

# The Pentagonal Number Theorem and classification of two dimensional lattices

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## 1 Introduction

- Partition of a positive number  $n$
- Pentagonal number theorem

## 2 Modular forms

- Elliptic integral
- The modular picture

# Warm-up problem

*Given a positive number  $n$ , how many ways can we write  $n$  in the form*

$$a_1 + a_2 + \dots + a_k?$$

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There are in total  $2^{n-1}$  of them, however, there are a lot of repetitions.

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Table: Initial partitions

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So you will get more and more repetitions listing this way. For example:

$$\begin{aligned}4 &= 1 + 1 + 1 + 1 \\&= 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 \\&= 2 + 2 \\&= 3 + 1 = 1 + 3\end{aligned}$$



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$n$	1	2	3	4	5	6
$p(n)$	1	2	3	5	7	12

Table: First few values of  $p(n)$

# Euler's first formulae

Euler came up with the following generating function for  $p(n)$

## Theorem 1

We have the following identity

$$(1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots)(\dots) = \sum p(n)x^n$$

## Proof.

This is just combinatorics, we actually counts the number smaller than  $n$  that appears the partition. □

A similar method can be used to count the number of partitions that contains a specific set of given number.

# Problems?

This method is very slow if one wants to compute  $p(n)$  explicitly for  $n$  large.

A programme in *Mathematica* following this method takes around 50s to compute the first fifty values of  $p(n)$ .

# A faster method

## Ring of formal series

Let

$$\mathcal{U} = \left\{ 1 + a_1x + a_2x^2 + \dots = \sum_{n \geq 0} a_n x^n \right\}$$

Then we can define a multiplication by

$$(1 + a_1x + a_2x^2 + \dots)(1 + b_1x + b_2x^2 + \dots) = 1 + c_1x + c_2x^2 + \dots$$

where  $c_k = a_k + a_{k-1}b_1 + \dots + b_k$

# A faster method

## Theorem 2

The set  $\mathfrak{U}$  with this product forms a group.

## Proof.

Left to the audiences  $\Rightarrow$ )). □

In particular, we have that

$$\frac{1}{1 - x^k} = \sum_{n \geq 0} x^{nk} = 1 + x^k + x^{2k} + \dots$$



## Theorem 3

We have the following generating function for  $p(n)$

$$\frac{1}{\prod_{k \geq 1} (1 - x^k)} = \sum_{n \geq 1} p(n) x^n$$

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## Proof.

This follows from theorem 1 by taking the inverses of the series on the left hand sides. □

# The pentagonal number theorem

Euler hoped to find a pattern that emerges from the denominator on the left hand side. He did that by multiplying everything and get the following

$$\prod_{k \geq 1} (1 - x^k) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \dots$$

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Note that we obviously have

$$(1 - x - x^2 + x^5 + x^7 - x^{12} - \dots)(1 + p(1)x + p(2)x^2 + p(3)x^3 + \dots) = 1$$

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So we have a recurrence relation

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15) + \dots = 0$$

Using this relation, it is much faster to compute the partition number.

# Why Pentagonal?

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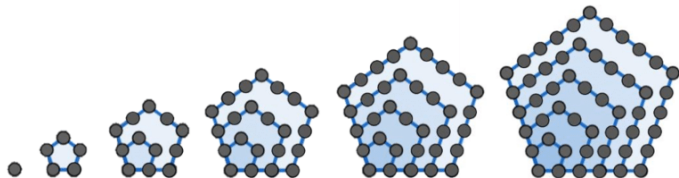
$$1, 5, 12, 22, 35, \dots$$

All the terms showed up here are the pentagonal number! In particular, they are given by the formulae

$$n = \frac{k(3k-1)}{2}, \quad k \geq 1$$



# Why Pentagonal



# Pentagonal number theorem

It turns out that the numbers at even positions is also given by pentagonal number formula, with opposite sign

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This is later proved algebraically by Euler in 1750 and again by Franklin in 1881.

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$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We know from calculus that the total length can be computed by

$$\int \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

# Length of an ellipse

Solve for the equation of ellipse in term of  $y$ , we get

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

which implies that the total length of an ellipse is

$$4a \int_0^1 \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} dt = 4 \int_0^1 \frac{1 - k^2 t^2}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} dt$$

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where  $k = \sqrt{\frac{a^2 - b^2}{a^2}}$ .

For  $k = 0$ , this is just the perimeter of the circle. The general case is called *elliptic integral*.

# A case study of Gauss

Gauss essentially tried to compute the elliptic integral by looking at the following integral

$$F(x) = \int_0^x \frac{1}{\sqrt{1-z^4}} dz$$

Mimicking the case for the integral

$$\int_0^x \frac{1}{\sqrt{1-z^2}} dz = \sin^{-1}(x)$$

Gauss tried to find an inverse function of  $F$ . This inversion is called *elliptic functions*.



# Weierstrass $\wp$ functions

Abel, and later Weierstrass, created an elliptic functions from scratch using the lattice.

## Lattice

A lattice  $L \subset \mathbb{R}^2$  is a set of the form

$$L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$$

where  $e_1, e_2$  are linearly independent over  $\mathbb{R}$ .

# Weierstrass $\wp$ functions

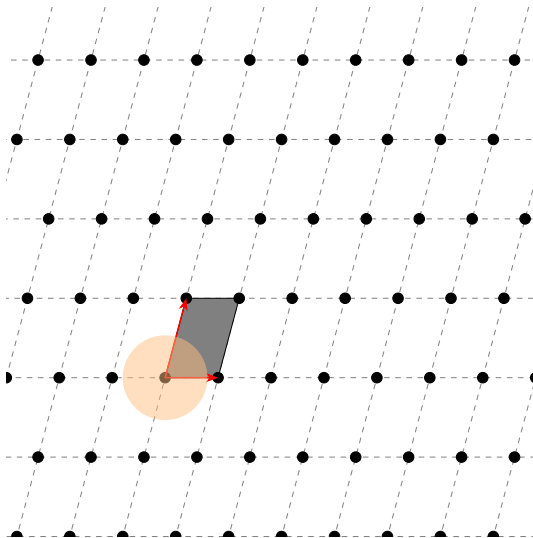


Figure: Example of a lattice

# Weierstrass $\wp$ functions

Weierstrass defined his function as

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n \neq 0} \left( \frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right)$$

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This function is holomorphic and doubly periodic. Differentiating both sides yields another elliptic functions

$$\wp'(z) = -2 \sum_{m,n} \frac{1}{(z - m\omega_1 - n\omega_2)^3}$$

# Weierstrass $\wp$ functions

## Theorem : Doubly periodic functions with prescribed periods

Every doubly periodic function with periods  $\omega_1, \omega_2$  can be written uniquely in the form

$$R_1(\wp(z)) + R_2(\wp(z))\wp'(z).$$

In other words, any doubly periodic function with periods  $\omega_i$  is an element of the function field  $\mathbb{C}(\wp, \wp')$

In particular, there are not many double periodic functions.

# Weierstrass $\wp$ functions

## Theorem

We have

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

# Classification of lattices

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**Answer:** Up to magnification, rotation and change of basis, the answer is yes.



# Fundamental domain

Up to rotations and magnifications, we can reduce a lattice

$$L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$$

to a lattice of the form

$$L_z = \mathbb{Z}z \oplus \mathbb{Z}, \quad \Im(z) > 0$$

So the upper half-plane parametrizes the 2 dimensional lattices.

## Classification of unit lattices

The map  $z \mapsto \mathbb{Z}z \oplus \mathbb{Z}$  induces a bijection

$$\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \cong \{ \text{lattices} \} / \mathbb{C}^\times$$

# Fundamental domain

So we reduce to study the space of lattices by looking the action of  $SL_2(\mathbb{Z})$  on the upper half plane. Geometrically, the domain is given by

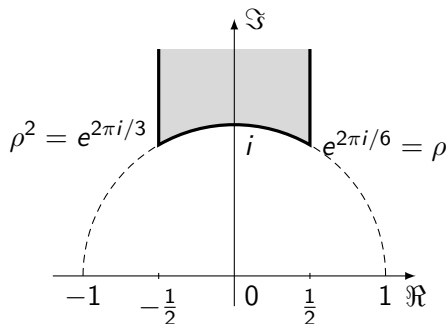
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# A brief look at modular forms

**Goal:** we have to assign each point  $z \in \mathfrak{D}$  a number  $j(z)$  such that  $j$  detects the  $z$  corresponding to distinct lattices.

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## Generators of $SL_2(\mathbb{Z})$

As a group,  $SL_2(\mathbb{Z})$  is generated by two elements

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So we are after a function  $j: \mathbb{H} \rightarrow \mathbb{C}$  such that

$$j(Sz) = j(Tz) = j(z)$$

# A brief look at modular forms

If  $g(z)$  is a holomorphic function on the unit disk, then  $g(e^{2i\pi\tau})$  is a holomorphic function over the upper half plane and have a period  $p = 1$ . So for each holomorphic function on the unit disk, we get a candidate. The problem is to find a function  $f(\tau) = g(e^{2i\pi\tau})$  such that

$$f(\tau) = f\left(\frac{1}{-\tau}\right)$$

We have the *Eisenstein series* defined as

$$g_4(\tau) = 60 \sum_{m,n \neq 0} \frac{1}{(m\tau + n)^4}$$

and

$$g_6(\tau) = 140 \sum_{m,n \neq 0} \frac{1}{(m\tau + n)^6}$$

# A brief look at modular forms

It can be shown that  $g_4, g_6$  are holomorphic functions over the unit disk. We formalize the notion of modular forms as follows

## Definitions of modular forms

A modular form of level  $k$  is a function  $f(\tau)$  on the upper half plane associated with a power series  $g(z)$  by the formula  $f(\tau) = g(e^{2i\pi\tau})$  that satisfies

$$f(-1/\tau) = \tau^k f(\tau)$$

We denote  $M_k$  the set of such weight  $k$  modular forms.



# A brief look at modular forms

## Theorem

The  $M_k$  are finite dimensional vector spaces. When  $k$  is odd, they contain only the zero vector.

- 1 If  $k$  is even and  $k \equiv 2 \pmod{12}$ , then  $\dim M_k = \lfloor \frac{k}{12} \rfloor$ .
- 2 If  $k$  is even and  $k \not\equiv 2 \pmod{12}$ , then  $\dim M_k = \lfloor \frac{k}{12} \rfloor + 1$ .
- 3 Thus  $M_0, M_2, M_4, M_6, M_8, M_{10}$ , and  $M_{12}$  have dimensions 1, 0, 1, 1, 1, 1, 2.
- 4 The product of an element of  $M_k$  and an element of  $M_l$  is an element of  $M_{k+l}$ .
- 5  $g_2 \in M_4$  and  $g_3 \in M_6$ .
- 6  $M_0$  only contains constants.

We know that over  $M_{12}$  we have at least two linearly independent modular forms - denoted by  $h_1$  and  $h_2$ . Then

$$j(\tau) = \frac{h_1}{h_2}$$

Seems to be the right function.

# Dedekind $\eta$ function

Now recalled the inverse of the partition generating function discovered by Euler

$$g(z) = 1 - z - z^2 + z^5 + z^7 - z^{12} - z^{15} + \dots$$

This is a holomorphic function over the unit disk. Let  $f(\tau) = g(e^{2i\pi\tau})$ . Then

## Dedekind's theorem

Let  $\eta(\tau) = e^{2i\pi\tau/24} f(\tau)$ . Then

$$\eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau)$$

# The $j$ –invariant

Let us define

$$j(\tau) = \frac{g_4^2(\tau)}{\eta^{24}(\tau)}$$

Then  $j$  has only a pole at  $\infty$  and  $j: \mathfrak{D} \rightarrow \mathbb{C}$  is a bijection.

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## Main theorem

Two lattices are equivalent under magnification, rotation, and base change if and only if they have the same  $j$ -invariant.