CHAPTER :SEMI-STABLE LATTICE IN HIGHER RANK

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In this chapter, we will establish the notion of semi-stable lattice. Heuristically, this is the lattice that achieve all the successive minima at the same time, see [**?**].
We will provide two different definitions of semi -stable lattice: one is geometric - which follows Grayson's idea of utilizing the canonical plot, and one is purely algebraic, which make use of the maximal standard parabolic subgroups. The toy model will be the moduli space of 2-dimensional lattice, which is essential the upper half plane in the complex field. At the end, we will show that the two definitions coincide.

1 Lattices in higher rank

For each z with $\Im(z) > 0$, we can attach to z a lattice structure $L_z = \mathbb{Z}z \oplus \mathbb{Z}$. Roughly speaking a lattice is a discrete subgroup that is generated by a $k-$ basis of the k-space V. In particular, we will only work with the real vector space V. Grayson works with lattice over a ring of algebraic integers, but we will restrict to just the lattice that has the underlying structure as a $\mathbb{Z}-$ module.

1.1 First definition of lattices

Definition 1.1 (Abstract $\mathbb{Z}$-lattices)**.** *Let L be a finitely generated $\mathbb{Z}$-module. In particular, it is a free $\mathbb{Z}$-module of finite rank. Suppose that L is endowed with a real-valued positive definite*[1] *quadratic form $Q\colon L \to \mathbb{R}$, such that the set*

$$\{x \in L : Q(x) \leqslant r\}$$

is finite for any real number r. We will call the pair (L, Q) a ***abstract $\mathbb{Z}$-lattice****.*

An easy example is to take $L = \mathbb{Z}^n$ and choose our quadratic form to be the standard one. namely

$$\langle x, y\rangle = x \cdot y = \sum_{i=1}^{n} x_i y_i$$

Here the multiplication is just the usual dot product between 2 vectors. In term of matrix, this quadratic form is assigned to the identity matrix I_n.
If there is no further confusion, we can just denote a Euclidean lattice by L, without specifying the bilinear form Q. The lattice L determines a full-rank lattice inside $L_{\mathbb{R}}$, namely, the rank of the lattice L is equal to the dimension of $L_{\mathbb{R}}$.

1.2 An alternative definition of lattices

For the sake of computation, we also usually adopt another definition of the lattice. In particular, we view lattice as a free $\mathbb{Z}-$ module of rank n that is isomorphic to $\mathbb{R}^n$ via base changing.

[1] The non-degenerate implicity state that rank L is the same as $dim L_{\mathbb{R}}$

Definition 1.2. *A lattice in $\mathbb{R}^n$ is a subset $L \subset \mathbb{R}^n$ such that there exists a basis $b_1, \ldots, b_n$ of $\mathbb{R}^n$ such that*

$$L = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \ldots \mathbb{Z}b_n$$

If we put the vector $b_1, b_2, \ldots, b_n$ in columns, with respect to the standard basis, namely

$$g = [b_1|b_2| \ldots |b_n],$$

then $L = g\mathbb{Z}^n$.

In the second definition, we can just identify L with the standard lattice $\mathbb{Z}^n$ and the symmetric positive definite form is $g^t g$. So an Euclidean $\mathbb{Z}$-lattice is an abstract lattice with the standard positive definite quadratic form.

1.3 Equivalence between two definitions of lattices

In this subsection, we will show that every abstract $\mathbb{Z}$- lattice is isomorphic to an Euclidean $\mathbb{Z}-$ lattice. This will be helpful in visualizing the abstract lattices, as we are just looking at concrete lattices with deformation by a linear transformation.
First we need to specify the notion of isomorphic lattices - in the first definition

Definition 1.3. *A map $f\colon (L, Q) \to (L', Q')$ is an* ***isomorhism*** *between lattices if it is a group isomorphism and for all $x \in L$, we have*

$$Q(x) = Q'(f(x))$$

Proposition 1.4. *Any abstract lattice is isomorphic to a Euclidean $\mathbb{Z}-$ lattice.*

Proof. Let (L, Q) be an arbitrary lattice. We define a bilinear form as

$$\langle x, y\rangle := \frac{Q(x+y) - Q(x-y)}{4}$$

We will show that this bilinear form defines an inner product over the real vector space $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$. Clearly we have $\langle x, x\rangle = 4Q(x)/4 = Q(x) \geqslant 0$ for all $x \in L \backslash \{0\}$. Now the extended bilinear form is defined as

$$\begin{aligned} \langle \cdot, \cdot \rangle : L_{\mathbb{R}} \times L_{\mathbb{R}} &\to \mathbb{R} \\ (x \otimes a, y \otimes b) &\mapsto ab\,\langle x, y\rangle \end{aligned}$$

It is immediate that the extended bilinear form is inner product. So we have proved that $L_{\mathbb{R}}$ is a Euclidean space containing L. Moreover, L is embedded injectively in $L_{\mathbb{R}}$ as $\mathbb{R}$ is a flat $\mathbb{Z}$ module. The condition that

$$\#\{x \in L : Q(x) \leqslant r\} < \infty$$

implies L can be identified with a discrete in $L_{\mathbb{R}}$. But this implies that there exists a basis $\{b_1, \ldots, b_n\} \subset L_{\mathbb{R}}$ such that

$$L = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \ldots \mathbb{Z}b_n$$

Hence we are done. ☐

fix the proof so that we use the definition 1.4.

1.4 Covolume of a lattice

Now that for every abstract lattice L we can find an invertible matrix g such that

$$L \cong g\mathbb{Z}^n$$

The number n is called the **rank** of the lattice L.
Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of $L_{\mathbb{R}} \cong \mathbb{R}^n$ and

$$g = [b_1|b_2| \ldots |b_n].$$

The covolume of the lattice L is defined as

Definition 1.5. *The covolume of L is given by the formulae*

$$\mathrm{vol}(L_{\mathbb{R}}/L) = |\det(b_i \cdot e_j)|$$

The rank and covolume are invariant numerical values of L, as they don't depend on the choice of basis. Indeed, two bases of a rank n lattice L are related by a transformation $g \in \mathrm{GL}_n(\mathbb{Z})$. Clearly this preserves the volume and the rank as a $\mathbb{Z}-$module.

1.5 Sublattices

To work with semi-stable lattice, we need to consider all the lattices containing inside

Definition 1.6 (sublattice)**.** *Let (L, Q) be a Euclidean $\mathbb{Z}$-lattice. We say that a $\mathbb{Z}-$submodule M of L a **sublattice** if and only if L/M is torsion free.*

From this definition, we can prove that M is a sublattice of L if it satisfies one of the following equivalent properties:

1. M is a summand of L.
2. every basis of M can be extended to a basis of L.
3. The group M is an intersection of L with a rational subspace of $L_{\mathbb{R}}$.

We refer to the [**?**] for a proof of these equivalences.

Example 1.7. *If $L = \mathbb{Z}^2$, then any sublattice of L is a primitive vector $u = (a, b)$, i.e $\gcd(a, b) = 1$. Indeed, $u = (a, b)$ is a sublattice of $\mathbb{Z}^2$ if and only if there exists a vector $v \in \mathbb{Z}^2$ such that $L = \mathbb{Z}u \oplus \mathbb{Z}v$. With respect to the usual inner product on $\mathbb{R}^2$, we have*

$$1 = \mathrm{vol}(\mathbb{Z}^2) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

This happens if and only if $\gcd(a, b) = 1$.

2 Semi-stable lattices: two definitions

2.1 Grayson's definition

In this section, we introduce the idea of Grayson in defining *semi-stable* lattices. In particular, he associates every lattices a plot and its convex hull - called *profiles*. An easy observation is that, if $M \subset L$ is a sublattice, then the space $M_{\mathbb{R}} = M \otimes \mathbb{R}$ is a subspace of $L_{\mathbb{R}}$, equipped with the restriction of the positive definite symmetric form Q of L, hence M is also a lattice of rank not exceeding rank of L.

Definition 2.1 (slope)**.** *The slope of a non-zero lattice L is the number*

$$\mu(L) = \frac{\log \mathrm{vol}(L)}{\dim L}$$

Definition 2.2. *Suppose we have a lattice L. For any sublattice $M \subset L$, we assign M to a point*

$$l(M) = (\dim M, \log \mathrm{vol}(M))$$

*in the plane $\mathbb{R}^2$. The collection of all points $l(M)$ where M ranges over all sublattices of L is called **the canonical plot** of the lattice L. By convention, we assign the lattice of zero rank to the origin of the plane.*

Example 2.3. *Add example about computing the volume of sublattices.*

The following lemma asserts that, for each vertical axis $x = i$, there is a lowest point.

Lemma 2.4. *Given a lattice L and a number c, there exists only a finite number of sublattices $M \subset L$ such that $\mathrm{vol}(M) < c$.*

Proof. We will prove by induction on the rank of the sublattices.

- For $r = 1$, the collection of all rank 1 sublattices of L is just the set of all vectors in L. So we reduce to show that for any $c > 0$, the set $B(0, c) \cap L$ has finitely many elements. But this follows immediately from the fact that L is a discrete subset of $L_{\mathbb{R}}$.

- Assume that the lemma holds for $r > 1$. Assume that
$$M = \mathbb{Z}m_1 \oplus \dots \mathbb{Z}m_r$$
is a sublattice of the lattices L of rank n. Consider the wedge product $\bigwedge^r L$, then clearly $m_1 \wedge m_2 \dots \wedge m_r$ is a vector in the lattice $\bigwedge^r L$. By the previous case, there are finitely many vectors with bounded length inside lattice. So we only need to show that the map
$$M \mapsto \bigwedge^r M$$
is finite to one, then we are done. But this is clear.

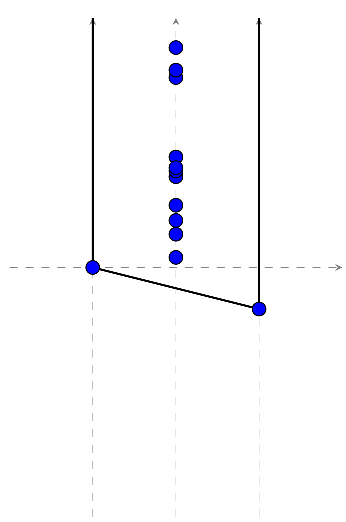
So the canonical plot is bounded below. $\square$

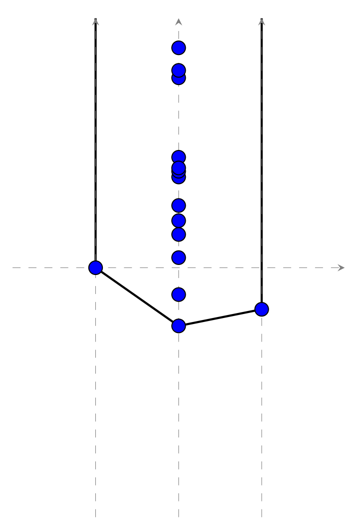
Definition 2.5. *The boundary polygon of the convex hull of the canonical plot is called* ***profile*** *of the lattice L.*

In theory, we can compute the profile by searching for the shortest vector in each of its exterior product, but this computation is infeasible when the dimension of the lattice grows. Since there are lattices with arbitrarily large volume of any rank smaller than that of L, we add to the side the point $(0, \infty)$ and (n, ∞). The sides of the profile are therefore two vertical lines. The bottom is just the convex polygonal connecting the origin with the point $l(L) = (n, \log \mathrm{vol}(L))$, where n is the rank of L.

Definition 2.6. *If the bottom of the profile contains only two points $(0, 0)$ and $(n, \log \mathrm{vol}\, L)$, then the lattice L is said to be* ***semi-stable****. Otherwise L is said to be* ***unstable****.*

Here are the picture of two lattices. The one on the left is semi-stable while the one on the right is unstable.





Visually, a lattice is called **semi-stable** if it satisfies the other equivalent conditions: If M is an arbitrary sublattice of L then $\mu(M) \geqslant \mu(L)$.

2.2 Parabolic k-subgroups

We will first recall what is a $k-$ parabolic subgroups for the general linear group GL_n for $n \geqslant 2$, over an arbitrary field k. Let $e_1, e_2, \ldots, e_n$ be a standard basis for the vector space k^n. From linear algebra, we know that each linear map $T\colon k^n \to k^n$ can be identified with a $n \times n$ matrix. In particular we obtain an identification between the group $\mathrm{GL}_n(k)$ with $\mathrm{GL}(k^n)$ of $k-$ automorphisms of k^n.

Definition 2.7. *A flag $\mathcal{F}$ of k^n is a chain of linear subspaces*

$$\mathcal{F}\colon 0 \subset F_1 \subset F_2 \subset \ldots \subset F_r \subset k^n$$

Let $d_i = \dim F_i$, then we call the ordered r-tuple $(d_1, d_2, \ldots, d_r)$ the type of the flag $\mathcal{F}$.

A parabolic subgroup of $\mathrm{GL}(k^n)$ is the stabilizer $P_{\mathcal{F}} = P$ of a flag $\mathcal{F}$. A parabolic subgroup P is call *minimal* if it stabilizes a flag of type $(1, 2, \ldots, n)$.
Let $e_1, \ldots, e_n$ be the standard basis for the k-vector space k^n. For any $1 \leqslant i \leqslant n$, define V_i to be $e_1 + \cdots + e_i$. We call a flag $\mathcal{V}$ by the chain

$$\mathcal{V} : 0 \subset V_{d_1} \subset V_{d_2} \subset \cdots \subset V_{d_s} \subset k^n,$$

a standard flag in k^n. Let $d_0 = 0$ and $d_{s+1} = n$. We define $r_j := d_j - d_{j-1}$, where $j = 1, \ldots, s+1$. Then $\rho = (r_1, \ldots, r_{s+1})$ is an ordered partition of n into positive integers, i.e., an ordered sequence of positive integers so that $r_1 + \cdots + r_{s+1} = n$. The corresponding standard parabolic subgroup $P_{\mathcal{V}} := P_\rho$ consists of all matrices in $\mathrm{GL}_n(k)$ admitting a block decomposition whose diagonal blocks are $(r_j \times r_j)$-matrices in $\mathrm{GL}_{r_j}(k)$, $j = 1, \ldots, r+1$, the lower entries are 0, and the other entries are arbitrary. Every parabolic subgroup of $\mathrm{GL}_n(k)$ is conjugate to a subgroup of this type.

Definition 2.8. *The maximal standard parabolic subgroups in $GL_n(k)$ corresponds to the stabilizer of the flag of type $\rho_i = (i, n-i)$, where $i = 1, \ldots, n-1$ of n. We will further denote $Q_i = P_{\rho_i}$ and* ***MaxParSt*** *the collection of such maximal parabolic subgroups.*

Example 2.9. *Below we list all the standard parabolic subgroup in $GL_3(\mathbb{R})$ and $GL_4(\mathbb{R})$.*

- *For $GL_3(\mathbb{R})$, there are three standard parabolic subgroups corresponding to three partitions of* 3, *namely*

  $$3 = 1 + 1 + 1, \quad 3 = 1 + 2, \quad 3 = 2 + 1$$

  For a partition $(r_1, \ldots, r_{s+1})$, we denote $P_{(r_1,\ldots,r_{s+1})}$ the corresponding parabolic subgroups. Thus we have

  $$P_{1,1,1} = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\}, \quad P_{1,2} = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\}$$

  $$P_{2,1} = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$$

  Clearly $\boldsymbol{MaxParSt} = \{P_{2,1}, P_{1,2}\}$.

- *For $GL_4(\mathbb{R})$, there are seven standard parabolic subgroups for seven partitions*

  $$4 = 1 + 1 + 1 + 1, \quad 4 = 1 + 1 + 2, \quad 4 = 1 + 2 + 1$$

  $$4 = 2 + 1 + 1, \quad 4 = 1 + 3, \quad 4 = 2 + 1, \quad 4 = 3 + 1$$

Explicitly, we have the following subgroups

$$P_{1,1,1,1} = \left\{\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}\right\}, \qquad P_{1,1,2} = \left\{\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}\right\}$$

$$P_{1,2,1} = \left\{\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix}\right\}, \qquad P_{2,1,1} = \left\{\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}\right\}$$

$$P_{1,2,1} = \left\{\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix}\right\}, \qquad P_{2,1,1} = \left\{\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}\right\}$$

$$P_{1,3} = \left\{\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}\right\}, \qquad P_{3,1} = \left\{\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix}\right\}$$

$$P_{2,2} = \left\{\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}\right\}$$

Clearly ***MaxParSt*** $= \{P_{1,3}, P_{3,1}, P_{2,2}\}$.

2.3 On the function H_P

Ask professor Manish about how to transfer H_P to H_B

2.4 ρ- definition of semi-stability

We are now ready to define the ρ-definition of semi-stable lattice. Recall that we define the space of lattices of rank n by $X_n := K\backslash \mathrm{GL}_n(\mathbb{R})$, where K is the orthogonal subgroup.

Definition 2.10 (ρ-definition)**.** *Let $x \in X_n$ be an arbitrary lattice, then the lattice x is called* ***semi-stable*** *if and only if its degree of instability* $\deg_{inst}(x) \geqslant 0$*, where*

$$\deg_{inst}(x) := \min_{Q \in MaxParSt, \gamma \in GL(\mathbb{Q})/Q_i(\mathbb{Q})} \langle \rho_Q, H_Q(x\gamma) \rangle$$

A simple observation is that - a lattice x is semi - stable if for all maximal standard parabolic subgroups Q_i, we have

$$\min_{\gamma \in \mathrm{GL}_n(\mathbb{Q})/Q_i(\mathbb{Q})} \langle \rho_Q, H_Q(x\gamma) \rangle \geqslant 0$$

Note that, in the definition of degree of instability, we can further replace H_Q with H_B. This implies that, if

$$x = kan, \quad k \in K, a \in A, n \in N,$$

as in Iwasawa decomposition, then $H_B(x) = H$ where $H = \exp(a)$. In particular, if

$$a = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}$$

then

$$\langle \rho_{Q_i}, H_B(x) \rangle = a_1 a_2 \dots a_i.$$

Thus, to check for the semi-stability of a lattice x, we just need to look at the A-coordinate of x, and verify whether the system

$$\begin{cases} a_1 \geqslant 0 \\ a_1 a_2 \geqslant 0 \\ \dots \\ a_1 a_2 \dots a_n \geqslant 0 \end{cases}$$

2.5 The equivalent between definitions of semi-stable lattices

So far we have two distinct definitions of semi-stability. The following theorem asserts that they are equivalent:

Proposition 2.11. *Let $x \in X_n = K \backslash SL_n(\mathbb{R})$ - the space of unit lattice. Then x is semi-stable if one of the following equaivalent conditions holds*

1. *The bottom of the profile of x is a line connect solely two points: the origin and $(n, 0)$.*
2. *The degree of instability of x is nonnegative, namely, $\deg_{inst}(x) \geqslant 0$.*

Proof.
If we can prove there is a correspondence between $\gamma \in \mathrm{GL}_n(\mathbb{Q})/Q_i(\mathbb{Q})$ and a sublattice of rank i of x, then we are done. We first need a slight reduction - we identified the quotient $\mathrm{GL}_n(\mathbb{Q})/Q_i(\mathbb{Q})$ with the quotient $\mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z}))$. Now let x be an arbitrary lattice of rank n.
We will first show the following correspondence

$$\mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})) \longleftrightarrow \{ \text{ sublattice of rank } i \text{ of } \mathbb{Z}^n\}$$

We define the map from the collection of sublattices of rank i to the cosets space as follows: For any sublattice $M \subset \mathbb{Z}^n$, there exists a basis of M, denoted by

$$\{v_1, v_2, \dots, v_i\}$$

we can extend this basis to get a basis of $\mathbb{Z}^n$

$$\mathfrak{B}' = \{v_1, v_2, \dots, v_n\}$$

Clearly in $\mathbb{Z}^n$ we have the standard basis $\mathfrak{B} = \{e_1, e_2, \dots, e_n\}$. Clearly there exists a map $\gamma \in \mathrm{GL_n}(\mathbb{Z})$ such that

$$\gamma \cdot e_k = v_k \quad \forall k = 1, 2, \dots, n$$

So we define the map

$$\begin{aligned} \varphi \colon \ \{\text{sublattices of rank } i \text{ of } \mathbb{Z}^n\} &\to \mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})) \\ M &\mapsto [\gamma] \end{aligned}$$

where $[\gamma]$ denoted the equaivalent class of γ in the quotient space. This is a well-defined map. Indeed, Assume that we extend the basis $\mathfrak{B}'$ in a different way to get the basis

$$\mathfrak{B}_1 = \left\{v_1, \dots, v_k, v'_{k+1}, \dots, v'_n\right\}$$

As above, there also exists $\gamma' \in \mathrm{GL_n}(\mathbb{Z})$ such that

$$\gamma' e_k = v_k \quad \forall k \leqslant i, \qquad \text{and} \qquad \gamma' e_k = v'_k \quad \forall k > i$$

But this implies that

$$(\gamma^{-1})\gamma' \cdot e_k = \gamma^{-1} v_k = e_k \quad \forall k \leqslant i$$

So in particular, we have $[\gamma] = [\gamma']$. The inverse map is given by

$$[\gamma] \mapsto \bigoplus_{k=1}^{i} \mathbb{Z}(\gamma \cdot e_i) = M$$

This generalizes in the obvious way for lattice $x = g\mathbb{Z}^n$ for some $g \in \mathrm{GL}_n(\mathbb{R})$. Indeed, we just define the map

$$\begin{aligned} \phi_g \colon \{\text{sublattices of rank } i \text{ of } g\mathbb{Z}^n\} &\to \mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})) \\ M_g = gM = g\bigoplus_{k=1}^{i} \mathbb{Z}v_i &\mapsto [\gamma] \end{aligned}$$

where $\gamma e_k = v_k$ in $\mathbb{Z}^n$ for all $k \leqslant i$ and $\phi_g^{-1}([\gamma]) = g\bigoplus_{k=1}^{i} \mathbb{Z}(\gamma \cdot e_i)$. $\square$