Normal Families and the Riemann Mapping theorem

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This note is used to list every theorems in chapter 9 of the book **Complex made simple**.

1 Quasi-metrics

Definition 1. *A function $d\colon X \times X \to [0, \infty]$ satisfying the condition*

- $d(x, x) = 0 (x \in X)$
- $d(x, y) = d(y, x)(x, y \in X)$
- $d(x, z) \leqslant d(x, y) + d(y, z)(x, y, z)$

then it is called a quasi-metric on space X. We can see that d is almost the same as a metric except that $d(x, y)$ can be zero for distinct x, y.

One can construct a metric $\overline{d}$ from quasi-metric d, noting that d is an equivalence relation on X.
Now we introduction the notion of *concave function*: The function $\psi\colon I \to \mathbb{R}$ is said to be *concave* if

$$\psi(tx + (1-t)y) \leqslant t\psi(x) + (1-t)\psi(y),$$

for all $x, y \in X$ and $0 \leqslant t \leqslant 1$. It can be inferred from the definition that ψ is concave iff $-\psi$ is convex. Now we have the following lemma

Lemma 1. *Suppose that $\psi\colon I \to \mathbb{R}$ is concave and $a_1, a_2, b_1, b_2 \in I$ such that $a_1 < b_1, a_2 < b_2, a_2 \geqslant a_1, b_2 \geqslant b_1$. Then*

$$\frac{\psi(b_2) - \psi(a_2)}{b_2 - a_2} \leqslant \frac{\psi(b_1) - \psi(a_1)}{b_1 - a_1}$$

Geometrically speaking, the slope of the segment joining two points on the graph of ψ decreases as the points moves to the right.
Applying this lemma for $a_1 = 0, b_1 = x = a_2, b_2 = x + y$, we get the following result:

Lemma 2. *Suppose that $\psi\colon [0, \infty] \to \mathbb{R}$ is concave and $\psi(0) = 0$. Then*

$$\psi(x + y) \leqslant \psi(x) + \psi(y),$$

for all $x, y \geqslant 0$.

Lemma 3. *Suppose that $\psi\colon [0, \infty] \to \mathbb{R}$ is concave and*

$$\psi(0) = 0.$$

*Suppose further that ψ is nondecreasing, $\psi(t) > 0$ for $t > 0$ and this function is continuous at 0.
If d is a quasi-metric on X then $\tilde{d} = \psi \circ d$ is also a quasi-metric on X such that*

- $\tilde{d}(x, y) = 0$ *iff* $d(x, y) = 0$.
- $\tilde{d}(x_n, y_n) \to 0$ *iff* $d(x_n, y_n) \to 0$.

Now choosing ψ to be a bounded function, we can further assume that $\tilde{d}$ is bounded. Two traditional choices are:

$$\psi_1(t) = \frac{t}{t+1}$$

and

$$\psi_2(t) = \begin{cases} t & (0 \leqslant t \leqslant 1) \\ 1 & (t > 1) \end{cases}$$

One might choose ψ_1 since it is a smooth function, or choose ψ_2 as it is easy to compute. The main reason for choosing a new equivalent bounded quasi-metric is because it is easy to add up. For example, in the next lemma, showing that given a countable family of quasi-metrics (d_j), there exists a single quasi-metric d such that $d(x_n, y_n) \to 0$ if and only if $d_j(x_n, y_n) \to 0$ for every j, begins by converting the original family of quasi-metrics to a family of bounded quasi-metrics.

Lemma 4. *Suppose that d_j is a quasi-metric on X for $j = 1, 2 \ldots$ Define*

$$d(x, y) = \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(x, y)}{1 + d_j(x, y)} \quad (x, y \in X)$$

Then d is a quasi-metric on X with the property that $d(x_n, y_n) \to 0$ as $n \to \infty$ if and only if $d_j(x_n, y_n) \to 0$ for every j. Furthermore, d is a metric on X if and only if for every $x, y \in X$ with $x \neq y$ there exists a positive integer j such that $d_j(x, y) > 0$.