

Introduction to Schemes

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August 11, 2024

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1 Why Schemes?

1.1 Summary of affine varieties

Let k be an algebraic closed field. The main idea of classical algebraic geometry is that we have a correspondence

$$\begin{aligned} \{\text{subsets of } k^n \text{ cutout by polynomials}\} &\leftrightarrow \{\text{finitely generated reduced } k\text{-algebras}\} \\ \text{Geometry} &\leftrightarrow \text{Algebra} \end{aligned}$$

In particular, the above correspondence can be given as follows:

- $I \subset k[x_1, \dots, x_n]$ ideal: Then we define

$$X := Z(I) = \{a \in k^n \mid f(a) = 0 \quad \forall f \in I\}$$

sometimes we use the notation $V(I)$ instead of $Z(I)$. This kind of set is an affine variety.

- \mathbb{A}^n : n -dimensional affine space. As a set, it is just k^n , but we equip this set with *Zariski* topology - where the closed subsets are generated by $Z(I)$.

- $I(X) := \{f \in k[x_1, \dots, x_n] \mid f(x) = 0 \quad \forall x \in X\}$. Then the quotient replacing $k[X] := \frac{k[x_1, \dots, x_n]}{I(X)}$ is called *coordinate ring* of X .

- $k[X]$ parametrizes functions on X :

$$x \in X \rightsquigarrow \mathfrak{m}_x := \ker(\text{ev}_x : k[X] \rightarrow k)$$

and $\forall f \in k[x]$ gives

$$\begin{aligned} f : X &\rightarrow \mathbb{A}^1 = k \\ x &\mapsto f(x) = \overline{f} \in k[x]/\mathfrak{m}_x \end{aligned}$$

- Hilbert's weak Nullstellensatz:

$$\{\text{points of } X\} \leftrightarrow \{\text{maximal ideals of } k[X]\}$$

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- Hilbert Nullstellensatz: $I(Z(I)) = \sqrt{I} := \{f : f^n \in I \text{ for some } n\}$.
 - Morphisms: given X and $Y \in \mathbb{A}^n$ a morphism between two affine varieties is given by $\varphi = (f_1, \dots, f_n)$. This morphism induces a k -algebra homomorphism

$$\varphi^*: k[Y] \rightarrow k[X], \quad \varphi^*(\psi) = \psi \circ \varphi,$$

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & \searrow \varphi^* f & \downarrow f \\ & & \mathbb{A}^1 \end{array}$$

so $\text{Hom}(X, Y) = \text{Hom}(k[Y], k[X])$ - which gives the equivalence of categories as stated in the beginning.

1.2 Why varieties are not good enough?

Some possible reasons are:

1. embedding into \mathbb{A}^n shouldn't really be part of the data. It would be nice to have an intrinsic definition, since you can embed the same variety in different spaces.
2. for non-algebraic closed field, the Nullstellensatz doesn't work.
3. We can ask, on which topological space is $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$ naturally a functions space? Or $\mathbb{Z}[x]$? Or \mathbb{Z} ? This leads to the idea of considering all possible rings.
4. Nilpotent arises naturally when deforming varieties, so ignoring them is not a good option.