Lie theory - homework 2

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Problem 1

Proof. By definition of $\mathfrak{g}$-module, we need to check that:

1. $(ax+by)\cdot f = a(x\cdot f)+b(y\cdot f)$: But it is easy to see that for any $v\in V$:

$$\begin{aligned}[(ax+by)\cdot f](v) &= (ax+by)\cdot f(v) - f((ax+by)\cdot v)\\ &= a(x\cdot f)(v) + b(y\cdot f)(v) - af(x\cdot v) - bf(y\cdot v)\\ &= a(x\cdot f)(v) + b(y\cdot f)(v)\end{aligned}$$

2. Similarly, we can also check that $x\cdot(af+bg) = a(x\cdot f)+b(y\cdot g)$.

3. $[xy]\cdot f = x\cdot(y\cdot f) - y\cdot(x\cdot f)$: Evaluating at $v\in V$, we get

$$\begin{aligned}([xy]\cdot f)(v) &= [xy]f(v) - f([xy]\cdot v)\\ &= x\cdot(y\cdot f(v)) - y\cdot(x\cdot f(v)) - f(x\cdot(y\cdot v) - y\cdot(x\cdot v))\\ &= (x\cdot(y\cdot f)(v)) - (y\cdot(x\cdot f)(v))\\ &= (x\cdot y\cdot f)(v) - (y\cdot x\cdot f)(v)\end{aligned}$$

   This shows that $\mathrm{Hom}(V,W)$ is indeed a $\mathfrak{g}$-module.

Now we want to show that $\mathrm{Hom}(V,W)$ is isomorphic to $V^*\otimes W$ as vector spaces. Since for any $v\in V$, $f(v)$ is a scalar for $f\in V^*$, we define a bilinear map from $V^*\times W$ to $\mathrm{Hom}(V,W)$ as follows:

$$\begin{aligned}\overline{f}\colon V^*\times W &\to \mathrm{Hom}(V,W)\\ (f,w) &\mapsto f(*)\cdot w\end{aligned}$$

Clearly this is a well-defined biliear map, and thus there exists a linear map f such that

$$\begin{aligned}f\colon V^*\otimes W &\to \mathrm{Hom}(V,W)\\ h\otimes w &\mapsto (v\mapsto h(v)\cdot w)\end{aligned}$$

Assume that V has dimension n, then so does V^*. Let $\{e_i\}_{i=1}^n$ be a basis of V and denote $\{e_i^*\}_{i=1}^n$ be its dual basis. Then for any $g\in\mathrm{Hom}(V,W)$ we define a map

$$\begin{aligned}g\colon\ \mathrm{Hom}(V,W) &\to V^*\otimes W\\ u &\mapsto \sum e_i^*\otimes u(e_i)\end{aligned}$$

Clearly g is also well-defined and we have that

$$(f\circ g(u))(v) = f\left(\sum e_i^*\otimes u(e_i)\right)(v) = \sum e_i^*(v)\cdot u(e_i) = u(v)$$

and

$$g\circ f(h\otimes w) = g(h()\cdot w) = \sum e_i^*\otimes h(e_i)w = (\sum h(e_i)e_i^*)\otimes w = h\otimes w$$

So g and f are inverse of each other, which implies the desired isomorphism. □

Problem 2

Proof. Without lost of generalization, we assume that we are working over algebraically closed field. Thus an element x is semisimple if it is diagonalizable. Clearly if x, y are nilpotent, then there are $n, m \in \mathbb{Z}$ such that $x^n = 0$ and $y^m = 0$. Since $xy = yx$ by hypothesis, we have that

$$(x+y)^{m+n} = \sum_{k=1}^{m+n} x^k y^{m+n-k} = 0$$

Claim: If two matrices are commute, they are simultaneously diagonalizable.
Using the above claim, we can choose a basis of V such that, with respect to that basis, x, y are two diagonal matrices. Then it is clear that $x + y$ is also diagonal. From the obvervation above, if $x = x_s + x_n$ and $y = y_s + y_n$, then we must have $x_s + y_s$ is semisimple and $x_n + y_n$ is nilpotent. Moreover

$$(x_s + y_s) + (x_n + y_n) = x + y$$

By the uniqueness of decomposition into semisimple and nilpotent part, we conclude that

$$(x+y)_s = x_s + y_s, \quad (x+y)_n = x_n + y_n$$

A proof of the claim can be found here: https://math.stackexchange.com/questions/2905474/regarding-a-proof-of-if-a-b-in-m-n-mathbbk-are-diagonalizable-and-commu □

Problem 3

Proof. Given the Killing form is nondegenerate, the corresponding Lie algebra will be semisimple. The proof of this part is identical to the proof in class (or in Humphrey's textbook). We will show that the converse is not always true, i.e. a semisimple Lie algebra can have degenerate Killing form. Using the hint, we look at the quotient algebra of $\mathfrak{sl}_3(\mathrm{F})$ of its center over the field F of characteristic 3.
First we show that the quotient is semisimple. Indeed, let denote this quotient by L. Any solvable ideals of L corresponds to a solvable ideals of $\mathfrak{sl}_3(\mathrm{F})$ containing the center. But note that we are working over ground field F with characteristic 3, thus $\mathfrak{sl}_3(\mathrm{F})$ has a non-trivial center, which is also a non-trivial ideal of $\mathfrak{sl}_3(\mathrm{F})$. Therefore, consider

$$\mathfrak{psl}_3(F) = \mathfrak{sl}_3(\mathrm{F}) / Z(\mathfrak{sl}_3(\mathrm{F}))$$

Then this algebra is simple, hence semisimple. Indeed, assume that we are given a non zero ideal I of $\mathfrak{psl}_3(F)$, then by repeating taking the Lie bracket like in $\mathfrak{sl}_2$, we will get the whole algebra $\mathfrak{psl}_3(F)$.
Now the Gram matrix with respect to the Killing form over L is given by

$$\begin{bmatrix} 12 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \end{bmatrix}$$

which is identially 0 matrix over field of characteristic 3, hence the Killing form is degenerate.

Below I provided how to compute the first few entries of the Grammian matrix of the Killing form - is essentially the same for every other entries.
To get the matrix above, we chose a basis of the algebra of $\mathfrak{sl}_3(\mathrm{F})$ quotient the center as

$$H_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad E_{ij} = \begin{bmatrix} \delta_{ij} \end{bmatrix}_{1 \leqslant i,j \leqslant 3}$$

It can be verified that $\mathrm{ad}(H_{12})(E_{ij}) = (\omega_i - \omega_j)E_{ij}$ where $\omega = (1,-1,0)$ for H_{13}. So as a matrix, with respect to the ordered basis

$$\{H_{12}, H_{23}, E_{12}, E_{21}, E_{13}, E_{31|}, E_{23}, E_{32}\}$$

$\mathrm{ad}\, H_{12}$ is of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In a similar way, we have

$$\mathrm{ad}\, H_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

From this we have that $\kappa(H_{12}, H_{12}) = \kappa(H_{23}, H_{23}) = 12$ and $\kappa(H_{13}, H_{23}) = -6$. $\square$