STAT 3119

Week 13: 11/19/2019 @GWU

Outline

- Review: on inferences for Unbalanced Studies
- Chapter 25 on Random and Mixed effects model
- One-factor ANOVA model II: Estimation and inferences

Review: Inferences for Unbalanced Studies

For two-factor studies (Chapter 23), we discussed estimation and inferences for factor level and treatment level means.

- Balanced studies can be considered as special cases of unbalanced studies with $n_{ij} = n$. We can apply the formulas in the unbalanced studies for the balanced studies, and will obtain the same results, but not the other way around.

Approach 1:

- General formulas are listed in **Table 23.5**

$$\hat{\mu}_{i\cdot} = \frac{\sum_j \bar{Y}_{ij\cdot}}{b}$$

$$s^2\{\hat{\mu}_{i\cdot}\} = \frac{MSE}{b^2} \sum_j \frac{1}{n_{ij}}$$

$$\hat{D} = \hat{\mu}_{i\cdot} - \hat{\mu}_{i'\cdot}$$

$$s^2\{\hat{D}\} = \frac{MSE}{b^2} \sum_j \left(\frac{1}{n_{ij}} + \frac{1}{n_{i'j}} \right)$$

- For a simple example, consider a two-factor study with factor A (2 levels) and B (2 levels): want to make inference such as getting CI for $\mu_{1.}$ and $\mu_{1.} - \mu_{2.}$

	B: j=1	B: j=2
A: i =1	Y_{111}, …, $Y_{11,\, n11}$	Y_{121}, …, $Y_{12,\, n12}$
A: i =2	Y_{211}, …, $Y_{21,\, n21}$	Y_{221}, …, $Y_{22,\, n22}$

Unbalanced case

n_{11}, n_{12}, n_{21}, n_{22} , may not be the same

	B: j=1	B: j=2
A: i =1	$\bar{Y}_{11}$	$\bar{Y}_{12}$
A: i =2	$\bar{Y}_{21}$	$\bar{Y}_{22}$

Balanced case (chapter 19)

$n_{11}= n_{12}= n_{21}=n_{22} =n$

$\widehat{\mu_{1.}} = \bar{Y}_{1..} = \Sigma\, \Sigma Y_{1jk}/2n$

$\mathrm{Var}(\widehat{\mu_{1.}}) = \frac{MSE}{2n}$

$\widehat{\mu_{1.}} - \widehat{\mu_{2.}} = \bar{Y}_{1..} - \bar{Y}_{2..}$

$\mathrm{Var}(\widehat{\mu_{1.}} - \widehat{\mu_{2.}}) = \frac{MSE}{2n}(1+1) = \frac{MSE}{n}$

Estimator/SE ~ ***t*** distribution df= (n-1)ab

Unbalanced case (chapter 23)

$\widehat{\mu_{1.}} = (\bar{Y}_{11} + \bar{Y}_{12})/2$

$\mathrm{Var}(\widehat{\mu_{1.}}) = \frac{MSE}{4}(\frac{1}{n_{11}} + \frac{1}{n_{12}})$

$\widehat{\mu_{1.}} - \widehat{\mu_{2.}} = (\bar{Y}_{11} + \bar{Y}_{12})/2 - (\bar{Y}_{21} + \bar{Y}_{22})/2$

$\mathrm{Var}(\widehat{\mu_{1.}} - \widehat{\mu_{2.}}) = \frac{MSE}{4}(\frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}})$

Estimator/SE ~ ***t*** distribution df= n_T-ab

Approach 2: to get the inferences on $\mu_{1.} - \mu_{2.}$ based on factor effects model

By definition,

$$\mu_{1.} - \mu_{2.} = (\mu_{..} + \alpha_1) + (\mu_{..} + \alpha_2) = \alpha_1 - \alpha_2 = \alpha_1 - (-\alpha_1) = 2\alpha_1$$

From the regression model output for coefficient estimates, we can obtain the estimate and SE for $\hat{\alpha}_1$. Then

$$\hat{\mu}_{1.} - \hat{\mu}_{2.} = 2\hat{\alpha}_1$$

And

$$v\hat{a}r(\hat{\mu}_{1.} - \hat{\mu}_{2.}) = 4v\hat{a}r(\hat{\alpha}_1)$$

or

$$\hat{\sigma}(\hat{\mu}_{1.} - \hat{\mu}_{2.}) = 2\hat{\sigma}(\hat{\alpha}_1)$$

Approach 3: use software to get estimate and CI.

For unbalanced three-factor studies, with factor A, B, C, we have similar formulation to get the estimates from the data.

- To get the paired comparison for factor level A, we still need to get the treatment means for each i, j, k levels, then average over all levels of j, k and the variance needs to sum up the corresponding reciprocal of $n_{i,j,k}$.

$$D = \mu_{i..} - \mu_{i'..} \tag{24.48a}$$

$$\hat{D} = \hat{\mu}_{i..} - \hat{\mu}_{i'..} \qquad \text{where} \qquad \hat{\mu}_{i..} = \frac{\sum_j \sum_k \bar{Y}_{ijk.}}{bc} \tag{24.48b}$$

$$s^2\{\hat{D}\} = \frac{MSE}{b^2c^2} \sum_j \sum_k \left(\frac{1}{n_{ijk}} + \frac{1}{n_{i'jk}} \right) \tag{24.48c}$$

The appropriate degrees of freedom associated with MSE are $n_T - abc$.

- We can also use
$$\mu_{1..} - \mu_{2..} = 2\alpha_1$$
and obtain the inferences based on factor effects model using the regression output.

For example (homework problem 24.15), each of factors (A, B and C) has 2 levels and we denote with one indicator function each and fit the regression:

```
## lm(formula = response ~ Ind.A * Ind.B * Ind.C, data = HW6B)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.9333 -0.7667  0.0500  0.9750  3.7667
##
## Coefficients:
##                     Estimate Std. Error t value Pr(>|t|)
## (Intercept)         60.01667    0.42910 139.866  < 2e-16 ***
## Ind.A               -5.67500    0.42910 -13.225 2.66e-09 ***
## Ind.B               -8.06667    0.42910 -18.799 2.49e-11 ***
## Ind.C              -10.02500    0.42910 -23.363 1.30e-12 ***
## Ind.A:Ind.B          0.04167    0.42910   0.097    0.924
## Ind.A:Ind.C          0.15000    0.42910   0.350    0.732
## Ind.B:Ind.C         -0.40833    0.42910  -0.952    0.357
## Ind.A:Ind.B:Ind.C    0.10000    0.42910   0.233    0.819
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.88 on 14 degrees of freedom
```

Then we can get the

$$\hat{\mu}_{1..} - \hat{\mu}_{2..} = 2\hat{\alpha}_1 = 2 * (-5.675) = -11.35$$

and

$$\hat{\sigma}(\hat{\mu}_{1..} - \hat{\mu}_{2..}) = 2\hat{\sigma}(\hat{\alpha}_1) = 2 * 0.4291 = 0.8582.$$

If we need to estimate the variance of error terms, σ^2 or MSE, then it is equal to the square of the residual standard error in the regression output. Here $\hat{\sigma}^2 = (1.88)^2 = 3.534 = MSE$.

Introduction: Random and Mixed effects Models

Fixed Effects vs Random Effects

- So far we have been considering fixed effects models, in which the levels of each factor were fixed in advance of the experiment and we were interested in differences in response among those **specific** levels.
- Now In this chapter, we will consider **random effects** models, in which the factor levels are a sample from a larger population of potential factor levels. We are interested in whether that factor has a significant effect in explaining the response for the general population with many possible levels for the given factor.
- In general, in the *planning* stages of an experiment, the researcher should decide whether the levels of factors considered are to be set at *fixed* values or are to be chosen at *random* from many possible levels, depending on the objectives of the experiment: Are the results to be judged for those levels alone or are they to be extended to more levels of which those in the experiment are only a random sample?
- For examples:

1. In a single-factor study by a company that owns several hundred retail stores, seven of these stores were selected at random, and a sample of employees in each store was asked to evaluate the management of the store. In this case, the interested is to generalize the results to the entire population of stores, not the difference in the specific 7 stores, hence the factor retail stores in this example is considered a random factor. However, if this study is for a small company only has 7 stores, then the stores are a fixed factor levels.
2. Think for example of a random sample of school classes that were drawn from all school classes in a country. Another example could be machines that were randomly sampled from a large population of machines. Typically, we are interested in making a statement about some properties of the whole population and not of the observed school classes or machines.

Design Consideration:

- Some experimental factors such as temperature, time or pressure, it is usually desirable to pick fixed levels, often low, middle or high levels, because a random choice may not cover the range in which the researcher is interested.
- Other factors such as operators, days, batches, hospitals may often be a small sample of all possible operators, days, batches or hospitals. Therefore, the particular factor levels selected to perform the experiment may not be important, only whether or not the given factor in general affect the response or increase the variability of the experiment.
- To design a study using a random effect model, the decision should be made prior to run the experiment so the factor levels can be chosen from all possible levels by a random process.

Type of ANOVA Models

When we have both fixed and random effects to model the factors in a multi-factor study, we call it a **mixed effects** model.

Analysis of variance (ANOVA) models include

- ANOVA model I (fixed effects model) : all factors are fixed

- ANOVA model II (random effects model): all factors are random
- ANOVA model III (mixed effects model): some factors are fixed and some are random

Note: In some situations, once the data is collected, it may be OK to apply either fixed effects model or random effects model in the data analysis. However, the interpretation of the model results should match with the model used to making inference about the specific factor levels (fixed effects model) or the general population (random effects model when factor levels can be considered a random sample that is representative of the population of possible levels).

One- factor Studies with a random factor: Example

Apex Enterprises, a company that builds roadside restaurants carrying one of several promoted trade names, leases franchises to individuals to operate the restaurants, and provides management services.

- This company employs a large number of personnel officers who interview applicants for jobs in the restaurants. At the end of an interview, the personnel officer assigns a rating between 0 and 100 to indicate the applicant's potential value on the job.
- 5 personnel officers were **selected at random** (the random factor levels): each was assigned 4 candidates at random.
- In this case, the company did not wish to make inferences concerning the five personnel officers who happened to be selected but rather about the population of all personnel officers.(A repetition of the same study would involve a new random sample of personnel officers which would probably consist of different officers.)
- Questions of interest included: How great is the variation in ratings among all personnel officers? What is the mean rating by all personnel officers?

TABLE 25.1 **Ratings by Five Personnel Officers—Apex Enterprises Example.**

Officer i	Candidate (j) 1	2	3	4	Mean
A	76	65	85	74	$\bar{Y}_{1.} = 75.00$
B	59	75	81	67	$\bar{Y}_{2.} = 70.50$
C	49	63	61	46	$\bar{Y}_{3.} = 54.75$
D	74	71	85	89	$\bar{Y}_{4.} = 79.75$
E	66	84	80	79	$\bar{Y}_{5.} = 77.25$
Mean					$\bar{Y}_{..} = 71.45$

Single factor ANOVA model II

One factor study setting:

- One factor with levels $i = 1, ..., r$
- Y_{ij} is the jth observation in factor level i, $j = 1, ..., n_i$.
- We consider the balanced studies first with $n_i = n$.

1. Random cell means model

$$Y_{ij} = \mu_i + \epsilon_{ij} \qquad (25.1)$$

where:

- μ_i are independent $N(\mu_., \sigma^2_\mu)$,
- ϵ_{ij} are independent $N(0, \sigma^2)$
- μ_i and ϵ_{ij} are independent random variables, $i = 1, ..., r$, and $j = 1, ..., n$.

2. Random factor effects model

Equivalently, we can expressing each factor level mean μ_i as a individual deviation from the expected value (population mean) $\mu_.$ with $\tau_i = \mu_i - \mu_.$ Then

$$Y_{ij} = \mu_. + \tau_i + \epsilon_{ij} \qquad (25.38)$$

where

- $\mu_.$ is the constant (population mean)
- τ_i are independent $N(0, \sigma^2_\mu)$ (with a mean of zero), representing the effect of the ith factor levels (its random deviation from the population mean response)
- ϵ_{ij} are are independent $N(0, \sigma^2)$
- τ_i and ϵ_{ij} are independent random variables, $i = 1, ..., r$, and $j = 1, ..., n$.

Difference between ANOVA model II and model I

- Unlike fixed effects models I (16.2) & (16.62) (where μ_i and τ_i are constant parameters), μ_i and τ_i are random variables with a common mean in random effects models II (25.1) & (25.38).
- The variability of the factor level mean μ_i is measured by its variance σ^2_μ. If all μ_i are the same, then its variance is 0. Thus, the question of μ_i all the same can now be addressed by considering testing if its variance $\sigma^2_\mu = 0$.

Illustration of random effects model

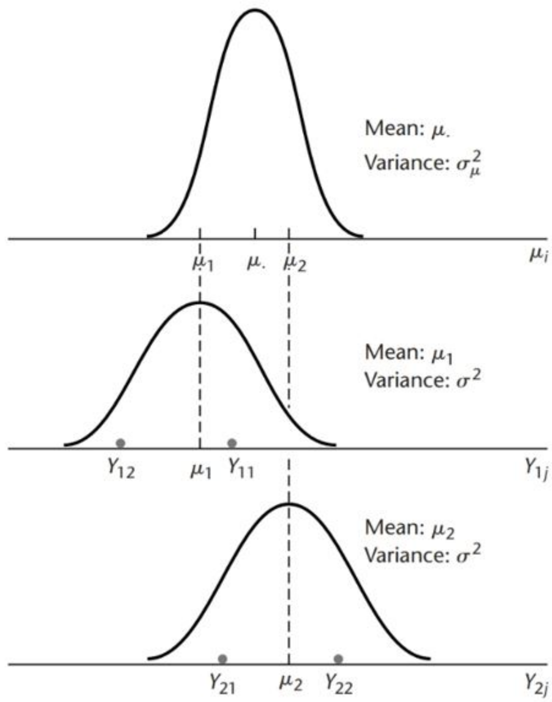


FIGURE 25.1 **Representation of ANOVA Model II.**

Mean responses for factor levels $\mu_1, \mu_2, \ldots, \mu_r$ follows a normal distribution with variance σ^2_μ; Or $\mu_1, \mu_2, \ldots$ are randomly selected from $N(\mu_., \sigma^2_\mu)$.

Given a factor level, μ_i , the observations Y_{ij}, within the factor level are i.i.d $N(\mu_i, \sigma^2)$, centered at different mean μ_i .

Properties of ANOVA model II

1. The expected value of a response Y_{ij} is
$$E\{Y_{ij}\} = \mu_.$$
2. The variance of Y_{ij} (total variance),is:
$$\sigma_Y^2 = \sigma_\mu^2 + \sigma^2$$
All observations Y_{ij} have the same variance, which is the sum of two components (variability of the μ_i and random measurement errors).
3. Y_{ij} are normally distributed $\sim N(\mu_., \sigma_Y^2)$, but not all of Y_{ij} are independent.

- The observations are independent, when they are from different factor levels.
- The observations are correlated when they are from the same factor level, their covariance is $cov(Y_{ij}, Y_{ij'}) = \sigma_\mu^2$; their correlation is
$$\rho(Y_{ij}, Y_{ij'}) = \frac{\sigma_\mu^2}{\sigma_\mu^2 + \sigma^2}$$
This correlation is also called the intraclass correlation coefficient (ICC), representing the proportion of the total variability of the Y_{ij} that is accounted for by the variability of the μ_i.

Note:

1. The responses from the same factor level are correlated because the responses are expected to be similar and they have the same random component μ_i (factor level mean) and will differ only because of the error terms. However, once the factor level is selected, the observations from the levels are "conditionally independent".

2. ICC =0 if $\sigma_\mu^2 = 0$ and no variability of μ_i; ICC is near 1 if σ^2 is much bigger than random error σ^2 and much of the total variability is accounted for by differences in factor level means.

3. When ANOVA model II is appropriate, there is usually no interest in inferences about the particular μ_i included in the study, such as which is the largest or smallest, the inference is usually focused on the entire population of μ_i, i.e., its mean $\mu_.$ and variance σ_μ^2.

ANOVA Table and testing whether $\sigma_\mu^2 = 0$

We first consider how to test whether all μ_i are equal. This is equivalent to test

$$H_0 : \sigma_\mu^2 = 0 \quad vs.$$

$$H_a : \sigma_\mu^2 > 0 \quad vs.$$

H_0 implies that all μ_i are equal; H_a implies that the μ_i differ.

- The terms and layout of the one-way ANOVA table are the same as what we used for the fixed effects in ANOVA model I as in Table 16.3

TABLE 16.3 ANOVA Table for Single-Factor Study.

Source of Variation	SS	df	MS
Between treatments	$SSTR = \sum n_i(\overline{Y}_i. - \overline{Y}..)^2$	$r-1$	$MSTR = \frac{SSTR}{r-1}$
Error (within treatments)	$SSE = \sum\sum(Y_{ij} - \overline{Y}_i.)^2$	$n_T - r$	$MSE = \frac{SSE}{n_T - r}$
Total	$SSTO = \sum\sum(Y_{ij} - \overline{Y}..)^2$	$n_T - 1$	

- However, the expected mean squares (EMS) are different.

$$E\{MSE\} = \sigma^2 \tag{25.6}$$

$$E\{MSTR\} = \sigma^2 + n\sigma_\mu^2 \tag{25.7}$$

- Compare the EMS, we expect $E\{MSTR\}$ is larger than $E\{MSE\}$ if $\sigma_\mu^2 > 0$. The F-test statistic F^*= MSTR/MSE follows a F-distribution $F(r-1, n_T - r)$. . Hence, we reject H_0 when F^* is large.

Apex enterprises example: ANOVA test

1. read the data

```r
Apex  =read.table(
        url("https://raw.githubusercontent.com/npmldabook/Stat3119/master/Week-13/CH25TA01.txt"))
names(Apex) =  c("Response", "Officers", "units")

# make  categorical variables for factor A and B
Apex$Officers =  as.factor(Apex$Officers)
str(Apex)
```

```
## 'data.frame':    20 obs. of  3 variables:
##  $ Response: int  76 65 85 74 59 75 81 67 49 63 ...
##  $ Officers: Factor w/ 5 levels "1","2","3","4",..: 1 1 1 1 2 2 2 2 3 3 ...
##  $ units   : int  1 2 3 4 1 2 3 4 1 2 ...
```

2. ANOVA test if $\sigma_\mu^2 = 0$.

```r
# the usual ANOVA fit for a balanced one-factor fixed effects study
summary( aov(Response~ Officers, data= Apex) )
```

```
##             Df Sum Sq Mean Sq F value Pr(>F)
## Officers     4   1580   394.9   5.389 0.0068 **
## Residuals   15   1099    73.3
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Results: We obtained the one-way ANOVA table for balanced study as usual. $F = 5.39$ with P-value=0.0068. Thus we reject H_0 and conclude that the mean ratings of the personnel officers differ.

Note: Although the ANOVA test and results are the same for fixed or random effects model. The only difference is the conclusion. If H_0 is rejected, we conclude there is a difference among 5 officers in their ratings for fixed model; for the random models, we conclude there is a difference among all officers of which the 5 officers are but a random sample. In addition, based on the random or fixed effects models, the model parameters and their inferences such as confidence intervals are different. We will illustrate this in the example.

Estimation of model parameters

When ANOVA model II is applicable, we are interested in estimating the overall mean $\mu_.$ and the variance components (σ_μ^2 and error σ^2) and ICC (intraclass correlation coefficient).

Estimation and inference formula:

Parameters	Estimate	Variance	Confidence Intervals
$\mu.$	$\bar{Y}..$	$\frac{MSTR}{rn}$	$\bar{Y}.. \pm t(1-\alpha/2; r-1)s\{\bar{Y}..\}$
$\sigma^2{}_\mu$	(MSTR- MSE)/n	$\frac{MSTR - MSE}{n}$	Easier using CI based on MLE (from software)
σ^2	MSE	$\frac{r(n-1)MSE}{\chi^2[1-\alpha/2; r(n-1)]} \leq \sigma^2 \leq \frac{r(n-1)MSE}{\chi^2[\alpha/2; r(n-1)]}$	
ICC: $\frac{\sigma_\mu^2}{\sigma_\mu^2 + \sigma^2}$	Plug in the estimates of $\sigma^2{}_\mu$, σ^2	$U^* = \frac{U}{1+U}$ $L^* = \frac{L}{1+L}$	where $L = \frac{1}{n}\left[\frac{MSTR}{MSE}\left(\frac{1}{F[1-\alpha/2; r-1, r(n-1)]}\right) - 1\right]$ $U = \frac{1}{n}\left[\frac{MSTR}{MSE}\left(\frac{1}{F[\alpha/2; r-1, r(n-1)]}\right) - 1\right]$

Note:

1) For the model parameters, it is easy to implement with software to get point estimate and CI based on the likelihood functions, which would be more efficient and applicable to the more general unbalanced cases. Therefore, we will skip the page 1043-1046 as these approximation methods have very limited practice use.

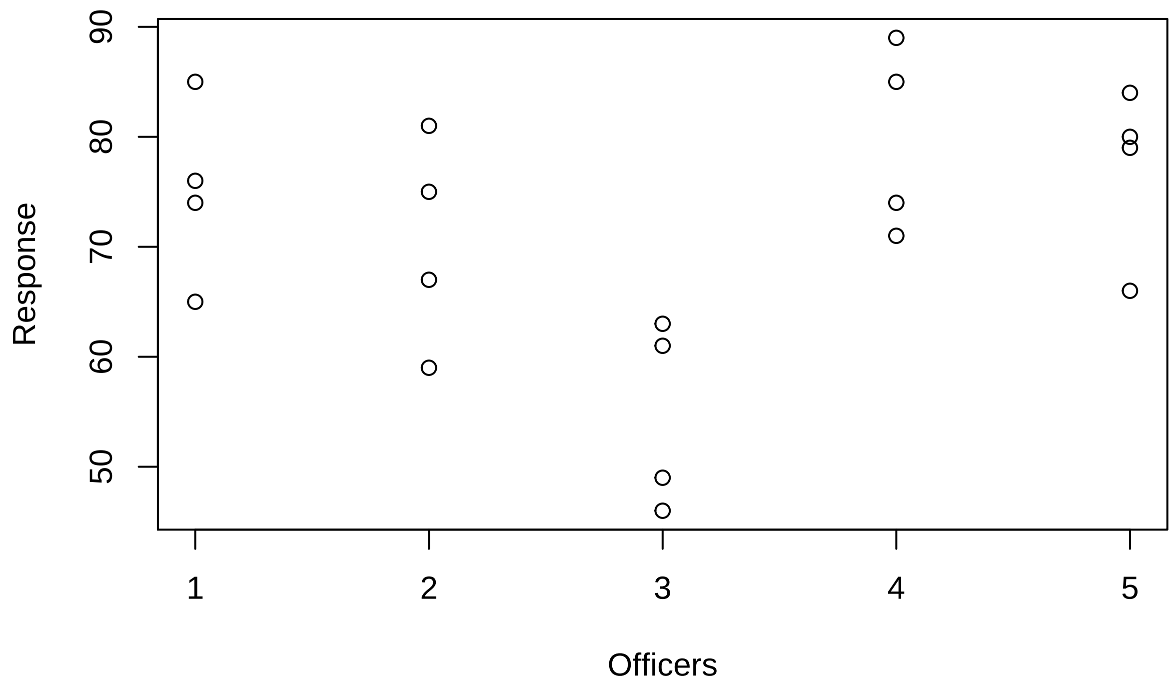
2) For ICC, if the lower limit is less than 0 from the formula. Since the parameter is always non-negative, we will use the $max(0, L^*)$ as the lower limit.

Apex enterprises example: estimation and inference

In *R*, there are several packages that can fit random or mixed effect models, we will consider **lme4** to estimate the overall means and variance components. This also applies to the general unbalanced cases. We use (1|factor) in the model to specify the random effects.

1. Apex example: Visualize data by factor levels

```
stripchart(Response ~ Officers, vertical = TRUE, pch = 1, xlab = "Officers", data = Apex)
```



2. Fit random effect models with lmer() function

```
Rpackage= "lme4"
if (! Rpackage %in% installed.packages()) install.packages(Rpackage)
library(lme4)
```

```
## Loading required package: Matrix
```

```
# We want to have a random effect per Officer
Ranfit.Apex <- lmer(Response ~ (1 | Officers), data = Apex)

# get a overall summary of the model
summary(Ranfit.Apex)
```

```
## Linear mixed model fit by REML ['lmerMod']
## Formula: Response ~ (1 | Officers)
##    Data: Apex
##
## REML criterion at convergence: 145.2
##
## Scaled residuals:
##     Min      1Q  Median      3Q     Max
## -1.3841 -0.8901  0.2620  0.6496  1.2605
##
```

```
## Random effects:
##  Groups   Name        Variance Std.Dev.
##  Officers (Intercept) 80.41    8.967
##  Residual             73.28    8.561
## Number of obs: 20, groups:  Officers, 5
##
## Fixed effects:
##             Estimate Std. Error t value
## (Intercept)   71.450      4.444   16.08
```

Results: 1) From the **Fixed effects** output, we can the estimate for the overall mean $\mu_.$ is 71.45 with estimate standard error 4.44 (same as results in page 1039.) This would be the expected rating of a randomly selected officer from the population of all officers in the company.

2) From the **Random effects** output, we can obtain $\hat{\sigma}^2_\mu$(Officers) = 80.41 and $\hat{\sigma}^2$(random residual error) = 73.28. Then the total variance = 80.41+ 73.28= 153.69, and the ICC = 80.41/153.69= 52.32%, i.e. about half the variance in rating is explained by Officers.

3. get the confidence intervals for σ_μ, σ and $\mu_.$.

```
# default is 95% CI
confint(Ranfit.Apex, oldNames = FALSE)
```

```
## Computing profile confidence intervals ...
```

```
##                              2.5 %   97.5 %
## sd_(Intercept)|Officers  2.174396 18.88895
## sigma                    6.217559 12.84920
## (Intercept)             61.894247 81.00575
```

```
confint(Ranfit.Apex, oldNames = FALSE, level=.90)
```

```
## Computing profile confidence intervals ...
```

```
##                                5 %     95 %
## sd_(Intercept)|Officers  3.307848 16.10241
## sigma                    6.515244 11.95074
## (Intercept)             63.919857 78.98014
```

Results: To compare with the results in the textbook, we have 95% CI for $\mu_.$ as [62, 81]. And 90% CI for σ_μ as [3.3, 16.1]; and 90% CI for σ as [6.5, 12.0]. (For variance components, the estimated 95% CIs are slightly different because the results were obtained from different methods.)

4. Compare with parameter estimation and inference in fixed effects model

```
options(contrasts = c("contr.sum", "contr.poly"))
fixedfit.Apex <- aov(Response~ Officers, data= Apex)
confint(fixedfit.Apex)
```

```
##                   2.5 %    97.5 %
## (Intercept)  67.369976 75.530024
## Officers1    -4.610047 11.710047
## Officers2    -9.110047  7.210047
## Officers3   -24.860047 -8.539953
## Officers4     0.139953 16.460047
```

Results: Although the ANOVA test results for the factor level difference is the same. The estimate and inference for fixed vs. random effects model are difference.

- With these contrast options above, here in the fixed effects model, we are estimating the overall means and its CI with the zero sum constraints for the fixed effects.

$$\mu_. = \frac{1}{5}\sum_{i=1}^{5}\mu_i$$

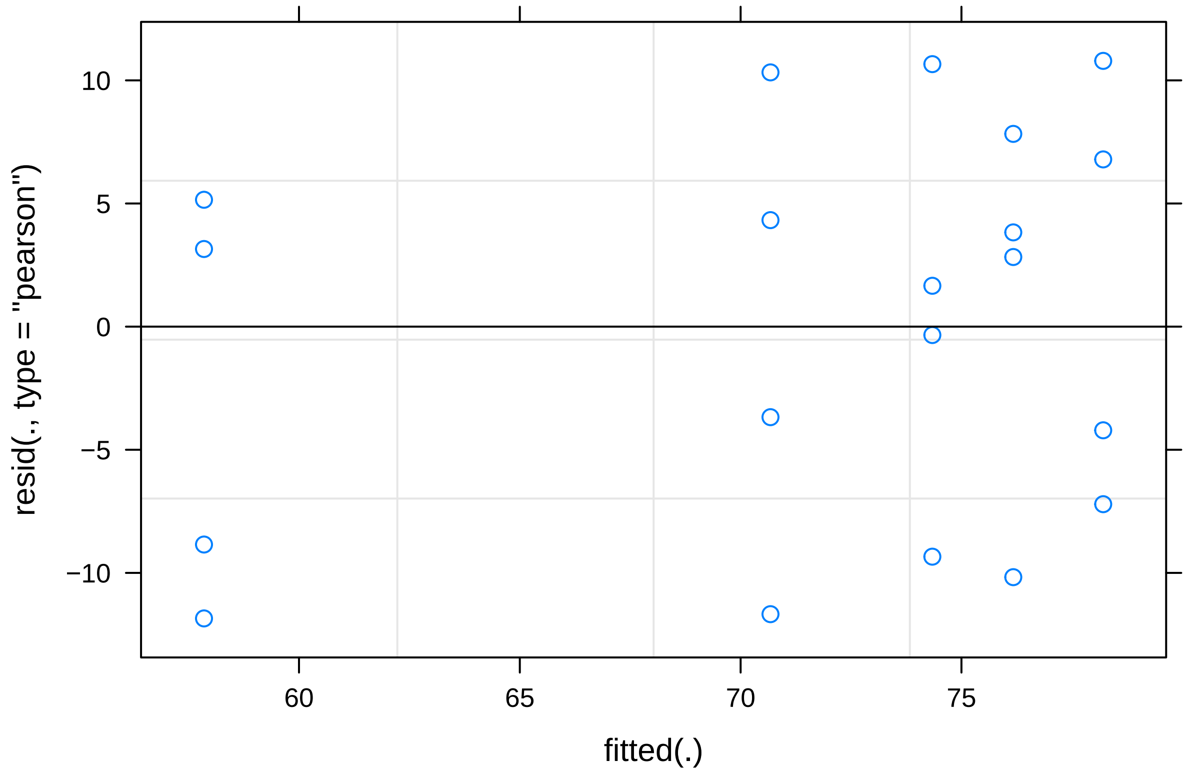
  as the average of constant factor mean levels for the five specific officers. The 95% CI for $\mu_.$ is [67.4, 75.5].

- We can see the 95% CI for $\mu_.$ [62, 81] based on the random effects model was wider, because random effects model allows to make inference about the population of all officers by adding variability from the random factor levels, while the fixed effects model only allows to make inference about these five specific officers.

Apex enterprises example: model checking

1. Get the residual vs. fitted plot to check constancy of variance

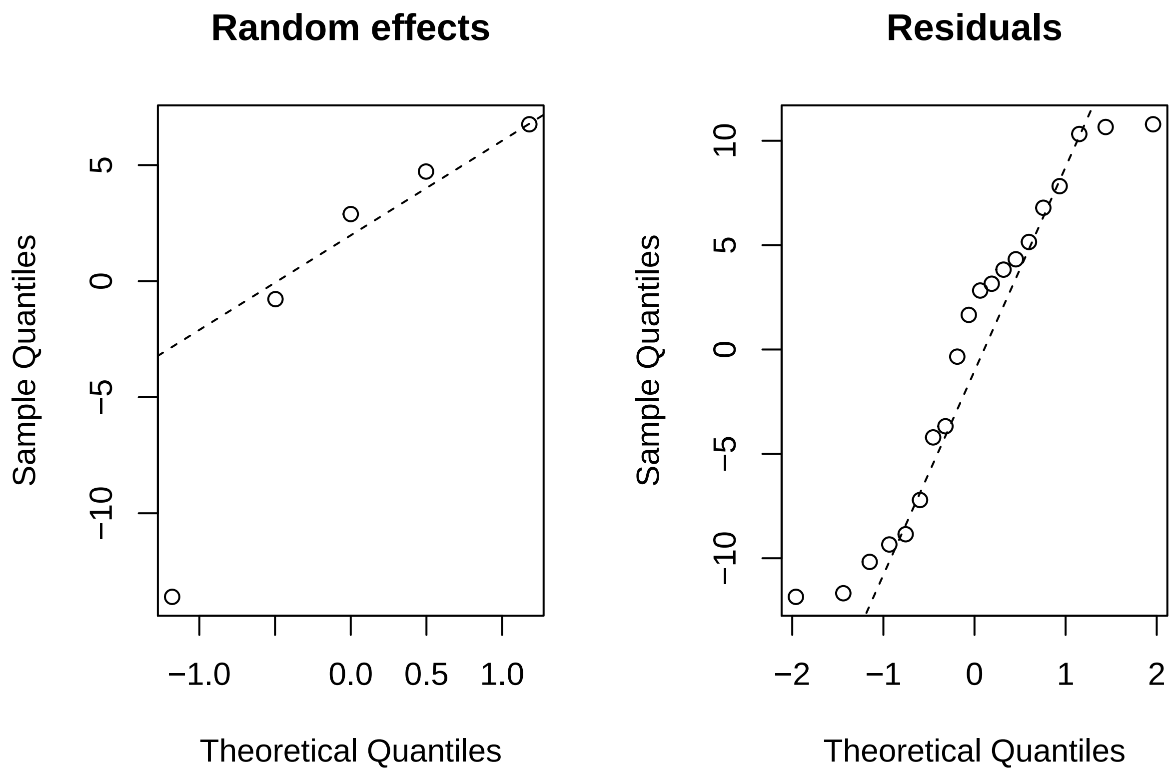
```
plot(Ranfit.Apex)
```



2. use QQ-plots to check the normality of random effects and the residuals

```
par(mfrow = c(1, 2))
qqnorm(ranef(Ranfit.Apex)$Officers[, "(Intercept)"], main = "Random effects")
qqline(ranef(Ranfit.Apex)$Officers[, "(Intercept)"], lty=2)

qqnorm(resid(Ranfit.Apex), main = "Residuals")
qqline(resid(Ranfit.Apex), lty=2)
```



Summary

- Reading Chapter 25.1
- Quiz this Thursday : covering chapter 23-24 (we drop the lowest non-zero score of the quiz 1-5 to caculate the final grade)