CHAPTER II :ROOTS AND WEIGHTS FOR $\mathrm{SL_n}(\mathbb{R})$

May 15, 2025

In this chapter, we review some basis theory of roots and weight. We will first recall the general theory and compute explicitly the examples for $\mathrm{SL_n}(\mathbb{R})/\mathrm{GL_n}(\mathbb{R})$.

1 Structure theory

1.1 The Cartan subalgebra

First we need the notion of Cartan subalgebra

Definition 1.1. *For any Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be Cartan algebra if it is*

- $\mathfrak{h}$ *is a nilpotent subalgebra.*
- *It is self normalizing. In particular, we have $\mathfrak{h} = \{x \in \mathfrak{g} : [x, \mathfrak{g}] \subset \mathfrak{g}\}$.*

When $\mathfrak{g}$ is a semisimple Lie algebra, we have the following theorem

Theorem 1.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field k of charateristic 0 with a subalgebra $\mathfrak{h}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ if and only if it is a maximal toral subalgebra, i.e. is is maximal among all subalgebras contanining only semisimple elements.*

add citation.

1.2 Root space decomposition

With respect to some choice of Cartan subalgebra, we have a root space decomposition. In particular, there is a finite set $\Phi \subset \mathfrak{h}^*$ of linear forms on H, whose elements are called **roots**, such that

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right),$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \forall h \in \mathfrak{h}\}$ for any $\alpha \in \Phi$.

1.3 A specific example: root space decomposition for $\mathfrak{sl}_n(\mathbb{R})$

For the semisimple Lie algebra $\mathfrak{sl}_\mathrm{n}(\mathbb{R})$, a typical choice of the Cartan subalgebra is the set

$$\mathfrak{h} = \left\{ H = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}, a_1 + a_2 + \ldots + a_n = 0 \right\}$$

With respect to this Cartan subalgebra, we can define the linear function

$$L_i \colon \mathfrak{h} \to \mathbb{R}, \quad H \mapsto L_i(H) = a_i$$

Then the roots are given by $\alpha_{ij} := L_i - L_j$ for distinct i, j. We have the root space decomposition for $\mathfrak{sl}_\mathrm{n}(\mathbb{R})$ as follows

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus \mathfrak{g}_{\alpha_{ij}} \right).$$

For the sake of brevity, we will denote $\alpha_{i,i+1}$ by α_i - these are called **simple roots**.

1.4 Roots at group level

Since the main object in this thesis is the Lie groups, we want to understand how the roots behave at group level. The analog for the Cartan subalgebra is the maximal torus

$$T = \left\{ t = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} : a_i \neq 0 \right\},$$

Then T acts on $\mathfrak{g}$ by conjugation. Explicitly, we can check that

$$\mathrm{Ad}(t)(E_{ij}) = t_i t_j^{-1} E_{ij}$$

Therefore, at the group level, the character $\alpha_{ij}(\mathrm{diag}(t_1, \dots, t_n)) = t_i t_j^{-1}$ is a root whenever $i \neq j$. The set

$$\Delta = \left\{ \alpha_i \mid i = \overline{1, n} \right\}$$

where

$$\alpha_i \colon T \mapsto \mathbb{R}, t \mapsto \frac{t_i}{t_{i+1}}$$

is the set of **simple roots**. We can decompose the set of root into to disjoint subsets, namely

$$\Phi = \{\alpha_{ij}, i \neq j\} = \Phi_+ \coprod \Phi_-$$

where the set Φ_+ comprises of α_{ij} for $i < j$ and the remaining roots are in Φ_-. The former comprises of **positive roots** while the latter contains **negative roots**. We have the following lemma

Lemma 1.3. *Each $\alpha \in \Phi$ can be written uniquely as a linear combination*

$$\alpha = m_1\alpha_1 + \ldots + m_d\alpha_d$$

with all $m_i \in \mathbb{Z}_{\geqslant 0}$ or $m_i \in \mathbb{Z}_{\leqslant 0}$. If $\alpha \in \Phi_+$ then all $m_i \geqslant 0$, otherwise $m_i \leqslant 0$ for all i.

1.5 Weights

Another class of linear forms that we are interested in are the **fundamental weights**. For each fundamental weights λ_i, we define

define in term of Lie algebra

$$\lambda_i \colon T \to \mathbb{R}, \quad \lambda_i(t) = a_1 \ldots a_i$$

We have the following

Lemma 1.4. *We can write*

$$\lambda_i := r_1\alpha_1 + r_2\alpha_2 + \ldots + r_d\alpha_d$$

where r_i's are rational number such that $r_i \geqslant 0$.add proof

Example 1.5. *When $n = 3$, we have the following relationsadd picture*

$$\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \quad \lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$$

Definition 1.6. *A weight λ is called **dominant** if it satisfies $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geqslant 0}$ for all α*

Clearly by lemma 1.3, the weight λ is dominant if and only if $\langle \lambda, \alpha_i^\vee \rangle$ for all fundamental root α_i. It is also clearly that the set of fundamental weight is given by addition of the fundamental weights, namely

$$\Lambda^+ := \{c_1\lambda_1 + \ldots + c_d\lambda_d \mid c_i \in \mathbb{Z}_{\geqslant 0}\}$$

The set of dominant weights is denoted Λ^+. A weight $\lambda = \sum n_i \lambda_i$ is called strongly dominant if $n_i > 0$ for all i. One important example is the minimal strongly dominant weight given by

$$\rho = \sum \lambda_i$$

This is called **Weyl vector** and is characterized in several ways:

1. $\langle \rho, \alpha_i^\vee \rangle = 1$ for all i.
2. $$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

To prove the last equation we use the action of the Weyl group W. Let $\mu = \frac{1}{2} \sum \alpha$. Apply the simple reflection s_i given by

$$s_i(x) = x - \langle x, \alpha_i^\vee \rangle \alpha_i$$

We know that s_i sends α_i to $-\alpha_i$ and permutes the other positive roots. So:

$$s_i(\mu) = \mu - \langle x, \alpha_i^\vee \rangle \alpha_i$$

Therefore, $(\mu, \alpha_i) = \mu(h_i) = 1$ for all i. So, $\mu = \rho$.

1.6 Weyl groups