

# Semi-stable lattices in higher rank

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# Outline

- 1 Introduction
- 2 In 2 dimensions
- 3 In dimension at least 3

# Historical motivation

Serre and Quillen used the notion of semistable vector bundle on an algebraic curve to study  $SL_n(\mathcal{O})$  when  $\mathcal{O}$  is a Dedekind domain finitely generated over a finite field. Stuhler then realized he can use the same method to adapt some work of Harder and Narasimhan on stable vector bundles to yields new facts about lattices in a Euclidean space.

# Definition of two-dimensional lattices

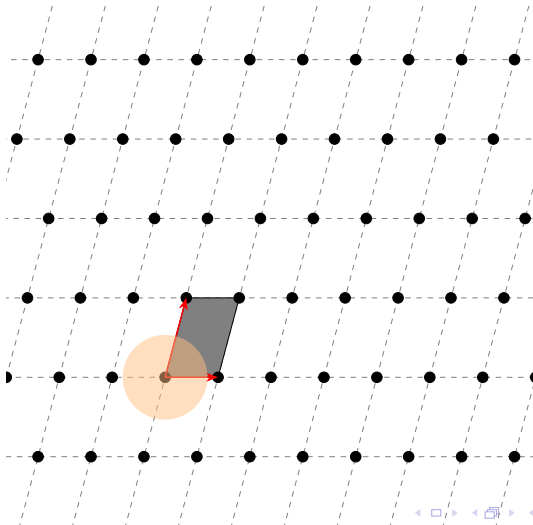
## Lattice

A lattice  $L \subset \mathbb{R}^2$  is a set of the form

$$L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$$

where  $e_1, e_2 \in \mathbb{R}^2$  are linearly independent over  $\mathbb{R}$ .

# Example of a 2-dim lattice



# Classification of lattices

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**Answer:** Up to rescaling, rotation and change of basis, the answer is yes.

# Fundamental domain

Up to rotations and rescaling, we can reduce a lattice

$$L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$$

to a lattice of the form

$$L_z = \mathbb{Z}z \oplus \mathbb{Z}, \quad \Im(z) > 0$$

So the upper half-plane parametrizes the 2 dimensional lattices.

## Classification of unit lattices

The map  $z \mapsto L_z = \mathbb{Z}z \oplus \mathbb{Z}$  induces a bijection

$$\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \cong \{ \text{lattices} \} / \mathbb{C}^\times$$



# Fundamental domain

So we reduce the study of the space of lattices by looking the action of  $SL_2(\mathbb{Z})$  on the upper half plane. Geometrically, the fundamental domain for  $SL_2(\mathbb{Z}) \backslash \mathbb{H}$  is given by

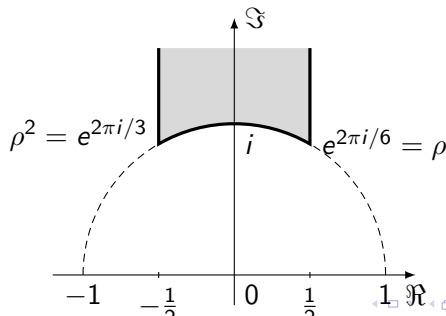
$$\mathcal{D} = \{z = x + iy \in \mathbb{H} : |z| \geq 1, -1/2 \leq x \leq 1/2\}$$

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# Canonical plot

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The process is as follows:

- 1 Put  $(0, 0)$  in the plot.
- 2 For each primitive vector  $v \in L$ , he assigns the point  $(1, \log(\|v\|))$  to the plot.
- 3 Put the point  $(2, \log(\text{vol}(L)))$  in the plot.

# Canonical plot

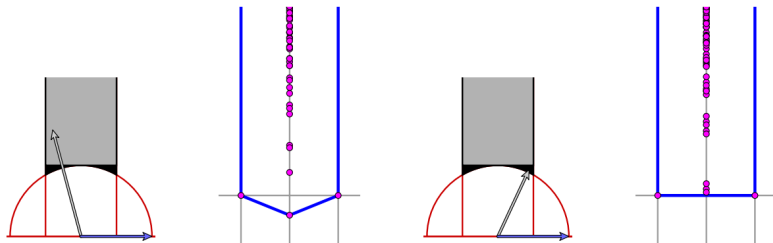


Figure: Casselman - The figure on the left corresponds to  $z = -2/5 + 3i/2$  and on the right corresponds to  $z = 7/16 + 15i/16$

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Grayson called the set of points plotted above as **canonical plot**.  
The convex hull of the collection of the plot points is called **profile**.

For any  $z \in \mathbb{H} = \{\text{Im}(z) > 0\}$ , we can assign to it a lattice of covolume 1 as follows

$$z \mapsto L_z = \mathbb{Z} \frac{e_1}{\sqrt{y}} + \mathbb{Z} \left( \frac{x}{\sqrt{y}} e_1 + \sqrt{y} e_2 \right)$$

The shortest vector is then  $e_1/\sqrt{y}$ , with length  $\frac{1}{\sqrt{y}}$ . So for  $y < 1$ , the lowest point is below the horizontal axis.



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The element  $z$  corresponds to a lattice  $L_z$  such that its lowest point on the vertical line  $x = 1$  lies below the  $x$ -axis is called **semi-stable**, otherwise  $z$  is called **unstable**.

# Semi-stable locus in fundamental domain

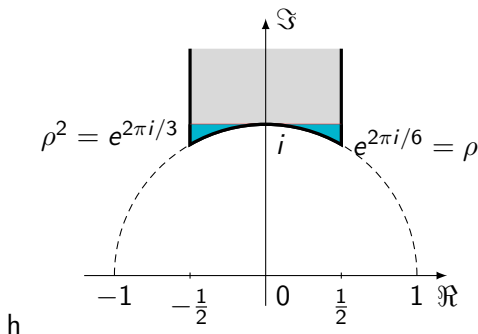
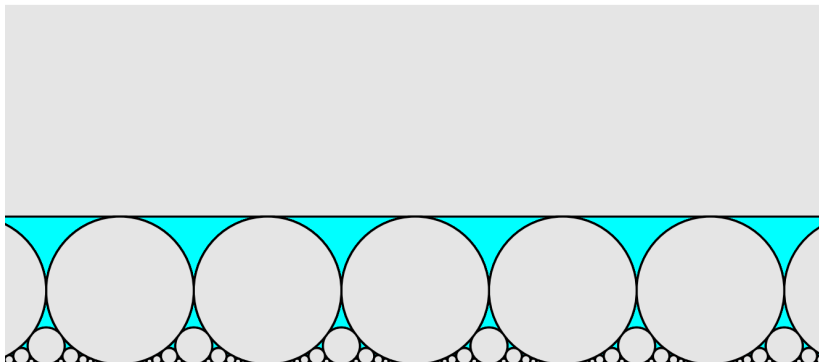


Figure: The blue part is the semistable locus in the fundamental domain

Since the semi-stability is preserved under the action of  $SL_2(\mathbb{Z})$ , the semi-stable locus in the upper half plane  $\mathbb{H}$  is as follows



**Figure:** Semi-stable locus over  $\mathbb{H}$  - it is the complement of the union of the gray area

# In higher dimensions

We work with the lattices of the form  $g\mathbb{Z}^n$  for  $g \in \mathrm{GL}_n(\mathbb{R})$  or  $g \in \mathrm{SL}_n(\mathbb{R})$ . The latter yields lattices with unit volumes. Even if we want to work with unit lattices, we still need to consider the sublattices of arbitrary volumes.

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## Sublattice

A discrete subgroup  $M$  of the lattice  $L$  is called **sublattice** if it satisfies one of the the following equivalent conditions:

- 1  $L/M$  is torsion-free.
- 2  $M$  is a direct summand in  $L$ .
- 3 Every basis of  $M$  can be extended to a basis in  $L$ .
- 4 The quotient  $L/M$  is a free  $\mathbb{Z}$ -module.

# Volume of lattice

The volume of  $L = g\mathbb{Z}^n$  is just  $\det(g)$ . Assume that  $M$  is a sublattice of  $L$  of rank  $k \leq n$  with a basis

$$\{v_1, v_2, \dots, v_k\}$$

Let  $e_1, e_2, \dots, e_n$  be the standard basis in  $L \otimes \mathbb{R} \cong \mathbb{R}^n$ . We can form a matrix of size  $k \times n$

$$A = [\langle v_i, e_j \rangle]$$

The volume of  $M$  is defined to be the sum of the squares of the determinants of the  $k \times k$  minor matrices in the matrix  $A$ .

# Canonical plot in higher dimension

Grayson assigns to the lattice  $L$  a canonical plot as follows:

- 1 Put the point  $(0, 0)$  in the plot.
- 2 For each sublattice  $M \subset L$ , assign a point with coordinates  $I(M) = (\text{rank}(M), \log(\text{vol}(M)))$  to the plot.
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As before, we call the convex hull of this plot its **profile**.

We have the following proposition:

### Lemma - Grayson

Fix a lattice  $L$  of rank  $n$  and a positive number  $c$ . For each  $k \leq n$ , there are only finitely many sublattices  $M \subset L$  such that  $\text{vol}(M) < c$ .

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### Semi-stable lattice

A lattice  $L$  is called **semi-stable** if the bottom of the profile is just a line.

## Example of a higher rank profile

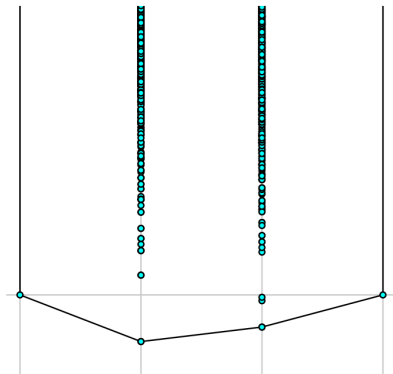


Figure: An unstable lattice

# Iwasawa decomposition

We recall the Iwasawa decomposition for  $G = \mathrm{GL}_n$ :

$$G = K \times A \times N$$

where:

- 1  $K$  is the orthogonal subgroup.
- 2  $A$  is the group of diagonal matrices with positive entries along the diagonal.
- 3  $N$  is the unipotent subgroup.

# Parabolic subgroups

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## Standard Parabolic subgroups of $GL_n$

For each partition

$$n = n_1 + n_2 + \dots + n_k$$

We denote  $P_{n_1, n_2, \dots, n_k}$  the standard parabolic subgroup of type  $(n_1, \dots, n_k)$  to be the subgroup of matrices of the form

$$P_{n_1, \dots, n_k} = \left\{ \begin{bmatrix} \mathfrak{m}_1 & * & \dots & * \\ 0 & \mathfrak{m}_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathfrak{m}_k \end{bmatrix} \right\}$$

where  $\mathfrak{m}_i$  is invertible of size  $n_i \times n_i$ .



# Degree of instability

Now we are ready to define the degree of instability

## Degree of instability, Chaudouard

For each  $x \in G$ , we define its degree of instability to be

$$\deg_{\text{inst}}(x) := \min_{P \in \text{ParSt}, \gamma \in G(\mathbb{Q})/P(\mathbb{Q})} \langle \rho_P, H_B(x\gamma) \rangle$$

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We define the notion of  $\rho$ -semistable as follows

## $\rho$ -semistable

A point  $x \in G$  is called **semi-stable** iff  $\deg_{\text{inst}}(x) \geq 0$ .

# Equivalent between two notions of semi-stable

We have

$$\langle \rho_{Q_i}, H_B(x\gamma) \rangle = a_1 a_2 \dots a_i$$

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We have

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where

$$x = k_x a_x n_x \in K \times A \times N,$$

in which

$$a_x = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}$$

# Equivalent between two notions of semi-stable lattices

We have the following Lemma

## Lemma - Chaudouard

The following are equivalent:

- 1  $\deg_{\text{inst}}(x) \geq 0$ ;
- 2 For every maximal parabolic subgroup  $P \subset G$ , every  $\delta \in G(\mathbb{Q})/P(\mathbb{Q})$ , and every  $\varpi \in \hat{\Delta}_P^G$ , we have:

$$\langle \varpi, H_B(x\delta) \rangle \geq 0.$$

This suggests that there should be a connection between the maximal parabolic subgroups of  $G$  and sublattices of  $L$ . Indeed we have

$$\mathrm{GL}_n(\mathbb{Z})/(Q_i(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})) \longleftrightarrow \{ \text{sublattices of rank } i \text{ of } \mathbb{Z}^n \}$$

So we have the main theorem

### Main theorem

Let  $x \in X_n = K \backslash \mathrm{GL}_n(\mathbb{R})$  - the space of lattices . Then  $x$  is semi-stable if one of the following equivalent conditions holds

- ❶ The bottom of the profile of the lattice corresponding to  $x$  is a straight line that connects the origin and  $(n, \log(\mathrm{vol}(L)))$ .
- ❷ The degree of instability of  $x$  is nonnegative, namely,  $\deg_{\mathrm{inst}}(x) \geq 0$ .

# References

- [1] Bill Casselman. Stability of lattices and the partition of arithmetic quotients. 2004.
- [2] Pierre-Henri Chaudouard. Sur une variante des troncatures d'arthur. In *Simons Symposium on the Trace Formula*, pages 85–120. Springer, 2016.
- [3] Daniel R Grayson. Reduction theory using semistability. *Commentarii Mathematici Helvetici*, 59(1):600–634, 1984.

*THANK YOU FOR YOUR ATTENTION.*