BLOCH'S THEOREM

Lemma 0.1. *Let f be analytic in $\Delta = \{|z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$. If $|f(z)| \le M$ for all $z \in \Delta$, then $f(\Delta)$ contains the disk $|w| \le (\sqrt{M+1} - \sqrt{M})^2$.*

Proof. By Schwartz's lemma, we implicitly have $M \ge 1$. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Using CIF, we have $|a_n| \le M$ for all n. Therefore,

$$\begin{aligned} |f(z)| &\ge |z| - \sum_{n=2}^{\infty} |a_n||z|^n \\ &= r - \frac{Mr^2}{1-r} \end{aligned} \tag{0.1}$$

for $|z| = r < 1$. Obviously, we can maximize the RHS of (0.1) by taking

$$r = \rho = 1 - \sqrt{\frac{M}{M+1}} \tag{0.2}$$

and correspondingly, $|f(z)| \ge (\sqrt{M+1} - \sqrt{M})^2$ for $|z| = \rho$.

For all $|w| < (\sqrt{M+1} - \sqrt{M})^2$, $|f(z) - (f(z) - w)| = |w| \le |f(z)|$ for all $|z| = \rho$. Therefore, $f(z) - w$ and $f(z)$ have the same number of zeros in $|z| < \rho$. It follows that $f(\Delta)$ contains the disk $|w| \le (\sqrt{M+1} - \sqrt{M})^2$. □

Obviously, by "scaling", we have the following:

Lemma 0.2. *Let f be an analytic function on $D = \{|z - a| < R\}$. If $|f(z) - f(a)| \le M$ for all $z \in D$, then $f(D)$ contains the disk $|w - f(a)| \le (\sqrt{M + |f'(a)R|} - \sqrt{M})^2$.*

Lemma 0.3. *An analytic function $f(z)$ on Δ is 1-1 if $|f'(z) - M| < |M|$ for all $z \in \Delta$ and a constant $M \in \mathbb{C}$.*

Proof. Let z_1 and z_2 be two distinct points in Δ and let γ be the line joining z_1 and z_2. Then

$$\begin{aligned} |f(z_1) - f(z_2)| &= \left| \int_\gamma f'(z) dz \right| \\ &= \left| \int_\gamma M - (M - f'(z)) dz \right| \\ &\ge \left| \int_\gamma M dz \right| - \left| \int_\gamma (f'(z) - M) dz \right| \\ &\ge |M| \int_\gamma |dz| - \int_\gamma |f'(z) - M||dz| > 0 \end{aligned} \tag{0.3}$$

and hence $f(z)$ is 1-1. Note: the third line is triangle inequality. □

Theorem 0.4 (Bloch's Theorem). *Let $f(z)$ be an analytic function on Δ satisfying $f'(0) = 1$. Then there is a positive constant B (called Bloch's constant), independent of f, such that there exists a disk $S \subset \Delta$ where f is 1-1 and whose image $f(S)$ contains a disk of radius B. In particular, $B > 1/72$.*

Proof. Obviously, it is enough to show this for $f'(z)$ bounded on Δ.

Let

$$m(r, g) = \max_{|z|=r} |g(z)|. \tag{0.4}$$

We let $0 \leq r_0 < 1$ be the largest number [1] such that $(1 - r_0)m(r_0, f') = 1$. Such r_0 exists since $f'(z)$ is bounded on Δ and $m(r, g)$ is continuous. Indeed, assume that $r_n \to r$ as $n \to \infty$ and denote $k_n = r_n/r$. We could assume that $m(r_n, g) = |g(z_n)|$ for every nonnegative integer n. Then clearly we have

$$|g(k_n z)| \leq g(z_n) \leq g(z) = m(r, g)$$

Since g is continuous, $g(k_n z) \to g(z)$, thus $g(z_n)$ also tends to $g(z)$. This implies $m(r, g)$ is continuous.

Then $(1 - r)m(r, f') < 1$ for $r > r_0$ and hence

$$|f'(z)| \leq \frac{1}{1 - |z|} \tag{0.5}$$

for $|z| \geq r_0$. And by principle of maximum modulus, we have

$$|f'(z)| \leq m(r_0, f') = \frac{1}{1 - r_0} \tag{0.6}$$

for $|z| \leq r_0$. In conclusion,

$$|f'(z)| \leq \frac{1}{1 - \max(r_0, |z|)} \tag{0.7}$$

for all $z \in \Delta$.

Let $a \in \Delta$ be a number such that $|a| = r_0$ and $|f'(a)| = 1/(1 - r_0)$.

For $0 < \rho < 1 - r_0$ and $|z - a| \leq \rho$, we have

$$|f'(z) - f'(a)| \leq \frac{1}{1 - r_0} + \frac{1}{1 - r_0 - \rho} \tag{0.8}$$

and hence

$$|f'(z) - f'(a)| \leq \frac{|z - a|}{\rho} \left(\frac{1}{1 - r_0} + \frac{1}{1 - r_0 - \rho} \right) \tag{0.9}$$

by Schwartz's lemma [2]. Therefore, $|f'(z) - f'(a)| < |f'(a)|$ for z in the disk

$$S = \left\{ |z - a| < \frac{\rho(1 - r_0 - \rho)}{2(1 - r_0) - \rho} \right\}. \tag{0.10}$$

The radius of S is founded by solving for $f'(a) >$ RHS 0.9.

[1] Here Xi Chen defined using max, but I think sup would be more accurate.

[2] See the end of the note.

By Lemma 0.3, f is 1-1 on S. Obviously, the radius of S is maximized when we set $\rho = (2-\sqrt{2})(1-r_0)$ and correspondingly,

$$S = \left\{ |z-a| < (3-2\sqrt{2})(1-r_0) \right\}. \tag{0.11}$$

Moreover, since

$$|f(z) - f(a)| \le \ln\left(\frac{\sqrt{2}+1}{2}\right) \tag{0.12}$$

for $z \in S$ by (0.7), we conclude that $f(S)$ contains a disk of radius

$$\left(\sqrt{\ln\left(\frac{\sqrt{2}+1}{2}\right) + (3-2\sqrt{2})} - \sqrt{\ln\left(\frac{\sqrt{2}+1}{2}\right)} \right)^2 > \frac{1}{72}. \tag{0.13}$$

□

Remark 0.5. The key to the proof of Bloch's theorem is the existence of $a \in \Delta$ and positive constants C_1 and C_2 such that $|f'(z)| \le C_2|f'(a)|$ for all $|z-a| \le C_1/|f'(a)|$.

Supplementary notes: A variant of Schwarz's lemma

Theorem 0.6 (A variant of Schwarz's lemma)**.** *Let* $f\colon \{|z| \le R\} \to \mathbb{C}$ *such that* $|f(z)| \le A$ *for all* z *and* $f(0) = 0$*. Then*

$$f(z) \le \frac{A|z|}{R}$$

Proof. Let define the function $g(y) := \dfrac{f(Ay)}{R}$. Then clearly $g\colon \Delta \to \Delta$ and satisfying $g(0) = 0$. By the usual Schwarz's lemma, we must have $|g(y)| \le |y|$. Change $y \to \frac{z}{R}$, we get the desired inequality. □