PROBLEM SET 1

(1) Prove that if f is an entire function, then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

(2) Consider the series

$$\sum_{k=0}^{\infty} z^{2^k} = z + z^2 + z^4 + \dots$$

Determine the singular points and the natural boundary (if exists) of the above series.

(3) Show that for any injective analytic map $f\colon \mathbb{C} \to \mathbb{C}$, it must have the form

$$f(z) = az + b, \quad a \neq 0, b \in \mathbb{C}$$

Similarly, show that all injective maps from $\mathbb{C}_\infty \to \mathbb{C}_\infty$ has the form

$$f(z) = \frac{az+b}{cz+d},$$

where $ad - bc \neq 0$.