STAT 3119

Week3: 9/10/2019 @GWU

Outline Week 3 (Chapter 16.9 & Chapter 17)

We have introduced the basic ANOVA models in two formulations, how to estimate and conduct the hypothesis testing with a F-test based on the ANOVA table and discussed the relationship with the standard regression model, and power/sample size analysis.

- A1. Quick questions for homework and Review for chapter 16
- A2. Randomization test (ch 16.9)
- A3. Analysis of factor levels means (Ch 17.1-3)
- B1. Simultaneous Inference Procedures

Review for the key concepts for Quiz#1 (Thursday)

- ANOVA models
- Estimation and testing, ANOVA tables
- Power and sample size

Randomization Test (ch 16.9)

Randomization test (permutation test) or other resample methods (e.g. Bootstrap) are widely used, not limited to ANOVA model. For ANOVA, it can provide the basis for making inferences without

1. requiring assumptions about the distribution of the error terms $\epsilon \sim N(0, \sigma^2)$.
2. requiring the sample size of the subjects/units per group to be reasonably large.

When the assumptions for the distribution of the test statistic may not hold or sample size is too small, the following procedure can be carried out to generate the *empirical distribution (Randomization distribution)* of the test statistic under the null hypothesis and p-value based on the observed sample

$$\mathbf{Z} = \{Y_{ij}, i = 1, ..., r; j = 1, ..., n_i\}$$

which had a test statistic F^* (with unknown distribution).

Because under H_0 (no treatment effects: $\mu_i's$ are the same), distribution of Y_{ij} will not depend on the factor level i, thus randomization will lead to an assignment of all Y_{ij} equally likely to different treatments (factor levels).

1. The 1st step is to generate all combinations of the Y_{ij} from the sample space $\mathbf{Z}$ but with different treatment assignments. (Under H_0, all treatment combinations of observations are equally likely with probability of $1/m$, where m is the # of combinations from $\mathbf{Z}$ with various treatment assignments.)
2. In step 2, for each new sample (one of the treatment combinations from $\mathbf{Z}$), we calculate the test statistic $F_k, k = 1, ..., m$ as usual. These F_k's will represent the empirical distribution (all possibilities) under H_0.

3. We calculate the empirical p-value (exact probability that $F_k \geq F^*$)= the percentage of $F_k \geq F^*$ out of m.

Example 1 (page 713): A single factor experiment with 2 treatments and 2 replications for each treatment.

Treatment 1 Y_{1j}	Treatment 2 Y_{2j}
3	8
7	10

This is the case that the sample size is too small to trust the F distribution approximation for the test-statistic. The overall sample space for the observations would be $\{3, 7, 8, 10\}$. If any observation is equally likely to be assigned to treatments 1 and 2, we will have the six possible sampling cases, because $\binom{4}{2}\binom{2}{2} = 6$.

Randomization	Treatment 1	Treatment 2	F^*	Probability
1	3, 7	8, 10	3.20	1/6
2	3, 8	7, 10	1.06	1/6
3	3, 10	8, 7	.08	1/6
4	8, 7	3, 10	.08	1/6
5	7, 10	3, 8	1.06	1/6
6	8, 10	3, 7	3.20	1/6

Therefore the empirical (exact) p-value for this example is $Pr(F \geq 3.20) = 2/6 = 0.33$, thus we cannot reject the null hypothesis that the treatment effects are different at the significant level of 0.05.

Example 2 (page 713): A single factor experiment with 3 treatments and 3 replications for each treatment.

TABLE 16.5 Randomization Samples and Test Statistics—Quality Control Example.

Randomization	Treatment 1	Treatment 2	Treatment 3	F^*	Probability
1	1.1, .5, −2.1	4.2, 3.7, .8	3.2, 2.8, 6.3	4.39	1/1,680
2	1.1, .5, −2.1	4.2, 3.7, 3.2	.8, 2.8, 6.3	3.74	1/1,680
3	1.1, .5, −2.1	4.2, 3.7, 2.8	3.2, .8, 6.3	3.67	1/1,680
...	...	...	...	...	...
1,680	3.2, 2.8, 6.3	4.2, 3.7, .8	1.1, .5, −2.1	4.39	1/1,680

The first row was the observed sample. Under the null, we can easily calculate the number of different treatment assignment combinations for the 9 observations $\{1.1, .5, -2.1, 4.2, 3.7, 0.8, 3.2, 2.8, 6.3\}$, would be $\binom{9}{3}\binom{6}{3}\binom{3}{3} = 84 * 20 * 1 = 1680$. Then we can compute the exact p-value for this case with software very quickly (if the F distribution is suspected due to normality assumption or the small sample sizes). The textbook calculated the exactly P-value =0.07 in this example by comparing with the F^* for the observed case with all the F-statistics from the 1680 permutated samples. Here is some idea how to do that in R.

```r
# R:  call R to compute the # of combinations
 choose(9, 3)* choose(6,3)
```

```
## [1] 1680
```

```r
# R:  call R can use combn to generate all combinations:
#  e.g. choose 2 cases to be treatment group 1: one case per column
combn(1:4, 2)
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    1    1    1    2    2    3
## [2,]    2    3    4    3    4    4
```

I will skip the programming detail and post my R-code for this example in the class website.

Note: Statisticians often run randomization test or statistical simulations when standard methods do not apply. That is why it is useful to learn programming such as R to solve more complex problems, besides that it can save us time with lots of build-in functions for standard models.

Introduction: Analysis of Factor Level Means (ch 17.1)

In Chapter 16, we discussed the F test for determining whether or not the factor level means μ_i differ. When the F test leads to the conclusion that the factor level means differ, a relation between the factor and the response variable is present.

In Chapter 17, we discuss how to run a thorough analysis of the nature of the factor level means after the F-test (for equality of the factor levels means) is rejected. This is done in two principal ways: estimation and testing.

- Estimation techniques : including constructing of two-sided confidence interval for an effect of interest
- Testing: e.g. testing certain treatment effect can be achieved or compare and contrast among parameters. When many related comparisons are to be made, testing often precedes estimation to determine the active or statistically significant set of comparisons.
- Estimation and testing are closely related, e.g. if we have 95% CI of a parameter, say θ, and we want to test $\theta = \theta_0$ with a type I error of 0.05, we can just check if the CI contains θ_0 or not: if θ_0 is outside of this CI, then we would reject the hythothesis of $\theta = \theta_0$ at the level 0.05. Similary, we can use 99% CI to decide if we can reject the hythothesis at level of 0.01. But to compute the p-value (how strongly the data support the alternative hypothesis), we still need to construct a test statistic.
- To make simultaneous inference about the several CIs or testing results, we need to use multiple comparison procedures to preserve the overall significant level or type I error.
- Finally, we will discuss sample size & power analysis based on estimation precision.

In this chapter, the usual single factor ANOVA is assumed as in Chapter 16.

The cell means version of this model was given in (16.2):

$$Y_{ij} = \mu_i + \varepsilon_{ij} \tag{17.1}$$

where:

μ_i are parameters
ε_{ij} are independent $N(0, \sigma^2)$

Kenton Food Company example

Studied in Lecture 2A, Example 16.4:

- The Kenton Food Company wished to test four different package designs for a new breakfast cereal. Twenty stores, with approximately equal sales volumes, were selected as the experimental units. Each store was randomly assigned one of the package designs, with each package design assigned to five stores. One store was dropped out due to fire. **Sales** (Y), in number of cases, were observed for the study period, and the results are recorded in Table 16.1. This study is a **completely randomized design** with package design as the **single, four-level factor**.

Package Design	Store (j)							Number of Stores
	1	2	3	4	5	Total	Mean	
i	Y_{i1}	Y_{i2}	Y_{i3}	Y_{i4}	Y_{i5}	$Y_{i.}$	$\bar{Y}_{i.}$	n_i
1	11	17	16	14	15	73	14.6	5
2	12	10	15	19	11	67	13.4	5
3	23	20	18	17		78	19.5	4
4	27	33	22	26	28	136	27.2	5
All designs						$Y_{..} = 354$	$\bar{Y}_{..} = 18.63$	19

- Based on a standard ANOVA analysis, we can obtain the following summary (in Ch 16),

TABLE 17.1 Summary of Results—Kenton Food Company Example.

	Package Design (i)				
	1	2	3	4	Total
n_i	5	5	4	5	19
$Y_{i.}$	73	67	78	136	354
$\overline{Y}_{i.}$	14.6	13.4	19.5	27.2	18.63

Source of Variation	SS	df	MS
Between designs	588.22	3	196.07
Error	158.20	15	10.55
Total	746.42	18	

Package Design	Characteristics
1	3 colors, with cartoons
2	3 colors, without cartoons
3	5 colors, with cartoons
4	5 colors, without cartoons

- From the ANOVA table, we obtain $F^* = MSTR/MSE = 196.07/10.55 = 18.6$ with a p-value =.00003, therefore we reject the H_0 that all designs have the same effects on the sales volume, suggesting there is a relation between package design and sales. This would lead to the further analysis to study the nature of these differences.

Plots of Estimated Factor Level Means (Ch 17.2)

Before undertaking formal analysis of the nature of the factor level effects, it is usually helpful to visually inspect these factor effects from a plot of the estimated factor level means, $\hat{\mu}_i = \overline{Y}_{i.}$ (sample mean for factor level i). The book showed 3 types of plots (1) line plot, (2) **bar graph**, and (3) **main effects plot**.

Bar graphs and main effects plots are frequently used to display and compare the estimated factor level means (so-called **main effects**) in two dimensions.

- In a bar graph, vertical bars are used to display the estimated factor level means .
- In a main effects plot, a scatter plot of the estimated factor level means is provided, and connected by straight lines, to visibly highlight potential trends in the cell means if the factor levels are ordered (such as low, middle or high, etc).
- An overall mean can be displayed using a horizontal line, permitting visual comparisons of the factor-level means with the overall mean.

The textbook showed you to plot with **MINITAB** software (Figure 17.2). Many statistical software, or the Excel can also be use to generate those plots. Here we use **R** for plotting and data analysis.

Step 1. We read the data into R then obtain the factor level means as before.

```r
# read data
Ex16 =  read.table(url("https://raw.githubusercontent.com/npmldabook/Stat3119/master/Week2/CH16_TA01.tx
names(Ex16) =  c("sales", "package", "stores")
Ex16$package =  as.factor(Ex16$package)

#  fit ANOVA model, get factor level and overall means
fit  =  aov(sales ~ package, data = Ex16)
Factor_means =  predict(fit, newdata = data.frame(package = factor(1:4)))
Overall_means = mean(Factor_means)

# list these statistics
list(Factor_means= Factor_means, Overall_means= Overall_means)
```

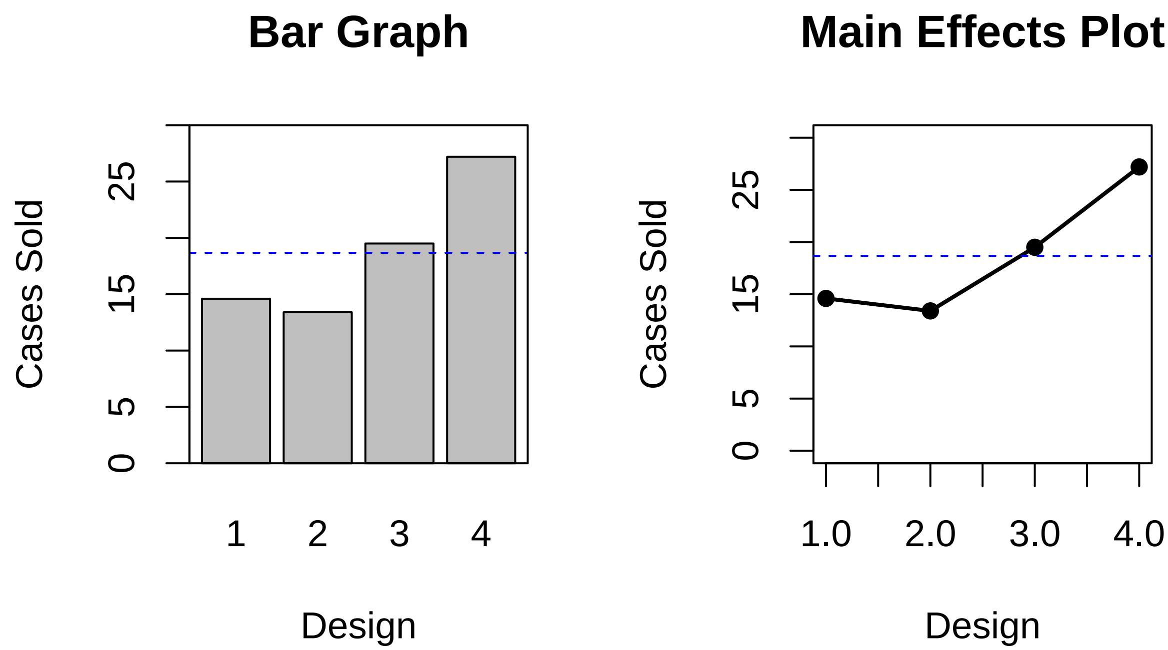
```
## $Factor_means
##    1    2    3    4
## 14.6 13.4 19.5 27.2
##
## $Overall_means
## [1] 18.675
```

Step 2. Generate a barchart and main effects plot.

```r
# note to choose fill color, ylim, labels
# mfrow to set 1 row 2 column in plot panel
#   cex=1.2 is increase font size from cex=1, pty is square box
par(mfrow=c(1,2), cex=1.2, pty='s')

# plot 1 : barchat
barplot(Factor_means, col='gray', ylim=c(0,30), xlab="Design", ylab="Cases Sold", main="Bar Graph")
abline(h= Overall_means, lty=2, col='blue')
box() # add a box around the barchart

# plot 2: main effects plot.
# note ? plot to see options, type='o' overplotted line and points
plot(1:4, Factor_means, type='o', pch=16, lwd=2,
      ylim=c(0,30), xlab="Design", ylab="Cases Sold", main="Main Effects Plot")
abline(h= Overall_means, lty=2, col='blue')
```



Estimation and Testing of Factor Level Means (Ch 17.3)

Statistical Inferences (estimate, standard errors and confidence interval) for factor level means are generally concerned with one or more of the following:

1. A single factor level mean (main effect) μ_i
2. A difference between two factor level means
3. A contrast among factor level means
4. A linear combination of factor level means

We discuss each of these types of inferences in turn.

Inferences for Single Factor Level Mean (page 739)

An unbiased point estimator of the factor level mean μ_i is

$$\hat{\mu}_i = \overline{Y}_{i.}$$

For this estimator and estimator $\hat{\sigma}^2$= MSE of , we then have

$$E(\overline{Y}_{i.}) = \mu_i$$
$$\text{var}(\overline{Y}_{i.}) = \sigma^2/n_i$$
$$\widehat{\text{var}}(\overline{Y}_{i.}) = MSE/n_i$$

It follows from the definition for the t distribution as a ratio of a standard normal variable and the square root of an independent chi-square variable, we then have

$$\frac{\overline{Y}_{i\cdot} - \mu_i}{s\{\overline{Y}_{i\cdot}\}} \text{ is distributed as } t(n_T - r) \text{ for ANOVA model (17.1)} \qquad \textbf{(17.5)}$$

where estimated standard deviation is

$$s\{\overline{Y}_{i.}\} = \sqrt{MSE/n_i}$$

Therefore,

1) we can obtain the $1-\alpha$ confidence interval for μ_i as

$$\overline{Y}_{i\cdot} \pm t(1-\alpha/2; n_T - r)s\{\overline{Y}_{i\cdot}\} \qquad \textbf{(17.7)}$$

2) For testing of $\mu_i = c$, we construct the test statistic t^* with a t-distribution with $df = (n_T - r)$.

Testing. The confidence interval based on the limits in (17.7) can be used to test a hypothesis of the form:

$$\begin{aligned} H_0&: \mu_i = c \\ H_a&: \mu_i \neq c \end{aligned} \qquad \textbf{(17.8)}$$

where c is an appropriate constant. We conclude H_0, at level of significance α, when c is contained in the confidence interval, and we conclude H_a when the confidence interval does not contain c. Equivalently, one can compute the test statistic:

$$t^* = \frac{\overline{Y}_{i\cdot} - c}{s\{\overline{Y}_{i\cdot}\}} \qquad \textbf{(17.9)}$$

Test statistic t^* follows a t distribution with $n_T - r$ degrees of freedom when H_0 is true, according to (17.5). Consequently, we conclude H_0 whenever $|t^*| \leq t(1-\alpha/2; n_T - r)$; otherwise, we conclude H_a.

Food Example: if we want to find out the confidence interval (CI) for μ_1.

- From the previous results, $\overline{Y}_{1.} = 14.6$, $n_1 = 5$, $MSE = 10.55$, then $s\{\overline{Y}_{i.}\} = \sqrt{10.55/5} = 1.453$.
- For 95% CI , $1-\alpha/2 = 0.975$, $n_T - r = 19 - 4 = 15$, and we have the critical value t(.975, 15)= 2.131 (from a t-probability table in the appendix or R), then we can calculate from formula (17.7), that $t(.975, 15) * s\{\overline{Y}_{i.}\} = 2.131 * 1.453 \approx 3.1$, then 95% CI is 14.6 ± 3.1= (11.5, 17.7).

This can be done easily with a calculator using the formula. If you want to use R, it is straightforward to get individual main effect estimates with 95% CI, df and standard errors (**SE**). We need to install "emmeans" package first with **install.packages("emmeans")**.

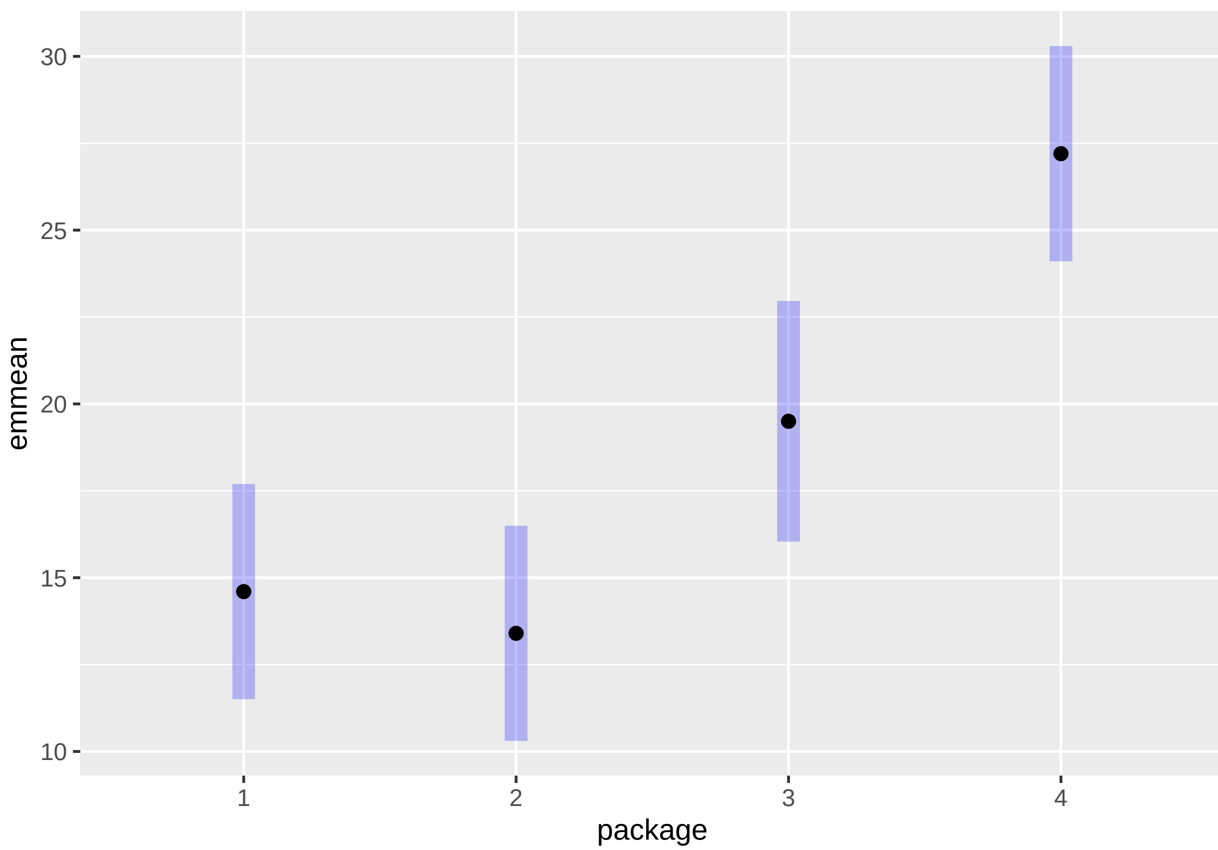
R example:

```
Rpackage= "emmeans"
if (! Rpackage  %in% installed.packages()) install.packages(Rpackage)
library(emmeans)

# get the estimate, SE, df and CI
(Est.mean<- emmeans(fit, ~ package))
```

```
##  package emmean   SE df lower.CL upper.CL
##  1         14.6 1.45 15     11.5     17.7
##  2         13.4 1.45 15     10.3     16.5
##  3         19.5 1.62 15     16.0     23.0
##  4         27.2 1.45 15     24.1     30.3
##
## Confidence level used: 0.95
```

```
# plot the main effects with 95% CIs
plot(Est.mean, horizontal=F)
```



Inferences for Difference between Two Factor Level Means (page 739)

In this situation, the parameter of interest is the difference between two factor level means:

$$D = \mu_i - \mu_{i'} \tag{17.10}$$

Such a difference between two factor level means is called a *pairwise comparison*. A point estimator of D in (17.10), denoted by $\hat{D}$, is:

$$\hat{D} = \overline{Y}_{i\cdot} - \overline{Y}_{i'\cdot} \tag{17.11}$$

The estimated variance of $\hat{D}$, denoted by $s^2\{\hat{D}\}$, is given by:

$$s^2\{\hat{D}\} = MSE\left(\frac{1}{n_i} + \frac{1}{n_{i'}}\right) \tag{17.14}$$

Then we have

$$\frac{\hat{D} - D}{s\{\hat{D}\}} \text{ is distributed as } t(n_T - r) \text{ for ANOVA model (17.1)} \tag{17.15}$$

Based on this t-distribution, we get the CI as follows:

Hence, the $1 - \alpha$ confidence limits for D are:

$$\hat{D} \pm t(1 - \alpha/2; n_T - r)s\{\hat{D}\} \tag{17.16}$$

And for the testing:

Testing. There is often interest in testing whether two factor level means are the same. The alternatives here are of the form:

$$\begin{aligned} &H_0: \mu_i = \mu_{i'} \\ &H_a: \mu_i \neq \mu_{i'} \end{aligned} \tag{17.17}$$

The alternatives in (17.17) can be stated equivalently as follows:

$$\begin{aligned} &H_0: \mu_i - \mu_{i'} = 0 \\ &H_a: \mu_i - \mu_{i'} \neq 0 \end{aligned} \tag{17.17a}$$

1. One way is to check if the confidence interval of D contained 0 or not to see if H_0 is reached at α level.
2. The second way is to construct a test statistic:

$$t^* = \frac{\hat{D}}{s\{\hat{D}\}} \tag{17.18}$$

Conclusion H_0 is reached if $|t^*| \leq t(1 - \alpha/2; n_T - r)$; otherwise, H_a is concluded.

Food Example: if we want to find out the confidence interval (CI) for $\mu_3 - \mu_4$.

$$\overline{Y}_{3.} = 19.5 \qquad n_3 = 4 \qquad MSE = 10.55$$
$$\overline{Y}_{4.} = 27.2 \qquad n_4 = 5$$

Hence:

$$\hat{D} = \overline{Y}_{3.} - \overline{Y}_{4.} = 19.5 - 27.2 = -7.7$$

The estimated variance of $\hat{D}$ is:

$$s^2\{\hat{D}\} = MSE\left(\frac{1}{n_3} + \frac{1}{n_4}\right) = 10.55\left(\frac{1}{4} + \frac{1}{5}\right) = 4.748$$

so that the estimated standard deviation of $\hat{D}$ is $s\{\hat{D}\} = 2.179$. We require $t(.975; 15) = 2.131$. The confidence limits therefore are $-7.7 \pm 2.131(2.179)$, and the desired 95 percent confidence interval is:

$$-12.3 \leq \mu_3 - \mu_4 \leq -3.1$$

For testing if the two means are the same:

$$H_0\colon \mu_3 - \mu_4 = 0$$
$$H_a\colon \mu_3 - \mu_4 \neq 0$$

Since the hypothesized difference zero in H_0 is not contained within the 95 percent confidence limits -12.3 and -3.1, we conclude H_a, that the presence of cartoons has an effect. We could also obtain test statistic (17.18):

$$t^* = \frac{\hat{D}}{s\{\hat{D}\}} = \frac{-7.7}{2.179} = -3.53$$

Since $|t^*| = 3.53 > t(.975; 15) = 2.131$, we conclude H_a. The two-sided P-value for this test is .003.

R example: pairs() function.

```
Est.mean<- emmeans(fit, ~ package)
pairs(Est.mean, adjust = "none" )
```

```
##  contrast estimate   SE df t.ratio p.value
##  1 - 2         1.2 2.05 15   0.584  0.5677
##  1 - 3        -4.9 2.18 15  -2.249  0.0399
##  1 - 4       -12.6 2.05 15  -6.135  <.0001
##  2 - 3        -6.1 2.18 15  -2.800  0.0135
##  2 - 4       -13.8 2.05 15  -6.719  <.0001
##  3 - 4        -7.7 2.18 15  -3.534  0.0030
```

As we can see this paired test yield test statistics for all the six pairwise comparisons, with the last row showing the exact results as in the textbook. Here **adjust = "none"** in the code indicates no adjustment for the multiple comparison is applied here. Each test is done at 0.05 level. We will discuss the multiple testing issues later in this chapter.

Inferences for Contrast of Factor Level Means (page 741)

Definition:

A *contrast* is a comparison involving two or more factor level means and includes the previous case of a pairwise difference between two factor level means in (17.10). A contrast will be denoted by L, and is defined as a linear combination of the factor level means μ_i where the coefficients c_i sum to zero:

$$L = \sum_{i=1}^{r} c_i \mu_i \qquad \text{where } \sum_{i=1}^{r} c_i = 0 \tag{17.19}$$

Examples to get the c_i in various contrasts. The comparison between any two means is a special case.

Q: write down the coefficient c_1, c_2, c_3, c_4 ?

- 1. $$L = \mu_1 - \mu_2$$
- 2. $$L = \frac{\mu_1 + \mu_2}{2} - \frac{\mu_3 + \mu_4}{2}$$
- 3. $$L = \frac{\mu_1 + \mu_3}{2} - \frac{\mu_2 + \mu_4}{2}$$
- 4. $$L = \mu_1 - \frac{\mu_1 + \mu_2 + \mu_3 + \mu_4}{4}$$
- 5. $$L = \frac{3\mu_1 + \mu_3}{4} - \frac{3\mu_2 + \mu_4}{4}$$

A:

$$L = \mu_1 - \mu_2$$

1. Here, $c_1 = 1$, $c_2 = -1$, $c_3 = 0$, $c_4 = 0$, and $\sum c_i = 0$.

$$L = \frac{\mu_1 + \mu_2}{2} - \frac{\mu_3 + \mu_4}{2}$$

2. Here, $c_1 = 1/2$, $c_2 = 1/2$, $c_3 = -1/2$, $c_4 = -1/2$, and $\sum c_i = 0$.

$$L = \frac{\mu_1 + \mu_3}{2} - \frac{\mu_2 + \mu_4}{2}$$

3. Here, $c_1 = 1/2$, $c_2 = -1/2$, $c_3 = 1/2$, $c_4 = -1/2$, and $\sum c_i = 0$.

$$L = \mu_1 - \frac{\mu_1 + \mu_2 + \mu_3 + \mu_4}{4}$$

4. Here, $c_1 = 3/4$, $c_2 = -1/4$, $c_3 = -1/4$, $c_4 = -1/4$, and $\sum c_i = 0$.

$$L = \frac{3\mu_1 + \mu_3}{4} - \frac{3\mu_2 + \mu_4}{4}$$

5. Here, $c_1 = 3/4$, $c_2 = -3/4$, $c_3 = 1/4$, $c_4 = -1/4$, and $\sum c_i = 0$.

For **Estimation**, we have estimate for the mean and variance as follows:

- $$\hat{L} = \sum_{i=1}^{r} c_i \overline{Y}_{i\cdot} \qquad \textbf{(17.20)}$$

- $$s^2\{\hat{L}\} = MSE \sum_{i=1}^{r} \frac{c_i^2}{n_i} \qquad \textbf{(17.22)}$$

Then we have the distribution of $\hat{L}$ and its CI:

$$\frac{\hat{L} - L}{s\{\hat{L}\}} \text{ is distributed as } t(n_T - r) \text{ for ANOVA model (17.1)} \qquad \textbf{(17.23)}$$

Consequently, the $1 - \alpha$ confidence limits for L are:

$$\hat{L} \pm t(1 - \alpha/2; n_T - r)s\{\hat{L}\} \qquad \textbf{(17.24)}$$

For **testing** of the contrast equal to 0 or not:

Testing. The confidence interval based on the limits in (17.24) can be used to test a hypothesis of the form:

$$H_0: L = 0$$
$$H_a: L \neq 0 \tag{17.25}$$

H_0 is concluded at the α level of significance if zero is contained in the interval; otherwise H_a is concluded. An equivalent procedure is based on the test statistic:

$$t^* = \frac{\hat{L}}{s\{\hat{L}\}} \tag{17.26}$$

If $|t^*| \leq t(1-\alpha/2; n_T - r)$, H_0 is concluded; otherwise, H_a is concluded.

Food Example: if we want to find out the confidence interval (CI) for a certain contrast:

- Estimation and CI:

$$L = \frac{\mu_1 + \mu_2}{2} - \frac{\mu_3 + \mu_4}{2}$$

The point estimate is (see data in Table 17.1):

$$\hat{L} = \frac{\overline{Y}_{1.} + \overline{Y}_{2.}}{2} - \frac{\overline{Y}_{3.} + \overline{Y}_{4.}}{2} = \frac{14.6 + 13.4}{2} - \frac{19.5 + 27.2}{2} = -9.35$$

Since $c_1 = 1/2$, $c_2 = 1/2$, $c_3 = -1/2$, and $c_4 = -1/2$, we obtain:

$$\sum \frac{c_i^2}{n_i} = \frac{(1/2)^2}{5} + \frac{(1/2)^2}{5} + \frac{(-1/2)^2}{4} + \frac{(-1/2)^2}{5} = .2125$$

and:

$$s^2\{\hat{L}\} = MSE \sum \frac{c_i^2}{n_i} = 10.55(.2125) = 2.242$$

so that $s\{\hat{L}\} = 1.50$.

For a 95 percent confidence interval, we require $t(.975; 15) = 2.131$. The confidence limits for L therefore are $-9.35 \pm 2.131(1.50)$, and the desired 95 percent confidence interval is:

$$-12.5 \leq L \leq -6.2$$

- For testing of the contrast $= 0$:

To test the hypothesis of no difference in mean sales for the 3-color and 5-color designs:

$$H_0\colon L = 0$$
$$H_a\colon L \neq 0$$

at the $\alpha = .05$ level of significance, we simply note that the hypothesized value zero is not contained in the 95 percent confidence interval. Hence, we conclude H_a, that the mean sales differ. To obtain a P-value of the test, test statistic (17.26) must be obtained. We find:

$$t^* = \frac{-9.35}{1.50} = -6.23$$

and the corresponding two-sided P-value is 0+.

R example: it is easy for user to define a given contrast or several contrasts by the vector $\{c_i\}$.

```
Est.mean<- emmeans(fit, ~ package)

#L1 was the same as last example#
L = list(
  L1=  c(0,0, 1, -1),
  L2 = c(1/2,  1/2, -1/2, -1/2))

contrast(Est.mean, L, adjust="none")
```

```
##  contrast estimate   SE df t.ratio p.value
##  L1          -7.70 2.18 15 -3.534  0.0030
##  L2          -9.35 1.50 15 -6.246  <.0001
```

Inferences for Linear Combination of Factor Level Means

- More generally, we may be interested in a linear combination of the factor level means that is not a contrast
- Define a linear combination ofthe factor level means μ_i as:

$$L = \sum_{i=1}^{r} c_i \mu_i$$

with no restrictions on the coefficients c_i. Special cases:

a. When $\sum c_i = 0$, L is a contrast;
b. When $\sum c_i = 1$, L is a weighted mean.

- Confidence limits (estimate, SE) and test statistics for a linear combination L are obtained in exactly the same way as those for a contrast, with slight modification for the test statistic.

$$L = \sum_{i=1}^{r} c_i \mu_i \qquad \cancel{\text{where} \sum_{i=1}^{r} c_i = 0}$$

$$\hat{L} = \sum_{i=1}^{r} c_i \overline{Y}_{i.} \qquad s^2\{\hat{L}\} = MSE \sum_{i=1}^{r} \frac{c_i^2}{n_i}$$

$$\frac{\hat{L} - L}{s\{\hat{L}\}} \text{ is distributed as } t(n_T - r)$$

$\Rightarrow$ $1 - \alpha$ confidence limits for L

$$\hat{L} \pm t(1 - \alpha/2; n_T - r) s\{\hat{L}\}$$

To test H_0: $\sum c_i \mu_i = c$

H_a: $\sum c_i \mu_i \neq c$

$\Rightarrow$

$$t^* = \frac{\hat{L} - c}{s\{\hat{L}\}}$$

follows a t distribution with $n_T - r$ of df , and rejection H_0 if $|t^*| \geq t(1 - \alpha/2; n_T - r)$.

Note: For these inference, because the denominator depends on MSE with $n_T - r$ df, all these t tests about the factor level means in one-factor ANOVA model follows the same t-distribution .

Summary of this class and homework

- Reading: Chapter 16.9; Chapter 17.1-17.3
- Quiz #1 this Thursday
- Homework#1(lask week) due this Thursday