Name:
Due date: 02/14

Problem 1

Let $q \geq 2$ be an integer and let χ be a Dirichlet character mod q. Consider the L

$$L(z;\chi) := \sum_{i=1}^{\infty} \frac{\chi(n)}{n^z}$$

which converges absolutely for $\Re(z) =: x > 1$ since

$$\sum_{n\geq 1} |\chi(n)n^{-z}| \geq \sum_{n\geq 1} n^{-x} < \infty,$$

by the $p-$test from FPM. Show that $L(z;\chi)$ defines a holomorphic function on $X = \{z \in \mathbb{C} : \Re(z) > 1\}$ and $L(z;\chi) \neq 0$ for all $z \in X$. Furthermore, show that

$$\frac{L'(z;\chi)}{L(z;\chi)} = -\sum_{p} \frac{\chi(p)\log(p)}{p^z - \chi(p)};$$

for $z \in X$.

Proof. Using Weierstrass M-test as in the statement in the problem, we have proved that the sequence

$$L_N(z;\chi) = \sum_{n=1}^{N} \frac{\chi(n)}{n^z}$$

converges uniformly to the function $L(z;\chi)$. In particular, $L(z;\chi)$ is continuous over the domain X. Clearly the sequence of functions L_N are holomorphic on X. Thus we have

$$\int_\gamma L_N(z;\chi)dz = 0,$$

for piecewise smooth closed curve inside X, by Cauchy's theorem. Furthermore, we have

$$\int_{\partial\Delta} L(z;\chi) = \lim_{N\to\infty} \int_{\partial\Delta} L_N(z,\chi) = 0$$

By Morera's theorem, we can conclude that $L(z,\chi)$ is holomorphic over the domain X.

Clearly the domain X is simply connected. Using proposition 2.3 in the note, we can rewrite the $L-$function as

$$L(z;\chi) = \sum_{i=1}^{\infty} \frac{\chi(n)}{n^z} = \prod_p \frac{1}{1-\chi(p)p^{-z}}$$

Note that over $X = \{z \in \mathbb{C} : \Re(z) > 1\}$, we have

$$\frac{1}{1-\chi(p)p^{-z}} = 1 + \chi(p)p^{-z} + (\chi(p)p^{-z})^2 + \ldots = 1 + \sum_{k\geq 1} \chi(p^k)p^{-kz}$$

where

$$\sum_{k\geq 1} |\chi(p^k)p^{-kz}| \leq \sum_{n\geq 1} n^{-\Re z} < \infty$$

Let denote $u_p(z) = \sum_{k\geq 1} \chi(p^k)p^{-kz}$, then

$$L(z;\chi) = \prod_p (1 + u_p(z))$$

Then $L(z;\chi) = 0$ if and only if $u_p(z) = -1$ for some p. But this is impossible as

$$|u_p(z)| \leq \sum_{k\geq 1} |\chi(p^k)p^{-kz}| = \sum_{k\geq 1} p^{-k\Re z} = \frac{p^{-\Re(z)}}{1 - p^{-\Re z}} = \frac{1}{p^{\Re z} - 1} < 1$$

Thus $L(z;\chi)$ never vanishes on the domain X.

Lastly, we will prove the identity for $\frac{L'(z;\chi)}{L(z;\chi)}$. From the discussion of the note, there exists a branch of the logarithm $\log L(z,\chi)$ on X and satisfies

$$\log L(z;\chi) = \sum_p \mathrm{Log}(1 - \chi(p)p^{-z})$$

We can then define the partial sum $S_m = \sum_{p\leq m} \mathrm{Log}(1 - \chi(p)p^{-z})$. Clearly this partial sum converges uniformly to $\log L(z;\chi)$ as $m \to \infty$ and are holomorphic. Thus

$$S_m'(z) = -\sum_{p\leq m} \frac{\chi(p)\log(p)}{p^z - \chi(p)} \to [\log(L(z;\chi))] = \frac{L'(z;\chi)}{L(z;\chi)}$$

In particular, for $z \in X$, we have

$$\frac{L'(z;\chi)}{L(z;\chi)} = -\sum_p \frac{\chi(p)\log(p)}{p^z - \chi(p)},$$

where the sum extends over all prime number p, as desired. $\square$

Problem 2

Let G be a finite abelian group. Show that $\#\hat{G} \leq \#G$.

Proof. First we will prove that the set S of maps $f\colon G \to \mathbb{C}$ has a vector space structure. The sum and the scalar multiplication of maps are defined as follows

- $(\chi_1 + \chi_2)(a) := \chi_1(a) + \chi_2(a)$ for all $a \in G$.
- $(c\chi)(a) := c \cdot \chi(a)$ for any $\chi \in S$ and $c \in \mathbb{C}$.

But we can check all the axioms for a set being a vector space pointwisely, with the zero beging the zero map. So S have a a structure of vector space. Let's compute the dimension of S as as $\mathbb{C}-$ vector space. Since G is finite, we can assume that, as a set

$$G = \{a_1, \ldots, a_n\}$$

Then we can define the map f_i to be the "dual" of a_i in the following sense: $f_i(a_j) = \delta_{ij}$, i.e. $f_i(a_i) = 1$ and vanishes at $a_j \neq a_i$. We claim that $\mathfrak{B} = \{f_i\}$ forms a basis of S. Indeed, let $f\colon G \to \mathbb{C}$ be arbitrary. Since G is finite, f is determined by its value at each $a_i \in G$. Assume that $c_i = f(a_i)$. Then for any $1 \leq i \leq n$, we have

$$f(a_i) = c_i = \sum_{i=1}^{n} c_i f_i(a_i)$$

In particular, we can rewrite the following identity as

$$f = c_1 f_1 + c_2 f_2 + \ldots + c_n f_n,$$

which implies $\mathfrak{B}$ spans S. On the other hand, if

$$0 = c_1 f_1 + \ldots + c_n f(n)$$

Applying both sides to a_i yields

$$0 = c_i f_i(a_i) = c_i$$

which means $\mathfrak{B}$ is a set of linearly independent vectors. Thus $\dim S = \#\mathfrak{B} = n$. In the note, we proved that the character $\chi_1, \chi_2, \ldots, \chi_m$ are mutually orthogonal. In particular, they are a subset of S comprising of linearly independent vectors. Thus

$$m = \#\hat{G} \leq n = \#G.$$

Remark: In fact, if we assume the theorem about the structure of finite abelian group, we can prove that the equality always happens, namely $\#\hat{G} = \#G$ $\square$

Problem 3

Consider the finite abelian group G_9 consisting of the invertible elements of $\mathbb{Z}/9\mathbb{Z}$. Find all the characteres of G_9. Make sure you prove that the characters of G_9 you find are all distinct and there are no others.

Proof. Using the Remark in the previous exercise, we predict that there are 6 characters in total. It can be checked easily that 2 generates G_9, so it is a cyclic group of order 6. Indeed, we have

$$\begin{aligned} 2^1 &\equiv 2 \pmod 9 \\ 2^2 &\equiv 4 \pmod 9 \\ 2^3 &\equiv 8 \pmod 9 \\ 2^4 &\equiv 7 \pmod 9 \\ 2^5 &\equiv 5 \pmod 9 \\ 2^6 &\equiv 1 \pmod 9 \end{aligned}$$

So the character is defined solely by determining where it sends $[2]$ in $\mathbb{C}$. Let $\chi \in \hat{G}_9$ be any character, we then have

$$\chi(2)^6 = \chi(2^6) = \chi(1) = 1 \in \mathbb{C}$$

thus implies $\chi(2)$ must be a $6-th$ root of unity. There are 6 choices in total, namely

$$\chi(2) = e^{\frac{2i\pi k}{6}}, 0 \leq k \leq 6$$

Clearly these are 6 distinct characters, and they are all possible characters of G_9.

We can further compute a character table as follows: $\chi(n) \pmod 9$

n	1	2	4	5	7	8
$\chi_1(n)$	1	1	1	1	1	1
$\chi_2(n)$	1	ω	ω^2	$-\omega^2$	$-\omega$	-1
$\chi_3(n)$	1	ω^2	$-\omega$	$-\omega$	ω^2	1
$\chi_4(n)$	1	-1	1	-1	1	-1
$\chi_5(n)$	1	$-\omega$	ω^2	ω^2	$-\omega$	1
$\chi_6(n)$	1	$-\omega^2$	$-\omega$	ω	ω^2	-1

where $\omega = e^{i\pi/3}$. $\square$

Problem 4

Show that there is exactly one real, non-principal Dirichlet character $\chi \pmod 9$. Find $\chi(916)$.

Proof. Continueing from the previous exercise, with a note that a character is called real character if its image lies entirely in $\mathbb{R}$, namely we have a group homomorphism $\chi\colon G_9 \to \mathbb{R}$. Since we know that all character $\chi \in \hat{G}$ satisfy

$$|\chi(a)| = 1, \text{ for all } a \in G_9,$$

we can deduce that a real character χ must satisfy $\chi(G_9) \subset \{\pm 1\}$. Since we want to find a non-principal character, the only choices is that $\chi(G_9) = \{\pm 1\}$. As shown above, χ is defined solely by its value at 2, so we must choose $\chi(2) = -1$. This implies that there is only one non-principal, real Dirichlet character over G_9.

Note that $916 \equiv 16 = 2^4 \pmod 9$. Thus

$$\chi(916) = \chi(2^4) = (\chi(2)^4) = (-1)^4 = 1.$$

And we are done. $\square$