Montel theorem and some related results

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In this expository note, I will try to explain explicitly how to compactify $\Gamma\backslash\mathbb{H}$ by adding points in two ways.

1 Some preparations

We will always denote Γ a subgroup of the group $SL_2(\mathbb{Z})$ of finite index, and this group acts on the upper half complex plane $\mathbb{H}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ z := \frac{az+b}{cz+d}$$

When z tends to infinity, we have

$$\lim_{z\to\infty} \frac{az+b}{cz+d} = \frac{a}{c},$$

so we add the rational line to define the action of this group at ∞. In particular, we consider the set

$$\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$$

Note that on the projective rational line, we define the action to be the multiplication of a 2×2 matrix with a 2×1 vector. Then under this action, we have the following lemma

Lemma 1. *$SL_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$.*

Proof. For each point in $\mathbb{P}^1(\mathbb{Q})$, we can choose the representative to be of the form $[a : b]$, where $\gcd(a, b) = 1$. Then there exists $x, y \in \mathbb{Z}$ such that

$$ax - by = 1$$

Thus we get the following equality

$$\begin{bmatrix} b & a \\ -x & y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This implies any points in $\mathbb{P}^1(\mathbb{Q})$ can be moved to $[0 : 1]$, and thus the action is transitive. $\square$

Corollary 2. *If Γ is a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$ then $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$ has only finite orbits.*

2 Compactification of $\Gamma\backslash\mathbb{H}$ by adding points.

We introduction a topology on $\overline{\mathbb{H}}$. For the usual upper half plane, the topology are the usual metric topology on $\mathbb{C}$, and we only define the system of neighborhood of $r \in \mathbb{P}^1(\mathbb{Q})$.

Let $S(c, \omega)$ be the circle that touches the real line at $\omega = p/q$ and has the radius $\frac{c}{2q^2}$. Then the collection of circle $D(c, \omega) = \bigcup_{0<c'\leqslant c} S(c', \omega)$ is called *Farey disk*. Let $c \to 0$, these disks define a neighborhood of ω. The Farey disks at ∞ is defined to be the region

$$D(T, \infty) = \{z : \Im z \geqslant T\}$$

It can be checked easily that the matrix under inversion, the Farey disk at ∞ is mapped to $D(1/T, 0)$. In general, if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \circ \infty = \omega$ then $D(T, \infty)$ is mapped to $D(1/T, \omega)$.

With the above topology on the extended upper half plane, we could show that

Lemma 3. $\Gamma\backslash\overline{\mathbb{H}}$ *is a compact set.*

Proof. By Corollary 2, we have

$$\Gamma\backslash\overline{\mathbb{H}} = \Gamma\backslash\mathbb{H} \cup \bigcup_{i=1}^{n} \Gamma_i,$$

where Γ_i's are orbits of $\mathbb{P}^1(\mathbb{Q})$ under the action of Γ. $\square$