CHAPTER II :ROOTS AND WEIGHTS FOR $\mathrm{SL_n}(\mathbb{R})$

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In this chapter, we review some basis theory of roots and weight. We will first recall the general theory and compute explicitly the exammples for $\mathrm{SL_n}(\mathbb{R})/\mathrm{GL_n}(\mathbb{R})$.

1 Structure theory

1.1 The Cartan subalgebra

First we need the notion of Cartan subalgebra

Definition 1.1. *For any Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be Cartan algebra if it is*

- *$\mathfrak{h}$ is a nilpotent subalgebra.*
- *It is self normalizing. In particular, we have $\mathfrak{h} = \{x \in \mathfrak{g} : [x, \mathfrak{g}] \subset \mathfrak{g}\}$.*

when $\mathfrak{g}$ is a semisimple Lie algebra, we have the following theorem

Theorem 1.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field k of charateristic 0 with a subalgebra $\mathfrak{h}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ if and only if it is a maximal toral subalgebra, i.e. is is maximal among all subalgebras contanining only semisimple elements.*

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1.2 Root space decomposition

With respect to some choice of Cartan subalgebra, we have a root space decomposition. In particular, there is a finite set $\Phi \subset \mathfrak{h}^*$ of linear forms on H, whose elements are called **roots**, such that

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right),$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h) x \forall h \in \mathfrak{h}\}$ for any $\alpha \in \Phi$.

1.3 A specific example: root space decomposition for $\mathfrak{sl}_n(\mathbb{R})$

For the semisimple Lie algebra $\mathfrak{sl}_\mathrm{n}(\mathbb{R})$, a typical choice of the Cartan subalgebra is the set

$$\mathfrak{h} = \left\{ H = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}, a_1 + a_2 + \ldots + a_n = 0 \right\}$$

With respect to this Cartan subalgebra, we can define the linear function

$$L_i \colon \mathfrak{h} \to \mathbb{R}, \quad H \mapsto L_i(H) = a_i$$

Then the roots are given by $\alpha_{ij} := L_i - L_j$ for distinct i, j. We have the root space decomposition for $\mathfrak{sl}_\mathrm{n}(\mathbb{R})$ as follows

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus \mathfrak{g}_{\alpha_{ij}} \right).$$

For the sake of brevity, we will denote $\alpha_{i,i+1}$ by α_i - these are called **fundamental roots**.

1.4 Roots at group level

Since the main object in this thesis is the Lie groups, we want to understand how the roots behave at group level. The analog for the Cartan subalgebra is the maximal torus

$$T = \left\{ \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \right\}$$

what the heck