STAT 3119

Week6: 10/3/2019 @GWU

Outline

- Review two-factor ANOVA table, estimation and test
- Interaction and transformation (p. 826)
- Analysis of factor means when factors do not interact (Ch 19.8)
- Analysis of factor means when interactions are important (Ch 19.9)
- Pooling SS in two-way ANOVA (Ch 19.10)
- Planning of Sample size for Two-Factor Studies (Ch 19.11)
- Quiz review for Quiz #3 (10/10)

Review: Two-factor ANOVA analysis:

- The response Y_{ijk} is continuous with subscript i denotes the level of the factor A, j denotes the level of the factor B, k denotes the kth observation for treatment (i, j), $k = 1$ to n, with $n \geq 1$.
- There are two categorical explanatory variables (factors), called Factor A (a levels) ad Factor B (b levels).
- A particular combination of levels is called a **treatment** or a **cell**. There are **ab** treatments.

1. Cell Means Models

$$Y_{ijk} = {}_{ij} + \epsilon_{ijk}$$

where

- μ_{ij} is the population mean or expected value of all observations in cell (i, j).
- ϵ_{ijk} are iid $N(0, \sigma^2)$

2. Factor effects model

$$Y_{ijk} = \mu.. + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

where

- $\mu_{..}$ overall mean,
- α_i and β_j are the main effect of factor A and B; $(\alpha\beta)_{ij}$ interaction effects.
- Zero sum constraints: $\sum_i \alpha_i = 0$, $\sum_j \beta_j = 0$, and

$$\sum_i (\alpha\beta)_{ij} = \sum_j (\alpha\beta)_{ij} = 0\,, \quad i = 1..., a\,;\, j = 1, ..., b$$

- ϵ_{ijk} are iid $N(0, \sigma^2)$

Estimation of model parameters based on sample means

	Parameter	Sum	Estimator (based on sample means)
Treatment (i,j) level mean	μ_{ij}	$Y_{ij\cdot} = \sum_{k=1}^{n} Y_{ijk}$	$\bar{Y}_{ij\cdot} = \frac{Y_{ij\cdot}}{n}$
Factor A *i*th level mean	$\mu_{i.}$	$Y_{i\cdot\cdot} = \sum_{j}^{b}\sum_{k}^{n} Y_{ijk}$	$\bar{Y}_{i\cdot\cdot} = \frac{Y_{i\cdot\cdot}}{bn}$
Factor B *j*th level mean	$\mu_{.j}$	$Y_{\cdot j\cdot} = \sum_{i}^{a}\sum_{k}^{n} Y_{ijk}$	$\bar{Y}_{\cdot j\cdot} = \frac{Y_{\cdot j\cdot}}{an}$
Overall mean	$\mu_{..}$	$Y_{\cdots} = \sum_{i}^{a}\sum_{j}^{b}\sum_{k}^{n} Y_{ijk}$	$\bar{Y}_{\cdots} = \frac{Y_{\cdots}}{nab}$
Main effect for factor A at *i*th *level*	α_i		$\bar{Y}_{i\cdot\cdot}$ - $\bar{Y}_{\cdots}$
Main effect for factor B at *j*th *level*	β_j		$\bar{Y}_{\cdot j\cdot}$ - $\bar{Y}_{\cdots}$
Interaction effect of A(*i*) and B(*j*)	$(\alpha\beta)_{ij}$		$\bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\cdots}$

ANOVA table and F-tests

$$SSTO = SSA + SSB + SSAB + SSE$$

TABLE 19.8 ANOVA Table for Two-Factor Study with Fixed Factor Levels.

Source of Variation	SS	df	MS
Factor A	$SSA = nb\sum(\bar{Y}_{i..} - \bar{Y}_{...})^2$	$a-1$	$MSA = \frac{SSA}{a-1}$
Factor B	$SSB = na\sum(\bar{Y}_{.j.} - \bar{Y}_{...})^2$	$b-1$	$MSB = \frac{SSB}{b-1}$
AB interactions	$SSAB = n\sum\sum(\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2$	$(a-1)(b-1)$	$MSAB = \frac{SSAB}{(a-1)(b-1)}$
Error	$SSE = \sum\sum\sum(Y_{ijk} - \bar{Y}_{ij.})^2$	$ab(n-1)$	$MSE = \frac{SSE}{ab(n-1)}$
Total	$SSTO = \sum\sum\sum(Y_{ijk} - \bar{Y}_{...})^2$	$nab-1$	

For factor A: $F^* = \frac{MSA}{MSE} \sim F[a-1, (n-1)ab]$

For factor B: $F^* = \frac{MSB}{MSE} \sim F[b\text{-}1, (n\text{-}1)ab]$

For Interaction $F^* = \frac{MSAB}{MSE} \sim F[(a-1)(b-1), (n-1)ab]$

We reject the null hypothesis of no effect for large *F*-values.

Interaction and transformation (p. 826)

- **Important** interactions: the treatment mean plots (A*B) shows effect curves are far from parallel (cross or intersect or different patterns); the estimated interaction effects $(\alpha\beta)_{ij}$ in the model are large; or F-test of interaction is significant. Consequently, one should not examine the effects of each factor separately because the effect of one factor depends on the certain levels of the other factor. Instead, a joint analysis for the two factors based on the treatment means μ_{ij} should be used.
- **Unimportant** interactions: the treatment mean plots shows effect curves are **almost** parallel; the estimated interaction effects $(\alpha\beta)_{ij}$ in the model are very small; or F-test of interaction is not significant. Consequently, one can proceed the simpler analysis as for the case of no interaction- Each factor can be studied separately.
- **Transformable** interaction
- 1. When factor effects that act multiplicatively,e.g.

$$\mu_{ij} = \mu_{..} * \alpha_i * \beta_j$$

  Then applying a **logarithmic** transformation will remove interaction.

$$\log \mu_{ij} = \log \mu_{..} + \log \alpha_i + \log \beta_j$$

- 2. When each interaction effect equals the product of functions of the main effects, e.g.

$$\mu_{ij} = \alpha_i + \beta_j + 2\sqrt{\alpha_i}\sqrt{\beta_j}$$

then

$$\mu_{ij} = (\sqrt{\alpha_i} + \sqrt{\beta_j})^2$$

Therefore applying the **square root** transformation will remove interaction.

- 3. Other simple transformation such as square , reciprocal or box-cox power transformation can also be considered to largely remove interaction.

- When interactions cannot be largely removed by a transformation, they are called **nontransformable** interactions.

Analysis of factor means when no or unimportant interaction (Ch 19.8) (1)

One can make inference on the factor level means $\mu_{i.j}$ and $\mu_{.j}$ separately (page 849-849).

	Estimator	Variance Estimators	(1- α) Confidence limits
Factor A *i*th level mean	$\hat{\mu}_{i.} = \bar{Y}_{i..}$	$s^2\{\bar{Y}_{i..}\} = \frac{MSE}{bn}$	$\bar{Y}_{i..} \pm t[1-\alpha/2; (n-1)ab]s\{\bar{Y}_{i..}\}$
Factor B *j*th level mean	$\hat{\mu}_{.j} = \bar{Y}_{.j.}$	$s^2\{\bar{Y}_{.j.}\} = \frac{MSE}{an}$	$\bar{Y}_{.j.} \pm t[1-\alpha/2; (n-1)ab]s\{\bar{Y}_{.j.}\}$
Linear combination or contrast $L = \sum c_i \mu_{i.}$	$\hat{L} = \sum c_i \bar{Y}_{i..}$	$s^2\{\hat{L}\} = \frac{MSE}{bn}\sum c_i^2$	$\hat{L} \pm t[1-\alpha/2; (n-1)ab]s\{\hat{L}\}$
$L = \sum c_j \mu_{.j}$	$\hat{L} = \sum c_j \bar{Y}_{.j.}$	$s^2\{\hat{L}\} = \frac{MSE}{an}\sum c_j^2$	$\hat{L} \pm t[1-\alpha/2; (n-1)ab]s\{\hat{L}\}$

- Note: the linear combination includes the contrast ($\sum c_i = 0$), or pairwise difference or single mean as special cases.

- The df for the t statistics depend on the variance estimator MSE, $df = (n-1)ab$ obtained from two-way ANOVA table.

No/unimportant interaction: Multiple Comparison procedures (2)

Similar to what we discussed for **one-factor studies**, we can choose to use 3 difference procedures:

- Tukey procedure applies for all the pairwise comparisons; Scheffe procedure applies for the set of all possible contrasts and Bonferroni procedure applies for any sets of comparisons.

- If all or a large number of pairwise comparisons among the factor level means (for factor A or B) are to be made, the **Tukey** procedure is appropriate.
- When only a few pairwise comparisons are to be made that are **pre-specified** in advance of the analysis, the **Bonferroni** procedure may be best.
- When a large number of comparisons among the factor-level means is of interest, the **Scheffe** method is usually preferred.
- For simultaneous CI: All three procedures are of the form: "Estimator +/- multiplier × SE".
- **Better** procedures means: better estimate precision, or "narrower CI". We can decide by comparing the multiples to find which procedure has the smallest width around the point estimate.
- Often, we can conduct tests for differences between pairs of factor level means before the construction of interval estimates for active comparisons or construct certain meaningful contrasts or linear combinationscc of the factor level means.

1. Estimates based on treatment means (page 853)

Although there is no interaction, if there is interest in particular treatment means μ_{ij} such as paired differences or contrasts. The same inference of one-factor studies can be used, with $r = ab$ and $n_T - r = nab - ab = (n-1)ab$, and plus in $\hat{\mu}_{ij} = \overline{Y}_{ij.}$ and $s^2\{\overline{Y}_{ij.}\} = \frac{MSE}{n}$ in the CI formulas.

Bonferroni multiple (for g comparisons) $t(1-\alpha/2g; n_T - r)$

Tukey multiple $1/\sqrt{2} * q(1-\alpha; r, n_T - r)$

Scheffé multiple $\sqrt{(r-1)F(1-\alpha; r-1, n_T - r)}$

2. Estimates based on factor level means (page 850-853)

Since there is no interaction, we can analyze the two factors (A & B) separately. Use the previous table above, we can get the estimator and its estimated stand deviation for a single factor level mean, a difference between two means or linear combination.

1) If the multiple comparisons only involve factor A:

The CI formula: "Estimator +/- multiplier × SE", we replace df for MSE and $r = a$ in the formula for one-factor studies.

Bonferroni multiple (for g comparisons) $B = t(1-\alpha/(2g); df = (n-1)ab)$

Tukey multiple $T = 1/\sqrt{2} * q(1-\alpha; df_1 = a, df_2 = (n-1)ab)$

Scheffé multiple $S = \sqrt{(a-1)F(1-\alpha; df_1 = a-1, df_2 = (n-1)ab)}$

2) If the multiple comparisons only involve factor B:

The CI formula:: "Estimator +/- multiplier × SE", we replace df for MSE and $r = b$ in the formula for one-factor studies.

Bonferroni multiple (for g comparisons) $B = t(1 - \alpha/(2g); df = (n-1)ab)$

Tukey multiple $T = 1/\sqrt{2} * q(1-\alpha; df_1 = b, df_2 = (n-1)ab)$

Scheffé multiple $S = \sqrt{(b-1)F(1-\alpha; df_1 = b-1, df_2 = (n-1)ab)}$

3) Simultaneous inferences on both factor A and B:

1. We use Bonferroni procedure alone, and control the family confidence coefficient $1-\alpha$ based on g (the number of total comparisons), and divide α into individual g comparisons, e.g. $\alpha_1 + ... + \alpha_g = \alpha$.
2. Combine Bonferroni procedure and Tukey or Scheffe method. If both factor A and B has multiple comparisons, we can divide type I error α by 2 and to use $\alpha/2$ for factor A comparison with either Tukey or Scheffe method; and the other $\alpha/2$ for factor B comparison with either Tukey or Scheffe method, to control overall type I error $\leq \alpha$.

No interaction case: Example 1 (page 853-855)

Example: Castle Bakery, we showed the estimated treatment means plot and the ANOVA analysis in the last class that no interaction effects are present and that display width may not have any effect. Now we want to examine the nature of the display height effects in more detail.

1. read the data

```
# read data from week5 folder online
Ex19  =read.table(
        url("https://raw.githubusercontent.com/npmldabook/Stat3119/master/Week6/CH19TA07.txt"))
names(Ex19) =  c("response", "height", "width", "units")

# make  categorical variables for factor A and B
Ex19$height =  as.factor(Ex19$height)
Ex19$width =  as.factor(Ex19$width)

str(Ex19)
```

```
## 'data.frame':    12 obs. of  4 variables:
##  $ response: int  47 43 46 40 62 68 67 71 41 39 ...
##  $ height  : Factor w/ 3 levels "1","2","3": 1 1 1 1 2 2 2 2 3 3 ...
##  $ width   : Factor w/ 2 levels "1","2": 1 1 2 2 1 1 2 2 1 1 ...
##  $ units   : int  1 2 1 2 1 2 1 2 1 2 ...
```

2. get paired comparison

```r
## pairwise comparison in the two factor studies with Tukey method
fit <- aov(response ~ height*width, data = Ex19)
summary(fit)
```

```
##              Df Sum Sq Mean Sq F value   Pr(>F)    
## height        2   1544   772.0  74.710 5.75e-05 ***
## width         1     12    12.0   1.161    0.323    
## height:width  2     24    12.0   1.161    0.375    
## Residuals     6     62    10.3                     
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```r
library(emmeans)
fit.emm <- emmeans(fit, ~ height)
```

```
## NOTE: Results may be misleading due to involvement in interactions
```

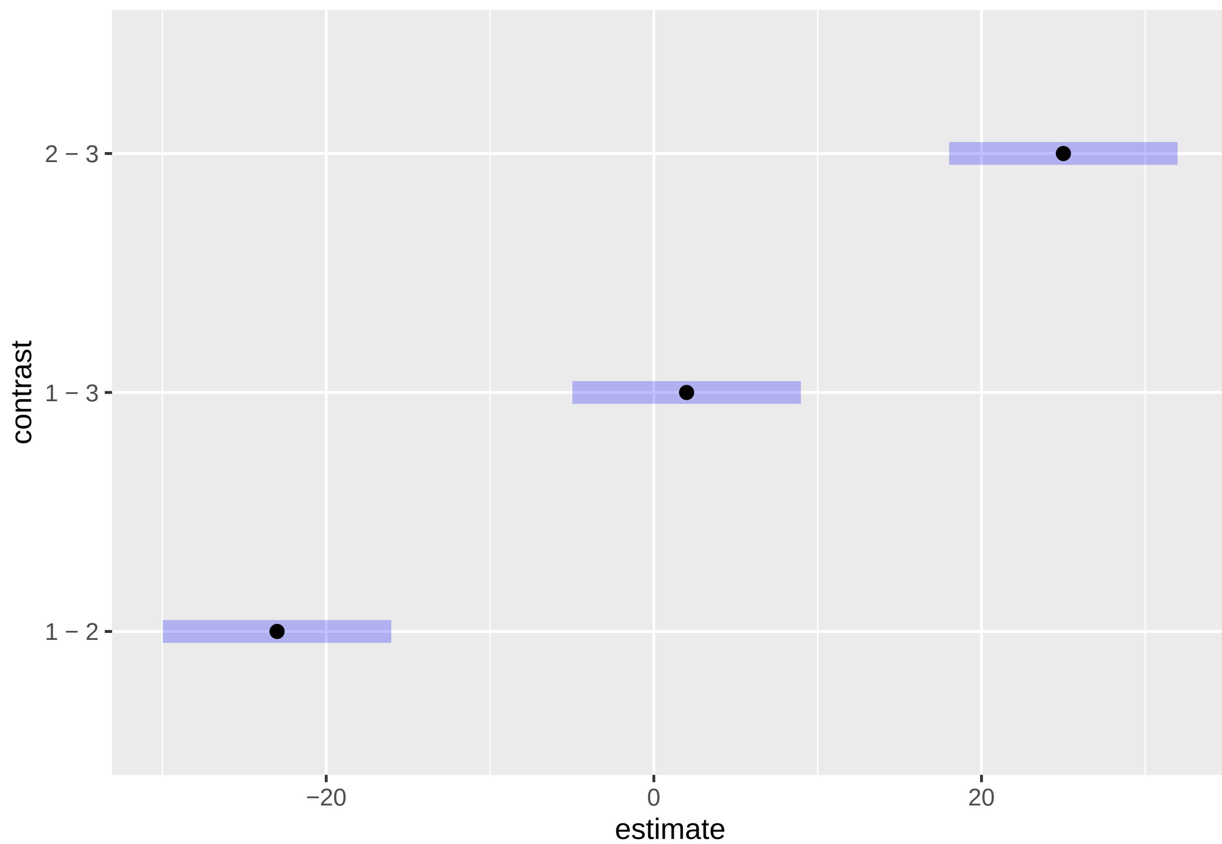
```r
pairs(fit.emm, adjust = "Tukey")
```

```
##  contrast estimate   SE df t.ratio p.value
##  1 - 2         -23 2.27  6 -10.119 0.0001
##  1 - 3           2 2.27  6   0.880 0.6714
##  2 - 3          25 2.27  6  10.999 0.0001
## 
## Results are averaged over the levels of: width 
## P value adjustment: tukey method for comparing a family of 3 estimates
```

Note:

- We can still use the *emmeans* package to make paired comparison for one factor. It gives a warning for an interaction. But for this case, we find no interaction likely present, we can just ignore this message.
- The estimated SE for the difference in this case $= \sqrt{(MSE/4)*(1+1)} = \sqrt{MSE/2} = \sqrt{10.3/2} =$ 2.27 based on formula but we can get it directly from the output table.
- This shows the results of pairwise test based on Tukey method, which suggests the mean for height level 2 were significantly greater than the mean for other 2 levels (1 and 3). We can compare the p-value with 0.05 since this is adjusted p-value.

```r
plot(pairs(fit.emm))
```




```r
confint(pairs(fit.emm), level = 0.95)
```

```
##  contrast estimate   SE df lower.CL upper.CL
##  1 - 2         -23 2.27  6   -29.97   -16.03
##  1 - 3           2 2.27  6    -4.97     8.97
##  2 - 3          25 2.27  6    18.03    31.97
##
## Results are averaged over the levels of: width
## Confidence level used: 0.95
## Conf-level adjustment: tukey method for comparing a family of 3 estimates
```

Note: We can then use *plot()* and *confint()* function to get the confidence intervals for the pairwise difference. It shows the similar results than height level 2 has signficantly higher mean response than the other two factors, but height level 1 and height level 3 had similar mean response (with the CI betwen means for level A1 and level A3 covering 0).

No interaction case: Example 2 (page 855)

Example 2: Estimation of Treatment Means: The manager of a supermarket that has sales volume and customers (similar to the supermarkets included in the Castle Bakery study) only has room for the **regular** shelf (width: level 1) display width, and wishes to obtain estimates of mean sales for the **middle** and top **shelf** heights (height: levels 2 and 3). The would like to get *interval estimates with a 90 percent family confidence coefficient.*

Solutions: Since only two intervals for μ_{21} and μ_{31}, we can only use Bonferroni procedure. (Tukey and Scheffe methods are used for pairwise difference or contrast).

To calculate using R, we can directly use formulas to get sample means for the treatment and MSE from ANOVA table to get the estimator and estimated SE.

- step 1:

```
(treatment.mean = with(Ex19, by(response, list(height,width), mean )))
```

```
## : 1
## : 1
## [1] 45
## --------------------------------------------------------
## : 2
## : 1
## [1] 65
## --------------------------------------------------------
## : 3
## : 1
## [1] 40
## --------------------------------------------------------
## : 1
## : 2
## [1] 43
## --------------------------------------------------------
## : 2
## : 2
## [1] 69
## --------------------------------------------------------
## : 3
## : 2
## [1] 44
```

We then get estimated mean(height=2, width=1)= 65 and mean(height=3, width=1)= 40.

- step 2:

```
summary(fit)
```

```
##              Df Sum Sq Mean Sq F value   Pr(>F)
## height        2   1544   772.0  74.710 5.75e-05 ***
## width         1     12    12.0   1.161    0.323
## height:width  2     24    12.0   1.161    0.375
## Residuals     6     62    10.3
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

We then get the estimated standard error $= \sqrt{MSE/n} = \sqrt{10.3/2} = 2.27$.

- step 3: get Bonferroni mutiple =2.447 as below

```r
#Bonferroni multiple, df=n_T- r = 30 +20 -4 = 46
g = 2 ; df= 6
alpha= 1-0.9
qt(1- alpha/(2*g), df)
```

```
## [1] 2.446912
```

- step 4: we obtain the CIs μ_{21} and μ_{31} with 90 percent family confidence coefficient:

$$65 \pm 2.447(2.27) \qquad 40 \pm 2.447(2.27)$$

and the desired confidence intervals are:

$$59.4 \leq \mu_{21} \leq 70.6 \qquad 34.4 \leq \mu_{31} \leq 45.6$$

using the Bonferroni procedure.

Analysis of factor means when interactions are important (Ch 19.9)

- When important interactions exist that cannot be made unimportant by a simple transformation, the analysis of factor effects generally must be based on the treatment means μ_{ij}.
- Typically, this analysis will involve estimation of multiple comparisons of treatment means or compares the levels of one factor for a given level of the other factor.
- The inference and multiple comparison procedures based on μ_{ij} is also same as the results for one factor study: no new formulas, in the CI formulas: "Estimator +/- multiplier × SE".

Bonferroni multiple (for g comparisons) $\quad t(1-\alpha/2g; n_T - r)$

Tukey multiple $\quad 1/\sqrt{2} * q(1-\alpha; r, n_T - r)$

Scheffé multiple $\quad \sqrt{(r-1)F(1-\alpha; r-1, n_T - r)}$

except for

$$r = ab\,, \quad n_T - r = nab - ab = (n-1)ab$$

and

$$\hat{\mu}_{ij} = \overline{Y}_{ij.}\,, \quad s^2\{\overline{Y}_{ij.}\} = \frac{MSE}{n}$$

Important interaction: Example (1)

Example 1 (page 857): A junior college system studied the effects of teaching method (factor A) and student's quantitative ability (factor B) on learning of college mathematics.

- Teaching (factor A): standard and abstract , 2 levels
- Ability (factor B) determined from a standard test: g excellent, good, or moderate, 3 levels
- $n = 21$ at each combination of Teaching and Ability combination.

In this example, we are only give the summary table 19.11 instead of the original observations, but this is **sufficient** for us to make inferences about the population. This is because for the normal distribution, the sample mean and sample variance are **sufficient statistics** about the data distribution, containing all the information needed about the distribution.

TABLE 19.11 Results—Mathematics Learning Example.

(a) Mean Learning Scores ($n = 21$)

Teaching Method i	Quantitative Ability (j) Excellent	Good	Moderate
Abstract	92 ($\bar{Y}_{11\cdot}$)	81 ($\bar{Y}_{12\cdot}$)	73 ($\bar{Y}_{13\cdot}$)
Standard	90 ($\bar{Y}_{21\cdot}$)	86 ($\bar{Y}_{22\cdot}$)	82 ($\bar{Y}_{23\cdot}$)

(b) ANOVA Table

Source of Variation	*SS*	*df*	*MS*
Factor *A* (teaching methods)	504	1	504
Factor *B* (quantitative ability)	3,843	2	1,921.5
AB interactions	651	2	325.5
Error	3,360	120	28
Total	8,358	125	

- **Interpretation of the ANOVA table**: obtain F-stat from ratio of MS and get p-value. Use $\alpha = 0.01$.

1. Testing interaction effect AB

```r
# Interaction AB, MSAB/MSE, F(df1,df2)= F(2,120)
Fstat.AB = 325.5/28
Ftail.AB = qf(.99, 2,120 )

# get the Pr(F>= Fstat.AB)
Pv.AB  =1- pf(Fstat.AB, 2, 20)

data.frame(Fstat.AB, Ftail.AB, Pv.AB)
```

```
##   Fstat.AB Ftail.AB        Pv.AB
## 1   11.625  4.78651 0.0004471352
```

2. Testing effect of factor A

```r
# Factor A, MSA/MSE, F(df1,df2)= F(1,120)
Fstat.A = 504/28
Ftail.A = qf(.99, 1,120)

# get the Pr(F>= Fstat.A)
Pv.A  =1- pf(Fstat.A, 1, 20)

data.frame(Fstat.A, Ftail.A, Pv.A)
```

```
##   Fstat.A  Ftail.A         Pv.A
## 1      18 6.850893 0.0003989052
```

3. Testing effect of factor B

```r
# Factor B, MSB/MSE, F(df1,df2)= F(2,120)
Fstat.B = 1921.5/28
Ftail.B = qf(.99, 2, 120 )

# get the Pr(F>= Fstat.B)
(Pv.B  =1- pf(Fstat.B, 1, 20))
```

```
## [1] 6.771465e-08
```

```r
data.frame(Fstat.B, Ftail.B, Pv.B)
```

```
##   Fstat.B Ftail.B         Pv.B
## 1  68.625 4.78651 6.771465e-08
```

ANOVA Results: At $\alpha = 0.01$, the interaction effect, main effect of A and B are all significant.

Important interaction: Example (2)

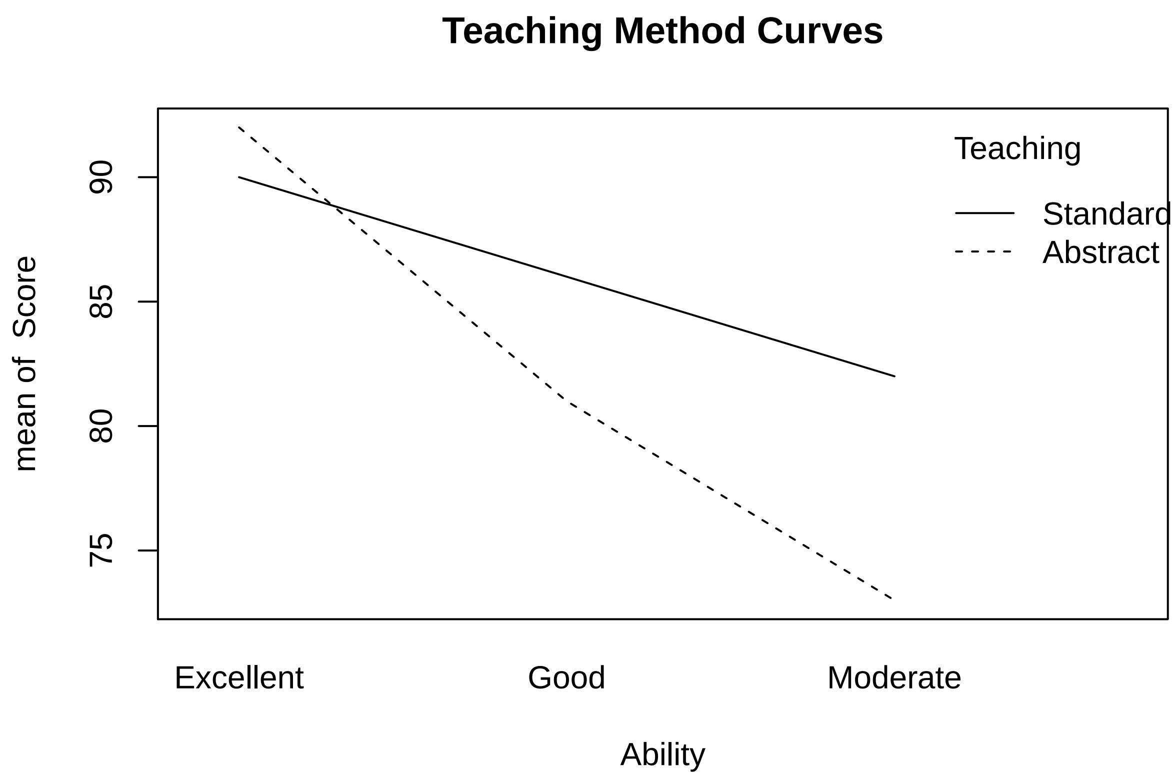
We can generate the treatment plot (interaction plot), which is to display the mean (i,j) by the two factors.

- If the effects of one factor vary for different levels of the other factor, then it shows a possible interaction effect.
- From the plot, we can also compare the treatment means for different (i,j) treatment, or comparing the treatment means by holding one factor as a given level.

```r
# Input the summary data 19.11 into R
# Get the factor level combination , match the estimate mean
( Ex19.11   =  data.frame( Teaching= rep(c("Abstract","Standard"), c(3,3)),
                      Ability=rep( c("Excellent",  "Good","Moderate"),2),
                      Score= c(92,81,73,90,86,82)))
```

```
##   Teaching   Ability Score
## 1 Abstract Excellent    92
## 2 Abstract      Good    81
## 3 Abstract  Moderate    73
## 4 Standard Excellent    90
## 5 Standard      Good    86
## 6 Standard  Moderate    82
```

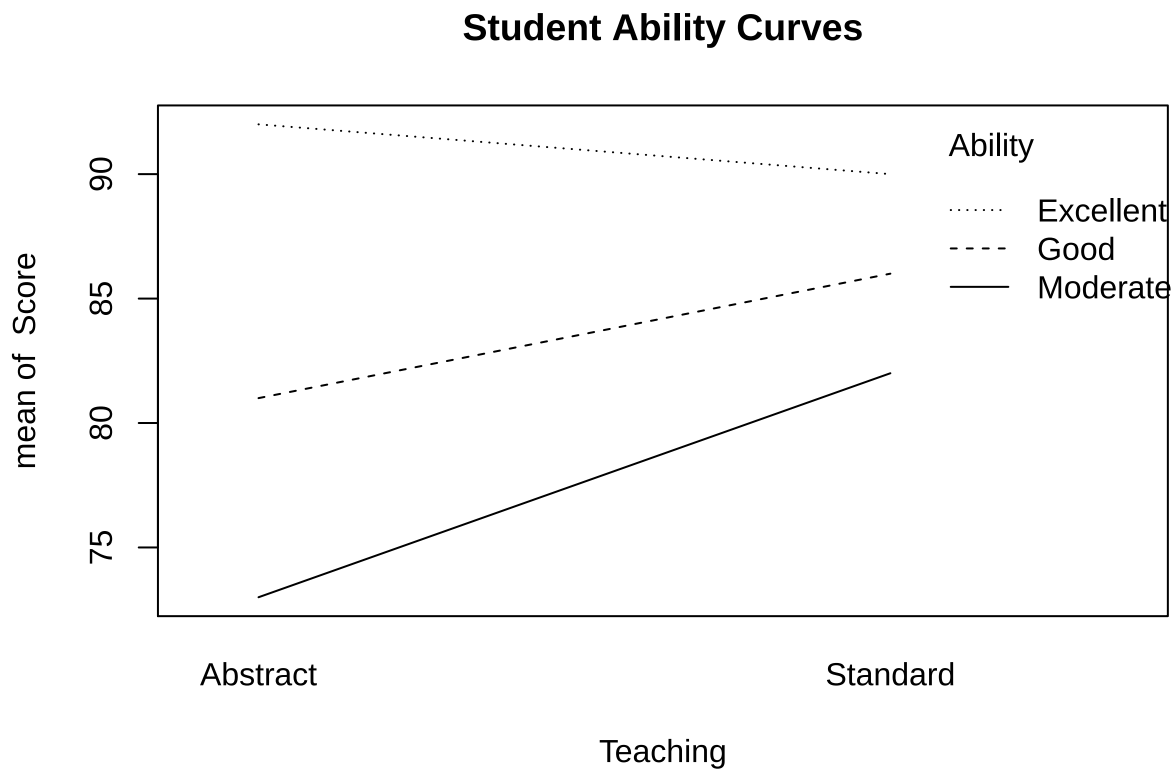
```
with(Ex19.11, interaction.plot(x.factor = Ability , trace.factor =Teaching , response = Score,
                                 main= "Teaching Method Curves"))
```



- First plot: Students with excellent ability are but little affected by teaching method (perhaps doing slightly better with the abstract method); students with good or moderate abilities learn much better with the standard teaching method.

We exchange the two factors in the R function.

```
with(Ex19.11, interaction.plot(x.factor = Teaching , trace.factor = Ability, response = Score,
                                 main= "Student Ability Curves"))
```



- Second plot: Students' mean scores differ by their ability levels, but these difference were larger given "Abstract teaching methods" compared to to the "standard teaching methods".

What we can find from these treatment mean (interaction) plot?

Next, we can investigate the nature of the interaction effects.

- If the researcher is interested to know the difference of teaching method (A_1 vs. A_2) given the same student ability (B_j), that is, to make inference for

$$D_1 = \mu_{11} - \mu_{21}\,, \quad D_2 = \mu_{12} - \mu_{22}\,, \quad D_3 = \mu_{13} - \mu_{23}$$

Then we can

1. obtain the estimator :

$$\hat{D}_1 = 92 - 90 = 2\,, \quad \hat{D}_2 = 81 - 86 = -5\,, \quad D_3 = 73 - 82 = -9$$

2. obtain the SE of the estimator: which is the same

$$s(\hat{D}_1) = s(\hat{D}_2) = s(\hat{D}_3) = \sqrt{2 * MSE/n} = \sqrt{2 * 28/21} = 1.633$$

3. If Bonferroni is used, with $\alpha = 0.05$, we can get Bonferroni multiple,

```r
#Bonferroni multiple, df=n_T- r = 120
g = 3 ; df= 120
alpha= 0.05
qt(1- alpha/(2*g), df)
```

```
## [1] 2.428004
```

4. Calculate the simultaneous CI for D1, D2 and D3, and make the interpretation of results (discuss the effect of factor A and B jointly):

$$2 \pm 2.428(1.633) \qquad -5 \pm 2.428(1.633) \qquad -9 \pm 2.428(1.633)$$

and the 95 percent confidence intervals for the family of comparisons are:

$$-1.96 \le \mu_{11} - \mu_{21} \le 5.96$$
$$-8.96 \le \mu_{12} - \mu_{22} \le -1.04$$
$$-12.96 \le \mu_{13} - \mu_{23} \le -5.04$$

For this family of confidence intervals, the following conclusions may be drawn with family confidence coefficient of 95 percent: (1) For students with excellent quantitative ability, the mean learning scores with the two teaching methods do not differ. (2) For students with either good or moderate quantitative abilities, the mean learning score with the abstract teaching method is lower than that with the standard method. The superiority of the standard teaching method may be particularly strong for students with moderate quantitative ability.

Important interaction: Example (3)

Example 2- Contrasts of Treatment Means (page 860): In the mathematics learning example, a school administrator also wished to know whether the amount of learning gain with the standard teaching method over the abstract method is **greater** for students with moderate quantitative ability than for students with good quantitative ability. **This question had been raised before the study began.**

1. To answer this question, We shall estimate the single contrast (difference of two paired difference):

$$L = D_3^{'} - D_2^{'} = (\mu_{23} - \mu_{13}) - (\mu_{22} - \mu_{12})$$

2. The estimator of L is

$$\hat{L} = (\hat{\mu}_{23} - \hat{\mu}_{13}) - (\hat{\mu}_{22} - \hat{\mu}_{12}) = (82 - 73) - (86 - 81) = 9 - 5 = 4$$

The estimate SE is

$$s(\hat{L}) = \sqrt{MSE/n(1+1+1+1)} = \sqrt{4 * 28/21} = 2.309$$

Since they only want to know if $L > 0$ or not, only **one-sided** CI, the $P(L > \text{CI lower limit}) = .95$

For one-sided 95 percent CI, the critical value $t(.05, df = 120)$ is

```r
qt(.95, 120)
```

```
## [1] 1.657651
```

Then the lower confidence limit is $4 - 1.658(2.309) = 0.17$. Therefore, we conclude, therefore, with 95 percent confidence coefficient that the gain in learning with the standard teaching method over the abstract method is greater for students with moderate quantitative ability than for students with good quantitative ability, with the difference in the mean gain being at least .17 point.

(Note: This is just an example to illustrate the additional analysis based on a certain contrast if interaction is present. If we test the contrast is zero or not, the resulting two-sided CI will cover zero, not suggesting a significant difference at 0.05 level. In practice, before the study is done, we almost always plan to use **two-sided test** for evaluate the difference or contrast without specifying the direction $(D > 0)$ or $(D < 0)$, then we will calculate the two-sided CI. This is because the one-sided test will have no power if we specify the wrong direction for the test.)

Pooling SS in two-way ANOVA (Ch 19.10)

In the ANOVA analysis based on the full model

$$Y_{ijk} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

When the test for interaction effects leads to the conclusion that no interactions are present. A reduced model without interaction term could then be used with only main effect of A and B

$$Y_{ijk} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + \epsilon_{ijk}$$

Then since the total SS (SSTO) is the same, this will lead to slightly different SS partition.

Model	total	A	B	A:B	Error
Full (A+B+A:B)	SSTO	SSA	SSB	SSAB df=(a-1)(b-1)	SSE df= ab(n-1)
Reduced (A+B)	SSTO	SSA	SSB	-	SSE*=SSE+SSAB df= ab(n-1)+(a-1)(b-1)

Therefore, in the reduced model, the error SS and df are pooled from the SSAB and SSE together compared the full model.

In general, when the interaction is not important, it is valid to use either full model or the reduced model to run two-way ANOVA. In the textbook, there are some practical suggestions for using the reduced model or pooling if

1. the degrees of freedom associated with MSE are small, perhaps 5 or less,
2. the test statistic MSAB/MSE falls substantially below the action limit of the decision rule, perhaps when MSAB/MSE < 2 for $\quad = .05$.

Part (1) of this rule is designed to limit pooling to cases where the gains may be substantial, while part (2) is designed to give reasonable assurance that there are indeed no interactions.

Planning of Sample size for Two-Factor Studies (Ch 19.11)

Power approach:

- When planning for a two-factor study, it is rarely based on the power for testing the interaction effect (harder to assume the targeted effect size for interaction, and often needs a much larger study to test interaction than main effects).
- Among two factor A and B, only one factor is the **primary** factor or more important, we can design power based on the primary comparison for that factor.
- When both factors are equally important, we can design the sample size based on both the power of detecting factor A main effects and the power of detecting factor B main effects.

1. One can first specify the minimum range of factor A level means for which it is important to detect factor A using what we learned in Chapter 16.10 for one factor studies with $r = a$. The resulting sample size is bn, from which n, the sample size per treatment (A & B combination) can be obtained readily.
2. In the same way, the minimum range of factor B level means can then be specified for which it is important to detect factor B main effects, and the needed sample sizes found as an for each level of B.
3. If the sample sizes obtained from the factor A and factor B power specifications differ substantially, a judgment will need to be made as to the final sample sizes.

- If type I error α is strictly controlled for the study and both factors are equally important, then $\alpha/2$ should be used for the power calculation for both A and B.

The estimation approach

- We can also planning sample sizes based on the estimation approach as we discussed in Chapter 17.8 for single-factor studies.
- Based on the number of comparisons , multiple comparison procedure and an assumption for the the standard deviation σ, we can we calculate the width of these CIs, and check various sample size to satisfy the pre-specified estimation precision.

Summary this week

- Reading: Chapter 19
- HW: 19.10 (a-d, skip e), 19.11, 19.30, 19.38 (due 10/10 by 6 pm before the class)
  - (For the inference on contrast and mutiple comparison, check the note for Ch17 may be helpful.)
- Quiz (#3) next Thurs (10/10): main concept of ch 18-19

1. ANOVA assumptions, diagnostics (residuals or tests), and remedial measures
2. Two-way ANOVA: 2 models, ANOVA table (ss partition, df, MS= SS/df), F-tests for A, B, and A:B, and intepretation of test results.