CHAPTER :SEMI-STABLE LATTICE IN HIGHER RANK

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In this chapter, we will establish the notion of semi-stable lattice. Heuristically, this is the lattice that achieve all the successive minima at the same time, see [?].
We will provide two different definitions of semi -stable lattice: one is geometric - which follows Grayson's idea of utilizing the canonical plot, and one is purely algebraic, which make use of the maximal standard parabolic subgroups. The toy model will be the moduli space of 2-dimensional lattice, which is essential the upper half plane in the complex field. At the end, we will show that the two definitions coincide.

1 Lattices in higher rank

For each z with $\Im(z) > 0$, we can attach to z a lattice structure $L_z = \mathbb{Z}z \oplus \mathbb{Z}$. Roughly speaking a lattice is a discrete subgroup that is generated by a $k-$ basis of the k-space V. In particular, we will only work with the real vector space V. Grayson works with lattice over a ring of algebraic integers, but we will restrict to just the lattice that has the underlying structure as a $\mathbb{Z}-$ module. The precise definition of a lattice is as follows:

Definition 1.1 (Euclidean $\mathbb{Z}$-lattices). *Let L be a finitely generated $\mathbb{Z}$-module. In particular, it is a free $\mathbb{Z}$-module of finite rank. Suppose that P is endowed with a real-valued symmetric positive definite*[1] *bilinear form, called Q. Then the space $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$ equipped with the bilinear form Q forms a real inner product space. We will call the pair (L, Q) a **Euclidean $\mathbb{Z}$-lattice**.*

add proof showing L is a lattice in the second definition

If there is no further confusion, we can just denote a Euclidean lattice by L, without specifying the bilinear form Q. The lattice l determines a full-rank lattice inside $L_{\mathbb{R}}$, namely, the rank of the lattice L is equal to the dimension of $L_{\mathbb{R}}$. We first recall the definition of discrete subgroup

Definition 1.2. *Let V be a finite-dimensional vector space over $\mathbb{R}$, endowed with the natural topology. A subgroup L of the additive group underlying the vector space V is said to be discrete if each point y in L has a neighbourhood in V whose intersection with L is $\{y\}$ or, equivalently, if, given a bounded set C in V, the set $C \cap L$ is finite.*

Thus, using the following Proposition, L has a structure of a discrete subgroup $V = L_{\mathbb{R}}$.

Proposition 1.3. *Given a finite-dimensional vector space V over $\mathbb{R}$, let L be a subgroup of the additive group V, and let m be the dimension of the $\mathbb{R}$-span of L in V. Then L is a discrete subgroup if and only if L is a free abelian group of rank m.*

A proof can be found in [?]. We now can define the notion of covolume of a lattice:

Definition 1.4 (Volume). *Let's assume that L is a full-rank lattice and has a basis*

$$L = \mathbb{Z}l_1 \oplus \ldots \oplus \mathbb{Z}l_n$$

[1] The non-degenerate implicity state that rank L is the same as $dimL_{\mathbb{R}}$

Then the volume of this lattice is defined to be the volume of the fundamental parallepiped. In particular, let $\{e_i\}$ be any orthonormal basis of the vector space $V = L_{\mathbb{R}}$. Then

$$\mathrm{vol}(L) := |\det Q(l_i, e_j)|$$

However, for the sake of computation, we also usually adopt another definition of the lattice. In particular, we view lattice as a free $\mathbb{Z}-$ module of rank n that is isomorphic to $\mathbb{R}^n$ via base changing. In more detail

Definition 1.5. *A lattice in $\mathbb{R}^n$ is a subset $L \subset \mathbb{R}^n$ such that there exists a basis $b_1, \ldots, b_n$ of $\mathbb{R}^n$ such that*

$$L = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \ldots \mathbb{Z}b_n$$

If we put the vector $b_1, b_2, \ldots, b_n$ in columns, with respect to the standard basis, namely

$$g = [b_1|b_2| \ldots |b_n],$$

then $L = g\mathbb{Z}^n$.

In the second sense, we can just identify L with the standard lattice $\mathbb{Z}^n$ and the symmetric positive definite form is $g^t g$.
Now, the basic problem we want to deal with is to classify "isomorphic" classes of lattice. Here we say two lattices L_1 and L_2 are isomorphic if and only if there is a map $\gamma \in \mathrm{GL_n}(\mathbb{Z})$ such that

$$\gamma \cdot g_1 = g_2,$$

From the first point of view, we identify L_i with the $\mathbb{Z}-$ module $\mathbb{Z}^n$ associated to the form $g_i^t g_i$. If we define X_n the space of all symmetric positive definite bilinear forms, then we are looking at the space $\mathrm{GL_n}(\mathbb{Z}) \backslash X_n$. We can also regard $L_i \otimes \mathbb{R} \cong \mathbb{R}^n$. From this point of view, the problem of classification isomorphic classes of lattices is the same as looking for discrete subgroups of $\mathbb{R}^n$ of rank n, modulo rotation. We will interchange these equivalent points of view depend on the situation.
As Bill Casselman note in his expository, even if we normalize the lattice to get a unimodular lattice, we will still have to work with arbitrary lattices in the smaller rank. This means we are embedding several copies of $\mathrm{GL}_m(\mathbb{Z})$ along the diagonal of $\mathrm{SL_n}(\mathbb{Z})$. Therefore it is not necessary to normalize the volume of the lattice.

2 Semi-stable lattices: two definitions

In this section, we introduce the idea of Grayson in defining *semi-stable* lattices. In particular, he associates every lattices a plot and its convex hull - called *profiles.* To understand what this means, we must first introduce the notation of *sublattice.*

Definition 2.1 (sublattice)**.** *Let (L, Q) be a Euclidean $\mathbb{Z}$-lattice. We say that a $\mathbb{Z}-$submodule M of L a **sublattice** if and only if L/M is torsion free.*

From this definition, we can prove that M is a sublattice of L if it satisfies one of the following equivalent properties:

1. M is a summand of L.
2. every basis of M can be extended to a basis of L.
3. L/M is torsion free.
4. The group M is an intersection of L with a rational subspace of $L_{\mathbb{R}}$.

Example 2.2. *If $L = \mathbb{Z}^2$, then a sublattice of L is a primitive vector $u = (a, b)$, i.e* $\gcd(a, b) = 1$.

add proof

An easy observation is that, if $M \subset L$ is a sublattice, then the space $M_{\mathbb{R}} = M \otimes \mathbb{R}$ is a subspace of $L_{\mathbb{R}}$, equipped with the restriction of the positive definite symmetric form Q of L,hence M is also a lattice of rank not exceeding rank of L.

As stated in definition 1.4, we can computed a volume of a lattice by base changing and choose an orthonormal basis. However, if we view the lattice L as $\mathbb{Z}^n$ under an action of $g \in \mathrm{GL}_n(\mathbb{R})$ as in definition 1.5, it is more convenient to define volume use wedge product. Suppose L has rank n, then L has a basis $b_1, b_2, \ldots, b_n$ such that

$$b_i = g \cdot e_i, \quad g \in \mathrm{GL}_n(\mathbb{R}),$$

where $e_1, e_2, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. let $\bigwedge^* \mathbb{R}^n$ denote the corresponding exterior algebra. If $\bigwedge^p \mathbb{R}^n$ denotes the pth exterior power of $\mathbb{R}^n$, the products $f_v := e_{v_1} \wedge \cdots \wedge e_{v_p}$, where v ranges over the ordered p-tuples $(v_1, \ldots, v_p)$ subject to the condition $1 \leqslant v_1 < \cdots < v_p \leqslant n$, then form a basis $\{f_v\}$ of $\bigwedge^p \mathbb{R}^n$. In a natural way, $\bigwedge^p \mathbb{R}^n$ permits the Euclidean norm, $\|\cdot\|$ defined by $\|f_v\| = 1$ and $\|\sum_v \lambda_v f_v\| = (\sum_v \lambda_v^2)^{1/2}$.

The group $\mathrm{GL}_n(\mathbb{R})$ operates on the exterior power $\bigwedge^p \mathbb{R}^n$, $p = 1, \ldots, n$, via

$$g(e_{v_1} \wedge \cdots \wedge e_{v_p}) = g(e_{v_1}) \wedge \cdots \wedge g(e_{v_p})$$

and linear extension.