CHAPTER I : $\mathrm{SL}_2(\mathbb{R})$

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In this chapter, I will give an exposition on the structure of $\mathrm{SL}_2(\mathbb{R})$ as the spaces of lattice, this space plays the role of a toy model before exploring the space of lattice in the higher rank. The exposition follows the paper [?] and [?] closely.

1 $\mathrm{SL}_2(\mathbb{R})$ and its action on the upper half plane $\mathfrak{H}$

A priori, the upper half plane

$$\mathfrak{H} = \{z : \Im z > 0\} \subset \mathbb{C}$$

has no group structure on its. However, we will show below that it can identify topologically with the space with the space of cosets $\mathrm{SO}_2(\mathbb{R}) \backslash \mathrm{SL}_2(\mathbb{R})$, and thus we can study the spaces $\mathfrak{H}$ via the spac of lattices $\mathrm{SO}_2(\mathbb{R}) \backslash \mathrm{SL}_2(\mathbb{R})$. We define the action of $G = \mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{H}$ as follows

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ (z) = \frac{az+b}{cz+d}$$

Proposition 1.1. *The group $SL_2(\mathbb{R})$ stabilizes $\mathfrak{H}$ and acts transitively on it. In particular,*

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} (i) = x + iy \quad (\textit{for } x \in \mathbb{R},\ y > 0)$$

Further, for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ *and* $z \in \mathfrak{H}$,

$$\Im\, g(z) = \frac{\Im\, z}{|cz+d|^2}.$$

Proof. The first formula is clear. The second formula would imply that the upper half-plane is stabilized. Compute directly:

$$\begin{aligned} 2i \cdot \Im\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) \right) &= \frac{az+b}{cz+d} - \frac{d\overline{z}+b}{c\overline{z}+d} = \frac{(az+b)(c\overline{z}+d) - (a\overline{z}+b)(cz+d)}{|cz+d|^2} \\ &= \frac{adz - bc\overline{z} - bcz + ad\overline{z}}{|cz+d|^2} = \frac{z-\overline{z}}{|cz+d|^2} \end{aligned}$$

since $ad - bc = 1$. □

The point $z = i$ is special, in the sense that its stability group is the orthogonal group $K = \mathrm{SO}_2(\mathbb{R})$. Indeed, for any $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ we have that

$$g \circ i = i \Leftrightarrow \frac{ai+b}{ci+d} = i \Leftrightarrow a = d \text{ and } b = -c$$

Combinining with the fact that $ad - bc = 1$, we must have $a^2 + b^2 = 1$. This implies that there is a θ such that $a = \cos\theta$ and $b = \sin\theta$. Since G acts on $\mathfrak{H}$ transitively, we know from group theory that there is a bijection between the collection of cosets of $\mathrm{Stab}(i)$ in G and the orbits of i. In particular

Proposition 1.2. *We have an isomorphism of $SL_2(\mathbb{R})$-spaces*

$$\mathrm{SO}_2(\mathbb{R}) \setminus SL_2(\mathbb{R}) \approx \mathfrak{H} \quad \textit{via} \quad SO(2)g \to g^{-1}(i)$$

That is, the map respects the action of $SL_2(\mathbb{R})$, in the sense that

$$(\mathrm{SO}_2(\mathbb{R})\, g) \cdot h \;\longrightarrow\; h^{-1}(g^{-1}i)$$

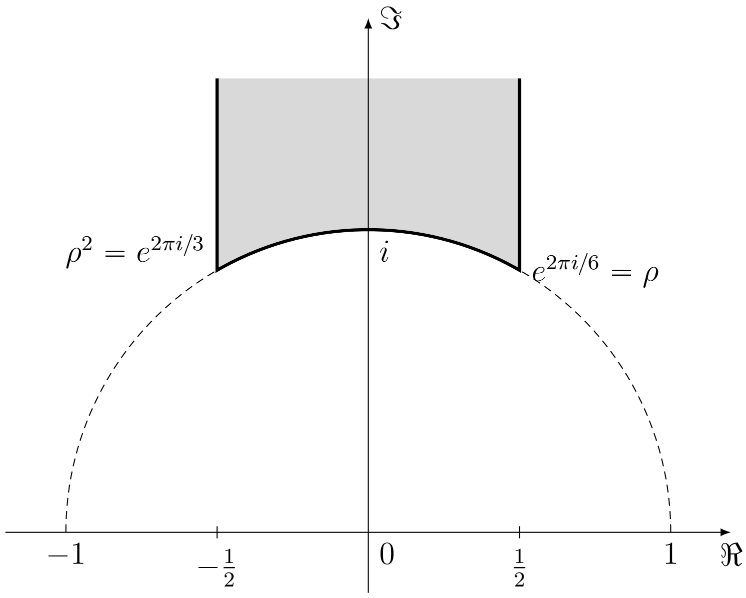
Proof. This is because of *associativity*:

$$(\mathrm{SO}_2(\mathbb{R})\, g) \cdot h = (\mathrm{SO}_2(\mathbb{R})) \cdot (gh) \;\longrightarrow\; (gh)^{-1}(i) = h^{-1}(g^{-1}(i))$$

giving the result. □

2 Fundamental domain for $\Gamma = SL_2(\mathbb{Z})$ on $\mathfrak{H}$

Here is a picture of the fundamental domain $\mathfrak{H}/\Gamma$.



The goal of this section is to prove that under the action of the $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, we can "move" every points on the upper half plane to a domain, under an equivalence given by a specific action. This is similar to the fundamental domain given by the translation action of $\mathbb{Z}$ to $\mathbb{R}$ is the half-open unit interval $[0, 1)$. In general, this give a simpler description to the homogenous space of lattice. Not that when we try to compute the fundamental domain of $\mathbb{Z}\backslash\mathbb{R}$, we have $\mathbb{Z}$ plays a role of "discrete" subset of $\mathbb{R}$. We give a precise definition of discreteness as follows

Definition 2.1. *Let a group G act continuously on a topological space X. A subset $\Gamma \subset G$ is called **discrete** if for any two compact subse A, B in X, there are only finitely many $g \in \Gamma$ such that $g \circ A \cap B \neq \varnothing$.*

We will prove that the set

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\}$$

is a discrete subgroup of $G = \mathrm{SL}_2(\mathbb{R})$. To prove this, we first need the following lemma

Lemma 2.1. *Fix a real number $r > 0$ and $0 < \delta < 1$. We denote $R_{r,\delta}$ the rectangle*

$$R_{r,\delta} = \left\{z = x + iy : -r \leqslant x \leqslant r, 0 < \delta \leqslant y \leqslant \delta^{-1}\right\}$$

Then for any $\epsilon > 0$ and any fixed set $\mathbb{S}$ of coset representatives for $\Gamma_\infty \backslash \Gamma$, there are finitely many $g \in \mathbb{S}$ such that $\Im(g \circ z) > \epsilon$ for some $z \in R_{r,\delta}$.

In the above lemma, the notation Γ_∞ is defined to be the set

$$\Gamma_\infty = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}.$$

It can be seen easily that this is the stability group of ∞ in $\mathfrak{H}$.

Proof. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then for $z \in R_{r,\delta}$,

$$\mathrm{Im}(g \circ z) = \frac{y}{c^2y^2 + (cx + d)^2} < \epsilon$$

if $|c| > (y\epsilon)^{-\frac{1}{2}}$. On the other hand, for $|c| \leqslant (y\epsilon)^{-\frac{1}{2}} \leqslant (\delta\epsilon)^{-\frac{1}{2}}$, we have

$$\frac{y}{(cx + d)^2} < \epsilon$$

if the following inequalities hold:

$$|d| > |c|r + (y\epsilon^{-1})^{\frac{1}{2}} \geqslant |c|r + (\epsilon\delta)^{-\frac{1}{2}}.$$

Consequently, $\Im(g \circ z) > \epsilon$ only if

$$|c| \leqslant (\delta\epsilon)^{-\frac{1}{2}} \quad \text{and} \quad |d| \leqslant (\epsilon\delta)^{-\frac{1}{2}}(r + 1),$$

and the total number of such pairs (not counting $(c, d) = (0, \pm 1), (\pm 1, 0)$) is at most $\frac{4(r+1)}{(\epsilon\delta)}$. This proves the lemma. $\square$

It follows from Lemma 2.1 that $\Gamma = SL(2, \mathbb{Z})$ is a discrete subgroup of $SL(2, \mathbb{R})$. This is because:

1. It is enough to show that for any compact subset $A \subset \mathfrak{H}$ there are only finitely many $g \in SL(2, \mathbb{Z})$ such that $(g \circ A) \cap A \neq \phi$;

2. Every compact subset of $A \subset \mathfrak{H}$ is contained in a rectangle $R_{r,\delta}$ for some $r > 0$ and $0 < \delta < \delta^{-1}$;

3. $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$, except for finitely many $\alpha \in \Gamma_\infty$, $g \in \Gamma_\infty \backslash \Gamma$.

To prove (3), note that Lemma 2.1 implies that $(g \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$ except for finitely many $g \in \Gamma_\infty \backslash \Gamma$. Let $S \subset \Gamma_\infty \backslash \Gamma$ denote this finite set of such elements g. If $g \notin S$, then Lemma 1.1.6 tells us that it is because $\Im(g \circ z) < \delta$ for all $z \in R_{r,\delta}$. Since $\Im(\alpha g \circ z) = \Im(g \circ z)$ for $\alpha \in \Gamma_\infty$, it is enough to show that for each $g \in S$, there are only finitely many $\alpha \in \Gamma_\infty$ such that $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} \neq \phi$. This last statement follows from the fact that $g \circ R_{r,\delta}$ itself lies in some other rectangle $R_{r',\delta'}$, and every $\alpha \in \Gamma_\infty$ is of the form $\alpha = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$ $(m \in \mathbb{Z})$, so that

$$\alpha \circ R_{r',\delta'} = \{x + iy \mid -r' + m \leqslant x \leqslant r' + m,\ 0 < \delta' \leqslant \delta''^{-1}\},$$

which implies $(\alpha \circ R_{r',\delta'}) \cap R_{r,\delta} = \phi$ for $|m|$ sufficiently large.