# Compactification in low dimension

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July 15, 2024

In this expository note, I will try to explain explicitly how to compactify $\Gamma\backslash\mathbb{H}$ by adding points in two ways.

## 1 Some preparations

We will always denote $\Gamma$ a subgroup of the group $SL_2(\mathbb{Z})$ of finite index, and this group acts on the upper half complex plane $\mathbb{H}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ z := \frac{az+b}{cz+d}$$

When $z$ tends to infinity, we have

$$\lim_{z\to\infty} \frac{az+b}{cz+d} = \frac{a}{c},$$

so we add the rational line to define the action of this group at $\infty$. In particular, we consider the set

$$\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$$

Note that on the projective rational line, we define the action to be the multiplication of a $2 \times 2$ matrix with a $2 \times 1$ vector. Then under this action, we have the following lemma

**Lemma 1.** *$SL_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$.*

*Proof.* For each point in $\mathbb{P}^1(\mathbb{Q})$, we can choose the representative to be of the form $[a : b]$, where $\gcd(a, b) = 1$. Then there exists $x, y \in \mathbb{Z}$ such that

$$ax - by = 1$$

Thus we get the following equality

$$\begin{bmatrix} b & a \\ -x & y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This implies any points in $\mathbb{P}^1(\mathbb{Q})$ can be moved to $[0 : 1]$, and thus the action is transitive. □

**Corollary 2.** *If $\Gamma$ is a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$ then $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$ has only finite orbits.*

## 2 Compactification of $\Gamma\backslash\mathbb{H}$ by adding points.

We introduction a topology on $\overline{\mathbb{H}}$. For the usual upper half plane, the topology is the usual metric topology on $\mathbb{C}$, and we only define the system of the neighborhood of $r \in \mathbb{P}^1(\mathbb{Q})$.

Let $S(c,\omega)$ be the circle that touches the real line at $\omega = p/q$ and has the radius $\frac{c}{2q^2}$. Then the collection of circle $D(c,\omega) = \bigcup_{0<c'\leqslant c} S(c',\omega)$ is called *Farey disk*. Let $c \to 0$, these disks define a neighborhood of $\omega$. The Farey disks at $\infty$ are defined to be the region

$$D(T,\infty) = \{z : \Im z \geqslant T\}$$

It can be checked easily that the matrix under inversion, the Farey disk at $\infty$ is mapped to $D(1/T,0)$. In general, if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \circ \infty = \omega$ then $D(T,\infty)$ is mapped to $D(1/T,\omega)$.

With the above topology on the extended upper half plane, we could show that

**Lemma 3.** $\Gamma\backslash\overline{\mathbb{H}}$ *is a compact set.*

*Proof.* We first prove for the case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. It is well known that the quotient space $\Gamma\backslash\mathbb{H}$ is identical to the set

$$\mathcal{F} = \{z \in \mathbb{H} : \Re z \in [-1/2, 1/2), |z| \geqslant 1 \text{ and } |z| > 1 \text{ if } \Re z > 0\}$$

By lemma 1, the projective rational line "shrinks" to a point under the action of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, Thus we can identify $\Gamma\backslash\overline{\mathbb{H}}$ with the set $\tilde{\mathcal{F}} = \mathcal{F} \cup \{\infty\}$. Consider an open cover $\{U_i\}_{i\in I}$ of $\tilde{\mathcal{F}}$ and the natural projection $\pi\colon \overline{\mathbb{H}} \to \tilde{\mathbb{F}}$. Then the set $\{\pi^{-1}(U_i)\}_{i\in I}$ forms an open cover of $\overline{\mathbb{H}}$. There must be an index $i_0$ such that $\pi^{-1}(U_{i_0})$ contains a neighborhood of $\infty$, namely contains a Farey disk $D(T,\infty)$ for some $T > 0$. Since $\overline{\mathcal{F}} - D(T,\infty)$ is a compact set, its image under $\pi$ is compact, hence it can be covered by $U_{i_1}, \ldots, U_{i_m}$. Altogether, $\tilde{\mathcal{F}}$ admits a finite subcover $U_{i_0}, \ldots, U_{i_m}$.

Now we proceed to the general case. Note that

$$\overline{\mathbb{H}} = \mathrm{SL}_2(\mathbb{Z}) \circ \tilde{\mathcal{F}} = \bigcup \Gamma a_i \circ \tilde{\mathcal{F}}$$

by corollary 2. Then under the surjective map $\pi\colon \overline{\mathbb{H}} \to \Gamma\backslash\overline{\mathbb{H}}$, we have

$$\Gamma\backslash\overline{\mathbb{H}} = \bigcup \pi\left(\Gamma a_i \circ \tilde{\mathcal{F}}\right),$$

which shows that the set $Y(\Gamma) = \Gamma\backslash\overline{\mathbb{H}}$ is compact as it is the union of compact sets. □

The orbit of $\mathbb{P}^1(\mathbb{Q})$ under the action of $\Gamma$ is called *cusps*. We have the obvious equality that

$$\Gamma\backslash\overline{\mathbb{H}} = \Gamma\backslash\mathbb{H} \cup \underbrace{\Gamma\backslash\mathbb{P}^1(\mathbb{Q})}_{\text{cusps}}$$

So in fact lemma 3 tells us that we only need to add finite cusp to get a compact domain. That means we only need to consider the action of $\Gamma$ on the projective rational line. By the orbit-stabilizer theorem, we get the decomposition

$$\Gamma\backslash\bigcup_{\omega} D(c,\omega) = \bigcup \Gamma_{\omega_i}\backslash D(c,\omega_i)$$

where $\omega_i$ is the set of representative for the action of $\Gamma$ on $\mathbb{P}^1(\mathbb{Q})$ and $\Gamma_{\omega_i}$ are the stabilizer of $\omega_i \in \Gamma$.