The Pentagonal Number Theorem and Modular Forms

Tri Nguyen

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Outline

Warm-up problem

Given a positive number n, how many ways can be write n in the form

$$a_1 + a_2 + \ldots + a_k?$$

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There are in total 2^{n-1} of them, however, there are a lot of repetitions.

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So you will get more and more repetitions listing this way. For example

$$\begin{aligned} 4 &= 1+1+1+1 \\ &= 2+1+1 = 1+2+1 = 1+1+2 \\ &= 2+2 \\ &= 3+1 = 1+3 \end{aligned}$$

Partition

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n	1	2	3	4	5	6
$p(n)$	1	2	3	5	7	12

Table: A first few value

Euler's first formulae

Euler came up with the following generating function for $p(n)$

Theorem 1

We have the following identity

$$(1+x+x^2+\dots)(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)(\dots) = \sum p(n)x^n$$

Proof.

This is just combinatorics, we actually counts the number smaller than n that appears the partition. $\square$

A similar method can be used to count the number of partitions that contains a specific set of given number.

Problems?

This method is very slow if one wants to compute $p(n)$ explicitly for n larges.
A programme in *Mathematica* following this method takes around 50s to compute the first fifty values of $p(n)$.

A faster method

Ring of formal series

Let

$$\mathfrak{U} = \left\{ 1 + a_1 x + a_2 x^2 + \ldots = \sum_{\geq 0} a_n x^n \right\}$$

Then we can define a multiplication by

$$(1 + a_1 x + a_2 x^2 + \ldots)(1 + b_1 x + b_2 x^2 + \ldots) = 1 + c_1 x + c_2 x^2 + \ldots$$

where $c_k = a_k + a_{k-1} b_1 + \ldots + b_k$

A faster method

Theorem 2

The set $\mathfrak{U}$ with this product forms a group.

Proof.

Left to the audiences =)). $\square$

In particular, we have that

$$\frac{1}{1-x^k} = \sum_{n\geq 0} x^{nk} = 1 + x^k + x^{2k} + \dots$$

A faster method

Theorem 3

We have the following generating function for $p(n)$

$$\frac{1}{\prod_{k\geq 1}(1-x^k)} = \sum_{n\geq 1} p(n)x^n$$

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Proof.

This follows from theorem 1 by taking the inverses of the series on the left hand sides. $\square$

The pentagonal number theorem

Euler hoped to find a pattern that emerges from the denominator on the left hand side. He did that by multiplying everything and get the following

$$\prod_{k\geq 1}(1-x^k) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \dots$$

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Note that we obviously have

$$(1 - x - x^2 + x^5 + x^7 - x^{12} - \dots)(1 + p(1)x + p(2)x^2 + p(3)x^3 + \dots) = 1$$

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So we have a recurrence relation

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15) + \dots = 0$$

Using this relation, it is much faster to compute the partition number.

Why Pentagonal?

If you list the exponents, you get a sequence of number

$$1, 2, 5, 7, 12, 15, 22, 26, 35, 40, \ldots$$

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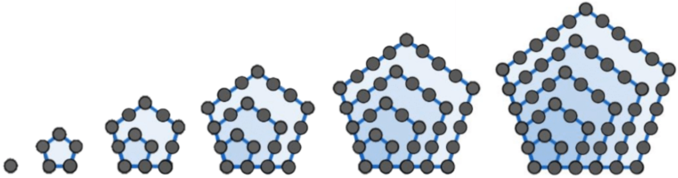
All the terms showed up here are the pentagonal number! In particular, they are given by the formulae

$$n = \frac{k(3k-1)}{2}, \quad k \geq 1$$

Why Pentagonal



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It turns our that the numbers at even positions is also given by pentagonal number formula, with oppositive sig

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This is later proved algebraically by Euler in 1750 and again by Franklin in 1881.

Length of ellipse

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$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We know from calculus that the total length can be computed by

$$\int \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Length of an ellipse

Solve for the equation of ellipse in term of y, we get

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

which implies that the total length of an ellipse is

$$4a\int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}dt = 4\int_0^1 \frac{1-k^2t^2}{\sqrt{(1-t^2)(1-k^2t^2)}}dt$$

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For $k = 0$, this is just the perimeter of the circle. The general case is called *elliptic integral*.

A case study of Gauss

Gauss essentially tried to compute the elliptic integral by looking at the following integral

$$F(x) = \int_0^x \frac{1}{\sqrt{1-z^4}} dz$$

Mimicking the case for the integral

$$\int_0^x \frac{1}{\sqrt{1-z^2}} dz = \sin^{-1}(x)$$

Gauss tried to find an inverse function of F. This inversion is called *elliptic functions*.

Weierstrass $\wp$ functions

Abel, and later Weierstrass, created an elliptic functions from scratch using the lattice.

Lattice

A lattice $L \subset \mathbb{R}^2$ is a set of the form

$$L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$$

where e_1, e_2 are linearly independent over $\mathbb{R}$.

Weierstrass $\wp$ functions

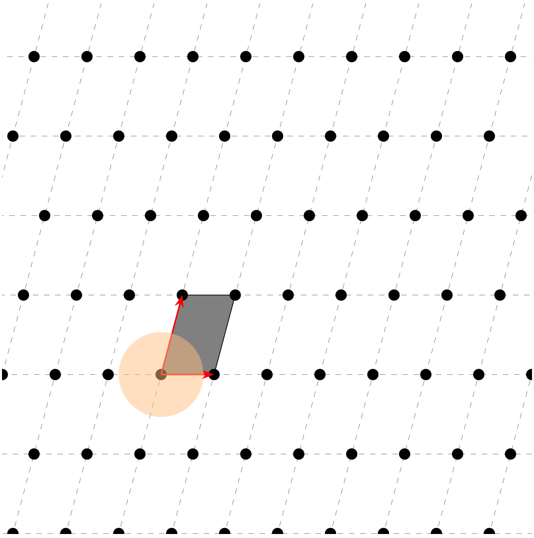

Figure: Example of a lattice

Weierstrass $\wp$ functions

Weierstrass defined his function as

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n\neq 0}\left(\frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2}\right)$$

This function is holomorphic and doubly periodic.

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This function is holomorphic and doubly periodic. Differentiating both sides yields another elliptic functions

$$\wp'(z) = -2 \sum_{m,n} \frac{1}{(z - m\omega_1 - n\omega_2)^3}$$

Theorem : Doubly periodic functions with prescribed periods

Every doubly periodic function with periods ω_1, ω_2 can be written uniquely in the form

$$R_1(\wp(z)) + R_2(\wp(z))\wp'(z).$$

In other words, any doubly periodic function with periods ω_i is an element of the function field $\mathbb{C}(\wp, \wp')$

In particular, there are not many double periodic functions.

Weierstrass $\wp$ functions

Theorem

We have

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

Classification of lattices

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Answer: Up to maginifcation, rotation and change of basis, the answer is yes.

Fundamental domain

Up to rorations and magnifications, we can reduce a lattice

$$L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$$

to a lattice of the form

$$L_z = \mathbb{Z}z \oplus \mathbb{Z}, \quad \Im(z) > 0$$

So the upper half-plane parametrizes the 2 dimensional lattices.

Classification of unit lattices

The map $z \mapsto \mathbb{Z}z \oplus \mathbb{Z}$ induces a bijection

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \cong \{\text{ lattices}\}/\mathbb{C}^\times$$

Fundamental domain

So we reduce to study the space of lattices by looking the action of $SL_2(\mathbb{Z})$ on the upper half plane. Geometrically, the domain is given by

$$\mathfrak{D} = \{z = x + iy \in \mathbb{H} : |z| \geq 1, -1/2 \leq x \leq 1/2\}$$

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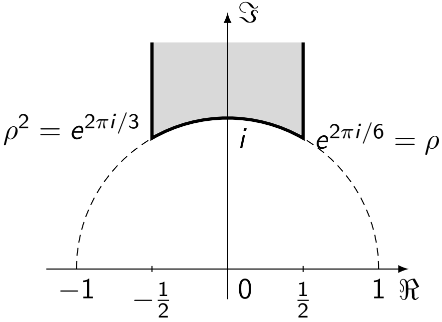

A brief look at modular forms

Goal: we have to assign each point $z \in \mathfrak{D}$ a number $j(z)$ such that j detects the z corresponding to distinct lattices.

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Generators of $\mathrm{SL}_2(\mathbb{Z})$

As a group, $\mathrm{SL}_2(\mathbb{Z})$ is generated by two elements

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So we are after a function $j\colon \mathbb{H} \to \mathbb{C}$ such that

$$j(Sz) = j(Tz) = j(z)$$

A brief look at modular forms

If $g(z)$ is a holomorphic function on the unit disk, then $g(e^{2i\pi\tau})$ is a holomorphic function over the upper half plane and have a period $p = 1$. So for each holomophic function on the unit disk, we get a candidate. The problem is to find a function $f(\tau) = g(e^{2i\pi\eta})$ such that

$$f(\tau) = f\left(\frac{1}{-\tau}\right)$$

We have the *Eisenstein series* defined as

$$g_4(\tau) = 60 \sum_{m,n\neq 0} \frac{1}{(m\tau + n)^4}$$

and

$$g_6(\tau) = 140 \sum_{m,n\neq 0} \frac{1}{(m\tau + n)^6}$$

A brief look at modular forms

It can be shown that g_4, g_6 are holomorphic functions over the unit disk. We formalize the notion of modular forms as follows

Definitions of modular forms

A modular form of level k is a function $f(\tau)$ on the upper half plane associated with a power series $g(z)$ by the formula $f(\tau) = g(e^{2i\pi\tau})$ that satisfies

$$f(-1/\tau) = \tau^k f(\tau)$$

We denote M_k the set of such weight k modular forms.

A brief look at modular forms

Theorem

The M_k are finite dimensional vector spaces. When k is odd, they contain only the zero vector.

1. If k is even and $k \equiv 2 \pmod{12}$, then $\dim M_k = \left\lfloor \frac{k}{12} \right\rfloor$.
2. If k is even and $k \not\equiv 2 \pmod{12}$, then $\dim M_k = \left\lfloor \frac{k}{12} \right\rfloor + 1$.
3. Thus $M_0, M_2, M_4, M_6, M_8, M_{10}$, and M_{12} have dimensions $1, 0, 1, 1, 1, 1, 2$.
4. The product of an element of M_k and an element of M_l is an element of M_{k+l}.
5. $g_2 \in M_4$ and $g_3 \in M_6$.
6. M_0 only contains constants.

Remark

We know that over M_{12} we have at least two linearly independent modular forms - denoted by h_1 and h_2. Then

$$j(\tau) = \frac{h_1}{h_2}$$

Seems to be the right function.

Dedekind η function

Now recalled the inverse of the partition generating function discovered by Euler

$$g(z) = 1 - z - z^2 + z^5 + z^7 - z^{12} - z^{15} + \dots$$

This is a holomorphic function over the unit disk. Let $f(\tau) = g(e^{2i\pi\tau})$. Then

Dedekind's theorem

Let $\eta(\tau) = e^{2i\pi\tau/24} f(\tau)$. Then

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$$

The $j-$invariant

Let us define

$$j(\tau) = \frac{g_4^2(\tau)}{\eta^{24}(\tau)}$$

Then j has only a pole at ∞ and $j\colon \mathfrak{D} \to \mathbb{C}$ is a bijection.

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Then j has only a pole at ∞ and $j : \mathfrak{D} \to \mathbb{C}$ is a bijection. Therefore we have the following theorem

Main theorem

Two lattices are equivalent under mangnification, rotation, and base change if and only if they have the same $j-$ invariant.