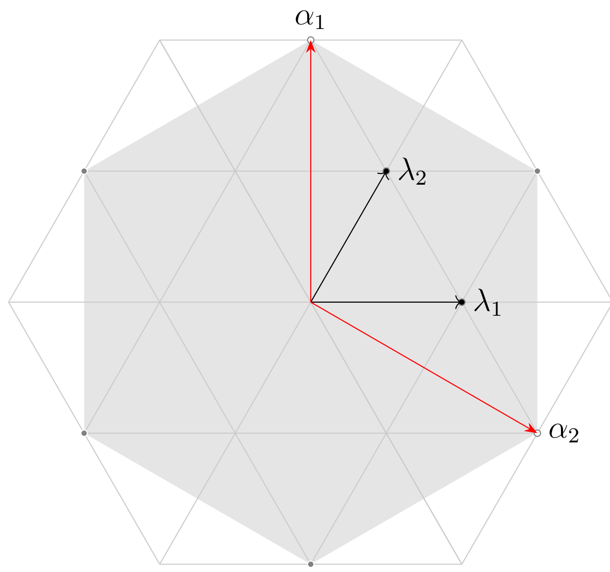


Figure 2.1: Roots and weights for the Lie group $\mathrm{SL}_3(\mathbb{R})$

1. $\langle \rho, \alpha_i^\vee \rangle = 1$ for all i.
2. $$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

To prove the last equation we use the action of the Weyl group W. Let $\mu = \frac{1}{2} \sum \alpha$. Apply the simple reflection s_i given by

$$s_i(x) = x - \langle x, \alpha_i^\vee \rangle \alpha_i$$

We know that s_i sends α_i to $-\alpha_i$ and permutes the other positive roots. So:

$$s_i(\mu) = \mu - \langle x, \alpha_i^\vee \rangle \alpha_i$$

Therefore, $(\mu, \alpha_i) = \mu(h_i) = 1$ for all i. So, $\mu = \rho$.
We of course also have an action of W on the weights. For example, one can verify that

$$s_i(\lambda_i) = \lambda_i - \alpha_i \quad \text{and} \quad s_j(\lambda_i) = \lambda_i \text{ for } i \neq j \ .$$

We have the following generalized action of Weyl group of ρ:

$$w\rho = \rho - \sum_{\alpha \in \Delta_{w^{-1}}} \alpha,$$

where check this explicitly

$$\Delta_\sigma := \{\alpha \in \Phi_+ \mid \sigma(\alpha) \in \Phi_-\}$$

Unlike Lemma 2.1.9, if we try to express the fundamental weights in terms of the fundamental roots, we do not always get positive coefficients. However, it is true that all the coefficients must be integer. In particular, we have

$$\alpha_j = \sum_{n_j} n_j \lambda_j, \quad n_j \in \mathbb{Z}.$$

To put it another way, the root lattice $\mathbb{Z}\Delta$ is contained inside the weight lattice $\mathbb{Z}\Pi$.

2.1.10 Cartan matrix

We fix a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_d\}$ is defined to be the matrix

$$A = \left[\left\langle \alpha_i, \alpha_j^\vee \right\rangle\right]$$

If we let $a_{ij} = \left\langle \alpha_i, \alpha_j^\vee \right\rangle$ then the Cartan matrix has the following simple properties:

Lemma 2.1.12.

- *For any i, we have $a_{ii} = 2$.*
- *For any $i \neq j$, a_{ij} is a non-positive integer, i.e. $a_{ij} \in \mathbb{Z}_{\leqslant 0}$.*

We give an explicit example for $\mathfrak{sl}_n(\mathbb{R})$, which has the root system A_n. The corresponding Cartan matrix is

$$A_n = \begin{bmatrix} 2 & -1 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & \ldots & 0 & -1 & 2 \end{bmatrix}$$

The Cartan matrix provides a linear combination presentation of the fundamental roots α_j's in terms of fundamental weights λ_i, as stated in the following lemma

Lemma 2.1.13. *Fix a number n and consider the set of simple roots α_j as well as the set of weights λ_i of $SL_n(\mathbb{R})$, we have the following relations*

$$\alpha_i = \begin{cases} 2\lambda_1 - \lambda_2, & \text{if } i = 1 \\ -\lambda_{i-1} + 2\lambda_i - \lambda_{i+1}, & \text{if } 1 < i < n \\ - - \lambda_{n-1} + 2\lambda_n, & \text{if } i = n \end{cases}.$$

Proof. Note that we have

$$\left\langle \alpha_i, \alpha_j^\vee \right\rangle = \begin{cases} 2, \text{ if } i = j \\ -1, \text{ if } |i - j| = 1 \\ 0, \text{ otherwise} \end{cases}.$$

So if we let $\alpha_j = \sum_{i=1}^n a_{ij}\lambda_i$ and compute $\left\langle \alpha_i, \alpha_j^\vee \right\rangle$, the result follows immediately. □

We will also need the inverse of the Cartan matrices of type A_n. The following formulae for the inverse matrices of $\mathfrak{sl}_n(\mathbb{R})$ can be found in [?]

Theorem 2.1.14. *The inverse of the Cartan matrices*

$$A_n = \begin{bmatrix} 2 & -1 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & \ldots & 0 & -1 & 2 \end{bmatrix},$$

is the matrix $(A_n)^{-1}$ with the entries given by the following formula

$$(A_n)^{-1}_{ij} = \min\{i, j\} - \frac{ij}{n+1}$$

As a consequence, we can see that the entries for the inverse matrix $(A_n)^{-1}$ is positive. Indeed, assume that $i \geqslant j$, we have

$$(A_n)^{-1}_{ij} = \frac{j(n+1-i)}{n+1} > 0$$

In particular, we just proved lemma 2.1.9.

2.2 Parabolic subgroups

We shall provide two equivalent viewpoints on parabolic subgroups. They will play different roles in defining different notions of semi-stability in the next chapter.

2.2.1 Parabolic subgroups I: An explicit description

For our purposes, it is enough to define the standard parabolic subgroups. There exists a bijection between each parabolic subgroup of $\mathrm{SL_n}(\mathbb{R})$ and each partition of n. We can therefore define the parabolic subgroup explicitly as follows:

Definition 2.2.1. *The standard parabolic subgroup associated to the partition $n = n_1+n_2+\cdots+n_r$ is denoted $P_{n_1,\ldots,n_r}$ and is defined to be the group of all matrices of the form*

$$\begin{bmatrix} M_{n_1} & 0 & \ldots & 0 \\ 0 & M_{n_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M_{n_k} \end{bmatrix},$$

*where $M_{n_i} \in \mathrm{GL}(n_i, \mathbb{R})$ for $1 \leqslant i \leqslant r$. The integer $r-1$ is called the rank of the parabolic subgroup $P_{n_1,\ldots,n_r}$. We denote the collection of such subgroups **ParSt**.*

In this form, we can choose the standard basis $\{e_i\}$ of $\mathbb{R}^n$. Let

$$d_i = n_1 + n_2 + \ldots + n_i$$

Then it is clearly that any element in $P_{n_1,\ldots,n_r}$ stabilizes the chain of vector spaces

$$0 \subset \mathbb{R}^{d_1} \subset \mathbb{R}^{d_2} \subset \ldots \subset \mathbb{R}^{d_k} = \mathbb{R}^n$$

Here R^k is generated by the linearly independent subset $\{e_1, \ldots, e_k\}$ of the standard basis. We call such chain of vector spaces a **flag** of type $(n_1, \ldots, n_k)$.

Definition 2.2.2. *The maximal standard parabolic subgroups in $GL_n(k)$ corresponds to the stabilizer of the flag of type $\rho_i = (i, n-i)$, where $i = 1, \ldots, n-1$ of n. We will further denote $Q_i = P_{\rho_i}$ and **MaxParSt** the collection of such maximal parabolic subgroups.*

Example 2.2.3. *Below we list all the standard parabolic subgroup in $GL_3(\mathbb{R})$. For $GL_3(\mathbb{R})$, there are three standard parabolic subgroups corresponding to three partitions of 3, namely*

$$3 = 1+1+1, \quad 3 = 1+2, \quad 3 = 2+1$$

For a partition $(r_1, \ldots, r_{s+1})$, we denote $P_{(r_1,\ldots,r_{s+1})}$ the corresponding parabolic subgroups. Thus, we have

$$P_{1,1,1} = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\}, \quad P_{1,2} = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\}$$

$$P_{2,1} = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$$

*Clearly **MaxParSt** $= \{P_{2,1}, P_{1,2}\}$.*

2.2.2 Parabolic subgroups II: Using BN-pairs

We first introduce the BN-pairs

Definition 2.2.4 (BN-pairs)**.** *A **BN-pairs** is a 4-tuple (G, B, N, R) where G is a group generated by subgroups B and N. The subgroup $H = B \cap N$, R is a finite set of involutions which generate the Weyl group $W = N/H$. Moreover, the following theorem holds*

- *If $r \in R$ and $w \in W$, then $rBw \subset BwB \cup BrwB$.*
- *If $r \in R, rBr \neq B$.*

For our purpose, it is enough to concentrate on the following example

Example 2.2.5. *Let $G = GL_{\mathrm{n}}(\mathbb{R})$, then the sets B, N, R are given explicitly as follows*

- *$B =$ upper triangular matrices*
- *$N =$ monomial matrices, namely, matrices that have exactly one non-zero entry in each row and colur*
- *From the above, it is clear that $H = B \cap N$ is the diagonal group, and this group is normal in N.*
- *It can be shown that $W = N/H \cong S_n$, and we can choose $R = \{(i, i-1)\}$ - the set of transpositions.*

Let $J \subset R$, we define W_J to be the subgroup of W generated by the involutions $r \in J$. We call it **standard parabolic subgroup** of W. Set $P_J = BW_JB$ as in the notation of BN-pairs. We have the following theorem

Theorem 2.2.6.

- *P_J is a subgroup if G. In particular, we have*
$$G = BWB,$$
*which is called **Bruhat decomposition** of G.*
- *If $P_I = P_J$ then we have $I = J$.*
- *All subgroups of G containing B arises in this way.* Add proofs?

The above theorem leads to the following definition of parabolic subgroups

Definition 2.2.7 (Parabolic subgroups)**.** *Using the same notation in the previous theorem, we call the subgroups P_J with $J \subset R$ standard parabolic subgroups of group G.*

We would like to explicitly describe the parabolic subgroup for $\mathrm{SL}_{\mathrm{n}}(\mathbb{R})$.

Example 2.2.8. *As introduced in previous section, the set $\Phi = \{\alpha_{ij}\}$ forms a root system for $SL_{\mathrm{n}}(\mathbb{R})$. The Borel subgroup is just $B \cap SL_{\mathrm{n}}(\mathbb{R})$, the group of upper triangular matrices with determinant 1. We also consider the set N of monomial matrices with determinant 1. Then it is easy to check that N/H is the set of all permutation matrices. That is, N/H is generated by the matrices of the form* see Cambridge

$$r_i := \begin{bmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{n-i-1} \end{bmatrix}$$

Via the identification $s_i \mapsto (i, i+1)$*, we identify the Weyl group* $W = N/H \cong S_n$*. So* $R = \{(i, i+1) | i = 1, 2, \ldots, n\}$*. We consider the set* $I = R \backslash \{r_i\}$ *for some* $i < n$*. The associated parabolic subgroup of* W *is*

$$W_I = \langle r_1, r_2, \ldots r_{i-1}, r_{i+1}, \ldots, r_{n-1} \rangle \cong S_i \times S_{n-i}$$

The corresponding parabolic subgroup is

$$P_{r_i} := P_I = \left\{ \begin{bmatrix} A & * \\ & B \end{bmatrix} \in SL_{\mathrm{n}}(\mathbb{R}) : A \in GL_i, B \in GL_{n-i} \right\}$$

2.2.3 Langlands decomposition

We fix a partition of n as

$$n = n_1 + n_2 + \ldots + n_k$$

and consider the parabolic subgroup of this type, i.e. the subgroup

$$P_{n_1,\ldots,n_k} = \left\{ \begin{bmatrix} \mathfrak{m}_1 & * & \ldots & * \\ 0 & \mathfrak{m}_2 & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathfrak{m}_k \end{bmatrix} \right\}$$

where $\mathfrak{m}_{\mathrm{i}}$ is invertible of size $n_i \times n_i$.
This group can be factored as

$$P_{n_1,\ldots,n_k} = M_{n_1,\ldots,n_k} \ltimes N_{n_1,\ldots,n_k}$$

where

$$N_{n_1,\ldots,n_k} = \left\{ \begin{bmatrix} I_1 & * & \ldots & * \\ 0 & I_2 & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_k \end{bmatrix} \right\} \quad (I_k \text{ is the } n_k \times n_k \text{ identity matrix})$$

and

$$M_{n_1,\ldots,n_k} = \left\{ \begin{bmatrix} M_1 & 0 & \ldots & 0 \\ 0 & M_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M_k \end{bmatrix} \right\}$$

The subgroup $M_{n_1,\ldots,n_k}$ is called **Levi component**. We can further factor this subgroup as

$$M_{n_1,\ldots,n_k} = M'_{n_1,\ldots,n_k} \cdot A_{n_1,\ldots,n_k}$$

with $A_{n_1,\ldots,n_k}$ plays the role of the connected center of $M_{n_1,\ldots,n_k}$:

$$A_{n_1,\ldots,n_k} = \left\{ \begin{bmatrix} t_1 I_1 & 0 & \ldots & 0 \\ 0 & t_2 I_k & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & t_k I_k \end{bmatrix} : t_i > 0 \right\}$$

and

$$M'_{n_1,\ldots,n_k} = \left\{ \begin{bmatrix} \mathfrak{m}'_1 & 0 & \ldots & 0 \\ 0 & \mathfrak{m}'_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathfrak{m}'_k \end{bmatrix} \right\},$$

where $\det(\mathfrak{m}'_i) = \pm 1$.

Definition 2.2.9. *For a given parabolic subgroup P, the factorization*

$$P = M'_P \times A_P \times N_P$$

*as above is called **Langlands decomposition**.*

2.2.4 Iwasawa decomposition and P-horospherical decomposition

We introduce two important ways to decompose a matrix in $\mathrm{GL_n}(\mathbb{R})$ into simple parts. This will be frequently used in the remaining part of this thesis.

Proposition 2.2.10. *Let*

$$K = \mathrm{O_n}(\mathbb{R}), \quad A = \left\{ \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix} : a_i > 0 \right\}, \quad U = \left\{ \begin{bmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \right\}$$

Then the natural product map

$$p\colon K \times A \times U \to GL_n(\mathbb{R}), \quad p(k, a, n) = kan$$

is an isomorphism.

Thus, the following decomposition makes sense

Definition 2.2.11. *The decomposition of $g = kan$ for $k \in K, a \in A$ and $n \in N$ is called **Iwasawa** decomposition.*

Using the Langlands decomposition as well as the Iwasawa decomposition $G = ANK = BK$, we observation the decomposition

$$G = KP = KM_PA_PN_P \cong KM_P \times A_P \times N_P$$

As a consequence, we get

$$X_G = K\backslash G = X_{M_P} \times A_P \times N_P$$

for $X_{M_P} := (K \cap M_P)\backslash M_P$.
This is P-**horospherical** decomposition of the lattice space X_G. In this way, we can identify $x \in X_G$ with the triple $(m_P(x), a_P(x), n_P(x))$.

Example 2.2.12. *When $n = 2$, the only nontrivial standard parabolic subgroup is the Borel subgroup $P = B$, so the P-horospherical decomposition is*

$$X_G = K\backslash G = A_B \times N_B$$

which is in fact just the Iwasawa decomposition.

Furthermore, if $Q \supset P$ is also a parabolic subgroup, it gives rise to a parabolic subgroup $*P := P \cap M_Q$ of M_Q. The space X_{M_Q} itself has the decomposition

$$X_{M_Q} = X_{*P} \times A(Q)_{*P} \times N(Q)_{*P}$$

where

$$X_{*P} = X_{M_P}, \quad A(Q)_{*P}A_Q = A_P, \quad N(Q)_{*P}N_Q = N_P.$$

This is called **relative** P-horospherical decomposition with respect to Q.

Example 2.2.13.

2.2.5 Parabolic sets and parabolic subalgebras

Definition 2.2.14. *Given a root system Δ. A* ***parabolic subset*** *Δ_P is a subset of Δ such that it satisfies the following conditions:*

1. *For any $\alpha \in \Delta$, at most one of the two elements $\alpha, -\alpha$ is contained in Δ_P.*
2. *It is closed, in the sense that, for any two root $\alpha, \beta \in \Delta_P$ such that $\alpha + \beta$ is a root, then $\alpha + \beta \in \Delta_P$.*

The parabolic set parametrizes the parabolic subalgebra with the root system Δ, as given in the following theorem

Theorem 2.2.15. *Given a semisimple Lie algebra $\mathfrak{g}$ with the root system Δ. There exists a correspondence between parabolic subset of Δ_P of Δ and subalgebras of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{b}$. The correspondence is given by*

$$\Delta_P \longleftrightarrow \mathfrak{p} := \mathfrak{b} \oplus \bigoplus_{\alpha \in \Delta \backslash \Delta_P} g_\alpha$$

Proof. We refer to [?] for a proof of this fact. □

Example 2.2.16. *We consider the case $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$. Let's denote $\Pi = \{\alpha, \beta\}$ a base for the root system of $\mathfrak{g}$. It is clear that the set of positive root is $\Delta_+ = \{\alpha, \beta, \gamma\}$. There are 4 parabolic sets, corresponding to 4 parabolic subalgebra given as follows*

$$\begin{aligned} \Delta_P = \Delta_+ &\longleftrightarrow \mathfrak{p} = \mathfrak{b} \\ \Delta_P = \Delta \cup \{-\alpha\} &\longleftrightarrow \mathfrak{p} = \mathfrak{b} \oplus \mathfrak{g}_{-\alpha} \\ \Delta_P = \Delta \cup \{-\beta\} &\longleftrightarrow \mathfrak{p} = \mathfrak{b} \oplus \mathfrak{g}_{-\beta} \\ \Delta_P = \Delta &\longleftrightarrow \mathfrak{p} = \mathfrak{g} \end{aligned}$$

2.2.6 On the function H_P

Recall that for the Lie group $\mathrm{SL}_n(\mathbb{R})$, we attach to it a root system $\Phi = \{\alpha_{i,j}\}$ with

$$\Delta = \{\alpha_i \mid i \in I\}, \quad I = \{1, 2, \ldots, n-1\}$$

as the set of fundamental roots. It is a fact that the Weyl group W satisfies

$$W = \langle r_{\alpha_i} : i \in I \rangle$$

For the sake of brevity, we denote $r_{\alpha_i} := r_i$. Then for each subset $J \subset I$, we define the following subset of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_n(\mathbb{R})$

$$\begin{aligned} \mathfrak{h}(J) &:= \mathrm{span}\{\alpha_i^\vee : i \in J\} \\ \mathfrak{h}_J &:= \{H \in \mathfrak{h} : \langle \alpha_i, H \rangle = 0 \text{ for } i \in J\} \end{aligned}$$

By the definition of weights, i.e. $\left\langle \lambda_i, \alpha_j^\vee \right\rangle = \delta_{ij}$, it is immediate that the subspace $\mathfrak{h}_J$ of $\mathfrak{h}$ contains the subspace

$$\mathrm{span}\{\lambda_i^\vee : i \notin J\}$$

It is also clear that the dimension of $\mathfrak{h}(J)$ is $|J|$ as the set $\{\alpha_i^\vee : i \in J\}$ is a linearly independent set. Assume that

$$\sum_{i \notin J} c_i \lambda_i^\vee = 0$$

Pairing both sides with α_i, we deduce that $c_i = 0$ for all $i \notin J$. In particular, the set $\{\lambda_i^\vee : i \notin J\}$ contains linearly independent vectors and thus spans a subspace of $\mathfrak{h}_J$ of dimension $|I| - |J|$. Therefore

$$\mathfrak{h}_J = \operatorname{span}\{\lambda_i^\vee : i \notin J\}$$

To put it another way, we have a direct sum decomposition

$$\mathfrak{h} = \mathfrak{h}(J) \oplus \mathfrak{h}_J$$

We also write $\lambda_{j,J}^\vee$ for the basis of $\mathfrak{h}(J)$ containing the fundamental weight, namely $\left\langle \alpha_k, \lambda_{j,J}^\vee \right\rangle = \delta_{kj}$ for $j, k \in J$. We have the following easy lemma

Lemma 2.2.17. *Let $H \in \mathfrak{g}$ be arbitrary. For $J \subsetneq I$ we have the orthogonal decomposition*

$$H = H(J) + H_J,$$

where $H(J) \in \mathfrak{h}(J)$ and $H_J \in \mathfrak{h}_J$. If $H = \sum_{i=1}^n p_i \lambda_i^\vee$ then

$$H(J) = \sum_{i \in J} p_i \lambda_{i,J}^\vee$$

Proof. Let $H(J) = \sum_{i\in J} c_i \lambda_{i,J}^\vee$, then for $i \in J$

$$p_j = \langle \alpha_i, H \rangle = \langle \alpha_i, H_J + H(J) \rangle = \sum_{i\in J} \left\langle \alpha_i, c_i \lambda_{i,J}^\vee \right\rangle = c_i$$

Hence $H(J) = \sum_{i\in J} p_i \lambda_{i,J}^\vee$. □

We know from previous section that for each standard parabolic subgroup of G, it must be of the form P_J for some subset $J \subset \{1, \ldots, n-1\}$. Therefore, we can define $\mathfrak{h}(P) := \mathfrak{h}(J)$ and $\mathfrak{h}_P := \mathfrak{h}_J$. we now define the function H_P as follows:

Definition 2.2.18. *Assume that for $x \in X$, we have a P-horospherical decomposition*

$$x = (m_P(x), a_P(x), n_P(x))$$

Then we define

$$\begin{aligned} H_P \colon X &\to \mathfrak{h}_P \\ x &\mapsto H_P(x) = \log(a_P(x)), \end{aligned}$$

where log *means we take the logarithm of the diagonal entries of $a_P(x)$.*

This definition is well-defined since $\log(a_P(x))$ is an element of $\mathfrak{h}_P$ by definition. We prove the following lemma

Lemma 2.2.19. *Let B the Borel standard subgroup of G, i.e. the minimal standard parabolic subgroup and P is any parabolic subgroup of G. Then we have*

$$H_B(x) = H_P(x) + H_{*B}(pr_{M_P}(x))$$

where $pr_{M_P}(x)$ stands for the natural projection of $x \in X$ on M_P

Proof. This follows immediately from the observation that

$$A_B = A_P A(P)_{*B}$$

and the definition 2.2.18 of the function H_P for parabolic subgroup P. □

Chapter 3

SEMI-STABLE LATTICE IN HIGHER RANK

In this chapter, we will establish the notion of semi-stable lattice. Heuristically, this is the lattice that achieve all the successive minima at the same time, see [**?**].
We will provide two different definitions of semi -stable lattice: one is geometric - which follows Grayson's idea of utilizing the canonical plot, and one is purely algebraic, which make use of the maximal standard parabolic subgroups. The toy model will be the moduli space of 2-dimensional lattice, which is essential the upper half plane in the complex field. At the end, we will show that the two definitions coincide.

3.1 Lattices in higher rank

For each z with $\Im(z) > 0$, we can attach to z a lattice structure $L_z = \mathbb{Z}z \oplus \mathbb{Z}$. Roughly speaking a lattice is a discrete subgroup that is generated by a $k-$ basis of the k-space V. In particular, we will only work with the real vector space V. Grayson works with lattice over a ring of algebraic integers, but we will restrict to just the lattice that has the underlying structure as a $\mathbb{Z}-$ module.

3.1.1 First definition of lattices

Definition 3.1.1 (Abstract $\mathbb{Z}$-lattices)**.** *Let L be a finitely generated $\mathbb{Z}$-module. In particular, it is a free $\mathbb{Z}$-module of finite rank. Suppose that L is endowed with a real-valued positive definite*[1] *quadratic form $Q\colon L \to \mathbb{R}$, such that the set*

$$\{x \in L : Q(x) \leqslant r\}$$

*is finite for any real number r. We will call the pair (L, Q) a **abstract $\mathbb{Z}$-lattice**.*

An easy example is to take $L = \mathbb{Z}^n$ and choose our quadratic form to be the standard one. namely

$$\langle x, y\rangle = x \cdot y = \sum_{i=1}^{n} x_i y_i$$

Here the multiplication is just the usual dot product between 2 vectors. In term of matrix, this quadratic form is assigned to the identity matrix I_n.
If there is no further confusion, we can just denote a Euclidean lattice by L, without specifying the bilinear form Q. The lattice L determines a full-rank lattice inside $L_{\mathbb{R}}$, namely, the rank of the lattice L is equal to the dimension of $L_{\mathbb{R}}$.

[1] The non-degenerate implicity state that rank L is the same as $dimL_{\mathbb{R}}$

3.1.2 An alternative definition of lattices

For the sake of computation, we also usually adopt another definition of the lattice. In particular, we view lattice as a free $\mathbb{Z}-$ module of rank n that is isomorphic to $\mathbb{R}^n$ via base changing.

Definition 3.1.2. *A lattice in $\mathbb{R}^n$ is a subset $L \subset \mathbb{R}^n$ such that there exists a basis $b_1, \ldots, b_n$ of $\mathbb{R}^n$ such that*

$$L = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \ldots \mathbb{Z}b_n$$

If we put the vector $b_1, b_2, \ldots, b_n$ in columns, with respect to the standard basis, namely

$$g = [b_1|b_2|\ldots|b_n],$$

then $L = g\mathbb{Z}^n$.

In the second definition, we can just identify L with the standard lattice $\mathbb{Z}^n$ and the symmetric positive definite form is $g^t g$. So an Euclidean $\mathbb{Z}$-lattice is an abstract lattice with the standard positive definite quadratic form.

3.1.3 Equivalence between two definitions of lattices

In this subsection, we will show that every abstract $\mathbb{Z}$- lattice is isomorphic to an Euclidean $\mathbb{Z}-$ lattice. This will be helpful in visualizing the abstract lattices, as we are just looking at concrete lattices with deformation by a linear transformation.
First we need to specify the notion of isomorphic lattices - in the first definition

Definition 3.1.3. *A map $f\colon (L,Q) \to (L',Q')$ is an* ***isomorhism*** *between lattices if it is a group isomorphism and for all $x \in L$, we have*

$$Q(x) = Q'(f(x))$$

Proposition 3.1.4. *Any abstract lattice is isomorphic to a Euclidean $\mathbb{Z}-$ lattice.*

Proof. Let (L,Q) be an arbitrary lattice. We define a bilinear form as

$$\langle x, y\rangle := \frac{Q(x+y) - Q(x-y)}{4}$$

We will show that this bilinear form defines an inner product over the real vector space $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$. Clearly we have $\langle x, x\rangle = 4Q(x)/4 = Q(x) \geqslant 0$ for all $x \in L\backslash\{0\}$. Now the extended bilinear form is defined as

$$\begin{aligned} \langle \cdot, \cdot\rangle : L_{\mathbb{R}} \times L_{\mathbb{R}} &\to \mathbb{R} \\ (x \otimes a, y \otimes b) &\mapsto ab\langle x, y\rangle \end{aligned}$$

It is immediate that the extended bilinear form is inner product. So we have proved that $L_{\mathbb{R}}$ is a Euclidean space containing L. Moreover, L is embedded injectively in $L_{\mathbb{R}}$ as $\mathbb{R}$ is a flat $\mathbb{Z}$ module. The condition that

$$\#\{x \in L : Q(x) \leqslant r\} < \infty$$

implies L can be identified with a discrete in $L_{\mathbb{R}}$. But this implies that there exists a basis $\{b_1, \ldots, b_n\} \subset L_{\mathbb{R}}$ such that

$$L = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \ldots \mathbb{Z}b_n$$

Hence we are done. ∎

fix the proof so that we use

3.1.4 Covolume of a lattice

Now that for every abstract lattice L we can find an invertible matrix g such that

$$L \cong g\mathbb{Z}^n$$

The number n is called the **rank** of the lattice L.
Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of $L_{\mathbb{R}} \cong \mathbb{R}^n$ and

$$g = [b_1|b_2|\ldots|b_n].$$

The covolume of the lattice L is defined as

Definition 3.1.5. *The covolume of L is given by the formulae*

$$\mathrm{vol}(L) = |\det(b_i \cdot e_j)|$$

The rank and covolume are invariant numerical values of L, as they don't depend on the choice of basis. Indeed, two bases of a rank n lattice L are related by a transformation $g \in \mathrm{GL}_n(\mathbb{Z})$. Clearly this preserves the volume and the rank as a $\mathbb{Z}-$module.

3.1.5 Sublattices

To work with semi-stable lattice L, we need to consider all the sublattices contained inside L.

Definition 3.1.6 (sublattice)**.** *Let (L, Q) be a Euclidean $\mathbb{Z}$-lattice. We say that a $\mathbb{Z}-$submodule M of L a **sublattice** if and only if L/M is torsion free.*

From this definition, we can prove that M is a sublattice of L if it satisfies one of the following equivalent properties:

1. M is a summand of L.

2. every basis of M can be extended to a basis of L.

3. The group M is an intersection of L with a rational subspace of $L_{\mathbb{R}}$.

We refer to the [**?**] for a proof of these equivalences.

Example 3.1.7. *If $L = \mathbb{Z}^2$, then any sublattice of L is a primitive vector $u = (a, b)$, i.e* $\gcd(a, b) = 1$. *Indeed, $u = (a, b)$ is a sublattice of $\mathbb{Z}^2$ if and only if there exists a vector $v \in \mathbb{Z}^2$ such that $L = \mathbb{Z}u \oplus \mathbb{Z}v$. With respect to the usual inner product on $\mathbb{R}^2$, we have*

$$1 = \mathrm{vol}(\mathbb{Z}^2) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

This happens if and only if $\gcd(a, b) = 1$.

3.2 Grayson's definition of semistablility

3.2.1 Grayson's definition

In this section, we introduce the idea of Grayson in defining *semi-stable* lattices. In particular, he associates every lattices a plot and its convex hull - called *profiles*. An easy observation is that, if $M \subset L$ is a sublattice, then the space $M_{\mathbb{R}} = M \otimes \mathbb{R}$ is a subspace of $L_{\mathbb{R}}$, equipped with the restriction of the positive definite symmetric form Q of L, hence M is also a lattice of rank not exceeding rank of L.

Definition 3.2.1 (slope)**.** *The slope of a non-zero lattice L is the number*

$$\mu(L) = \frac{\log \mathrm{vol}(L)}{\dim L}$$

Definition 3.2.2. *Suppose we have a lattice L. For any sublattice $M \subset L$, we assign M to a point*

$$\ell(M) = (\dim M, \log \mathrm{vol}(M))$$

in the plane $\mathbb{R}^2$. The collection of all points $\ell(M)$ where M ranges over all sublattices of L is called ***the canonical plot*** *of the lattice L. By convention, we assign the lattice of zero rank to the origin of the plane.*

Example 3.2.3. *Add example about computing the volume of sublattices.*

The following lemma asserts that, for each vertical axis $x = i$, there is a lowest point.

Lemma 3.2.4. *Given a lattice L and a number c, there exists only a finite number of sublattices $M \subset L$ such that* $\mathrm{vol}(M) < c$.

Proof. We will prove by induction on the rank of the sublattices.

- For $r = 1$, the collection of all rank 1 sublattices of L is just the set of all vectors in L. So we reduce to show that for any $c > 0$, the set $B(0, c) \cap L$ has finitely many elements. But this follows immediately from the fact that L is a discrete subset of $L_{\mathbb{R}}$.
- Assume that the lemma holds for $r > 1$. Assume that

  $$M = \mathbb{Z}m_1 \oplus \dots \mathbb{Z}m_r$$

  is a sublattice of the lattices L of rank n. Consider the wedge product $\bigwedge^r L$, then clearly $m_1 \wedge m_2 \dots \wedge m_r$ is a vector in the lattice $\bigwedge^r L$. By the previous case, there are finitely many vectors with bounded length inside lattice. So we only need to show that the map

  $$M \mapsto \bigwedge^r M$$

  is finite to one, then we are done. But this is due to the observation of $\bigwedge^r M$ determines the lattice $L \cap (M \otimes \mathbb{Q})$ and the index of M inside this lattice.

So the canonical plot is bounded below. $\square$

Definition 3.2.5. *The boundary polygon of the convex hull of the canonical plot is called* ***profile*** *of the lattice L.*

In theory, we can compute the profile by searching for the shortest vector in each of its exterior product, but this computation is infeasible when the dimension of the lattice grows. Since there are lattices with arbitrarily large volume of any rank smaller than that of L, we add to the side the point $(0, \infty)$ and (n, ∞). The sides of the profile are therefore two vertical lines. The bottom is just the convex polygonal connecting the origin with the point $\ell(L) = (n, \log \mathrm{vol}(L))$, where n is the rank of L.

Definition 3.2.6. *If the bottom of the profile contains only two points $(0, 0)$ and $(n, \log \mathrm{vol}\, L)$, then the lattice L is said to be* ***semi-stable****. Otherwise L is said to be* ***unstable****.*

Below are the picture of two lattices. Visually, a lattice is called **semi-stable** if it satisfies the other equivalent conditions: If M is an arbitrary sublattice of L then $\mu(M) \geqslant \mu(L)$.

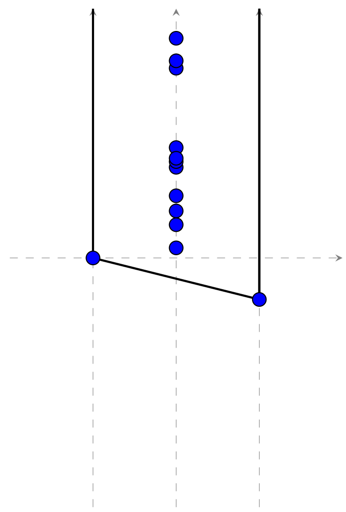


Figure 3.1: This is a semistable lattice

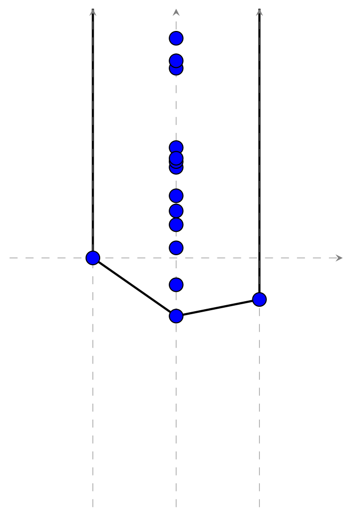


Figure 3.2: This is an unstable lattice

3.2.2 Canonical filtration

Given a lattice L and a sublattice $M \subset L$, the quotient group L/M have the structure of a lattice. Indeed, consider the exact sequence of lattices

$$0 \to M \to L \to L/M \to 0$$

By tensoring with $\mathbb{R}$ we get a short exact sequence of $\mathbb{R}$-vector subspaces

$$0 \to M_{\mathbb{R}} \to L_{\mathbb{R}} \to (L/M)_{\mathbb{R}} \to 0,$$

which is split. Thus we have the isomorphisms

$$(L/M)_{\mathbb{R}} \cong L_{\mathbb{R}}/M_{\mathbb{R}} \cong M_{\mathbb{R}}^{\perp}$$

Therefore, by restriction of the inner product over $L_{\mathbb{R}}$ to $M_{\mathbb{R}}^{\perp}$, we clearly see that L/M also inherits an inner product. In particular, it is a lattice.

Definition 3.2.7. *Given a lattice L containing a sublattice M, then L/M is a lattice. We call this lattice* ***quotient lattice****.*

Lemma 3.2.8. *If L is a lattice and $M \subset L$ is a sublattice, we have*

$$\mathrm{vol}(L) = \mathrm{vol}(M) \cdot \mathrm{vol}(L/M)$$

and if N is any sublattice of L that satisfies $N + M = L$ then

$$\mathrm{vol}(N) \geqslant \mathrm{vol}(L/M)$$

Proof. Assume that $\{m_i\}$ is a basis for the lattice M and $\{e_i\}$ be an orthonormal basis for the vector space $M_{\mathbb{R}}$. Since M is a sublattice of L, we can extend the basis $\{m_i\}$ to get a basis $\{m_i\} \cup \{n_j\}$ for the lattice L. Similarly, we can extend $\{e_i\}$ to get an orthonormal basis $\{e_i\} \cup \{f_j\}$ for the vector space $L_{\mathbb{R}}$. In particular, we would have $\langle m_i, f_j \rangle = 0$ for all i, j. By definition, we have

$$\begin{aligned} vol(L) &= \det \begin{bmatrix} \langle m_i, e_i \rangle & \langle n_j, e_i \rangle \\ \langle m_i, f_J \rangle & \langle n_j, f_j \rangle \end{bmatrix} \\ &= \det \begin{bmatrix} \langle m_i, e_i \rangle & \langle n_j, e_i \rangle \\ 0 & \langle n_j, f_j \rangle \end{bmatrix} \\ &= \mathrm{vol}(M) \cdot \mathrm{vol}(L/M) \end{aligned}$$

Hence we are done. The latter inequality follows from the fact that the volume decrease under the orthogonal projection and the observation that $N_{\mathbb{R}} \supset (L/M)_{\mathbb{R}}$. □

In the canonical plot, the import of lemma 3.2.8 is that the slope of the quotient lattice L/M appears as the slope of the line segment connecting the points corresponding to the sublattice M and the lattice L. This is due to the geometry fact that

$$(\mathrm{rank}(M), \log(\mathrm{vol}(M)) + (\mathrm{rank}(L/M), \log\mathrm{vol}(L/M))) = (\mathrm{rank}(L), \log\mathrm{vol}(L))$$

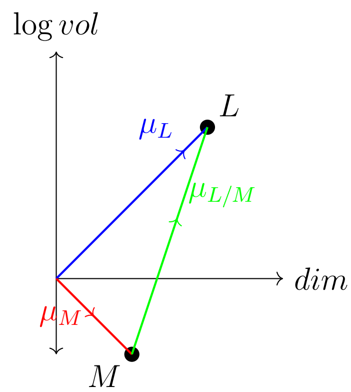


Figure 3.3: Geometric meaning of slope in canonical plot

Given two sublattices $M_1, M_2 \subset L$, if we apply lemma 3.2.8 to the sublattices $M_1/M_1 \cap M_2$ and $M_2/M_1 \cap M_2$ in $M_1 + M_2/M_1 \cap M_2$, we get

$$\mathrm{vol}(M_1/M_1 \cap M_2)\,\mathrm{vol}(M_2/M_1 \cap M_2) \geqslant \mathrm{vol}(M_1 + M_2/M_1 \cap M_2)$$

or equivalently

Lemma 3.2.9.

$$\mathrm{vol}(M_1 + M_2)\,\mathrm{vol}(M_1 \cap M_2) \leqslant \mathrm{vol}(M_1)\,\mathrm{vol}(M_2)$$

Grayson used the logarithm to express the above inequality in additive terms:

Proposition 3.2.10. *Suppose M_1, M_2 are sublattices of L then*

$$\log\mathrm{vol}(M_1) + \log(\mathrm{vol}(M_2)) \geqslant \log(\mathrm{vol}(M_1 + M_2)) + \log\mathrm{vol}(M_1 \cap M_2)$$

Proof. It follows immediately from the fact that log is an increasing function over $(0, \infty)$. □

Grayson called this parallelogram rule, as the geometric meaning of proposition 3.2.10 is illustrated in the figure 3.4. The **vertices** of the profile are the extremal/lowest points of the plot. Clearly the two points $(0, 0)$ and $(n, \log\mathrm{vol}(L))$ are vertices of the plot of the lattices L with $\mathrm{rank}(L) = n$. The following lemma states the situation for other vertices of the profile

Lemma 3.2.11. *Suppose that M_* is a lattice with with the point $\ell(M_*)$ as a vertex of the profile. If M is any other sublattice such that $\ell(M)$ is also a vertex of the profile, then we must have either $M \subset M_*$ or $M_* \subset M$.*

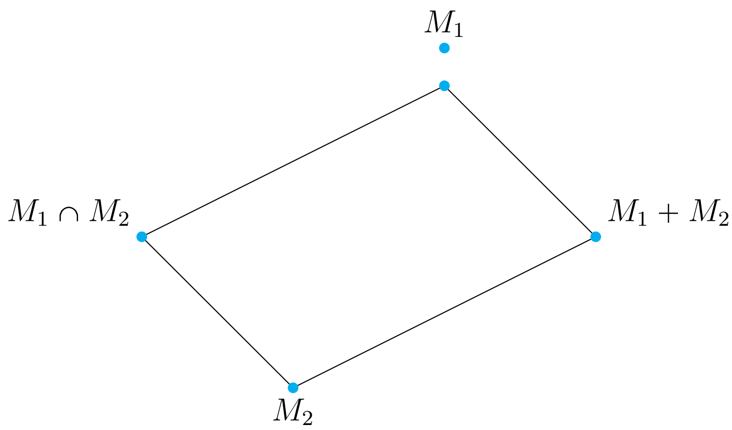


Figure 3.4: Grason's parallelogram rule

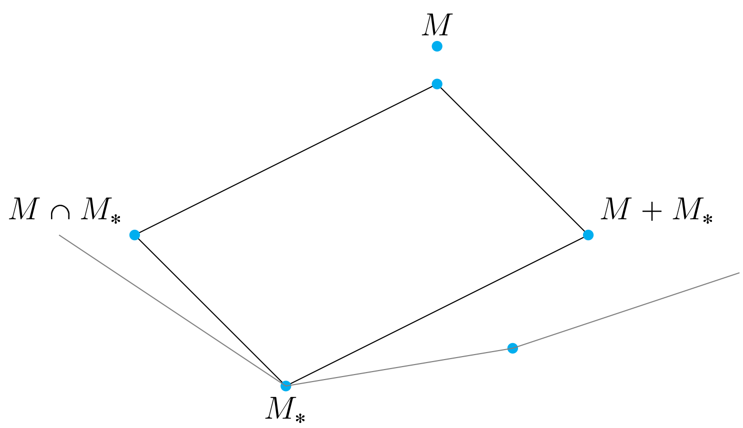


Figure 3.5

Proof. We refer to figure 3.5. Clearly we can see that the point $\ell(M \cap M_*)$ lies somewhere on the left of both $\ell(M)$ and $\ell(M_*)$. Similarly, the point $\ell(M + M_*)$ lies somewhere inclusively to the right of $\ell(M)$ and $\ell(M_*)$. By the paralellogram rule, the point $ell(M)$ will therefore be above the boundary points $\ell(M \cap M_*)$, $\ell(M_*)$ and $\ell(M + M_*)$. If M is also a vertex of the profile, the parallelogram must degenerate. In particular we either have $M \cap_M {}^* = M$, in which $M \subset M_*$, or $M + M_* = M$, in which $M_* \subset M$. □

An immediate consequence is that

Theorem 3.2.12. *The vertices of the profile of a lattice L are represented by unique sublattices, and they form a chain.*

For a given lattice L, we call the chain of sublattices in theorem 3.2.12 the **canonical filtration** of L. Assume that the canonical filtration for L is

$$\mathcal{F} : 0 = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_{k-1} \subset L_k = L$$

then it can be seen from the diagram that

1. L_i/L_{i-1} is semistable for all $1 \leqslant i \leqslant k$.

2. $\mu(L_i/L_{i-1}) \leqslant \mu(L_{i+1}/L_i)$ for all $1 \leqslant i \leqslant k-1$.

These two conditions is also sufficent for a chain to be a canonical filtration

Theorem 3.2.13. *Suppose*

$$\mathcal{F} : 0 = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_{k-1} \subset L_k = L$$

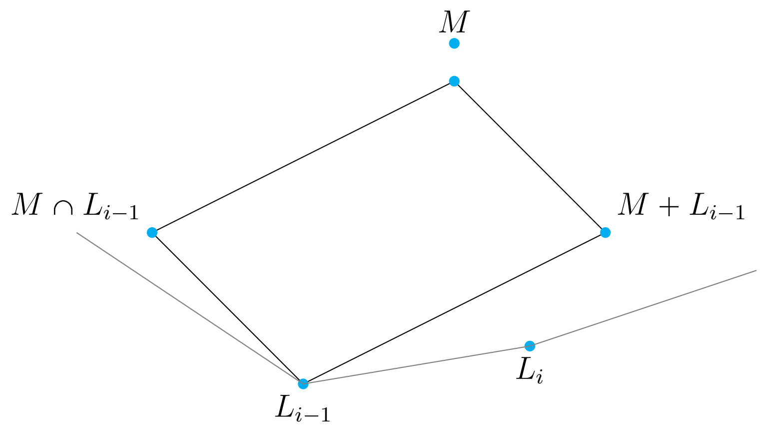


Figure 3.6

is a chain of lattices such that L_i/L_{i-1} is semistable and the slope L_i/L_{i-1} is not larger than the slope of L_{i+1}/L_i. Then this chain is the canonical filtration.

Proof. Suppose M to be any other sublattice of L. We want to know that $\ell(M)$ lies above the plot P of the $\ell(L_i)$. We prove by induction on the index i that if $M \subset L_i$, then $\ell(M)$ lies above the plot P.

- If $i = 1$ then $M \subset L_1$. Since $L_1$is semistable, we must have $\ell(M)$ lies above the line connecting $(0,0)$ and $\ell(L_1)$. Hence $\ell(M)$ lies above the plot P.
- Suppose that $M \subset L_i$ for $i > 1$. Then $M + L_{i-1} \subset L_i$ contains L_{i-1}. Therefore, it corresponds to the point lies above the line connecting $\ell(L_{i-1})$ and $\ell(L_i)$, thus lies above P. By induction, the fact that $(M \cap L_{i-1}) \subset L_{i-1}$ implies $\ell(M \cap L_{i-1})$ lies above the plot P. Using the parallelogram rule, the point $\ell(M)$ must then also lie above the plot P.

Thus any chain satisfies the conditions given in the theorem is a canonical filtration. □

3.3 ρ- definition of semi-stability

We are now ready to define the ρ-definition of semi-stable lattice. Recall that we define the space of lattices of rank n by $X_n := K\backslash \mathrm{GL}_n(\mathbb{R})$, where K is the orthogonal subgroup.

Definition 3.3.1 (ρ-definition)**.** *Let $x \in X_n$ be an arbitrary lattice, then the lattice x is called **semi-stable** if and only if its degree of instability $\deg_{inst}(x) \geqslant 0$, where*

$$\deg_{inst}(x) := \min_{Q \in ParSt, \gamma \in GL(\mathbb{Q})/Q_i(\mathbb{Q})} \langle \rho_Q, H_Q(x\gamma) \rangle$$

We first have an elementary lemma

Lemma 3.3.2. *Given $x \in X_n$, then the following are equivalent*

1. $\deg_{inst}(x) \geqslant 0$.
2. *For every stsandard parabolic subgroup $P \subset G$ and $\omega \in \widehat{\Delta}_P$ we have*

$$\langle \omega, H_P(x\delta) \rangle \geqslant 0$$

for each $\delta \in G(\mathbb{Q})/Q_i(\mathbb{Q})$.

3. *For every maximal parabolic subgroup $Q \subset G$ and $\omega \in \widehat{\Delta}_Q$ we have*

$$\langle \omega, H_Q(x\delta) \rangle \geqslant 0$$

for each $\delta \in G(\mathbb{Q})/Q_i(\mathbb{Q})$.

Proof.
First we prove $1 \Rightarrow 3$: This follows immediately as $\rho_Q = c\omega$ for some positive number c and $\omega \in \widehat{\Delta}_Q$. In particular

$$\langle \omega, H_Q(x\delta) \rangle = \frac{1}{c} \langle \rho_Q, H_Q(x\delta) \rangle \geqslant \frac{1}{c} \deg_{\text{inst}}(x) \geqslant 0$$

For $2 \Rightarrow 3$: We can choose a maximal standard parabolic Q such that $P \subset Q \subset G$. But then

$$\langle \omega, H_P(x\delta) \rangle = \langle \omega, H_Q(x\delta) \rangle \geqslant 0$$

Finally, we have $2 \Rightarrow 1$ as ρ_P is a positive linear combination of elements contained in $\widehat{\Delta}_P$. $\square$

From lemma 3.3.2, to check whether $x \in G$ is semistable, we just need to verify whether

$$\langle \omega, H_Q(x\delta) \rangle \geqslant 0.$$

A simple observation is that - a lattice x is semi - stable if for all maximal standard parabolic subgroups Q_i, we have

$$\min_{\gamma \in \mathrm{GL}_n(\mathbb{Q})/Q_i(\mathbb{Q})} \langle \rho_Q, H_Q(x\gamma) \rangle \geqslant 0$$

From lemma 2.2.17

$$H_B = H_Q + H(B)$$

where H_Q is a scalar multiple of $\lambda_i^\vee$ for $Q = Q_i$ and $H(B)$ is a linear combination of $\alpha_j^\vee$ for $j \neq i$. On the other hand, since Q is a maximal standard parabolic subgroup of G, ρ_Q is proportional to λ_i. Thus

$$\langle \rho_Q, H(B)(x\gamma) \rangle = 0.$$

In particular, we can replace H_Q by H_B in verifying the semistablility. This implies that, if

$$x\gamma = kan, \quad k \in K, a \in A, n \in N,$$

as in Iwasawa decomposition, then $H_B(x) = H$ where $H = \exp(a)$. In particular, if

$$a = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}$$

then

$$\langle \rho_{Q_i}, H_B(x\gamma) \rangle = \frac{n}{2} \log(a_1 a_2 \dots a_i)$$

Thus, to check for the semi-stability of a lattice x, we just need to look at the A-coordinate of $x\gamma$ for every $\gamma \in G(\mathbb{Q})/Q_i(\mathbb{Q})$, and verify whether the following system holds

$$\begin{cases} a_1 \geqslant 1 \\ a_1 a_2 \geqslant 1 \\ \dots \\ a_1 a_2 \dots a_n \geqslant 1 \end{cases}$$

3.4 Canonical pair

Consider a pair (P, δ) of a standard parabolic subgroup $P \subsetneq G$ and $\delta \in G(\mathbb{Q})/P(\mathbb{Q})$. Such a pair is called **destabilizing** for x if

$$\langle \rho_P, H_P(x\delta) \rangle = \deg_{\text{inst}}(x)$$

Such a pair is called **extremal** for x if for any standard parabolic $Q \subset P$ such that

$$\langle \rho_Q, H_Q(x\delta) \rangle = \langle \rho_P, H_P(x\delta) \rangle$$

then $Q = P$.

Definition 3.4.1. *A pair (P, δ) that is both destabilizing and extremal for x is called a **canonical pair** for x.*

A canonical pair for x, if exists, will be denoted by $\mathbf{cp}(\mathrm{x}) := (P, \delta)$. A priori, it is not clear whether $\mathbf{cp}(\mathrm{x})$ exists or not. We will show that it is in fact equivalent to the notion of canonical filtration introduced in the previous sections, and deduce that $\mathbf{cp}(\mathrm{x})$ must exist and it is unique.
We obtain the following lemma as a consequence of the definition of canonical pair

Lemma 3.4.2. *Let $x \in X_n$ and (P, δ) be a pair with $P \subsetneq G$ be standard parabolic subgroup and $\delta \in G(\mathbb{Q})/P(\mathbb{Q})$. Then*

1. *If (P, δ) is destabilizing for x, then $\deg^P_{inst}(x\delta) \geqslant 0$.*
2. *If (P, δ) is extremal for x, then $\langle \alpha, H_P(x\delta) \rangle < 0$ for any $\alpha \in \Delta_P$.*

Proof. For the first part, let $Q \subset P$ be any standard parabolic, then

$$\rho_Q = \rho_P + \rho_Q^P$$

Thus, for any $\eta \in P(\mathbb{Q})/Q(\mathbb{Q})$

$$\begin{aligned}\left\langle \rho_Q^P, H_Q(x\gamma\eta) \right\rangle &= \langle \rho_Q, H_Q(x\gamma\eta) \rangle - \langle \rho_P, H_P(x\gamma\eta) \rangle \\ &= \langle \rho_Q, H_Q(x\gamma\eta) \rangle - \deg_{inst}(x) \geqslant 0\end{aligned}$$

For the second part, we can pick a standard parabolic $Q \subset G$ containing P such that P is maximal in Q. In particular, we have $\Delta_P^Q = \{\alpha\}$. Then

$$\begin{aligned}\left\langle \rho_P^Q, H_P(x\delta) \right\rangle &= \langle \rho_P, H_P(x\gamma) \rangle - \langle \rho_Q, H_Q(x\gamma\eta) \rangle \\ &= \deg_{inst}(x) - \langle \rho_Q, H_Q(x\gamma\eta) \rangle < 0\end{aligned}$$

Since ρ_P^Q and α are proportional by a positive number, the result follows immediately. $\square$

Chapter 4

Equivalence between semistability and ρ-semistability

4.1 The equivalent between definitions of semi-stable lattices

So far we have two distinct definitions of semi-stability. The following theorem asserts that they are equivalent:

Proposition 4.1.1. *Let $x \in X_n = K\backslash GL_n(\mathbb{R})$ - the space of unit lattice. Then x is semi-stable if one of the following equaivalent conditions holds*

1. *The bottom of the profile of x is a line connect solely two points: the origin and $(n, 0)$.*
2. *The degree of instability of x is nonnegative, namely, $\deg_{inst}(x) \geqslant 0$.*

Proof.
If we can prove there is a correspondence between $\gamma \in \mathrm{GL}_n(\mathbb{Q})/Q_i(\mathbb{Q})$ and a sublattice of rank i of x, then we are done. We first need a slight reduction - we identified the quotient $\mathrm{GL}_n(\mathbb{Q})/Q_i(\mathbb{Q})$ with the quotient $\mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z}))$. Now let x be an arbitrary lattice of rank n.
We will first show the following correspondence

$$\mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})) \longleftrightarrow \{ \text{ sublattice of rank } i \text{ of } \mathbb{Z}^n\}$$

We define the map from the collection of sublattices of rank i to the cosets space as follows: For any sublattice $M \subset \mathbb{Z}^n$, there exists a basis of M, denoted by

$$\{v_1, v_2, \ldots, v_i\}$$

we can extend this basis to get a basis of $\mathbb{Z}^n$

$$\mathfrak{B}' = \{v_1, v_2, \ldots, v_n\}$$

Clearly in $\mathbb{Z}^n$ we have the standard basis $\mathfrak{B} = \{e_1, e_2, \ldots, e_n\}$. Clearly there exists a map $\gamma \in \mathrm{GL_n}(\mathbb{Z})$ such that

$$\gamma \cdot e_k = v_k \quad \forall k = 1, 2, \ldots, n$$

So we define the map

$$\varphi\colon \{\text{sublattices of rank } i \text{ of } \mathbb{Z}^n\} \to \mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z}))$$
$$M \mapsto [\gamma]$$

where $[\gamma]$ denoted the equivalent class of γ in the quotient space. This is a well-defined map. Indeed, Assume that we extend the basis $\mathfrak{B}'$ in a different way to get the basis

$$\mathfrak{B}_1 = \{v_1, \ldots, v_k, v'_{k+1}, \ldots, v'_n\}$$

As above, there also exists $\gamma' \in \mathrm{GL_n}(\mathbb{Z})$ such that

$$\gamma' e_k = v_k \quad \forall k \leqslant i, \quad \text{ and } \quad \gamma' e_k = v'_k \quad \forall k > i$$

But this implies that

$$(\gamma^{-1})\gamma' \cdot e_k = \gamma^{-1} v_k = e_k \quad \forall k \leqslant i$$

So in particular, we have $[\gamma] = [\gamma']$. The inverse map is given by

$$[\gamma] \mapsto \bigoplus_{k=1}^{i} \mathbb{Z}(\gamma \cdot e_i) = M$$

This generalizes in the obvious way for lattice $x = g\mathbb{Z}^n$ for some $g \in \mathrm{GL}_n(\mathbb{R})$. Indeed, we just define the map

$$\phi_g \colon \{\text{sublattices of rank } i \text{ of } g\mathbb{Z}^n\} \to \mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z}))$$
$$M_g = gM = g\bigoplus_{k=1}^{i} \mathbb{Z}v_i \mapsto [\gamma]$$

where $\gamma e_k = v_k$ in $\mathbb{Z}^n$ for all $k \leqslant i$ and $\phi_g^{-1}([\gamma]) = g\bigoplus_{k=1}^{i} \mathbb{Z}(\gamma \cdot e_i)$. □

Check the map in the generalization carefully

4.2 Canonical pair and Canonical filtration

We can further prove that, the equivalence between different notions of semistability comes from the equivalence between the notion of canonical pair and the bottom of the profile of a lattice - the canonical filtration. The main result is the following proposition

Proposition 4.2.1. *Given an $x \in X_n = K\backslash G$. Then (P, δ) is the canonical pair for x if and only if P is the stabilizer of the canonical filtration for the lattice $L_x = x\mathbb{Z}^n$.*

Recall that, from theorem 3.2.13 and lemma 3.4.2, both canonical pair and canonical filtration are characterized by two conditions. We will show that these two conditions are equivalent. Throughout this section, we fix an $x \in X_n$ and the corresponding lattice $L_x = x\mathbb{Z}^n$.

Lemma 4.2.2. *The the following conditions are equivalent*

1. *$\langle \alpha, H_P(x\delta) \rangle < 0$ for any $\alpha \in \Delta_P$ and for some $\delta \in G_{\mathbb{Q}}/P_{\mathbb{Q}}$.*

2. *L_x has a chain of vector spaces*

$$\mathcal{F} : 0 = L_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{k-1} \subset M_k = L_x$$

such that $\mu(M_i/M_{i-1}) < \mu(M_{i+1}/M_i)$ for all i. We also attach to the flag

$$\mathcal{F} \otimes \mathbb{R} \colon L_0 \subset M_1 \otimes \mathbb{R} \subset M_2 \otimes \mathbb{R} \subset \cdots \subset M_{k-1} \otimes \mathbb{R} \subset M_k \otimes \mathbb{R} = \mathbb{R}^n$$

a rational standard parabolic subgroup $P_{\mathbb{Q}}$.

Proof. We first prove (1) implies (2). Assume that (P, δ) is the canonical pair for x and

$$a(x\delta) = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}$$

in the Iwasawa decomposition of $x\delta$. We further assume that the standard parabolic P in the canonical pair is of type $(n_1, \dots, n_k)$. Let

$$d_i = n_1 + n_2 + \ldots + n_i$$

Then from the chain of standard lattice

$$0 \subset \mathbb{Z}^{d_1} \subset \mathbb{Z}^{d_2} \subset \cdots \subset \mathbb{Z}^{d_k}$$

we obtain a chain of sublattices of L_x as follows

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_k$$

where

$$M_i := \bigoplus_{m=1}^{d_i} \mathbb{Z}x\delta \cdot e_m$$

.

Note that for $M_k = x\delta\mathbb{Z}^{d_k} = x\mathbb{Z}^n = L_x$.
since the volume depend solely on the A-coordinate in the Iwasawa decomposition. In this way, it is immediate that the volume of the sublattice M_i is

$$\mathrm{vol}(M_i) = a_1 a_2 \cdots a_i,$$

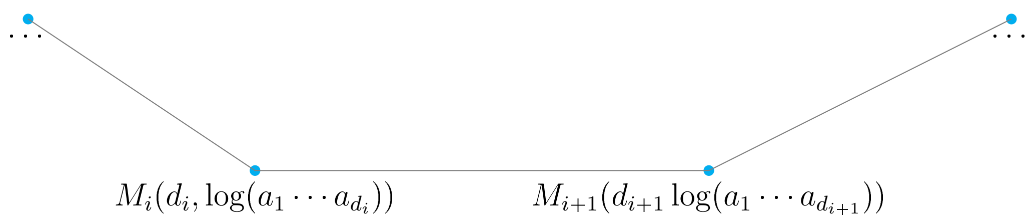


Figure 4.1

We will show that

$$H_P(x\delta) = m_1\lambda^\vee_{d_1} + m_2\lambda^\vee_{d_2} + \ldots + m_k\lambda^\vee_{d_k}$$

where

$$m_i = \frac{\log(a_{d_{i-1}} \cdots a_{d_i})}{n_i} - \frac{\log(a_{d_i+1} \cdots a_{d_{i+1}})}{n_{i+1}}$$

If we can prove this, we are done, as

$$\mu(M_i/M_{i-1}) - \mu(M_{i+1}/M_i) = m_i = \langle \alpha_{d_i}, H_P(x\delta) \rangle < 0$$

Hence we recover the second condition in Grayson's criterion, namely, $\mu(M_i/M_{i-1}) \leqslant \mu(M_{i+1}/M_i)$ for all $1 \leqslant i \leqslant k-1$.
To prove this, we first note that

$$\begin{aligned} H_B(x\delta) &= \log(a_1)\alpha^\vee_i + \log(a_1a_2)\alpha^\vee_2 + \ldots \log(a_1a_2\cdots a_{n-1})\alpha^\vee_{n-1} \\ &= \log\left(\frac{a_1}{a_2}\right)\lambda^\vee_1 + \ldots + \log\left(\frac{a_{n-1}}{a_n}\right)\lambda^\vee_{n-1} \end{aligned}$$

To compute $H_P(x\delta)$, we will use the Lemma 2.2.19 to get

$$H_B = H_P + H_{\star B}$$

and Lemma 2.2.17 to compute $H_{\star B}$ explicitly. Since P is a standard parabolic subgroup, we have $P = P_J$ and thus

$$H_{\star B}(x\delta) = \sum_{i \notin J} \log\left(\frac{a_i}{a_{i+1}}\right) \lambda_{i,J}^\vee$$

We want to write $\lambda_{i,J}^\vee$ as linear combination of $\alpha_j^\vee$'s. To do this, we use the theorem 2.1.14. Note that we can write $H_P(x\delta)$ in two ways

$$\begin{aligned} H_P(x\delta) &= H_B(x\delta) - H_{\star B}(x\delta) \\ &= \sum_{j=1}^{n-1} t_j \alpha_j^\vee \\ &= m_1 \lambda_{d_1}^\vee + m_2 \lambda_{d_2}^\vee + \ldots + m_k \lambda_{d_k}^\vee \end{aligned}$$

Then by 2.1.13, we must have

$$m_i = 2t_{d_i} - t_{d_i - 1} + t_{d_i + 1}$$

for $1 < i < k$. To compute t_{d_i}, we find the coefficient of $\alpha_{d_i}^\vee$. But this is just the coefficient of $\alpha_{d_i}^\vee$ in the linear combination of $H_B(x\delta)$. Thus

$$t_{d_i} = \log(a_1 \cdots a_{d_i})$$

The value of t_{d_i-1} and t_{d_i+1} is slightly more complicated. For t_{d_i-1}, it is the difference between the coefficient of $\alpha_{d_i-1}^\vee$ in $H_B(x\delta)$ and that of $\alpha_{d_i-1}^\vee$ in $H_{\star B}(x\delta)$. By lemma 2.1.14, we have

$$\begin{aligned} \sum_{s=p}^{q} \log\left(\frac{a_s}{a_{s+1}}\right) \lambda_s^\vee &= \begin{bmatrix} \log\left(\frac{a_p}{a_{p+1}}\right) & \cdots & \log\left(\frac{a_q}{a_{q+1}}\right) \end{bmatrix} \begin{bmatrix} \lambda_p^\vee \\ \vdots \\ \lambda_q^\vee \end{bmatrix} \\ &= \begin{bmatrix} \log\left(\frac{a_p}{a_{p+1}}\right) & \cdots & \log\left(\frac{a_q}{a_{q+1}}\right) \end{bmatrix} A^{-1} \begin{bmatrix} \lambda_p^\vee \\ \vdots \\ \lambda_q^\vee \end{bmatrix} \end{aligned}$$

So the coefficient of $\alpha_{d_i-1}^\vee$ in $H_{\star B}(x\delta)$ is

$$\sum_{u=1}^{n_i-1} \log\left(\frac{a_{d_{i-1}+u}}{a_{d_{i-1}+u+1}}\right)\left(\min\{u, n_i - 1\} - \frac{u(n_i-1)}{n_i}\right) = \sum_{u=1}^{n_i-1} \log\left(\frac{a_{d_{i-1}+u}}{a_{d_{i-1}+u+1}}\right)\frac{u}{n_i}$$

Argue similarly, we can also find the value of t_{d_i+1}. Overall, we have

$$\begin{cases} t_{d_i-1} = \log(a_1 \cdots a_{d_i-1}) - \displaystyle\sum_{u=1}^{n_i-1} \log\left(\frac{a_{d_{i-1}+u}}{a_{d_{i-1}+u+1}}\right)\frac{u}{n_i} \\ t_{d_i+1} = \log(a_1 \cdots a_{d_i+1}) - \displaystyle\sum_{u=1}^{n_{i+1}-1} \log\left(\frac{a_{d_i+u}}{a_{d_i+u+1}}\right)\left(1 - \frac{u}{n_{i+1}}\right) \end{cases}$$

which simplify to $m_i = \dfrac{\log(a_{d_{i-1}} \cdots a_{d_i})}{n_i} - \dfrac{\log(a_{d_i+1} \cdots a_{d_{i+1}})}{n_{i+1}}$ as desired. The cases $i = 1, k$ can be verified similarly.

Conversely, assume that we have a a chain

$$\mathcal{F} : 0 = L_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{k-1} \subset M_k = L_x$$

Then multiplying by x^{-1} on the left gives a chain of lattices in $\mathbb{Z}^n$.

$$\mathcal{F} : 0 = L_0 \subset N_1 \subset N_2 \subset \cdots \subset N_{k-1} \subset N_k^n = \mathbb{Z}$$

where $N_i := x^{-1}M_i$. Let us denote $n_i := \mathrm{rank}(N_i)$. Then there exists an ordered basis for the standard lattice $\mathbb{Z}^n$

$$\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$$

such that $\{v_1, \ldots, v_{n_i}\}$ is the basis for the lattice N_i. We clearly can find an element $\delta \in \mathrm{GL_n}(\mathbb{Z})$ such that $\delta \cdot v_i = e_i$. Repeat the argument in Theorem 4.1.1, we can see that $\delta\eta$ stabilizes the flag $\mathcal{F}$. So it does not change anything if we consider the coset $\delta P_{\mathbb{Q}}$. Consider the element $x\delta$ for $\delta \in G_{\mathbb{Q}}/P_{\mathbb{Q}}$ and do the same computation for $x\delta$ as in the $1 \Rightarrow 2$ part, we can see that

$$0 > \mu(M_i/M_{i-1}) - \mu(M_{i+1}/M_i) = m_i = \langle \alpha_{d_i}, H_P(x\delta)\rangle$$

Thus the lemma is proved. □

Remark. *From the proof the lemma 4.2.2, we explicitly construct an equivalence*

$$xG_{\mathbb{Q}}/P_{\mathbb{Q}} \longleftrightarrow \{ \textit{ flag of sublattices of } x\mathbb{Z}^n \textit{ of the same type of partition type of } P\}$$

Lemma 4.2.3. *Fixed the notations as in lemma 4.2.2, the following conditions are equivalent*

1. *M_i/M_{i-1} is semistable for all i, where the flag*

$$\mathcal{F} : 0 \subset M_1 \subset M_2 \subset \cdots \subset M_{k-1} \subset M_k = L_x$$

corresponds to $x\delta$.

2. *$x\delta$ is $P-$semistability for some $\delta \in G_{\mathbb{Q}}/P_{\mathbb{Q}}$.*

Proof.
We prove $1 \Rightarrow 2$ by contradiction. Assume not, then there exists a maximal parabolic subgroup $Q \subset P$ and an $\eta \in P/Q$ such that

$$\left\langle \rho_Q^P, H_Q(x\delta\eta)\right\rangle < 0$$

by lemma 3.3.2. We assume that P is a standard parabolic subgroup of type $(n_1, n_2, \ldots, n_k)$. Then Q must be of the type $(n_1, \ldots, n_{i,1}, n_{i,2}, \ldots, n_k)$ where $n_{i,1} + n_{i,2} = n_i$ for some $1 \leqslant i \leqslant k$. Argue as in lemma 4.2.2, we obtain a chain of sublattices of L_x

$$0 \subset M_1 \subset \cdots M_{i-1} \subset M' \subset M_{i+1} \subset \cdots \subset M_k = L_x$$

where $\mathrm{rank}(M_j) = d_j$ for $j \neq i$ and $\mathrm{rank}(M') = d_{i-1} + n_{i,1} = r$.
Using the same computation as in lemma 4.2.2, it can be seen that

$$0 > \left\langle \rho_Q^P, H_Q(x\delta\eta)\right\rangle = c\langle \alpha_r, H_Q(x\delta\eta)\rangle = c \cdot (\mu(M'/M_{i-1}) - \mu(M_{i+1}/M'))$$

In particular

$$\mu(M'/M_{i-1}) - \mu(M_{i+1}/M') < 0 \Leftrightarrow \mu(M'/M_{i-1}) < \mu(M_{i+1}/M')$$

But this contradicts the fact that M_i/M_{i-1} is a semistable for all i, i.e. there are no sublattices M' such that $\ell(M')$ lies below the line connectings $\ell(M_i)$ and $\ell(M_{i-1})$ for all i.
Conversely, to prove $2 \Rightarrow 1$, we also prove by contradiction. Assume that M_i/M_{i-1} is unstable for some index i. Then there exists a flag

$$0 \subset M^* \subset M_i/M_{i-1}$$

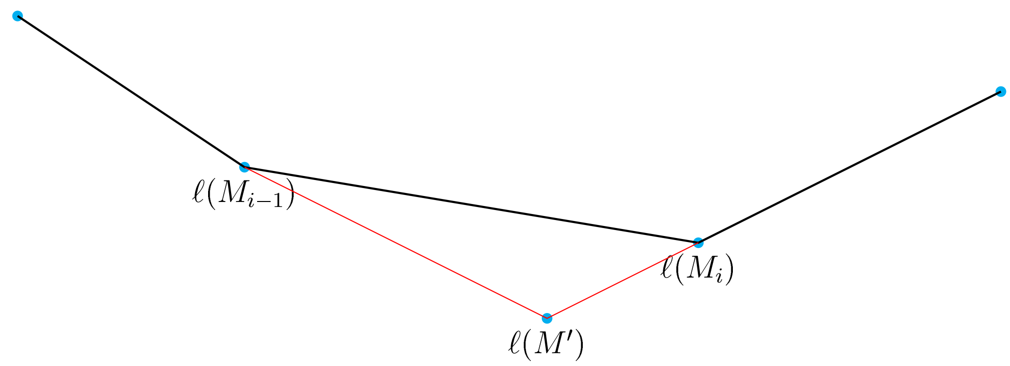


Figure 4.2

such that $\mu(M^*) < \mu(M_i/M_{i-1})$. Let $M' := M^* \oplus M_{i-1}$, then M' is a sublattice of M_i containing M_{i-1} such that

$$\mu(M'/M_{i-1}) < \mu(M_i/M_{i-1}) = \frac{rk(M'/M_{i-1})}{\mathrm{rank}(M_i/M_{i-1})}\mu(M'/M_{i-1}) + \frac{rk(M_i/M')}{\mathrm{rank}(M_i/M_{i-1})}\mu(M_i/M')$$

which implies that

$$\mu(M'/M_{i-1}) < \mu(M_i/M')$$

Since the chain

$$0 \subset M_1 \subset \cdots M_{i-1} \subset M' \subset M_{i+1} \subset \cdots \subset M_k = L_x$$

is a refinement of the flag $\mathcal{F}$, this gives rise to a pair (Q, δ') where $Q \subset P$ is a maximal standard parabolic subgroup of P and $\delta' P = \delta P$. But then again we have

$$\left\langle \rho_Q^P, H_Q(x\delta\eta) \right\rangle = c \left\langle \alpha_r, H_Q(x\delta\eta) \right\rangle = c \cdot (\mu(M'/M_{i-1}) - \mu(M_{i+1}/M')) < 0$$

which is a contradiction. □

Combinining Lemmas 4.2.3 and 4.2.2, we get a proof for proposotion 4.2.1 as follows

Proof. Assume that (P, δ) is a canonical pair for x, then there exists a flag

$$\mathcal{F} : 0 = L_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{k-1} \subset M_k = L_x$$

that corresponds to $x\delta$ and admits $P_{\mathbb{Q}}$ as its stabilizer. Lemmas 4.2.2 and 4.2.3 guarantees that this flag satisfies two condition in Grayson's criterion 3.2.13, thus $\mathcal{F}$ is the canonical filtration. Conversely, let $\mathcal{F}$ be the canonical filtration for x and denote Q the stabilizer of $\mathcal{F}$. If (P, δ) is the canonical pair, it also gives rise to a canonical flag $\mathcal{F}'$. Theorem 3.2.12 ensures that the canonical filtration is unique, thus $\mathcal{F} \equiv \mathcal{F}'$. Consenquently, we must have $Q = P$. □

Remark. *The construction above also shows that the canonical pair, must be unique and determined by the canonical filtration.*