Normal Families and the Riemann Mapping theorem

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March 11, 2024

This note is used to list every theorems in chapter 9 of the book **Complex made simple**.

1 Quasi-metrics

Definition 1. *A function* $d\colon X \times X \to [0, \infty]$ *satisfying the condition*

- $d(x, x) = 0 (x \in X)$
- $d(x, y) = d(y, x) (x, y \in X)$
- $d(x, z) \leqslant d(x, y) + d(y, z) (x, y, z)$

then it is called a quasi-metric on space X*. We can see that* d *is almost the same as a metric except that* $d(x, y)$ *can be zero for distinct* x, y*.*

One can construct a metric $\overline{d}$ from quasi-metric d, noting that d is an equivalence relation on X.
Now we introduction the notion of *concave function*: The function $\psi\colon I \to \mathbb{R}$ is said to be *concave* if

$$\psi(tx + (1 - t)y) \leqslant t\psi(x) + (1 - t)\psi(y),$$

for all $x, y \in X$ and $0 \leqslant t \leqslant 1$. It can be inferred from the definition that ψ is concave iff $-\psi$ is convex. Now we have the following lemma

Lemma 1. *Suppose that* $\psi\colon I \to \mathbb{R}$ *is concave and* $a_1, a_2, b_1, b_2 \in I$ *such that* $a_1 < b_1, a_2 < b_2, a_2 \geqslant a_1, b_2 \geqslant b_1$*. Then*

$$\frac{\psi(b_2) - \psi(a_2)}{b_2 - a_2} \leqslant \frac{\psi(b_1) - \psi(a_1)}{b_1 - a_1}$$

Geometrically speaking, the slope of the segment joining two points on the graph of ψ decreases as the points moves to the right.