STAT 3119

Week 12: 11/12/2019 @GWU

Outline

- Multi-factor studies (chapter 24)
- ANOVA model and estimation
- ANOVA table and testing

Introduction

- When three or more factors are studied simultaneously, the model and analysis employed are straightforward extensions of the two-factor case. We will illustrate mainly for 3-factor studies and we can apply the similar techniques for study with more than 3 factors.
- We will study the ANOVA model and formulas for 3 factor studies with balanced and unbalanced cases and how to implement the analysis with software.

ANOVA Model for Three-Factor Studies

We first consider the balanced cases, i.e. **all treatment sample sizes are equal and all treatment means are of equal importance** (Ch 24.1–24.5).

Notation (we need more subscripts for additional factors):

- Consider: Three factors, A, B, and C, are investigated at a, b, c levels, respectively. The mean response for the treatment when factor A is at the ith level ($i = 1, ..., a$), factor B is at the jth level ($j = 1, ..., b$), and factor C is at the kth level ($k = 1, ..., c$) is denoted by μ_{ijk}. The number of cases for each treatment is assumed to be constant, denoted by $n, n \geq 2$ (with replications within treatment).
- Since all treatment means are assumed to have equal importance, we use unweighted mean (simple average) to denote these following parameters:

1. The mean level of response when two factors (AB, AC, BC) are held at two given fixed levels as follows:

$$\mu_{ij\cdot} = \frac{\sum_k \mu_{ijk}}{c} \qquad \textbf{(24.1a)}$$

$$\mu_{i\cdot k} = \frac{\sum_j \mu_{ijk}}{b} \qquad \textbf{(24.1b)}$$

$$\mu_{\cdot jk} = \frac{\sum_i \mu_{ijk}}{a} \qquad \textbf{(24.1c)}$$

2. The mean level of response when one factor is held at a given level is :

$$\mu_{i..} = \frac{\sum_j \sum_k \mu_{ijk}}{bc} \tag{24.2a}$$

$$\mu_{.j.} = \frac{\sum_i \sum_k \mu_{ijk}}{ac} \tag{24.2b}$$

$$\mu_{..k} = \frac{\sum_i \sum_j \mu_{ijk}}{ab} \tag{24.2c}$$

Finally, the overall mean response is denoted by $\mu_{...}$ and is defined:

$$\mu_{...} = \frac{\sum_i \sum_j \sum_k \mu_{ijk}}{abc} \tag{24.3}$$

Two model formulations

Let Y_{ijkm} denote the observation for the mth case ($m = 1, ..., n$) for the treatment consisting of the ith level of A ($i = 1, ..., a$), the jth level of B ($j = 1, ..., b$), and the kth level of C ($k = 1, ..., c$). Thus, the total number of cases in the study is: $n_T = nabc$. (If the study is unbalanced, we have n_{ijk} with the factor level combinations- treatments level.)

1. **Cell Means Model**:

   The ANOVA model for a three-factor study in terms of the cell (treatment) means μ_{ijk} with fixed factor levels is:

   $$Y_{ijkm} = \mu_{ijk} + \varepsilon_{ijkm} \tag{24.12}$$

   where:

   μ_{ijk} are parameters
   ε_{ijkm} are independent $N(0, \sigma^2)$
   $i = 1, \ldots, a;\ j = 1, \ldots, b;\ k = 1, \ldots, c;\ m = 1, \ldots, n$

2. **Factor Effects Model**: An equivalent factor effects model can be developed that incorporates the factorial structure by expressing each treatment mean μ_{ijk} in terms of the various factor effects.

$$\mu_{ijk} \equiv \mu_{\cdots} + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} \qquad \textbf{(24.13)}$$

where:

$$\begin{aligned}
\mu_{\cdots} &= \frac{\sum\sum\sum \mu_{ijk}}{abc} \\
\alpha_i &= \mu_{i\cdot\cdot} - \mu_{\cdots} \\
\beta_j &= \mu_{\cdot j\cdot} - \mu_{\cdots} \\
\gamma_k &= \mu_{\cdot\cdot k} - \mu_{\cdots} \\
(\alpha\beta)_{ij} &= \mu_{ij\cdot} - \mu_{i\cdot\cdot} - \mu_{\cdot j\cdot} + \mu_{\cdots} \\
(\alpha\gamma)_{ik} &= \mu_{i\cdot k} - \mu_{i\cdot\cdot} - \mu_{\cdot\cdot k} + \mu_{\cdots} \\
(\beta\gamma)_{jk} &= \mu_{\cdot jk} - \mu_{\cdot j\cdot} - \mu_{\cdot\cdot k} + \mu_{\cdots} \\
(\alpha\beta\gamma)_{ijk} &= \mu_{ijk} - \mu_{ij\cdot} - \mu_{i\cdot k} - \mu_{\cdot jk} + \mu_{i\cdot\cdot} + \mu_{\cdot j\cdot} + \mu_{\cdot\cdot k} - \mu_{\cdots}
\end{aligned}$$

Hence, the equivalent factor effects ANOVA model for a three-factor study is:

$$Y_{ijkm} = \mu_{\cdots} + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \varepsilon_{ijkm} \qquad \textbf{(24.14)}$$

Model assumptions and constraints

1. **Error terms** ϵ_{ijkm} are independent $N(0, \sigma^2)$.

2. The **main effects for factor A, B and C** is defined as the difference of factor level means and overall mean.

$$\begin{aligned}
\alpha_i &= \mu_{i\cdot\cdot} - \mu_{\cdots} \\
\beta_j &= \mu_{\cdot j\cdot} - \mu_{\cdots} \\
\gamma_k &= \mu_{\cdot\cdot k} - \mu_{\cdots}
\end{aligned}$$

It follows from the definitions, the sums of the main effects are zero:

$$\sum_i \alpha_i = \sum_j \beta_j = \sum_k \gamma_k = 0$$

Therefore, for these factors, their degrees of freedow (dfs) are $a-1$, $b-1$ and $c-1$. We need $a-1$, $b-1$ and $c-1$ terms to describe their main effects and the last effect is equal to the negative sum of the other effects.

3. The **two-factor (two-way) interaction effects** in a three-factor study are defined in the same fashion as for a two-factor study, except that all means are averaged over the third factor.

$$(\alpha\beta)_{ij} = \mu_{ij\cdot} - \mu_{i\cdot\cdot} - \mu_{\cdot j\cdot} + \mu_{\cdot\cdot\cdot}$$
$$(\alpha\gamma)_{ik} = \mu_{i\cdot k} - \mu_{i\cdot\cdot} - \mu_{\cdot\cdot k} + \mu_{\cdot\cdot\cdot}$$
$$(\beta\gamma)_{jk} = \mu_{\cdot jk} - \mu_{\cdot j\cdot} - \mu_{\cdot\cdot k} + \mu_{\cdot\cdot\cdot}$$

It follows from the definitions, these *first-order interactions* have the constraints:

$$\sum_i (\alpha\beta)_{ij} = 0 \quad \text{for all } j \qquad \sum_j (\alpha\beta)_{ij} = 0 \quad \text{for all } i \qquad \textbf{(24.7a)}$$
$$\sum_i (\alpha\gamma)_{ik} = 0 \quad \text{for all } k \qquad \sum_k (\alpha\gamma)_{ik} = 0 \quad \text{for all } i \qquad \textbf{(24.7b)}$$
$$\sum_j (\beta\gamma)_{jk} = 0 \quad \text{for all } k \qquad \sum_k (\beta\gamma)_{jk} = 0 \quad \text{for all } j \qquad \textbf{(24.7c)}$$

Therefore, subject to these constraints, we need $(a-1)(b-1)$ terms to describe the interaction effects between A & B, $(a-1)(c-1)$ terms to describe the interaction effects between A & C, and $(b-1)(c-1)$ terms to describe the interaction effects between B & C.

4. In a three-factor study the **three-factor interaction** $(\alpha\beta\gamma)_{ijk}$ is defined as the difference between the treatment mean μ_{ijk} and the value that would be expected from main effects and first-order interactions, i.e., an additional adjustment term.

Hence, the *three-factor interaction* $(\alpha\beta\gamma)_{ijk}$, also called the *second-order interaction,* is defined as:

$$(\alpha\beta\gamma)_{ijk} = \mu_{ijk} - [\mu_{\cdot\cdot\cdot} + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk}] \qquad \textbf{(24.9a)}$$

or equivalently:

$$(\alpha\beta\gamma)_{ijk} = \mu_{ijk} - \mu_{ij\cdot} - \mu_{i\cdot k} - \mu_{\cdot jk} + \mu_{i\cdot\cdot} + \mu_{\cdot j\cdot} + \mu_{\cdot\cdot k} - \mu_{\cdot\cdot\cdot} \qquad \textbf{(24.9b)}$$

From the definition of the three-factor interactions, it follows that they sum to zero when added over any index:

$$\sum_i (\alpha\beta\gamma)_{ijk} = 0 \quad \text{for all } j, k \qquad \sum_j (\alpha\beta\gamma)_{ijk} = 0 \quad \text{for all } i, k \qquad \sum_k (\alpha\beta\gamma)_{ijk} = 0 \quad \text{for all } i, j \qquad \textbf{(24.10)}$$

If *all* three-factor interactions $(\alpha\beta\gamma)_{ijk}$ are zero, we say that there are no three-factor interactions among factors A, B, and C. If some $(\alpha\beta\gamma)_{ijk}$ are not zero, we say that three-factor interactions are present.

Therefore, for factor A, B & c, there are $(a-1)(b-1)(c-1)$ terms to describe the three-way interaction effects, etc.

Example 1 of Three-Factor Studies

We consider a study of the effects of gender, age, and intelligence level of college graduates on learning time for a complex task.

- Gender is factor A and has $a = 2$ levels (male, female). Age is factor B and is defined in terms of $b = 3$ levels (young, middle, old). Finally, intelligence is factor C and is defined in terms of $c = 2$ levels (high IQ, normal IQ).
- Table 24.1a shows the treatment means μ_{ijk} for all factor level combinations, as well as other **model parameters** for the ANOVA model. (This is not a table of observed data.)

TABLE 24.1 Mean Learning Times and ANOVA Model Parameters—Learning Time Example 1.

(a) Mean Learning Times (in minutes)

Factor A—Gender	Intelligence (factor C) and Age (factor B)											
	$k=1$ High IQ				$k=2$ Normal IQ				Average			
	$j=1$ Young	$j=2$ Middle	$j=3$ Old	Average	$j=1$ Young	$j=2$ Middle	$j=3$ Old	Average	$j=1$ Young	$j=2$ Middle	$j=3$ Old	Average
$i=1$ Male	9 (μ_{111})	12 (μ_{121})	18 (μ_{131})	13 ($\mu_{1\cdot1}$)	19 (μ_{112})	20 (μ_{122})	21 (μ_{132})	20 ($\mu_{1\cdot2}$)	14 ($\mu_{11\cdot}$)	16 ($\mu_{12\cdot}$)	19.5 ($\mu_{13\cdot}$)	16.5 ($\mu_{1\cdot\cdot}$)
$i=2$ Female	9 (μ_{211})	10 (μ_{221})	14 (μ_{231})	11 ($\mu_{2\cdot1}$)	19 (μ_{212})	20 (μ_{222})	21 (μ_{232})	20 ($\mu_{2\cdot2}$)	14 ($\mu_{21\cdot}$)	15 ($\mu_{22\cdot}$)	17.5 ($\mu_{23\cdot}$)	15.5 ($\mu_{2\cdot\cdot}$)
Average	9 ($\mu_{\cdot11}$)	11 ($\mu_{\cdot21}$)	16 ($\mu_{\cdot31}$)	12 ($\mu_{\cdot\cdot1}$)	19 ($\mu_{\cdot12}$)	20 ($\mu_{\cdot22}$)	21 ($\mu_{\cdot32}$)	20 ($\mu_{\cdot\cdot2}$)	14 ($\mu_{\cdot1\cdot}$)	15.5 ($\mu_{\cdot2\cdot}$)	18.5 ($\mu_{\cdot3\cdot}$)	16 ($\mu_{\cdot\cdot\cdot}$)

(b) ANOVA Model Parameters

$\mu_{\cdot\cdot\cdot} = 16.0$	$\beta_1 = -2.0$	$\gamma_1 = -4.0$	$(\alpha\beta)_{12} = 0.0$	$(\beta\gamma)_{11} = -1.0$	$(\alpha\beta\gamma)_{111} = -.5$
$\alpha_1 = .5$	$\beta_2 = -.5$	$(\alpha\beta)_{11} = -.5$	$(\alpha\gamma)_{11} = .5$	$(\beta\gamma)_{21} = -.5$	$(\alpha\beta\gamma)_{121} = 0.0$

By definition, we can compute 1+2+1 =4 main effect parameters:

$$\alpha_1 = \mu_{1..} - \mu_{...} = 16.5 - 16 = .5$$
$$\beta_1 = \mu_{.1.} - \mu_{...} = 14 - 16 = -2$$
$$\beta_2 = \mu_{.2.} - \mu_{...} = 15.5 - 16 = -.5$$
$$\gamma_1 = \mu_{..1} - \mu_{...} = 12 - 16 = -4$$

There are 1*2+2*1+1= 5 two-way interaction effects:

$$(\alpha\beta)_{11} = 14 - 16.5 - 14 \quad + 16 = \; -.5$$
$$(\alpha\beta)_{12} = 16 - 16.5 - 15.5 + 16 = \quad 0.0$$
$$(\alpha\gamma)_{11} = 13 - 16.5 - 12 \quad + 16 = \quad .5$$
$$(\beta\gamma)_{11} = \; 9 - 14 \quad - 12 \quad + 16 = -1.0$$
$$(\beta\gamma)_{21} = 11 - 15.5 - 12 \quad + 16 = \; -.5$$

And 1*2*1= 2 three-way interaction effects:

$$(\alpha\beta\gamma)_{ijk} = \mu_{ijk} - [\mu_{...} + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk}]$$

$$\Rightarrow (\alpha\beta\gamma)_{111} = 9 - (16 + .5 - 2 - 4 - .5 + .5 - 1) = -.5$$

$$(\alpha\beta\gamma)_{121} = 0.0$$

Interpretation of Interactions (Ch 24.2)

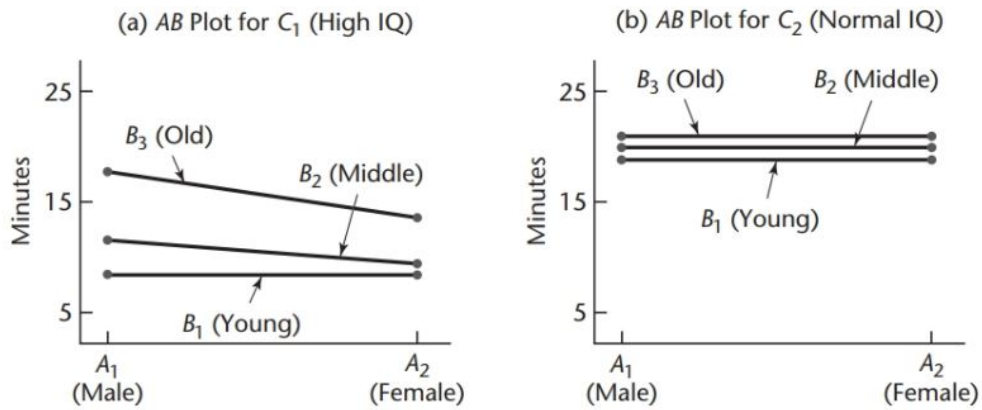
In general, we can use statistical tests in ANOVA models or regression models to assess if the two-way or three-way interactions are significant. The graphic display of the treatment means can also help us to check and understand the nature of the interactions.

Learning Time (Example 1): Interpretation of Three-Factor Interactions.

In a three-factor study, the presence of a three-factor interaction indicates that responses must be explained in terms of the combined effects of all three factors. Thus, no simplified explanation, for example in terms of main effects or first-order interactions, is possible.

- Any graphical presentation of cell means should display all of the individual cell means μ_{ijk}. A convenient way to do so is to create separate two-factor treatment mean (or interaction) plots for each level of a third factor.
- e.g. Learning time example: considers the effects of gender (factor A), age (factor B), and intelligence (factor C) on learning time

FIGURE 24.1 Cell Means Plot with ABC Interaction Present—Learning Time Example 1.



The plot shows:

1. For persons with normal IQ (C2 level), gender has no effect on mean learning time, and age has only a small effect leading to slightly longer learning times for older persons. For persons with high IQ (C1 level), on the other hand, females tend to learn more quickly than males for older persons but not for young persons, and older persons tend to require substantially longer learning times than young persons. [i.e. Given C1, there was interaction of A & B, but this interaction was not present for C2].
2. The slopes of the curves in the AB cell means plots are not the same for the two levels of C. For C1, the curves for middle-aged and older subjects have negative slopes, while these curves both are flat for C2. This lack of parallelism in the two plots suggests a possible three-factor interaction.
3. In other words, stratifying on one factor (e.g. factor C1, C2), if the pattern of the treatment means plot for the other two factors are very different, these results will indicator a possible three-way interaction, or we need more terms to explain the data.

In practice, three-factor interactions are difficult to explain, and most of the statistical models of many factors will not consider interactions involving more than three factors as higher order interactions are often quite small.

Fitting of ANOVA model (Ch 24.3)

Given the observed data, we can estimate the **ANOVA model parameters** by the math operations between treatment means and factor level means.

Parameter	Estimator	
$\mu_{\ldots}$	$\hat{\mu}_{\ldots} = \overline{Y}_{\ldots.}$	**(24.19a)**
α_i	$\hat{\alpha}_i = \overline{Y}_{i\ldots} + \overline{Y}_{\ldots.}$	**(24.19b)**
β_j	$\hat{\beta}_j = \overline{Y}_{.j..} + \overline{Y}_{\ldots.}$	**(24.19c)**
γ_k	$\hat{\gamma}_k = \overline{Y}_{..k.} + \overline{Y}_{\ldots.}$	**(24.19d)**
$(\alpha\beta)_{ij}$	$\widehat{(\alpha\beta)}_{ij} = \overline{Y}_{ij..} + \overline{Y}_{i\ldots} + \overline{Y}_{.j..} + \overline{Y}_{\ldots.}$	**(24.19e)**
$(\alpha\gamma)_{ik}$	$\widehat{(\alpha\gamma)}_{ik} = \overline{Y}_{i.k.} + \overline{Y}_{i\ldots} + \overline{Y}_{..k.} + \overline{Y}_{\ldots.}$	**(24.19f)**
$(\beta\gamma)_{jk}$	$\widehat{(\beta\gamma)}_{jk} = \overline{Y}_{.jk.} + \overline{Y}_{.j..} + \overline{Y}_{..k.} + \overline{Y}_{\ldots.}$	**(24.19g)**
$(\alpha\beta\gamma)_{ijk}$	$\widehat{(\alpha\beta\gamma)}_{ijk} = \overline{Y}_{ijk.} - \overline{Y}_{ij..} - \overline{Y}_{i.k.} - \overline{Y}_{.jk.} + \overline{Y}_{i\ldots} + \overline{Y}_{.j..} + \overline{Y}_{..k.} - \overline{Y}_{\ldots.}$	**(24.19h)**

where the sample means are simple average of observations for given treatment means, 2- factor level means, one factor level means and the overall mean.

$$Y_{ijk.} = \sum_m Y_{ijkm} \qquad \overline{Y}_{ijk.} = \frac{Y_{ijk.}}{n} \tag{24.15a}$$

$$Y_{ij..} = \sum_k \sum_m Y_{ijkm} \qquad \overline{Y}_{ij..} = \frac{Y_{ij..}}{cn} \tag{24.15b}$$

$$Y_{i.k.} = \sum_j \sum_m Y_{ijkm} \qquad \overline{Y}_{i.k.} = \frac{Y_{i.k.}}{bn} \tag{24.15c}$$

$$Y_{.jk.} = \sum_i \sum_m Y_{ijkm} \qquad \overline{Y}_{.jk.} = \frac{Y_{.jk.}}{an} \tag{24.15d}$$

$$Y_{i\ldots} = \sum_j \sum_k \sum_m Y_{ijkm} \qquad \overline{Y}_{i\ldots} = \frac{Y_{i\ldots}}{bcn} \tag{24.15e}$$

$$Y_{.j..} = \sum_i \sum_k \sum_m Y_{ijkm} \qquad \overline{Y}_{.j..} = \frac{Y_{.j..}}{acn} \tag{24.15f}$$

$$Y_{..k.} = \sum_i \sum_j \sum_m Y_{ijkm} \qquad \overline{Y}_{..k.} = \frac{Y_{..k.}}{abn} \tag{24.15g}$$

$$Y_{\ldots.} = \sum_i \sum_j \sum_k \sum_m Y_{ijkm} \qquad \overline{Y}_{\ldots.} = \frac{Y_{\ldots.}}{abcn} \tag{24.15h}$$

We also have the **fitted values and residuals** of the model:

Thus, the *fitted values* for the observations are the estimated treatment means:

$$\hat{Y}_{ijkm} = \overline{Y}_{ijk\cdot} \qquad (24.17)$$

and the *residuals* are the deviations of the observed values from the estimated treatment means:

$$e_{ijkm} = Y_{ijkm} - \hat{Y}_{ijkm} = Y_{ijkm} - \overline{Y}_{ijk\cdot} \qquad (24.18)$$

Evaluation of Appropriateness of multi-factor ANOVA Model

- Same assumptions as 1-factor or 2-factor ANOVA, i.e., **normality, constancy of error variance, and independence of error terms**, which can be examined by residual plots

  $$e_{ijkm} = Y_{ijk} - \overline{Y}_{ijk.}$$

  or the statistical tests for normality and constancy of error variance.

- Remedial measures discussed in earlier chapters on these topics apply completely to the multi-factor case.
  - Weighted least squares as usual is a standard remedial measure when the error variance is not constant but the distribution of the error terms is normal.
  - A transformation of the response variable may be helpful to stabilize the error variance, to make the error distributions more normal, and/or to make important interactions unimportant.

Example 2: Stress Test

Three factor study on on exercise tolerance (Y) were studied in a small-scale investigation of persons 25 to 35 years old.

- Gender of subject (factor A)
- Body fat of subject (measured in percent, factor B)
- Smoking history of subject (factor C)
- 2^3 factorial : $a = b = c = 2$, 8 treatment levels
- Exercise tolerance was measured in minutes until fatigue occurs while the subject is performing on a bicycle.
- Three subjects for each gender-body fat-smoking history group were given the exercise tolerance stress test.

Data Table:

TABLE 24.4 Sample Data and Estimated Treatment and Factor Level Means for Three-Factor Study—Stress Test Example.

(a) Data

	Smoking History	
	$k = 1$ Light	$k = 2$ Heavy
$j = 1$ Low fat:		
$i = 1$ Male	24.1 (Y_{1111})	17.6 (Y_{1121})
	29.2 (Y_{1112})	18.8 (Y_{1122})
	24.6 (Y_{1113})	23.2 (Y_{1123})
$i = 2$ Female	20.0 (Y_{2111})	14.8 (Y_{2121})
	21.9 (Y_{2112})	10.3 (Y_{2122})
	17.6 (Y_{2113})	11.3 (Y_{2123})
$j = 2$ High fat:		
$i = 1$ Male	14.6 (Y_{1211})	14.9 (Y_{1221})
	15.3 (Y_{1212})	20.4 (Y_{1222})
	12.3 (Y_{1213})	12.8 (Y_{1223})
$i = 2$ Female	16.1 (Y_{2211})	10.1 (Y_{2221})
	9.3 (Y_{2212})	14.4 (Y_{2222})
	10.8 (Y_{2213})	6.1 (Y_{2223})

Sample mean estimates can be obtained to estimate the model parameters and plot to check interactions.

(b) Estimated Means	$k = 1$	$k = 2$	All k
$j = 1$:			
$i = 1$	25.97 ($\bar{Y}_{111\cdot}$)	19.87 ($\bar{Y}_{112\cdot}$)	22.92 ($\bar{Y}_{11\cdot\cdot}$)
$i = 2$	19.83 ($\bar{Y}_{211\cdot}$)	12.13 ($\bar{Y}_{212\cdot}$)	15.98 ($\bar{Y}_{21\cdot\cdot}$)
All i	22.90 ($\bar{Y}_{\cdot 11\cdot}$)	16.00 ($\bar{Y}_{\cdot 12\cdot}$)	19.45 ($\bar{Y}_{\cdot 1\cdot\cdot}$)
$j = 2$:			
$i = 1$	14.07 ($\bar{Y}_{121\cdot}$)	16.03 ($\bar{Y}_{122\cdot}$)	15.05 ($\bar{Y}_{12\cdot\cdot}$)
$i = 2$	12.07 ($\bar{Y}_{221\cdot}$)	10.20 ($\bar{Y}_{222\cdot}$)	11.13 ($\bar{Y}_{22\cdot\cdot}$)
All i	13.07 ($\bar{Y}_{\cdot 21\cdot}$)	13.12 ($\bar{Y}_{\cdot 22\cdot}$)	13.09 ($\bar{Y}_{\cdot 2\cdot\cdot}$)
All j:			
$i = 1$	20.02 ($\bar{Y}_{1\cdot 1\cdot}$)	17.95 ($\bar{Y}_{1\cdot 2\cdot}$)	18.98 ($\bar{Y}_{1\cdot\cdot\cdot}$)
$i = 2$	15.95 ($\bar{Y}_{2\cdot 1\cdot}$)	11.17 ($\bar{Y}_{2\cdot 2\cdot}$)	13.56 ($\bar{Y}_{2\cdot\cdot\cdot}$)
All i	17.98 ($\bar{Y}_{\cdot\cdot 1\cdot}$)	14.56 ($\bar{Y}_{\cdot\cdot 2\cdot}$)	16.27 ($\bar{Y}_{\cdot\cdot\cdot\cdot}$)

Example 2: Interaction and residuals plots

1. Read the data

```
Ex24  =read.table(
        url("https://raw.githubusercontent.com/npmldabook/Stat3119/master/Week-12/CH24TA04.txt"))

head(Ex24)
```

```
##     V1 V2 V3 V4 V5
## 1 24.1  1  1  1  1
## 2 29.2  1  1  1  2
## 3 24.6  1  1  1  3
## 4 20.0  2  1  1  1
## 5 21.9  2  1  1  2
## 6 17.6  2  1  1  3
```

```
names(Ex24) =  c("Response", "Gender", "Body_Fat", "Smoking","units")


# make  categorical variables for factor A and B
Ex24$Gender =  as.factor(Ex24$Gender)
Ex24$Body_Fat =  as.factor(Ex24$Body_Fat)
Ex24$Smoking =  as.factor(Ex24$Smoking)
```

```r
# Add factor level labels
levels(Ex24$Gender) = c("M", "F")
levels(Ex24$Body_Fat) = c("Low", "High")
levels(Ex24$Smoking) = c("Light", "Heavy")

str(Ex24)
```

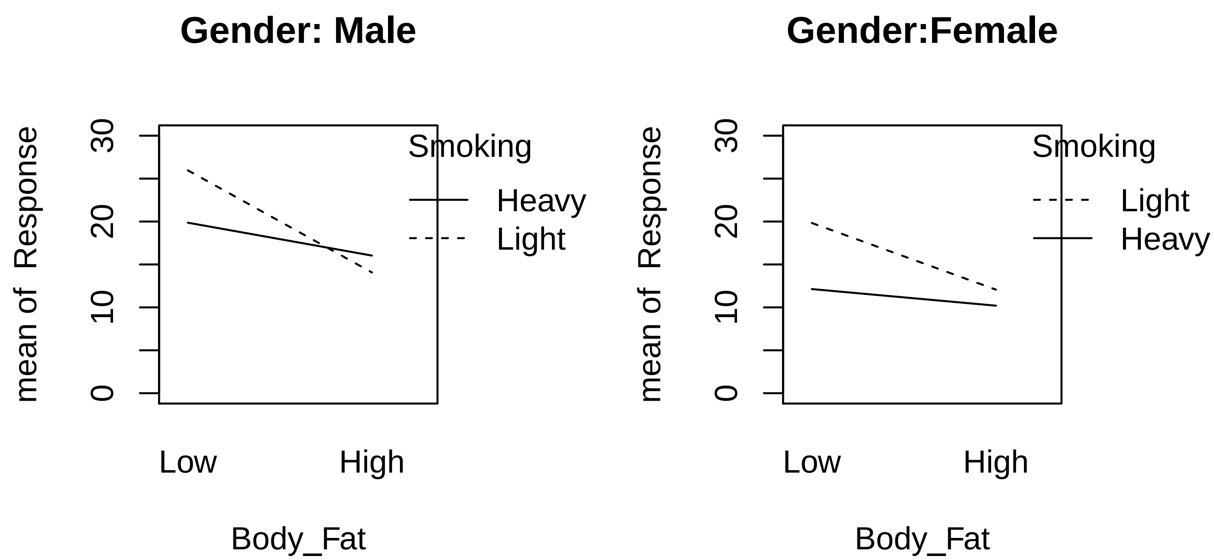
```
## 'data.frame':    24 obs. of  5 variables:
##  $ Response: num  24.1 29.2 24.6 20 21.9 17.6 14.6 15.3 12.3 16.1 ...
##  $ Gender  : Factor w/ 2 levels "M","F": 1 1 1 2 2 2 1 1 1 2 ...
##  $ Body_Fat: Factor w/ 2 levels "Low","High": 1 1 1 1 1 1 2 2 2 2 ...
##  $ Smoking : Factor w/ 2 levels "Light","Heavy": 1 1 1 1 1 1 1 1 1 1 ...
##  $ units   : int  1 2 3 1 2 3 1 2 3 1 ...
```

2. Plot the interactions

For the three factor studies, we can generate the interaction plot for two given factors, stratified by the third factor (holding it as a constant level separately).

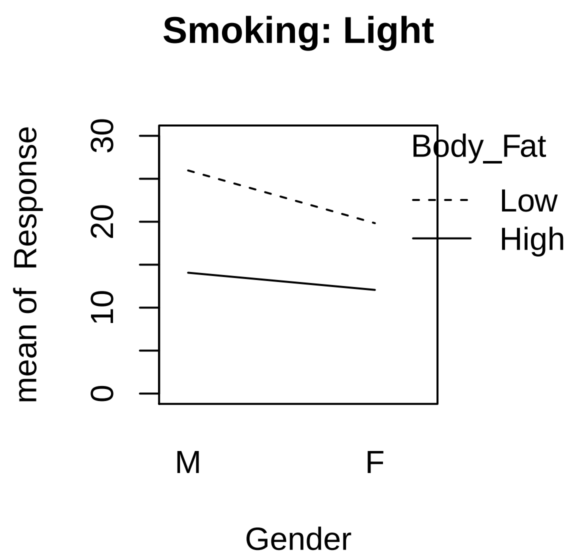
1) Body fat x Smoking history , stratified by Gender

```r
par(mfrow= c(1,2),  pty="s",mar=c(4,4,5,5))

 with( Ex24[Ex24$Gender=='M',],
  interaction.plot(x.factor = Body_Fat, trace.factor = Smoking , response = Response,
                   main= "Gender: Male" , ylim=c(0, 30)) )

 with( Ex24[Ex24$Gender=='F',],
   interaction.plot(x.factor = Body_Fat, trace.factor = Smoking , response = Response,
                   main= "Gender:Female", ylim=c(0, 30) ))
```



2) Gender x Body fat , stratified by Smoking history

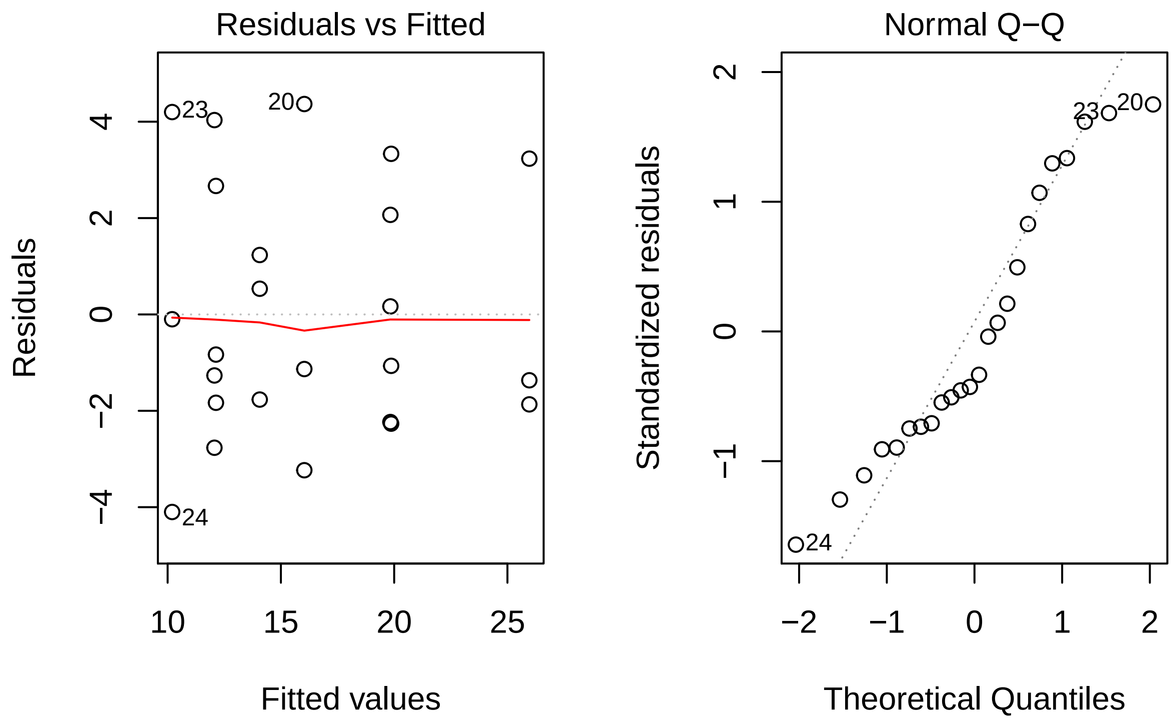
```r
par(mfrow= c(1,2), pty="s",  mar=c(4,4,5,5))

 with( Ex24[Ex24$Smoking=='Light',],
  interaction.plot(x.factor = Gender , trace.factor = Body_Fat , response = Response,
                   main= "Smoking: Light" , ylim=c(0, 30)) )

 with( Ex24[Ex24$Smoking=='Heavy',],
   interaction.plot(x.factor = Gender, trace.factor = Body_Fat , response = Response,
                   main= "Smoking:Heavy" , ylim=c(0, 30)))
```



From the interactions plots: It appears that some factors may interact in their effects on exercise tolerance and that gender, in particular, may affect the endurance in stress testing. (Stratifying on a third factor, the two 2-factor interaction plots do not show an additive structure - a location shift or parallel comparing one with the other.)

3. Residual Analysis.

```
par(mfrow= c(1,2))
 fit <- aov(Response~Gender*Body_Fat*Smoking,   data= Ex24)
 plot(fit, 1)
 plot(fit, 2)
```



From the residuals plots: It appears that the ANOVA assumptions (constant variance and normal errors) are reasonable with no significant departure.

ANOVA analysis (Ch 24.4) : partition the source of variation

1. First, if we neglect the factorial structure of the three-factor study and simply considering it to contain *abc* treatments, we obtain the usual breakdown of the total sum of squares:

$$SSTO = SSTR + SSE \tag{24.21}$$

where:

$$SSTO = \sum_i \sum_j \sum_k \sum_m (Y_{ijkm} - \bar{Y}_{....})^2 \tag{24.21a}$$

$$SSTR = n \sum_i \sum_j \sum_k (\bar{Y}_{ijk.} - \bar{Y}_{....})^2 \tag{24.21b}$$

$$SSE = \sum_i \sum_j \sum_k \sum_m (Y_{ijkm} - \bar{Y}_{ijk.})^2 = \sum_i \sum_j \sum_k \sum_m e^2_{ijkm} \tag{24.21c}$$

2. When the study are balanced (same n within the factor level combinations), we can further partition the $SSTR$ into the SS contributed out of the three factor effects and their two-way three interactions

(because different components are orthogonal and the cross-product terms drop out).

$$\underbrace{\overline{Y}_{ijk\cdot} - \overline{Y}_{\cdots\cdot}}_{\substack{\text{Estimated}\\ \text{treatment}\\ \text{mean deviation}}} = \underbrace{\overline{Y}_{i\cdots} - \overline{Y}_{\cdots\cdot}}_{A \text{ main effect}} + \underbrace{\overline{Y}_{\cdot j\cdot\cdot} - \overline{Y}_{\cdots\cdot}}_{B \text{ main effect}} + \underbrace{\overline{Y}_{\cdot\cdot k\cdot} - \overline{Y}_{\cdots\cdot}}_{C \text{ main effect}} + \underbrace{\overline{Y}_{ij\cdot\cdot} - \overline{Y}_{i\cdots} - \overline{Y}_{\cdot j\cdot\cdot} + \overline{Y}_{\cdots\cdot}}_{AB \text{ interaction effect}}$$

$$+ \underbrace{\overline{Y}_{i\cdot k\cdot} - \overline{Y}_{i\cdots} - \overline{Y}_{\cdot\cdot k\cdot} + \overline{Y}_{\cdots\cdot}}_{AC \text{ interaction effect}} + \underbrace{\overline{Y}_{\cdot jk\cdot} - \overline{Y}_{\cdot j\cdot\cdot} - \overline{Y}_{\cdot\cdot k\cdot} + \overline{Y}_{\cdots\cdot}}_{BC \text{ interaction effect}}$$

$$+ \underbrace{\overline{Y}_{ijk\cdot} - \overline{Y}_{ij\cdot\cdot} - \overline{Y}_{i\cdot k\cdot} - \overline{Y}_{\cdot jk\cdot} + \overline{Y}_{i\cdots} + \overline{Y}_{\cdot j\cdot\cdot} + \overline{Y}_{\cdot\cdot k\cdot} - \overline{Y}_{\cdots\cdot}}_{ABC \text{ interaction effect}}$$

When we square each side and sum over i, j, k, and m, all cross-product terms drop out and we obtain:

$$SSTR = SSA + SSB + SSC + SSAB + SSAC + SSBC + SSABC \qquad \textbf{(24.22)}$$

where:

$$SSA = nbc\sum_i(\overline{Y}_{i\cdots} - \overline{Y}_{\cdots\cdot})^2 \qquad \textbf{(24.22a)}$$

$$SSB = nac\sum_j(\overline{Y}_{\cdot j\cdot\cdot} - \overline{Y}_{\cdots\cdot})^2 \qquad \textbf{(24.22b)}$$

$$SSC = nab\sum_k(\overline{Y}_{\cdot\cdot k\cdot} - \overline{Y}_{\cdots\cdot})^2 \qquad \textbf{(24.22c)}$$

$$SSAB = nc\sum_i\sum_j(\overline{Y}_{ij\cdot\cdot} - \overline{Y}_{i\cdots} - \overline{Y}_{\cdot j\cdot\cdot} + \overline{Y}_{\cdots\cdot})^2 \qquad \textbf{(24.22d)}$$

$$SSAC = nb\sum_i\sum_k(\overline{Y}_{i\cdot k\cdot} - \overline{Y}_{i\cdots} - \overline{Y}_{\cdot\cdot k\cdot} + \overline{Y}_{\cdots\cdot})^2 \qquad \textbf{(24.22e)}$$

$$SSBC = na\sum_j\sum_k(\overline{Y}_{\cdot jk\cdot} - \overline{Y}_{\cdot j\cdot\cdot} - \overline{Y}_{\cdot\cdot k\cdot} + \overline{Y}_{\cdots\cdot})^2 \qquad \textbf{(24.22f)}$$

$$SSABC = n\sum_i\sum_j\sum_k(\overline{Y}_{ijk\cdot} - \overline{Y}_{ij\cdot\cdot} - \overline{Y}_{i\cdot k\cdot} - \overline{Y}_{\cdot jk\cdot} + \overline{Y}_{i\cdots} + \overline{Y}_{\cdot j\cdot\cdot} + \overline{Y}_{\cdot\cdot k\cdot} - \overline{Y}_{\cdots\cdot})^2 \qquad \textbf{(24.22g)}$$

Combining (24.21) and (24.22), we have thus established the orthogonal decomposition:

$$SSTO = SSA + SSB + SSC + SSAB + SSAC + SSBC + SSABC + SSE \qquad \textbf{(24.23)}$$

ANOVA Table

Table 24.5 contains the general ANOVA table for three-factor ANOVA model.

TABLE 24.5 General ANOVA Table for Three-Factor Study with Fixed Factor Levels.

Source of Variation	SS	df	MS	E{MS}
Factor A	SSA	$a-1$	MSA	$\sigma^2 + bcn\frac{\sum \alpha_i^2}{a-1}$
Factor B	SSB	$b-1$	MSB	$\sigma^2 + acn\frac{\sum \beta_j^2}{b-1}$
Factor C	SSC	$c-1$	MSC	$\sigma^2 + abn\frac{\sum \gamma_k^2}{c-1}$
AB interactions	$SSAB$	$(a-1)(b-1)$	$MSAB$	$\sigma^2 + cn\frac{\sum\sum (\alpha\beta)_{ij}^2}{(a-1)(b-1)}$
AC interactions	$SSAC$	$(a-1)(c-1)$	$MSAC$	$\sigma^2 + bn\frac{\sum\sum (\alpha\gamma)_{ik}^2}{(a-1)(c-1)}$
BC interactions	$SSBC$	$(b-1)(c-1)$	$MSBC$	$\sigma^2 + an\frac{\sum\sum (\beta\gamma)_{jk}^2}{(b-1)(c-1)}$
ABC interactions	$SSABC$	$(a-1)(b-1)(c-1)$	$MSABC$	$\sigma^2 + n\frac{\sum\sum\sum (\alpha\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$
Error	SSE	$abc(n-1)$	MSE	σ^2
Total	$SSTO$	$abcn-1$		

Note: The expected mean squares in Table 24.5 show that: MSA, MSB, MSC, MSAB, MSAC, MSBC, and MSABC all have expectations equal to σ^2 if their corresponding effects are not present. If such effects are present, each mean square has an expectation exceeding σ^2 and the larger the effect, the larger the expected value for the given MS. Therefore we can use the ratio of MS for each effect and MSE to construct the F-test.

F-tests for the effects

TABLE 24.6 Test Statistics for Three-Factor Study with Fixed Factor Levels.

Alternatives	Test Statistic	Percentile
H_0: all $\alpha_i = 0$ H_a: not all $\alpha_i = 0$	$F^* = \frac{MSA}{MSE}$	$F[1-\alpha; a-1, (n-1)abc]$
H_0: all $\beta_j = 0$ H_a: not all $\beta_j = 0$	$F^* = \frac{MSB}{MSE}$	$F[1-\alpha; b-1, (n-1)abc]$
H_0: all $\gamma_k = 0$ H_a: not all $\gamma_k = 0$	$F^* = \frac{MSC}{MSE}$	$F[1-\alpha; c-1, (n-1)abc]$
H_0: all $(\alpha\beta)_{ij} = 0$ H_a: not all $(\alpha\beta)_{ij} = 0$	$F^* = \frac{MSAB}{MSE}$	$F[1-\alpha; (a-1)(b-1), (n-1)abc]$
H_0: all $(\alpha\gamma)_{ik} = 0$ H_a: not all $(\alpha\gamma)_{ik} = 0$	$F^* = \frac{MSAC}{MSE}$	$F[1-\alpha; (a-1)(c-1), (n-1)abc]$
H_0: all $(\beta\gamma)_{jk} = 0$ H_a: not all $(\beta\gamma)_{jk} = 0$	$F^* = \frac{MSBC}{MSE}$	$F[1-\alpha; (b-1)(c-1), (n-1)abc]$
H_0: all $(\alpha\beta\gamma)_{ijk} = 0$ H_a: not all $(\alpha\beta\gamma)_{ijk} = 0$	$F^* = \frac{MSABC}{MSE}$	$F[1-\alpha; (a-1)(b-1)(c-1), (n-1)abc]$

All those test statistics F^* have similar properties: Under H_0 (the effects are not present), the test-statistic will follow a F distribution $F(df_1, df_2)$, where df_1 is the df for the numerator MS and $df_2 = n_T - abc = (n-1)abc$ (df for MSE). We compare the F^* with the critical value of the corresponding F distribution and reject H_0 for large F^* values.

Kimball Inequality

- Based on the fact that all those tests share a same denominator (MSE), the Kimball Inequality has a more accurate (less conservative) bound than the Bonferroni Inequality.
- The Kimball inequality for the family level of significance α in a three-factor study when the family consists ofthe combined set of k tests, including main effects, two-factor and three factor interactions is,

$$\alpha < 1 - (1-\alpha_1)(1-\alpha_2)...(1-\alpha_k)$$

where α_i is the level of significance for the ith test.

- We can use this bound to control for the family-wise error rate and set the significance level for individual tests.

Example 2 (continued): ANOVA analysis

In the stress test example, the researcher first wished to test for the various factor effects. The researcher desired to conduct the seven potential tests with a family level of significance of $\alpha = .10$. This will ensure that if in fact no factor effects are present, there will be only 10% for one or more of the seven tests to lead to the conclusion of the presence of factor effects (false positive).

Analysis:

1. Determine the significance level for individual test

Using the Kimball inequality and assuming same α_0 for each test, We solved the equation:

$$1 - (1-\alpha_1)(1-\alpha_2)...(1-\alpha_k) = (1-\alpha_i)^7 = 0.1$$

Using R to solve this equation:

```r
(alpha.i= 1- (1-0.1)^(1/7))
```

```
## [1] 0.01493879
```

Thus we will use $\alpha_i = 0.015$ for each test to ensure that the family level of significance will not exceed .10.

2. ANOVA analysis

```r
# full model with all interactions
fit <- aov(Response~Gender*Body_Fat*Smoking,   data= Ex24)
summary(fit)
```

```
##                         Df Sum Sq Mean Sq F value   Pr(>F)    
## Gender                   1 176.58  176.58  18.915 0.000497 ***
## Body_Fat                 1 242.57  242.57  25.984 0.000108 ***
## Smoking                  1  70.38   70.38   7.539 0.014357 *  
## Gender:Body_Fat          1  13.65   13.65   1.462 0.244143    
## Gender:Smoking           1  11.07   11.07   1.186 0.292299    
## Body_Fat:Smoking         1  72.45   72.45   7.761 0.013221 *  
## Gender:Body_Fat:Smoking  1   1.87    1.87   0.200 0.660434    
## Residuals               16 149.37    9.34                     
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Interpretation:

1. Test for Three-Factor Interactions:

- $$H_0\text{: all } (\alpha\beta\gamma)_{ijk} = 0$$
  $$H_a\text{: not all } (\alpha\beta\gamma)_{ijk} \text{ equal zero}$$

The F test statistic= 0.20 with a p-value= 0.66, thus we don't reject H_0 and we conclude that no ABC interactions are present.

2. Tests for Two-Factor Interactions.

- Test for AB interactions (Gender:Body fat): $F = 1.462$, P-value= 0.24, thus we conclude that no AB interactions are present.
- Test for AC interactions (Gender:Smoking): $F = 1.186$, P-value= 0.29, thus we conclude that no AC interactions are present.
- Test for BC interactions (Body_Fat:Smoking): $F = 7.761$, P-value= $0.013 < 0.015$, thus we conclude that BC interactions are significant.

3. Tests for Main Effects.

- Since factor A (gender) did not interact with the other two factors, we turn to testing for factor A main effects. $F = 18.915$, P-value= $0.0005 < 0.015$, thus we conclude that Main Effect of gender are significant.
- The factor B and factor C main effects were not tested at this point because BC interactions were found to be present. We should study the nature of the BC interaction effects before determining whether the factor B and factor C main effects are of any practical interest, separately, or just study the effects of BC combinations.

Summary

- Reading: Chapter 24.1-24.4