

# CHAPTER I : $SL_2(\mathbb{R})$

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In this chapter, I will give an exposition on the structure of  $SL_2(\mathbb{R})$  as the spaces of lattice, this space plays the role of a toy model before exploring the space of lattice in the higher rank. The exposition follows the paper [?] and [?] closely.

## 1 $SL_2(\mathbb{R})$ and its action on the upper half plane $\mathfrak{H}$

A priori, the upper half plane

$$\mathfrak{H} = \{z : \Im z > 0\} \subset \mathbb{C}$$

has no group structure on its. However, we will show below that it can identify topologically with the space with the space of cosets  $SO_2(\mathbb{R}) \backslash SL_2(\mathbb{R})$ , and thus we can study the spaces  $\mathfrak{H}$  via the space of lattices  $SO_2(\mathbb{R}) \backslash SL_2(\mathbb{R})$ . We define the action of  $G = SL_2(\mathbb{R})$  on  $\mathfrak{H}$  as follows

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \circ (z) = \frac{az + b}{cz + d}$$

**Proposition 1.1.** *The group  $SL_2(\mathbb{R})$  stabilizes  $\mathfrak{H}$  and acts transitively on it. In particular,*

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} (i) = x + iy \quad (\text{for } x \in \mathbb{R}, y > 0)$$

Further, for  $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$  and  $z \in \mathfrak{H}$ ,

$$\Im g(z) = \frac{\Im z}{|cz + d|^2}.$$

*Proof.* The first formula is clear. The second formula would imply that the upper half-plane is stabilized. Compute directly:

$$\begin{aligned} 2i \cdot \Im \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} (z) \right) &= \frac{az + b}{cz + d} - \frac{d\bar{z} + b}{c\bar{z} + d} = \frac{(az + b)(c\bar{z} + d) - (a\bar{z} + b)(cz + d)}{|cz + d|^2} \\ &= \frac{adz - bc\bar{z} - bcz + ad\bar{z}}{|cz + d|^2} = \frac{z - \bar{z}}{|cz + d|^2} \end{aligned}$$

since  $ad - bc = 1$ . □

The point  $z = i$  is special, in the sense that its stability group is the orthogonal group  $K = SO_2(\mathbb{R})$ .

Indeed, for any  $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$  we have that

$$g \circ i = i \Leftrightarrow \frac{ai + b}{ci + d} = i \Leftrightarrow a = d \text{ and } b = -c$$

Combining with the fact that  $ad - bc = 1$ , we must have  $a^2 + b^2 = 1$ . This implies that there is a  $\theta$  such that  $a = \cos \theta$  and  $b = \sin \theta$ . Since  $G$  acts on  $\mathfrak{H}$  transitively, we know from group theory that there is a bijection between the collection of cosets of  $\text{Stab}(i)$  in  $G$  and the orbits of  $i$ . In particular

**Proposition 1.2.** *We have an isomorphism of  $SL_2(\mathbb{R})$ -spaces*

$$SO_2(\mathbb{R}) \backslash SL_2(\mathbb{R}) \approx \mathfrak{H} \quad \text{via} \quad SO(2)g \rightarrow g^{-1}(i)$$

*That is, the map respects the action of  $SL_2(\mathbb{R})$ , in the sense that*

$$(SO_2(\mathbb{R})g) \cdot h \longrightarrow h^{-1}(g^{-1}i)$$

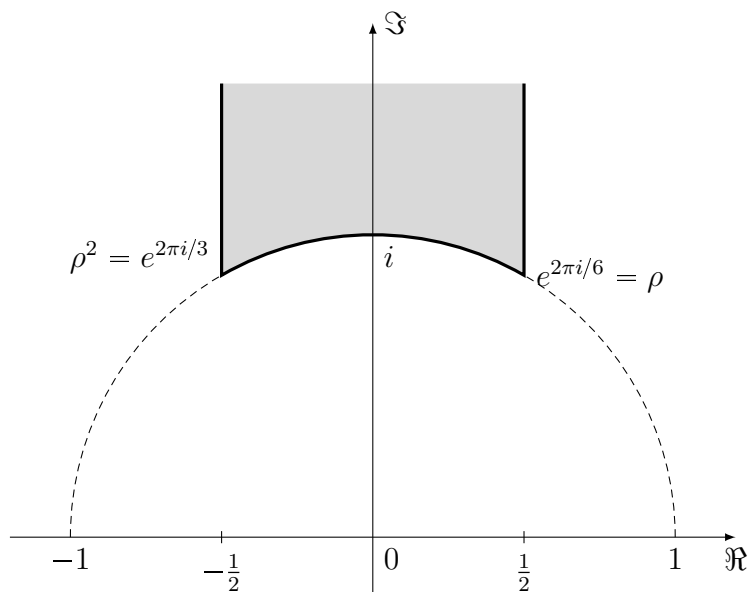
*Proof.* This is because of *associativity*:

$$(SO_2(\mathbb{R})g) \cdot h = (SO_2(\mathbb{R})) \cdot (gh) \longrightarrow (gh)^{-1}(i) = h^{-1}(g^{-1}(i))$$

giving the result. □

## 2 Fundamental domain for $\Gamma = SL_2(\mathbb{Z})$ on $\mathfrak{H}$

Here is a picture of the fundamental domain  $\mathfrak{H}/\Gamma$ .



The goal of this section is to prove that under the action of the  $\Gamma = SL_2(\mathbb{Z})$ , we can "move" every points on the upper half plane to a domain, under an equivalence given by a specific action. This is similar to the fundamental domain given by the translation action of  $\mathbb{Z}$  to  $\mathbb{R}$  is the half-open unit interval  $[0, 1)$ . In general, this give a simpler description to the homogenous space of lattice. Not that when we try to compute the fundamental domain of  $\mathbb{Z} \backslash \mathbb{R}$ , we have  $\mathbb{Z}$  plays a role of "discrete" subset of  $\mathbb{R}$ . We give a precise definition of discreteness as follows

**Definition 2.1.** *Let a group  $G$  act continuously on a topological space  $X$ . A subset  $\Gamma \subset G$  is called **discrete** if for any two compact subse  $A, B$  in  $X$ , there are only finitely many  $g \in \Gamma$  such that  $g \circ A \cap B \neq \emptyset$ .*

We will prove that the set

$$\Gamma = SL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\}$$

is a discrete subgroup of  $G = SL_2(\mathbb{R})$ . To prove this, we first need the following lemma

**Lemma 2.1.** Fix a real number  $r > 0$  and  $0 < \delta < 1$ . We denote  $R_{r,\delta}$  the rectangle

$$R_{r,\delta} = \{z = x + iy : -r \leq x \leq r, 0 < \delta \leq y \leq \delta^{-1}\}$$

Then for any  $\epsilon > 0$  and any fixed set  $\mathbb{S}$  of coset representatives for  $\Gamma_\infty \backslash \Gamma$ , there are finitely many  $g \in \mathbb{S}$  such that  $\Im(g \circ z) > \epsilon$  for some  $z \in R_{r,\delta}$ .

In the above lemma, the notation  $\Gamma_\infty$  is defined to be the set

$$\Gamma_\infty = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}.$$

It can be seen easily that this is the stability group of  $\infty$  in  $\mathfrak{H}$ .

*Proof.* Let  $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Then for  $z \in R_{r,\delta}$ ,

$$\Im(g \circ z) = \frac{y}{c^2 y^2 + (cx + d)^2} < \epsilon$$

if  $|c| > (y\epsilon)^{-\frac{1}{2}}$ . On the other hand, for  $|c| \leq (y\epsilon)^{-\frac{1}{2}} \leq (\delta\epsilon)^{-\frac{1}{2}}$ , we have

$$\frac{y}{(cx + d)^2} < \epsilon$$

if the following inequalities hold:

$$|d| > |c|r + (y\epsilon^{-1})^{\frac{1}{2}} \geq |c|r + (\epsilon\delta)^{-\frac{1}{2}}.$$

Consequently,  $\Im(g \circ z) > \epsilon$  only if

$$|c| \leq (\delta\epsilon)^{-\frac{1}{2}} \quad \text{and} \quad |d| \leq (\epsilon\delta)^{-\frac{1}{2}}(r + 1),$$

and the total number of such pairs (not counting  $(c, d) = (0, \pm 1), (\pm 1, 0)$ ) is at most  $\frac{4(r+1)}{(\epsilon\delta)}$ . This proves the lemma.  $\square$

It follows from Lemma 2.1 that  $\Gamma = SL(2, \mathbb{Z})$  is a discrete subgroup of  $SL(2, \mathbb{R})$ . This is because:

1. It is enough to show that for any compact subset  $A \subset \mathfrak{H}$  there are only finitely many  $g \in SL(2, \mathbb{Z})$  such that  $(g \circ A) \cap A \neq \emptyset$ ;
2. Every compact subset of  $A \subset \mathfrak{H}$  is contained in a rectangle  $R_{r,\delta}$  for some  $r > 0$  and  $0 < \delta < \delta^{-1}$ ;
3.  $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} = \emptyset$ , except for finitely many  $\alpha \in \Gamma_\infty$ ,  $g \in \Gamma_\infty \backslash \Gamma$ .

To prove (3), note that Lemma 2.1 implies that  $(g \circ R_{r,\delta}) \cap R_{r,\delta} = \emptyset$  except for finitely many  $g \in \Gamma_\infty \backslash \Gamma$ . Let  $S \subset \Gamma_\infty \backslash \Gamma$  denote this finite set of such elements  $g$ . If  $g \notin S$ , then Lemma 2.1 tells us that it is because  $\Im(g \circ z) < \delta$  for all  $z \in R_{r,\delta}$ . Since  $\Im(\alpha g \circ z) = \Im(g \circ z)$  for  $\alpha \in \Gamma_\infty$ , it is enough to show that for each  $g \in S$ , there are only finitely many  $\alpha \in \Gamma_\infty$  such that  $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} \neq \emptyset$ . This last statement follows from the fact that  $g \circ R_{r,\delta}$  itself lies in some other rectangle  $R_{r',\delta'}$ , and every  $\alpha \in \Gamma_\infty$  is of the form  $\alpha = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$  ( $m \in \mathbb{Z}$ ), so that

$$\alpha \circ R_{r',\delta'} = \{x + iy \mid -r' + m \leq x \leq r' + m, 0 < \delta' \leq y \leq \delta'^{-1}\},$$

which implies  $(\alpha \circ R_{r',\delta'}) \cap R_{r,\delta} = \emptyset$  for  $|m|$  sufficiently large.