A discretized point-hyperplane incidence bound in $\mathbb{R}^d$

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Abstract

Let P be a δ-separated (δ, s, C_P)-set of points in $B(0,1) \subset \mathbb{R}^d$ and Π be a δ-separated (δ, t, C_Π)-set of hyperplanes intersecting $B(0,1)$ in $\mathbb{R}^d$. Define

$$I_{C\delta}(P, \Pi) = \#\{(p, \pi) \in P \times \Pi \colon p \in \pi(C\delta)\}.$$

Suppose that $s, t \geq \frac{d+1}{2}$, then we have $I_{C\delta}(P, \Pi) \lesssim \delta|P||\Pi|$. The main ingredient in our argument is a measure theoretic result due to Eswarathansan, Iosevich, and Taylor (2011) which was proved by using Sobolev bounds for generalized Radon transforms. Our result is essentially sharp, a construction will be provided and discussed in the last section.

1 Introduction

We start with the following two definitions.

Definition 1.1 ((δ, s, C) set)**.** *Let $0 \leq s < \infty$ and $\delta \in (0,1)$ and a constant $C > 0$. Given a metric space (X, d), a bounded set $E \subset X$ is called (δ, σ)-set if for every $\delta \leq r \leq 1$ and for all ball $B \subset X$ of radius r, we have*

$$|E \cap B|_\delta \leq Cr^s |E|_\delta,$$

where $|E|_\delta$ denotes the δ-covering of E in the space (X, d).

We note that this definition is slightly different compared to the classical one introduced by Katz and Tao in [9].

Definition 1.2 (Katz-Tao (δ, s, C)-set)**.** *Let $0 \leq s < \infty$ and $\delta \in (0,1)$ and a constant $C > 0$. Given a metric space (X, d), a bounded set $E \subset X$ is called Katz-Tao (δ, s, C)-set if for every $\delta \leq r \leq 1$ and for all ball $B \subset X$ of radius r, we have*

$$|E \cap B|_\delta \leq C \left(\frac{r}{\delta}\right)^s.$$

Note that if $|E|_\delta \sim \delta^{-s}$, then the two above definitions are equivalent.

Let P be a δ-separated (δ, s, C_P)-set of points in $\mathbb{R}^d$ and Π be a δ-separated (δ, t, C_Π)-set of hyperplanes intersecting $B(0,1)$ in $\mathbb{R}^d$. The number of discretized point-plane incidences between P and Π is defined by

$$I_{C\delta}(P, \Pi) = \#\{(p, \pi) \in P \times \Pi \colon p \in \pi(C\delta)\},$$

where $\pi(C\delta)$ denotes the $C\delta$ neighborhood of the hyperplane π. In this paper, we treat C_P, C_Π, and other constants as absolutely positive bounded constants.

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Our initial motivation comes from the following recent theorem due to Orponen, Shmerkin, and Wang [11], and Fu and Ren [4] in two dimensions.

Theorem 1.3. *Let $0 \leq s,t \leq 2$. Then, for every $\epsilon > 0$, there exists $\delta_0 = \delta_0(\epsilon) > 0$ such that the following holds for all $\delta \in (0, \delta_0]$. Let $P \subset B(0,1) \subset \mathbb{R}^2$ be a δ-separated $(\delta, s, \delta^{-\epsilon})$-set of points and $\mathcal{T}$ be a δ-separated $(\delta, t, \delta^{-\epsilon})$-set of tubes intersecting $B(0,1)$.*

1. *If $1 \geq t \geq s$ or $1 \geq s \geq t$, then*
$$I_\delta(P,\mathcal{T}) \lesssim |P||\mathcal{T}|\delta^{\frac{st}{s+t}-O(\epsilon)}.$$
2. *If $t \geq 1 \geq s \geq t-1$, then*
$$I_\delta(P,\mathcal{T}) \lesssim |P||\mathcal{T}|\delta^{\frac{st}{1+s}-O(\epsilon)}$$
3. *If $s \geq 1 \geq t \geq s-1$, then*
$$I_\delta(P,\mathcal{T}) \lesssim |P||\mathcal{T}|\delta^{\frac{st}{1+t}-O(\epsilon)}.$$
4. *If $t > 1$ and $s > 1$, then*
$$I_\delta(P,\mathcal{T}) \lesssim |P||\mathcal{T}|\delta^{\kappa(s+t-1)-O(\epsilon)},\ \kappa = \min\{1/2,\ 1/(s+t-1)\}.$$

Throughout this paper, by $X \lesssim Y$ we mean that $X \leq CY$ for some constant C, and $X \sim Y$ if $X \lesssim Y \lesssim X$.

The main purpose of this paper is to extend Theorem 1.3 to higher dimensions, namely, in $\mathbb{R}^d$ with $d \geq 3$. Our first result is the following.

Theorem 1.4. *Let $C > 0$, $0 \leq s,t \leq d$. There exists $\delta_0 = \delta_0(C,s,t) > 0$ such that the following holds for $\delta \in (0,\delta_0)$. Let P be a δ-separated (δ, s, C_P)-set of points in $B(0,1) \subset \mathbb{R}^d$ and Π be a δ-separated (δ, t, C_Π)-set of hyperplanes intersecting $B(0,1)$ in $\mathbb{R}^d$. Define*

$$I_{C\delta}(P,\Pi) = \#\{(p,\pi) \in P \times \Pi \colon p \in \pi(C\delta)\}.$$

Suppose that $s,t > \frac{d+1}{2}$, then we have the sharp estimate that $I_{C\delta}(P,\Pi) \lesssim \delta|P||\Pi|$.

In the above theorem, the conditions $s > \frac{d+1}{2}$ and $t > \frac{d+1}{2}$ are required in the proof. When s,t are small, using an elementary geometric argument, we are able to prove the following non-trivial result.

Theorem 1.5. *Let $C > 0$, $0 \leq s,t \leq d$. There exists $\delta_0 = \delta_0(C,s,t) > 0$ such that the following holds for $\delta \in (0,\delta_0)$. Let P be a (δ, s, C_P) set of points in $B(0,1) \subset \mathbb{R}^d$ and Π be a (δ, t, C_Π) set of hyperplanes intersecting $B(0,1)$ in $\mathbb{R}^d$ with $d \geq 3$. We further assume that $s - d + 2 > 0$. Then, for any $\epsilon > 0$, we have*
$$I_{C\delta}(P,\Pi) \lesssim |P| \cdot |\Pi| \cdot \delta^{f(t)(s-d+2)-\epsilon},$$
where
$$f(t) = \begin{cases} 1/2, & \text{if } t \geq 1 \\ \dfrac{t}{1+t}, & \text{if } t < 1 \end{cases}.$$

Remark 1.1. *If $t - d + 2 > 0$, then we can use the dual arguments to obtain a similar result. While Theorem 1.4 is optimal, we do not have any constructions on the sharpness of this theorem.*

Main ideas and Comparisons: We first discuss the main idea in the proof of Theorem 1.3. Observe that if P is a (δ, s, C)-set, then P can be covered by at most $|P|\delta^{s-O(\epsilon)}$ Katz-Tao (δ, s)-sets. A detailed proof can be found in [11, Lemma 3.5]. With this observation, one can apply Theorem 1.4 and Theorem 1.5 due to Fu and Ren in [4]. To prove these two theorems, Fu and Ren used a geometric argument and some earlier results due to Guth, Solomon, and Wang [5] and a generalization due to Bradshaw [1].

In higher dimensions $d \geq 3$, to prove Theorem 1.4, we use a completely different approach. More precisely, we use a measure theoretic result due to Eswarathansan, Iosevich, and Taylor [3], which was proved by using Sobolev bounds for generalized Radon transforms. The proof of Theorem 1.5 is a combination of Cauchy-Schwartz argument and geometric results due to Hera, Keleti, and Mathe [6].

We also want to add a remark that the approach in the proof of Theorem 1.4 is similar to the mechanism introduced by Iosevich, Jorati, and Laba [8] when they studied incidences between a set of "tubes" and a homogeneous set of points with the same size. Since our proof uses Sobolev bounds for generalized Radon transforms, it can be generalized to a more general form, i.e. the equation of hyperplanes $x_d = a_1x_1 + \cdots + a_{d-1}x_{d-1} + a_d$ can be replaced by $\Psi(a_1, \ldots, a_d, x_1, \ldots, x_d) = 0$, where the function Ψ satisfies the Phong-Stein curvature condition (4) below.

If the set Π of planes only needs to satisfy the property that it is δ-separated, a recent work of Dabrowski, Orponen, and Villa [2] for the case of $(d-1)$-hyperplanes tells us that

$$I_{C\delta}(P, \Pi) \lesssim \delta^{-\epsilon}\delta^{\frac{(d-1)(s+1-d)}{2d-1-s}}|P||\Pi|^{\frac{d-1}{2d-1-s}}, \tag{1}$$

where P is a δ-separated (δ, s, C_P)-set with $s > 1$.

This result is weaker than Theorem 1.4 when $|\Pi| \leq \delta^{-d}$.

We now compare the estimate (1) and Theorem 1.5. Theorem 1.5 states that if $s - d + 2 > 0$, then

$$I_{C\delta}(P, \Pi) \lesssim |P| \cdot |\Pi| \cdot \delta^{f(t)(s-d+2)-\epsilon},$$

where

$$f(t) = \begin{cases} 1/2, & \text{if } t \geq 1 \\ \dfrac{t}{1+t}, & \text{if } t < 1 \end{cases}.$$

Assume $t \geq 1$, then by a direct computation, this bound is stronger than that of (1) when

$$\delta^{-t} \lesssim |\Pi| \leq \delta^{\frac{M(2d-1-s)}{d-s}},$$

where

$$M = \frac{(d-1)(s+1-d)}{2d-1-s} - \frac{s-d+2}{2}.$$

This range is non-empty when $2t \leq s + 1 < d$.

Assume $t < 1$, then Theorem 1.5 is stronger than that of (1) when

$$\delta^{-t} \lesssim |\Pi| \leq \delta^{\frac{M'(2d-1-s)}{d-s}},$$

where

$$M' = M - \left(\frac{t}{t+1} - \frac{1}{2}\right)(s - d + 2).$$

This range is non-empty when $d-2<s<d-1$.

2 Proof of Theorem 1.4

We first recall some notations that can be found in [2]. Let $\mathcal{A}(d,n)$ be the set of n-dimensional affine subspaces in $\mathbb{R}^d$. The metric $d_{\mathcal{A}}$ defined on $\mathcal{A}(d,n)$ ([10, page 53]) is defined by

$$d_{\mathcal{A}}(V,W)=||\pi_{V_0}-\pi_{W_0}||_{op}+|a-b|,$$

where $V=V_0+a$ and $W=W_0+b$, $V_0,W_0\in\mathcal{G}(d,n)$ (the set of n-dimensional subspaces), $a\in V_0^{\perp}$, $b\in W_0^{\perp}$, and $||\cdot||_{op}$ is the operator norm.

Let γ_1 and γ_2 be two $(d-1)$-planes defined by

$$\gamma_1\colon x_d=a_1x_1+\cdots+a_{d-1}x_{d_1}+a_d,$$

and

$$\gamma_2\colon x_d=b_1x_1+\cdots+b_{d-1}x_{d_1}+b_d.$$

The following formula is more useful in practise

$$d_{\mathcal{A}}\left(\gamma_1,\gamma_2\right)=\left|\frac{(a_1\ldots,a_{d-1},-1)}{|(a_1\ldots,a_{d-1},-1|}-\frac{(b_1\ldots,b_{d-1},-1)}{|(b_1\ldots,b_{d-1},-1|}\right|+\left|\frac{a_d}{|(a_1\ldots,a_{d-1},-1|}-\frac{b_d}{|(b_1\ldots,b_{d-1},-1)|}\right|.$$

This can be proved by a direct computation or can be found in [7, Lemma 2.5].

For each plane defined by $x_d=a_1x_1+\cdots+a_{d-1}x_{d-1}+a_d$, we call the point $(a_1,\ldots,a_d)$ its dual point in $\mathbb{R}^d$.

Let $D\colon\mathbb{R}^d\to\mathcal{A}(d,d-1)$ be the map defined by

$$D\colon(x_1,\ldots,x_d)\mapsto\left\{y_d=\sum_{i=1}^{d-1}x_iy_i+x_d\right\}.$$

There are some properties of D we should keep in mind, namely, the restriction of D to the ball $B(R)$, $0<R<\infty$, is bilipschitz onto its image, and the bilipschitz constant depends only on R and d. Thus, D is injective. The inverse map of D, denoted by $D^*\colon\mathtt{im}(D)\to\mathbb{R}^d$, defined by

$$D(x_1,\ldots,x_d)\mapsto(-x_1,\ldots,-x_{d-1},x_d).$$

There exists a dimensional constant $0<r_d<1$ such that the restriction of D^* to $B(V_0,r_d)$ is bilipschitz onto its image, here $V_0=D(0,\ldots,0)$.

With these notations, the next two lemmas can be proved with standard arguments.

Lemma 2.1. *Let S be a set of δ-separated hyperplanes intersecting the unit ball $B(0,1)$. Then the set of corresponding dual points is also $c\delta$-separated for some absolute constant c.*

Proof. If two hyperplanes γ_1 and γ_2 are δ-separated with the metric

$$d_{\mathcal{A}}\left(\gamma_1,\gamma_2\right)=\left|\frac{(a_1\ldots,a_{d-1},-1)}{|(a_1\ldots,a_{d-1},-1|}-\frac{(b_1\ldots,b_{d-1},-1)}{|(b_1\ldots,b_{d-1},-1|}\right|+\left|\frac{a_d}{|(a_1\ldots,a_{d-1},-1|}-\frac{b_d}{|(b_1\ldots,b_{d-1},-1)|}\right|,$$

where γ_1 and γ_2 are respectively defined by equations

$$(\gamma_1) : y_d = \sum_{i=1}^{d-1} a_i y_i + a_d,$$

and

$$(\gamma_2) : y_d = \sum_{i=1}^{d-1} b_i y_i + b_d,$$

then we want to prove that the usual Euclidean distance between two points $(a_1, \ldots, a_d)$ and $(b_1, \ldots, b_d)$ is at least $c\delta$ for some absolute constant c. For the sake of simplicity, we will denote $u = (a_1 \ldots, a_{d-1}, -1)$ and $v = (b_1 \ldots, b_{d-1}, -1)$. Then clearly $|u|, |v| \geq 1$. Since two hyperplanes γ_1, γ_2 are δ-separated, we have that $d_{\mathcal{A}}(\gamma_1, \gamma_2) \geq \delta$. With these assumptions, we attempt to prove that

$$\sqrt{\sum_{i=1}^{d} (a_i - b_i)^2} \gtrsim \delta$$

It should be noted that, if we view a_d, b_d as vectors $(0, \ldots, a_d), (0, \ldots, b_d)$ respectively, then the usual Euclidean distance between two points $(a_1, \ldots, a_d)$ and $(b_1, \ldots, b_d)$ can be rewritten as $\sqrt{|u - v|^2 + |a_d - b_d|^2}$.

Thus, the goal is to prove the following inequality

$$\left| \frac{u}{|u|} - \frac{v}{|v|} \right| + \left| \frac{a_d}{|u|} - \frac{b_d}{|v|} \right| \lesssim \sqrt{|u - v|^2 + |a_d - b_d|^2}. \tag{2}$$

Instead of proving 2, we prove a stronger version

$$\left| \frac{u}{|u|} - \frac{v}{|v|} \right| + \left| \frac{a_d}{|u|} - \frac{b_d}{|v|} \right| \lesssim |u - v| + |a_d - b_d|. \tag{3}$$

We first prove

$$\left| \frac{u}{|u|} - \frac{v}{|v|} \right| \leq |u - v|.$$

Squaring both sides gives

$$|u|^2 + |v|^2 - 2u \cdot v \geq 2 - \frac{2u \cdot v}{|u||v|}.$$

This is reduced to

$$2|u||v| - 2u \cdot v \geq 2 - \frac{2u \cdot v}{|u||v|}.$$

We denote $|u||v| = x, u \cdot v = y$, then the above can be represented as

$$(x - 1)(x - y) \geq 0.$$

This inequality is true since $|u|, |v| \geq 1$ and $x \geq y$ by the Cauchy-Schwarz inequality.

Now we estimate $\left| \frac{a_d}{|u|} - \frac{b_d}{|v|} \right|$. Since all hyperplanes intersect $B(0, 1)$, one has $\frac{|a_d|}{|u|}$ and $\frac{|b_d|}{|v|}$, i.e the distances from the origin to the hyperplanes γ_1, γ_2, are not larger than 1. Then

$$\left| \frac{a_d}{|u|} - \frac{b_d}{|v|} \right| = \left| \frac{a_d|v| - a_d|u| + a_d|u| - b_d|u|}{|u||v|} \right| \leq |u - v| + |a_d - b_d|.$$

This completes the proof. $\square$

Lemma 2.2. *Let Π be a set of δ-separated (δ, t, C_π)-hyperplanes intersecting $B(0,1)$. Then the set of dual points is $c\delta$-separated (δ, t, c')-set for some absolute constants $c, c' > 0$.*

Proof. We denote Π^* the set of dual points of hyperplanes Π. It is immediate that the set Π^* is $c\delta$-separated by Lemma 2.1. Hence, we only need to prove that, for arbitrary ball $B(x,r)$, we have

$$|\Pi^* \cap B(x,r)|_\delta \le c' r^t |\Pi^*|_\delta, \delta \le r \le 1,$$

for some constant c' that will be chosen later. It is sufficient to prove that

$$|\left\{x' \in \Pi^* : d(x',x) \le r\right\}| \le c' r^t |\Pi^*|_\delta, \delta \le r \le 1.$$

As mentioned earlier, the map D is bilipschitz onto its image, we have $|\{x' \in \Pi^* : d(x',x) \le r\}|$ is at most

$$\left|\left\{D(x') \in \Pi : d_{\mathcal{A}}(D(x'), D(x)) \le K_D r\right\}\right| \le C_\pi K_D^t r^t |\Pi^*|_\delta, \delta \le r \le 1.$$

Choose $c' = C_\pi K_D^t$, then the lemma follows. Note that K_D^t can be replaced by some constant that does not depend on t since $t \in (0,d)$. $\square$

2.1 Sobolev bounds for generalized Radon transforms and consequences

In this section, we recall some known results that make use the boundedness of general Radon transforms. Let $g \colon \mathbb{R}^d \to \mathbb{R}$ be a Schwartz function, $t \in \mathbb{R}$, $\psi \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a smooth cut-off function, and $\Psi(\mathbf{x}, \mathbf{y}) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a smooth function with some suitable assumptions. We define

$$T_{\Psi_t} g(\mathbf{x}) := \int_{\{\Psi(\mathbf{x},\mathbf{y})=t\}} g(\mathbf{y}) \psi(\mathbf{x},\mathbf{y}) d\sigma_{\mathbf{x},t}(\mathbf{y}),$$

where $d\sigma_{\mathbf{x},t}$ is the Lebesgue measure on the set $\{\mathbf{y} \colon \Psi(\mathbf{x},\mathbf{y}) = t\}$.

We denote the usual L^2-Sobolev space of L^2 functions with s generalized derivatives in $L^2(\mathbb{R}^d)$ by $L^2_s(\mathbb{R}^d)$. The following theorem was proved by Eswarathasan, Iosevich and Taylor in [3].

Theorem 2.3 ([3], Proposition 2.2)**.** *Let $E \subset \mathbb{R}^d$ be a compact set with $\dim_H(E) = \alpha$, and μ be the corresponding Frostman measure on E. Assume that T_{Ψ_t} maps L^2 to L^2_s with constants uniform in a small neighborhood of t, and $d - s < \alpha < d$. Then we have*

$$\mu \times \mu\{(\mathbf{x},\mathbf{y}) \in E \times E : t \le |\Psi(\mathbf{x},\mathbf{y})| \le t + \epsilon\} \lesssim \epsilon.$$

Remark 2.1. *It can be checked from Eswarathasan-Iosevich-Taylor's proof that the same result holds for two different sets $E \times F$, namely,*

$$\mu_E \times \mu_F\{(\mathbf{x},\mathbf{y}) \in E \times F : t \le |\Psi(\mathbf{x},\mathbf{y})| \le t + \epsilon\} \lesssim \epsilon,$$

if T_{Ψ_t} maps L^2 to L^2_s with $d - s < \alpha, \beta < d$, where $\alpha = \dim_H(E)$ and $\beta = \dim_H(F)$. Notice also that the sets E and F are not required to be A-D regular. Moreover in the proofs, the only thing that is needed associated with the set E is the existence of a probably measure μ supported on E that satisfies $\mu(B(x,r)) \lesssim r^\alpha$, the same applies for F.

To apply the above theorem in the proof of Theorem 1.4, we need to recall a celebrated result of Phong and Stein [12] stating that the operator T_{Ψ_t} is uniformly bounded from L^2 to L^2_s on a small neighborhood of t with $s = \frac{d-1}{2}$ if the so-called Phong-Stein rotational curvature condition

$$\det \begin{pmatrix} 0 & \nabla_{\mathbf{x}} \Psi \\ -\nabla_{\mathbf{y}} \Psi & \frac{\partial^2 \Psi}{\partial x_i \partial y_j} \end{pmatrix} \neq 0 \tag{4}$$

holds on the set $\{(\mathbf{x}, \mathbf{y}) \colon \Psi(\mathbf{x}, \mathbf{y}) = t\}$. This and Remark 2.1 imply the following theorem.

Theorem 2.4. *Let μ_E, μ_F be probability measures on compact sets $E, F \subset \mathbb{R}^d$, respectively, satisfying*

$$\mu_E(B(x,r)) \lesssim r^{\alpha}, \ \mu_F(B(x,r)) \lesssim r^{\beta},$$

for all $x \in \mathbb{R}^d$ and $r > 0$. Suppose that the Phong-Stein rotational curvature condition (4) *holds for the function Ψ, and $d - (d-1)/2 < \alpha, \beta < d$. Then, for $\epsilon > 0$, we have*

$$\mu_E \times \mu_F\{(\mathbf{x}, \mathbf{y}) \in E \times F : |\Psi(\mathbf{x}, \mathbf{y})| \leq \epsilon\} \lesssim \epsilon.$$

2.2 Proof of Theorem 1.4

Let Π^* be the set of dual points corresponding to planes in Π. As we proved in Lemma 2.2 that this set is $c\delta$-separated (δ, t, c')-set.

We first define two probability measures on $P(\delta)$ and $\Pi^*(c\delta)$, denoted by μ_P and μ_Π, respectively, as follows:

$$\mu_P(X) = \frac{|X \cap P(\delta)|}{|P(\delta)|},$$

and

$$\mu_{\Pi^*}(X) = \frac{|X \cap \Pi^*(c\delta)|}{|\Pi^*(c\delta)|}.$$

Since P is (δ, s, C_P)-set and Π^* is $(c\delta, t, c')$-set, we have

$$\mu_P(B(x,r)) \lesssim r^s,$$

and

$$\mu_{\Pi^*}(B(x,r)) \lesssim r^t,$$

for all $x \in \mathbb{R}^d$ and $r \geq \delta$.

For $r \leq \delta$, we have $|B(x,r) \cap P(\delta)| \lesssim r^d$. On the other hand, we have $|P(\delta)| \gtrsim \delta^{d-s}$, which follows from the assumption that P is a (δ, s, C_P)-set. For $r \leq \delta$ and $s < d$, we have $r^{d-s} \leq \delta^{d-s}$, this means that

$$\mu_P(B(x,r)) \lesssim \frac{r^d}{\delta^{d-s}} \lesssim r^s.$$

The same holds for Π^*. In other words, the two probability measures μ_P and μ_{Π^*} are Frostman measures with exponents s and t, respectively.

To proceed further, we may assume that all hyperplanes are defined by the equation of the form

$$a_1 x_1 + \cdots + a_{d-1} x_{d-1} + a_d = x_d. \tag{5}$$

This gives that the dual points are of the form $(a_1, \ldots, a_{d-1}, 1)$. Define

$$\Psi(x_1, \ldots, x_d, a_1, \ldots, a_d) = a_1 x_1 + \cdots + a_{d-1} x_{d-1} - x_d + a_d.$$

A direct computation shows that this function satisfies the curvature condition (4). We now observe that $(x_1,\dots,x_d)\in\pi(C\delta)$, where π is defined by (5), if $|\Psi(x_1,\dots,x_d,a_1,\dots,a_d)|\le C\delta$. On the other hand, if $|\Psi(x_1,\dots,x_d,a_1,\dots,a_d)|\le C\delta$, then $|\Psi(U,V)|\lesssim\delta$ for all $U\in B((x_1,\dots,x_d),\delta)$ and $V\in B((a_1,\dots,a_d),c\delta)$, when δ is small enough. This infers that

$$\frac{1}{|P||\Pi|}|I_\delta(P,\Pi)|\le\mu_P\times\mu_{\Pi^*}\left\{(U,V)\in P(\delta)\times\Pi^*(c\delta)\colon|\Psi(U,V)|\lesssim\delta\right\}.$$

Therefore, our result is reduced to show the following.

$$\mu_P\times\mu_{\Pi^*}\left\{(U,V)\in P(\delta)\times\Pi^*(c\delta)\colon|\Psi(U,V)|\lesssim\delta\right\}\lesssim\delta,$$

which follows from Theorem 2.4.

3 Proof of Theorem 1.5

In this section, we present an elementary argument to study the incidence problem.

For $p\in P$, let $I(p)$ be the set of hyperplanes $\pi\in\Pi$ such that $p\in\pi(C\delta)$. We observe that

$$I_{C\delta}(P,\Pi)=\sum_{p\in P}|I(p)|.$$

Thus, for $x,y>0$ and $x\ge y$, by the Hölder inequality, we have

$$I_{C\delta}(P,\Pi)\le|P|^{\frac{x}{x+y}}\left(\sum_p|I(p)|^{\frac{x+y}{y}}\right)^{\frac{y}{x+y}}.$$

This implies that

$$I_{C\delta}(P,\Pi)^{x+y}\le|P|^x\left(\sum_p|I(p)|^{\frac{x+y}{y}}\right)^{y}.$$

To proceed further, we need to estimate the sum $\sum_p|I(p)|^{1+x/y}$. We have

$$\sum_{p\in P}|I(p)|^{1+x/y}=\sum_{p\in P}\sum_{\pi\colon p\in\pi(C\delta)}|I(p)|^{x/y}=\sum_{\pi\in\Pi}J(\pi),$$

here $J(\pi)=\sum_{p\in\pi(C\delta)}|I(p)|^{x/y}$, which can be represented as follows

$$J(\pi)=\sum_{p\in\pi(C\delta)}\left(\sum_i|I(p)\cap J_{2^i\delta}(\pi)|\right)^{x/y}\lesssim\sum_{p\in\pi(C\delta)}\sum_{i=1}^{\log\delta^{-1}}|I(p)\cap J_{2^i\delta}(\pi)|^{x/y},$$

where $J_{2^i\delta}(\pi)$ is the set of hyperplanes π' such that $d_{\mathcal{A}}(\pi,\pi')\sim 2^i\delta$.

Since the set of planes is (δ,t,C_Π), we have

$$|J_{2^i\delta}(\pi)|\lesssim(2^i\delta)^t|\Pi|.$$

Lemma 3.1. *Let π and π' be two hyperplanes in $\mathcal{A}(d,d-1)$. Assume these two planes both intersect the unit ball $B(0,1)$ and $d_{\mathcal{A}}(\pi,\pi')=w>\delta$, then the intersection $\pi(\delta)\cap\pi'(\delta)\cap B(0,1)$*

can be covered by at most $\delta^{-(d-2)}$ cubes of parameters $\frac{\delta}{w} \times \delta \times \cdots \times \delta$ in $\mathbb{R}^d$.

Proof. The proof is essentially a combination of a number of results from [6]. Let $e_0 = (0, \cdots, 0)$ and $\{e_1, \cdots, e_d\}$ be the standard basis of $\mathbb{R}^d$. If we denote $H_0 = \langle e_d \rangle$ then H_0 is a line and $\pi \cap H_0$ is a point of the form $(0, \cdots, 0, a_0)$. We also denote $H_i = e_i + H_0$ so that $\pi \cap H_i$ is also a point of the form $(0, \cdots, 1, \cdots, a^i)$. Let $b_i = a_i - a_0$, then we refer to a_0 as the vertical intercept and b_i as the slopes of hyperplane π.

For each hyperplane π, we associate it to a point $x = x(\pi) = (a_0, b_1, \cdots, b_{d-1}) \in \mathbb{R}^d$ and note that the map $\varphi \colon \Pi \mapsto x(\Pi)$ is well-defined and injective. The space generated by this map is called **code space**. We endow this space with the maximum metric, namely,

$$||x - x'|| := \max \left(|a_0 - a_0'|, \max_{i=1,\ldots,d-1} \left(|b_i - b_i'| \right) \right).$$

As noted in [6, Remark 4.2], the maximum metric on the code space is strongly equivalent to the metric between hyperplanes $d_{\mathcal{A}}$, in the sense that $||\varphi(\pi) - \varphi(\pi')|| \sim d_{\mathcal{A}}(\pi, \pi')$.

Fix two hyperplanes π, π', and denote their two corresponding codes by $x(\pi)$ and $x(\pi')$. Then we can present the plane π by the equation

$$a_0 + b_1 t_1 + \cdots + b_{d-1} t_{d-1} = t_d, (t_1, \ldots, t_{d-1}) \in \mathbb{R}^{d-1},$$

where $x(\pi) = (a_0, b_1, \cdots, b_{d-1})$. Similarly, we can define π' from its code coordinate. As presented in the proof of [6, Lemma 4.3] that there is a constant c independent on δ such that:

$$\pi(\delta) \subset \pi + (\{0\} \times (-c\delta, c\delta)).$$

Therefore, we have

$$\pi(\delta) \cap \pi'(\delta) \cap \mathcal{C} \subset \left\{ (t, u) \in \mathbb{R}^d : u \in B(t_d, c\delta) \cap B(t_d', c\delta) \right\},$$

where $\mathcal{C} = \mathcal{C}_{d-1} \times L^1$ and $\mathcal{C}_{d-1}$ is the convex hull of $(0, \ldots, 0), e_1, \ldots, e_{d-1}$, and L^1 is the unit segment centered at the origin in $< e_d >$.

If $|a_0 - a_0'| > \max_{i=1,\ldots,d-1} (|b_i - b_i'|) + D\delta$, then choosing $D = 2c$ implies $B(t_d, c\delta) \cap B(t_d', c\delta)$ is empty. This infers that $\pi(\delta) \cap \pi'(\delta) \cap \mathcal{C} = \emptyset$.

If $\max_{i=1,\ldots,d-1}(|b_i - b_i'|) > 0$ and $B(t_d, c\delta) \cap B(t_d', c\delta) \neq \emptyset$, we set $N = \{t \in \mathcal{C}_{d-1} : |t_d - t_d'| < 2c\delta\}$, then we have

$$N = \{t \in \mathcal{C}_{d-1} : p_-(t) \leq t_{d-1} \leq p_+(t)\}$$

where

$$p_-(t) = p_-(t_1, \cdots, t_{d-2}) = \frac{-2c\delta - (a_0 - a_0') - \sum_{i=1}^{d-2} t_i (b_i - b_i')}{b_{d-1} - b_{d-1}'},$$

and

$$p_+(t) = p_+(t_1, \cdots, t_{d-2}) = \frac{2c\delta - (a_0 - a_0') - \sum_{i=1}^{d-2} t_i (b_i - b_i')}{b_{d-1} - b_{d-1}'}.$$

This implies that N is the intersection of $\mathcal{C}_{d-1}$ and the strip between two parallel hyperplanes $\{t_{d-1} = p_-(t)\}$ and $\{t_{d-1} = p_+(t)\}$. By a direct computation, the distance between these two hyperplanes is equal to

$$d(p_-(t),p_+(t))=\frac{2c\delta}{\sqrt{\sum_{i=1}^{d-2}(b_i-b_i')^2}}\lesssim\frac{\delta}{\max_{i=1,\dots,d-1}|b_i-b_i'|}.$$

Hence, N is contained in a rectangular box that has the shortest side of length $d\lesssim\delta/w$ and the other $d-2$ sides of length at most $diam(\mathcal{C})=\sqrt{2}$.

To conclude the proof, we do a rotation if needed to assume that the hyperplane π has e_d as normal vector. This gives

$$\pi:f(t)=a_0.$$

The hyperplane π' has the equation of the form

$$\pi':g(t)=a_0'+b_1't_1+\cdots+b_{d-1}'t_{d-1}.$$

Under these hypotheses, if (t,u) and (t',u') are two elements inside $\pi(\delta)\cap\pi'(\delta)\cap\mathcal{C}$, then triangle inequality gives

$$|u-u'|=|u-f(t)+f(t')-u'|\le|u-f(t)|+|u-f(t')|<2\delta$$

Therefore, the d-th coordinate of the intersection part is contained in a box of dimension $\lesssim\delta$. In other words, the intersection $\pi(\delta)\cap\pi'(\delta)\cap\mathcal{C}$ can be covered by roughly δ^{2-d} dyadic boxes of size $\frac{\delta}{w}\times\underbrace{\delta\times\cdots\delta}_{d-1\text{ times}}$. This completes the proof. $\square$

Lemma 3.2. *Fix $\pi\in\Pi$. For any $\delta<w\ll1$, we have*

$$\sum_{p\in\pi(c\delta)}|I(p)\cap J_w(\pi)|\lesssim|J_w(\pi)|\cdot|P|\delta^s\cdot\frac{1}{w^{\min\{s,1\}}}\cdot\frac{1}{\delta^{d-2}}.$$

Proof. Fix $\pi\in J_w(\pi)$, then it follows from Lemma 3.1 that $\pi\cap\pi'$ is contained in the union of δ^{2-d} boxes with parameters $\frac{\delta}{w}\times\delta\times\cdots\times\delta$. This means that the diameter of each box is $\sim\delta/w$. We now bound the above sum in two ways:

Since P is δ-separated, each box contains at most $\lesssim\frac{1}{w}$ elements from P. This gives

$$\sum_{p\in\pi(c\delta)}|I(p)\cap J_w(\pi)|\lesssim|J_w(\pi)|\cdot\frac{1}{w}\cdot\frac{1}{\delta^{d-2}}.$$

We also observe that each box is contained in a ball of radius δ/w, this infers each box contains at most $C_P|P|\delta^s\frac{1}{w^s}$ balls of P since P is (δ,s,C_P) set. In total, one has

$$\sum_{p\in\pi(c\delta)}|I(p)\cap J_w(\pi)|\lesssim|J_w(\pi)|\cdot|P|\delta^s\cdot\frac{1}{w^s}\cdot\frac{1}{\delta^{d-2}}.$$

Note that by assumption we have $|P|\delta^s\ge1$ so that combining these two estimates, we get

$$\sum_{p\in\pi(c\delta)}|I(p)\cap J_w(\pi)|\lesssim|J_w(\pi)|\cdot|P|\delta^s\cdot\frac{1}{w^{\min\{s,1\}}}\cdot\frac{1}{\delta^{d-2}}.$$

This completes the proof of the lemma. $\square$

We now continue the proof of the incidence estimate. In particular,

$$\begin{aligned} J(\pi) &\lesssim \sum_{p\in\pi(C\delta)}\sum_i |I(p)\cap J_{2^i\delta}(\pi)|^{x/y} \\ &\lesssim \sum_i \sum_{p\in\pi(C\delta)} |I(p)\cap J_{2^i\delta}(\pi)|\cdot|J_{2^i\delta}(\pi)|^{x/y-1} \\ &\lesssim \sum_i |(2^i\delta)^t|\Pi||^{x/y}\cdot|P|\delta^s\cdot\frac{1}{(2^i\delta)^{\min\{s,1\}}}\cdot\frac{1}{\delta^{d-2}}. \end{aligned}$$

We now fall into the following cases:

If $s\geq 1$ and $tx/y\geq 1$, then

$$J(\pi)\lesssim|\Pi|^{x/y}\cdot|P|\delta^s\cdot\frac{1}{\delta^{d-2}}\cdot\sum_i(2^i\delta)^{\frac{tx}{y}-1}\lesssim|\Pi|^{x/y}\cdot|P|\delta^s\cdot\frac{1}{\delta^{d-2}}.$$

If $s\geq 1$ and $tx/y<1$, then

$$J(\pi)\lesssim|\Pi|^{x/y}\cdot|P|\delta^s\cdot\frac{1}{\delta^{d-2}}\cdot\sum_i(2^i\delta)^{\frac{tx}{y}-1}\lesssim|\Pi|^{x/y}\cdot|P|\delta^{s+\frac{tx}{y}-1}\cdot\frac{1}{\delta^{d-2}}.$$

If $s<1$ and $tx/y\geq s$, then

$$J(\pi)\lesssim|\Pi|^{x/y}\cdot|P|\delta^s\cdot\frac{1}{\delta^{d-2}}\cdot\sum_i(2^i\delta)^{\frac{tx}{y}-s}\lesssim|\Pi|^{x/y}\cdot|P|\delta^s\cdot\frac{1}{\delta^{d-2}}.$$

If $s<1$ and $tx/y<s$, then

$$J(\pi)\lesssim|\Pi|^{x/y}\cdot|P|\delta^s\cdot\frac{1}{\delta^{d-2}}\cdot\sum_i(2^i\delta)^{\frac{tx}{y}-s}\lesssim|\Pi|^{x/y}\cdot|P|\delta^{\frac{tx}{y}-d+2}.$$

Plugging these estimates into $I_{C\delta}(P,\Pi)$, one has

Case 1: If $s\geq 1$ and $tx/y\geq 1$, then

$$I_{C\delta}(P,\Pi)\lesssim|P||\Pi|\delta^{\frac{y}{x+y}(s-d+2)}.$$

If $s-d+2<0$ then we only have the trivial upper bound. Thus, this upper bound is valid in the range $s-d+2>0$. By choosing $x=y$, we obtain the upper bound

$$I_{C\delta}(P,\Pi)\lesssim|P||\Pi|\delta^{\frac{1}{2}(s-d+2)}.$$

Case 2: If $s\geq 1$ and $tx/y<1$, then

$$I_{C\delta}(P,\Pi)\lesssim|P||\Pi|\delta^{\frac{y}{x+y}\left(s+\frac{tx}{y}-d+1\right)}.$$

We observe that $t\leq 1$ since $x\geq y$. If $s-d+2<0$, then we only have the trivial upper bound. If

$s-d+2>0$, then we can choose $y=xt+\epsilon$, $\epsilon>0$, to get the upper bound

$$I_{C\delta}(P,\Pi)\lesssim |P||\Pi|\delta^{\frac{t}{1+t}(s-d+2)-\epsilon}.$$

Case 3: If $s<1$ and $tx/y\geq s$, then

$$I_{C\delta}(P,\Pi)\lesssim |P||\Pi|\delta^{\frac{y}{x+y}(s-d+2)}.$$

In this case, since $s<1$ and $d\geq 3$, we have $s-d+2$ is always negative, so only trivial upper bound is obtained.

Case 4: If $s<1$ and $tx/y<s$, then

$$I_{C\delta}(P,\Pi)\lesssim |P||\Pi|\delta^{\frac{y}{x+y}\left(\frac{tx}{y}-d+2\right)}.$$

Since $tx/y<s$, $s<1$, and $d\geq 3$, we only have the trivial upper bound in this case.

Putting these cases together, Theorem 1.5 follows.

4 Sharpness of Theorem 1.4

In this section we provide an example to show the sharpness of our result in Theorem 1.4. The construction is mainly adapted from the two dimensional example due to Fu and Ren in [4]. For the reader's convenience, we sketch the ideas here.

We first recall that Construction 4 in [4] shows that in the plane when $s+t\geq 3$, the incidence between balls and tubes is $\sim \delta^{-(s+t-1)}=\delta|P||\mathcal{T}|$. We will see that their example can be extended to higher dimensions which gives us a sharp upper bound.

Note that the incidences results of Fu and Ren [4] in the plane are applied to the sets of balls and tubes satisfying Definition 1.2 (Katz-Tao (δ,s) sets). However, as we mentioned before, when the set is of size $\sim\delta^{-s}$ then Definitions 1.1 and 1.2 are equivalent. Moreover, in their sharpness example, what they constructed are actually (δ,s,C) sets. Therefore, we can extend their construction to higher dimensional spaces. In the rest, we denote $D=\delta^{-1}$ for convenience.

4.1 Construction in $\mathbb{R}^3$

The idea is to construct a configuration such that the intersection between this configuration and the plane O_{xy} is exactly Construction 4 in [4]. Thus, we can reduce to the two-dimensional case when we consider the intersection between δ-hyperplanes and δ-balls with O_{xy}. Then we move the plane O_{xy} vertically by spacing 2δ (two consecutive planes have distance 2δ) to get the desired bound.

We consider balls of form:

$$(x-a_1)^2+(y-a_2)^2+z^2=\delta^2.$$

Clearly these are δ-balls in $\mathbb{R}^3$ whose intersection with O_{xy} are also δ-balls in dimension two. For each tube in Construction 4 in [4], we just extend the tube vertically to the $\mathbb{R}^3$ which becomes a plane. Thus, if we put balls by spacing 2δ in $\mathbb{R}^3$ (the centers of two consecutive balls have distance of 2δ), then the incidence between these balls and planes is $\sim\delta^{-(s+t-1)}\cdot\delta^{-1}=\delta\cdot\delta^{-s-1}\cdot\delta^{-t}$ which is equal to $\delta|P||\Pi|$.

The next step is to check that the set of balls and hyperplanes constructed above satisfy Definition 1.1. Since the hyperplanes are just the tubes expanding vertically, the number of δ-hyperplanes is equal to that of δ-tubes, i.e, $\sim D^t$. Moreover, as already shown in the paper of Fu and Ren [4] (page 6) that the set of δ-tubes is a (δ, t, C) set which gives that our set of δ-hyperplanes is also a (δ, t, C) set.

Now for each copy of O_{xy}, there are $\sim D^s$ balls, as stated in [4], and there are δ^{-1} copies in total. So $|P| \sim D^{s+1}$. Moreover, as shown in [4], for each copy of O_{xy}, a ball of radius w in the plane contains at most $Cw^s\delta^{-s}$ δ-balls. So, a ball B_w in $\mathbb{R}^3$ contains at most $Cw^s\delta^{-s-1}$ δ-balls in P. This gives that

$$|\{p \in P : p \subset B_w\}| \leq Cw^s\delta^{-s-1} \sim Cw^s|P|.$$

Therefore, the set P is a (δ, s, C)-set.

4.2 Construction in $\mathbb{R}^d$, $d \geq 4$

For dimension $d > 3$, the construction is inductively constructed. For instance, we consider $d = 4$. We directly extend each hyperplane from above 3-dimensional construction to a hyperplane in $\mathbb{R}^4$ by just adding a variable x_4. Precisely, assume a hyperplane in above 3-dimensional construction is defined by $\{(x_1, x_2, x_3) : ax_1 + bx_2 + cx_3 = e\}$, then the extension in $\mathbb{R}^4$ is defined by $\{(x_1, x_2, x_3, x_4) : ax_1 + bx_2 + cx_3 + 0x_4 = e\}$. We now consider balls defined by the equation:

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + x_3^2 + x_4^2 = \delta^2.$$

These balls are moved by spacing 2δ in x_3 and x_4 directions. Thus, we see that when we fix the last coordinate x_4, the number of incidences between these hyperplanes and balls is the same as we constructed above in 3-dimension which is $\delta\delta^{-s-1}\delta^{-t}$. Moreover, we have δ^{-1} number of copies in x_4 direction which implies that the number of incidences is $\delta\delta^{-s-1}\delta^{-t}\delta^{-1} = \delta|P||\Pi|$. It is not difficult to prove that the sets of balls and hyperplanes are (δ, s, C) and (δ, t, C) sets, respectively. For other dimensions $d > 4$, the construction works in the same way. Thus, the number of incidences $I(P, \Pi) \sim |\{\text{incidences in the plane}\}| \cdot \delta^{2-d} \sim \delta|P||\Pi|$.

For the sharpness of Theorem 1.5, one might think of the same idea by extending Constructions 1, 2, 3 in [4] to higher dimensions, but it does not match the upper bounds in Theorem 1.5 at least in the way we tried.

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