CHAPTER :SEMI-STABLE LATTICE IN HIGHER RANK

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In this chapter, we will establish the notion of semi-stable lattice. Heuristically, this is the lattice that achieve all the successive minima at the same time, see [?].
We will provide two different definitions of semi -stable lattice: one is geometric - which follows Grayson's idea of utilizing the canonical plot, and one is purely algebraic, which make use of the maximal standard parabolic subgroups. The toy model will be the moduli space of 2-dimensional lattice, which is essential the upper half plane in the complex field. At the end, we will show that the two definitions coincide.

1 Lattices and semi-stable lattice in higher rank

For each z with $\Im(z) > 0$, we can attach to z a lattice structure $L_z = \mathbb{Z}z \oplus \mathbb{Z}$. Roughly speaking a lattice is a discrete subgroup that is generated by a $k-$ basis of the k-space V. In particular, we will only work with the real vector space V. Grayson works with lattice over a ring of algebraic integers, but we will restrict to just the lattice that has the underlying structure as a $\mathbb{Z}-$ module. The precise definition of a lattice is as follows:

Definition 1.1 (Euclidean $\mathbb{Z}$-lattices)**.** *Let L be a finitely generated $\mathbb{Z}$-module. In particular, it is a free $\mathbb{Z}$-module of finite rank. Suppose that P is endowed with a real-valued symmetric positive definite*[1] *bilinear form, called Q. Then the space $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$ equipped with the bilinear form Q forms a real inner product space. We will call the pair (L, Q) a **Euclidean $\mathbb{Z}$-lattice**.*

add proof showing L is a lattice in the second definition

If there is no further confusion, we can just denote a Euclidean lattice by L, without specifying the bilinear form Q. The lattice l determines a full-rank lattice inside $L_{\mathbb{R}}$, namely, the rank of the lattice L is equal to the dimension of $L_{\mathbb{R}}$. We first recall the definition of discrete subgroup

Definition 1.2. *Let V be a finite-dimensional vector space over $\mathbb{R}$, endowed with the natural topology. A subgroup L of the additive group underlying the vector space V is said to be discrete if each point y in L has a neighbourhood in V whose intersection with L is $\{y\}$ or, equivalently, if, given a bounded set C in V, the set $C \cap L$ is finite.*

Thus, using the following Proposition, L has a structure of a discrete subgroup $V = L_{\mathbb{R}}$.

Proposition 1.3. *Given a finite-dimensional vector space V over $\mathbb{R}$, let L be a subgroup of the additive group V, and let m be the dimension of the $\mathbb{R}$-span of L in V. Then L is a discrete subgroup if and only if L is a free abelian group of rank m.*

A proof can be found in [?]. We now can define the notion of covolume of a lattice:

Definition 1.4 (Volume)**.** *Let's assume that L is a full-rank lattice and has a basis*

$$L = \mathbb{Z}l_1 \oplus \ldots \oplus \mathbb{Z}l_n$$

[1] The non-degenerate implicity state that rank L is the same as $dimL_{\mathbb{R}}$

Then the volume of this lattice is defined to be the volume of the fundamental parallepiped. In particular, let $\{e_i\}$ *be any orthonormal basis of the vector space* $V = L_{\mathbb{R}}$*. Then*

$$\mathrm{vol}(L) := |\det Q(l_i, e_j)|$$

However, for the sake of computation, we also usually adopt another definition of the lattice. In particular, we view lattice as a free $\mathbb{Z}-$ module of rank n that is isomorphic to $\mathbb{R}^n$ via base changing. In more detail

Definition 1.5. *A lattice in* $\mathbb{R}^n$ *is a subset* $L \subset \mathbb{R}^n$ *such that there exists a basis* $b_1, \ldots, b_n$ *of* $\mathbb{R}^n$ *such that*

$$L = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \ldots \mathbb{Z}b_n$$

If we put the vector $b_1, b_2, \ldots, b_n$ *in collumn, with respect to the standard basis, namely*

$$g = [b_1|b_2| \ldots |b_n],$$

then $L = g\mathbb{Z}^n$*.*

In the second sense, we can just identify L with the standard lattice $\mathbb{Z}^n$ and the symmetric positive definite form is $g^t g$.
Now, the basic problem we want to deal with is to classify "isomorphic" classes of lattice. Here we say two lattices L_1 and L_2 are isomorphic if and only if there is a map $\gamma \in \mathrm{GL_n}(\mathbb{Z})$ such that

$$\gamma \cdot g_1 = g_2,$$

Note that here we use the second point of view, i.e. we identify L_i with the $\mathbb{Z}-$ module $\mathbb{Z}^n$ associated to the form $g_i^t g_i$. If we define X_n the space of all symmetric positive definite bilinear form, then we are looking at the space $\mathrm{GL_n}(\mathbb{Z}) \backslash X_n$