Semi-stable lattices in higher rank

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Outline

Historical motivation

Serre [1977] and Quillen [see Grayson, 1982] used the notion of semistable vector bundle on an algebraic curve to study $\mathrm{SL}_n(\mathcal{O})$ when $\mathcal{O}$ is a Dedekind domain finitely generated over a finite field. Stuhler then realized he can used the same method to adapt some work of Harder and Narasimhan on stable vector bundles to yields new facts about lattices in a Euclidean space.

Due to [?] , it is heuristical that the semi-stable lattices are the lattices in which the successive minima are closed.

Definition of two-dimensional lattices

Lattice

A lattice $L \subset \mathbb{R}^2$ is a set of the form

$$L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$$

where e_1, e_2 are linearly independent over $\mathbb{R}$.

Example of a lattice

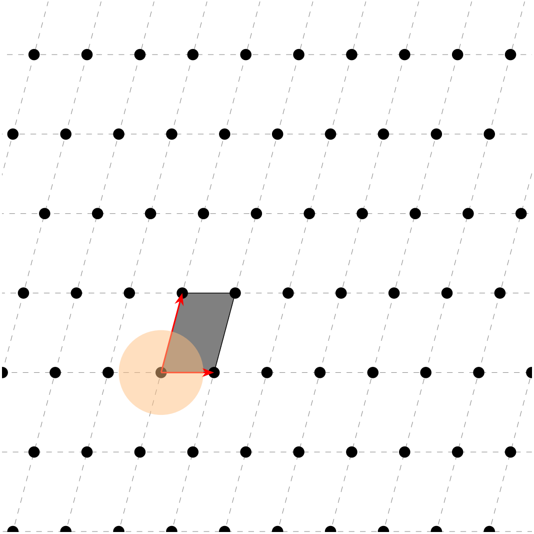

Figure: Example of a lattice

Classification of lattices

Do we know all the possible 2 dimensional "lattice shape"?

Classification of lattices

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Answer: Up to maginifcation, rotation and change of basis, the answer is yes.

Fundamental domain

Up to rorations and magnifications, we can reduce a lattice

$$L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$$

to a lattice of the form

$$L_z = \mathbb{Z}z \oplus \mathbb{Z}, \quad \Im(z) > 0$$

So the upper half-plane parametrizes the 2 dimensional lattices.

Classification of unit lattices

The map $z \mapsto \mathbb{Z}z \oplus \mathbb{Z}$ induces a bijection

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \cong \{\text{ lattices}\}/\mathbb{C}^\times$$

Fundamental domain

So we reduce to study the space of lattices by looking the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane. Geometrically, the domain is given by

$$\mathfrak{D} = \{z = x + iy \in \mathbb{H} : |z| \geq 1, -1/2 \leq x \leq 1/2\}$$

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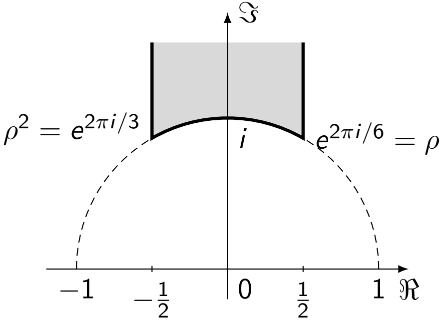

Canonical plot

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1. Put $(0, 0)$ to the plot.
2. For each primitive vector $v \in L$, he assigns the point $(1, \log(||v||))$ to the plot.
3. Put the point $(2, \log(vol(L)))$ to the plot.

Canonical plot

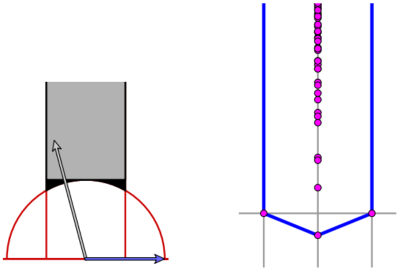
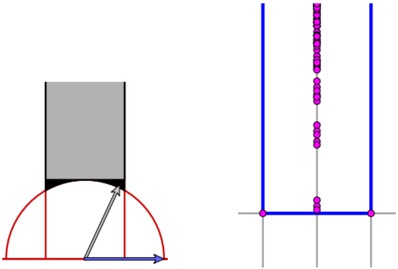

Figure: The figure on the left corresponds to $z = -2/5 + 3i/2$ and on the right corresponds to $z = 7/16 + 15i/16$

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Grayson called the set of points plotted above as **Canonical plot**. The convex hull of the collection of the plot point is called **profile**.

For any $z \in \mathbb{H} = \{\mathrm{Im}(z) > 0\}$, we can assign to it a unit lattices

$$z \mapsto L_z = \mathbb{Z}\frac{e_1}{\sqrt{y}} + \mathbb{Z}\left(\frac{x}{\sqrt{y}}e_1 + \sqrt{y}e_2\right)$$

The shortest vector is then $e_1/\sqrt{y}$, with length $\dfrac{1}{\sqrt{y}}$. So for $y < 1$, the lowest point is below the horizontal axis.

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The element z corresponds to the lattice L_z such that its lowest point on the vertical line $x = 1$ lies below the x-axis is called **semi-stable**, otherwise it is called **unstable**.

Since the semi-stability is preserved under the action of $\mathrm{SL}_2(\mathbb{Z})$, the semi-stable locus in the upper half plane $\mathbb{H}$ is as follows

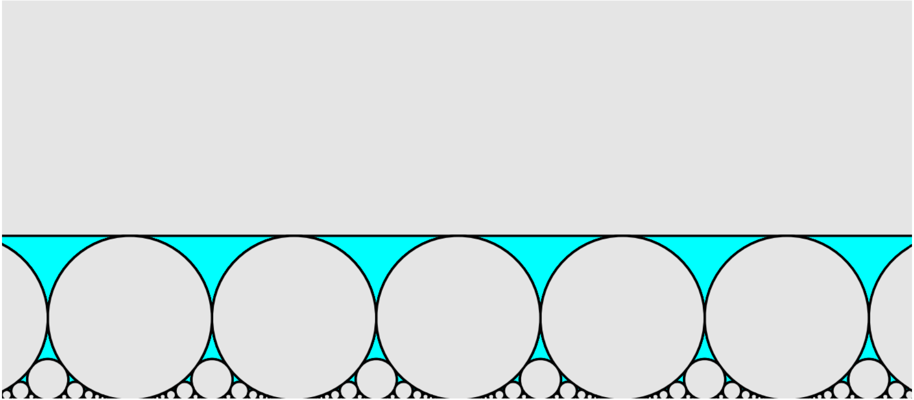

Figure: Semi-stable locus over $\mathbb{H}$

In higher dimensional

THANK YOU FOR YOUR ATTENTION.