Introduction to Schemes

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1 Why Schemes?

1.1 Summary of affine varieties

Let k be an algebraic closed field. The main idea of classical algebraic geometry is that we have a correspondence

$$\{\text{subsets of } k^n \text{ cutout by polynomials}\} \leftrightarrow \{\text{finitely generated reduced } k- \text{ algebras }\}$$
$$\text{Geometry} \leftrightarrow \text{Algebra}$$

In particular, the above correspondence can be given as follows:

- $I \subset k[x_1, \ldots, x_n]$ ideal: Then we define

  $$X := Z(I) = \{a \in k^n | f(a) = 0 \quad \forall f \in I\}$$

  sometimes we use the notation $V(I)$ instead of $Z(I)$. This kind of set is an affine variety.

- $\mathbb{A}^n$: $n-$ dimensional affine space. As a set, it is just k^n, but we equip this set with *Zariski* topology - where the closed subsets are generated by $Z(I)$.

- $I(X) := \{f \in k[x_1, \ldots, x_n] | f(x) = 0 \quad \forall x \in X\}$. Then the quotient replacing $k[X] := \dfrac{k[x_1, \ldots, x_n]}{I(X)}$ is called *coordinate ring* of X.

- $k[X]$ parametrizes functions on X:

  $$x \in X \rightsquigarrow \mathfrak{m}_x := \ker(ev_x : k[X] \to k)$$

  and $\forall f \in k[x]$ gives

  $$\begin{aligned} f \colon X &\to \mathbb{A}^1 = k \\ x &\mapsto f(x) = \overline{f} \in k[x]/\mathfrak{m}_x \end{aligned}$$

- Hilbert's weak Nullstellensatz:

$$\{ \text{ points of } X\} \leftrightarrow \{ \text{ maximal ideals of } k[X]\}$$

- Hilbert Nullstellensatz: $I(Z(I)) = \sqrt{I} := \{f : f^n \in I \text{ for some } n\}$.

- Morphisms: given X and $Y \in \mathbb{A}^{\text{n}}$ a morphism between two affine varieties is given by $\varphi = (f_1, \ldots, f_n)$. This morphism induces a $k-$ algebra homomorphism

$$\varphi^* : k[Y] \to k[X], \quad \varphi^*(\psi) = \psi \circ \varphi, \qquad \begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, dashed, "\varphi^* f"'] & Y \arrow[d, "f"] \\ & \mathbb{A}^1 \end{tikzcd}$$

  so $\text{Hom}(X, Y) = \text{Hom}(k[Y], k[X])$ - which gives the equivalence of categories as stated in the beginning.

1.2 Why varieties are not good enough?

Some possible reasons are:

1. embedding into $\mathbb{A}^{\text{n}}$ shouldn't be part of the data. It would be nice to have an intrinsic definition, since you can embed the same variety in different spaces.

2. for non-algebraic closed field, the Nullstellensatz doesn't work: $I = (x^2 + y^2 + 1)$ is a prime ideal in $\mathbb{R}[x, y]$, hence is a radical ideal. But $Z(I) = \varnothing$, so $I(Z(I)) = \mathbb{R}[x, y]$.

3. We can ask, on which topological space is $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$ naturally a functions space? Or $\mathbb{Z}[x]$? Or $\mathbb{Z}$? This leads to the idea of considering all possible rings.

4. Nilpotent arises naturally when deforming varieties, so ignoring them is not a good option.

2 The prime spectrum

In the last sections, we have an equivalence between categories

$$\text{affine varieties over } k = \overline{k} \cong_{op} \text{ reduced finitely generated } k- \text{ algebras}$$

Now we want to extend this equivalent relation as follows:

$$\text{affine schemes } \cong_{op} \text{ commutative ring with unit}$$

This generalization allows to study arithmetic phenomena by geometric methods, by taking rings to be $\mathbb{Z}, \mathbb{Z}_p$, etc.

Definition 1. *Given a ring R, its spectrum is defined to be*

$$\operatorname{Spec} R := \{\mathfrak{p} | \mathfrak{p} \subset R \textit{ is a prime ideal}\}$$

This way ,$x \in \operatorname{Spec} R \leftrightsquigarrow \mathfrak{p}_x \in R$.
NB: in general, we cannot think about $f \in R$ as functions with values in a fixed field k. However, there is a more general notion.

Definition 2. *Let $x \in \operatorname{Spec} R$ correspond to $\mathfrak{p} \subset R$. The* ***residue field*** *of x (or $\mathfrak{p}$) is*

$$\kappa(x) = \kappa(\mathfrak{p}) := R_{\mathfrak{p}} / \mathfrak{p} R_{\mathfrak{p}}$$

Every $f \in R$ has a "value"

$$f(x) := f \mod \mathfrak{p}_x \in \kappa(x), \quad \forall x \in \operatorname{Spec} R$$

and the codomain depends on the choice of x. By definition, $f(x) = 0$ if $f \in \mathfrak{p}_x$. The moral of the story is that $\operatorname{Spec} R$ will be the space on which R is the ring of functions: the affine scheme corresponding to R.