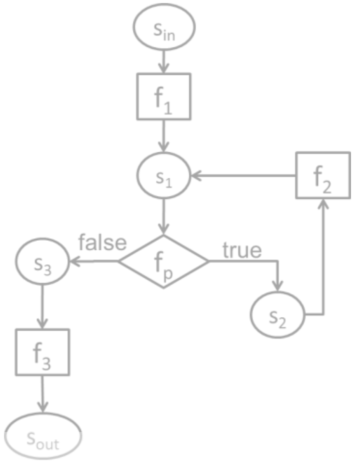
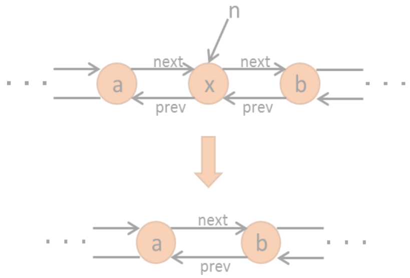
$$\exists c \forall in\ Q(c, in)$$

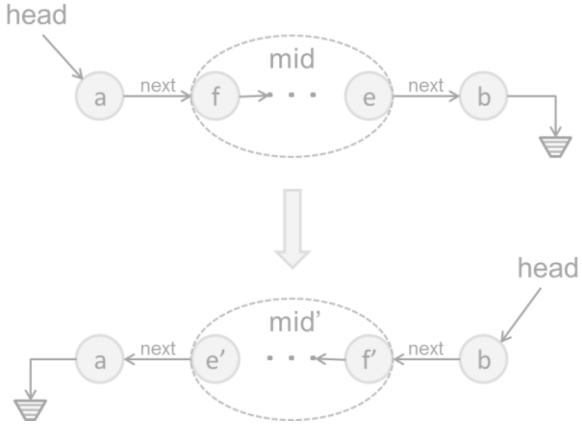
```
/* Average of x and y without using x+y (avoid overflow)*/
int avg(int x, int y){
  int t = expr({x/2, y/2, x%2, y%2, 2 }, {PLUS, DIV});
  assert t == (x+y)/2;
  return t;
}
```

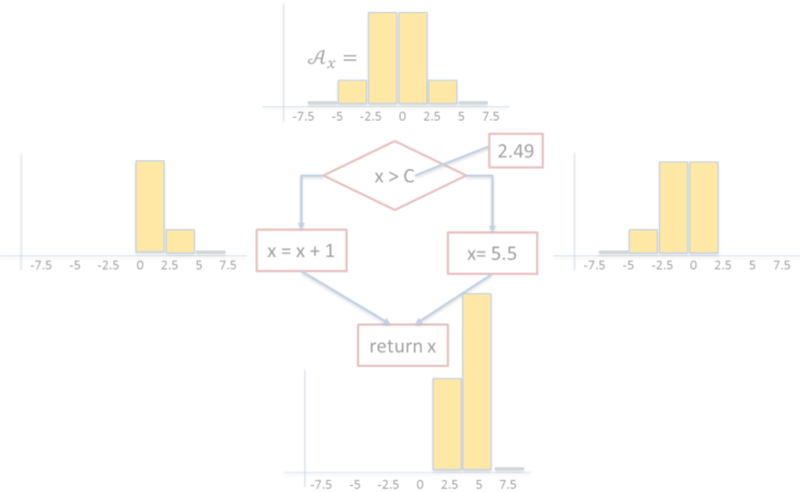



```
{
  s = n.succ;
  p = n.pred;
  p.succ = s;
  s.pred = p;
}
```

# Module I: Synthesizing Simple Programs




$$\varphi(p)$$

$$Sk[c](in)$$

# Lecture 2
# Syntax-Guided Synthesis and Enumerative Search

*Nadia Polikarpova*

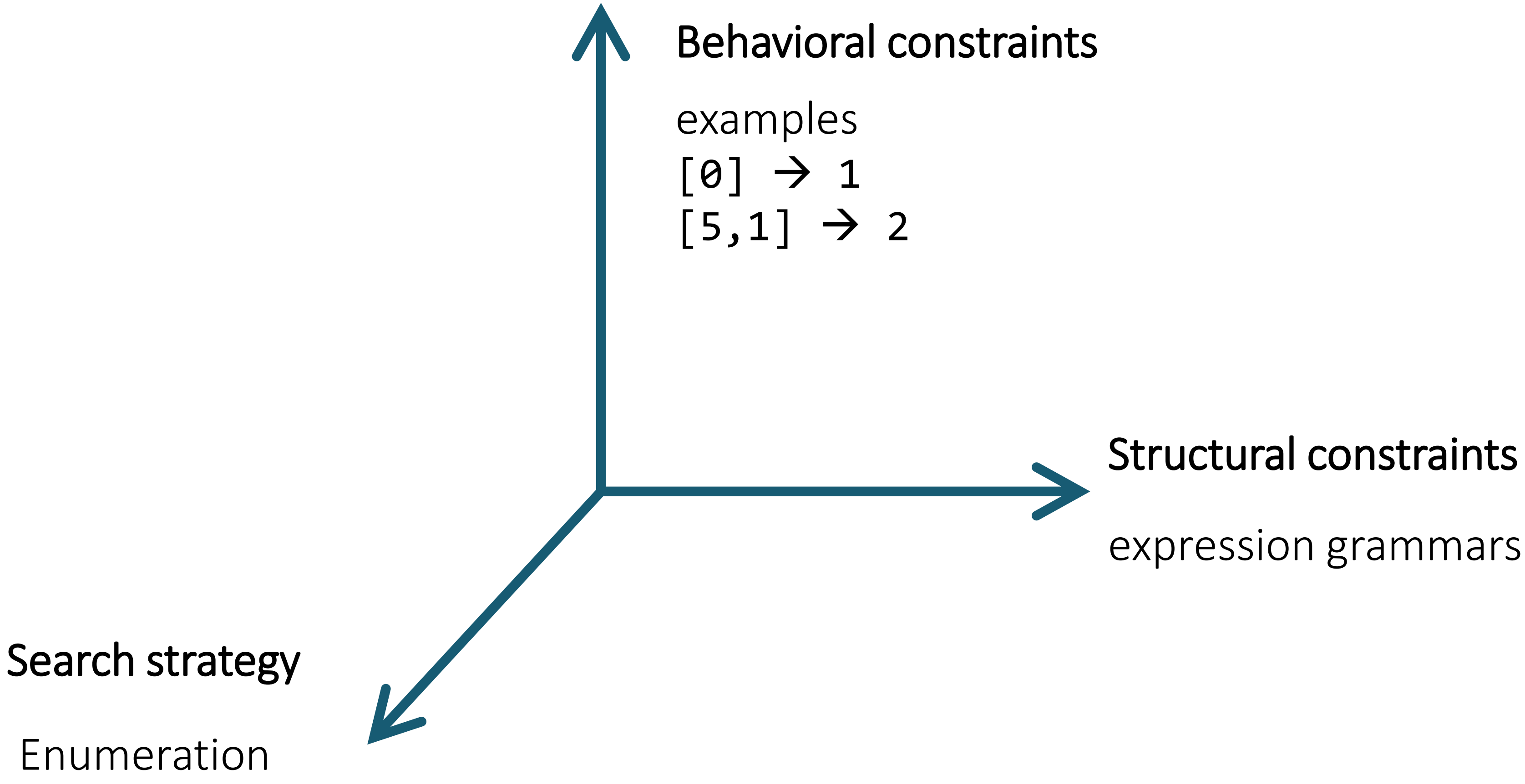
# Logistics

Shared Google folder

- Does everyone have access?
- Register your team by next Friday

Other questions?

# Week 1-2


Behavioral constraints
examples
[0] → 1
[5,1] → 2
Structural constraints
expression grammars
Search strategy
Enumeration

# Today

Synthesis from examples

Syntax-guided synthesis

- expression grammars as structural constraints
- the SyGuS project

Enumerative search

- enumerating all programs generated by a grammar
- bottom-up vs top-down

# Synthesis from examples

# Synthesis from Examples

=

Programming by Example

=

Inductive Programming

Inductive Learning

# A little bit of history: inductive learning

MIT/LCS/TR-76

LEARNING STRUCTURAL DESCRIPTIONS FROM EXAMPLES

Patrick H. Winston

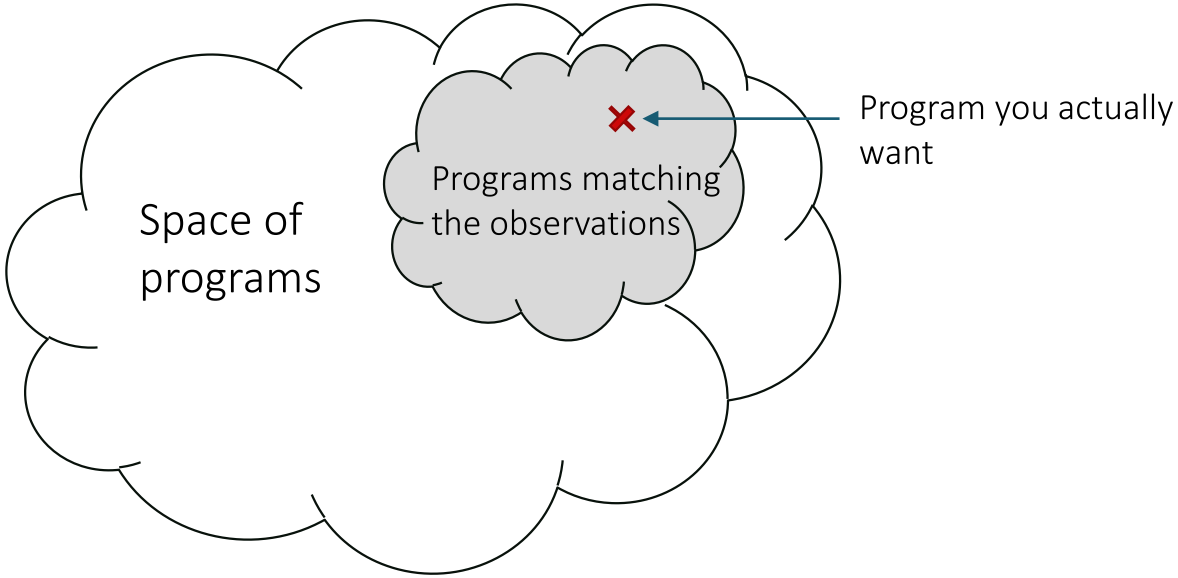
September 1970

Patrick Winston

Explored the question of generalizing from a set of observations

Became the foundation of machine learning

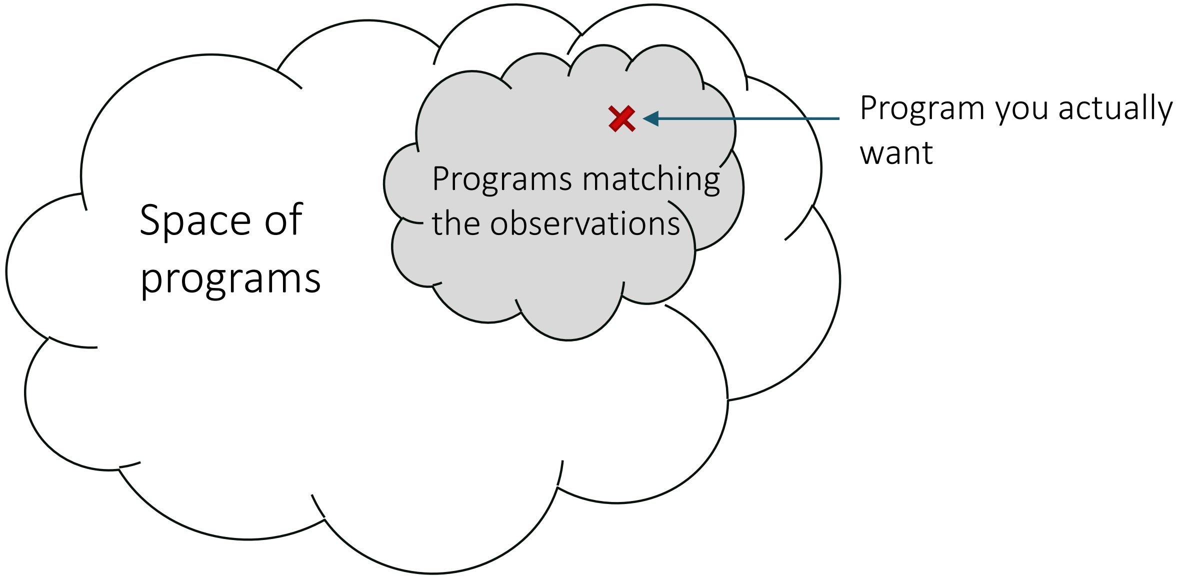
# Key issues in inductive learning




(1) How do you find a program that matches the observations?

(2) How do you know it is the program you are looking for?

# Key issues in inductive learning



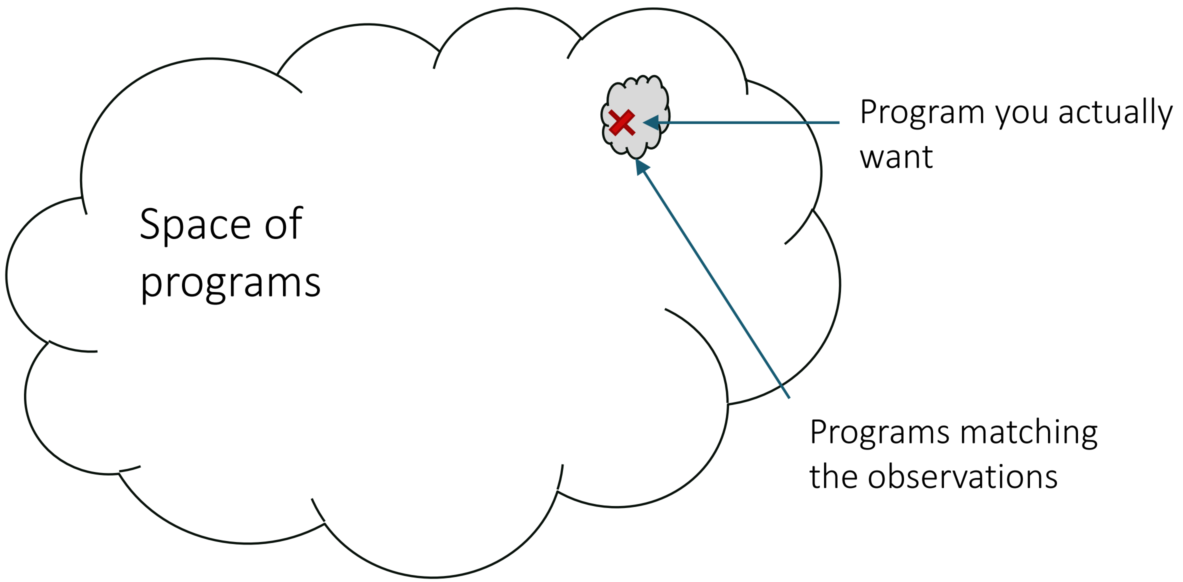

Traditional ML:

- Fix the space so that (1) is easy
- (2) becomes the main challenge

(1) How do you find a program that matches the observations?

(2) How do you know it is the program you are looking for?

# The synthesis approach



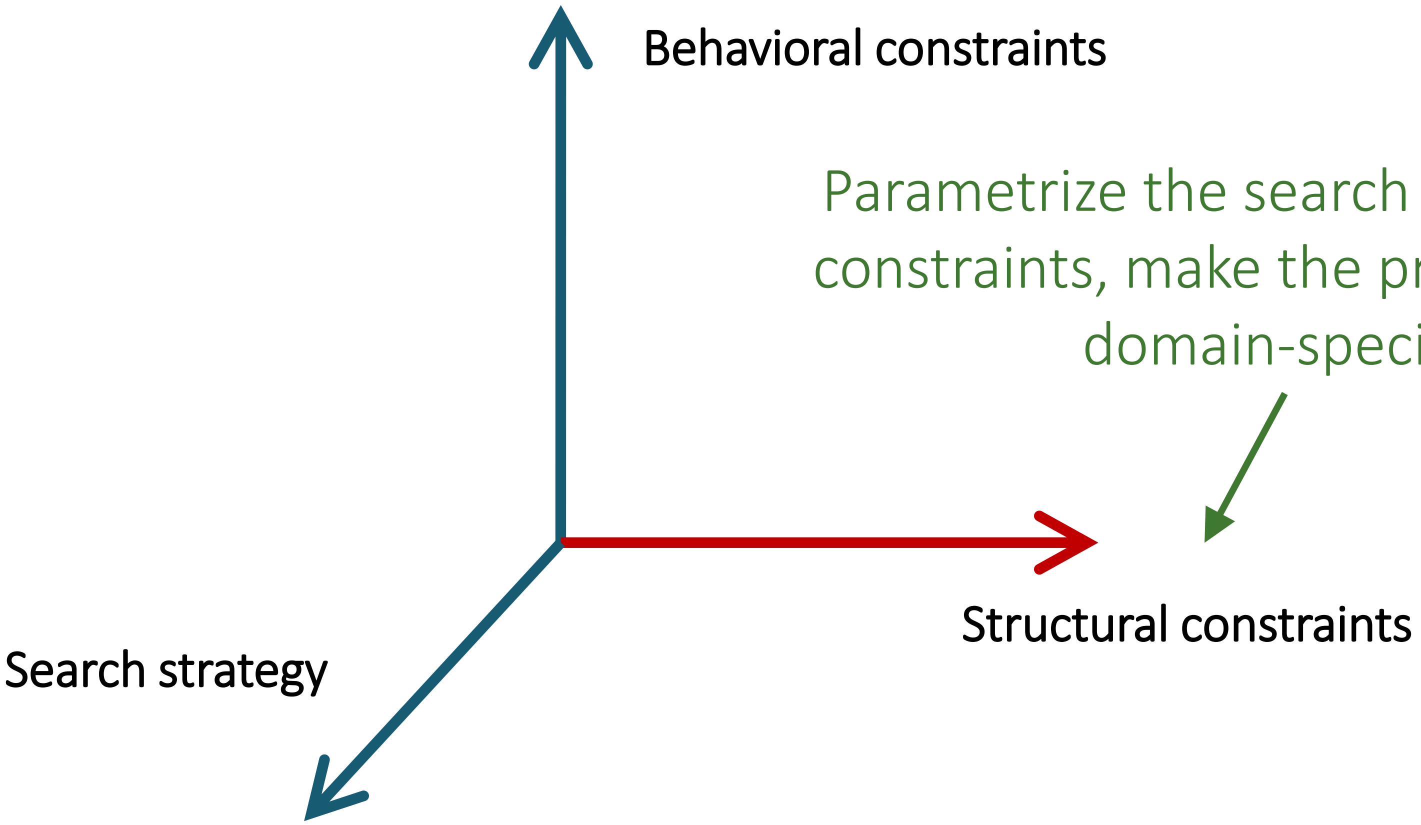

Program synthesis:

- Customize the space so that (2) becomes easier
- (1) is now the main challenge

(1) How do you find a program that matches the observations?

(2) How do you know it is the program you are looking for?

# Key idea




Parametrize the search by structural constraints, make the program space domain-specific

# Syntax-Guided Synthesis

# Example

[1,4,7,2,0,6,9,2,5,0] → [1,2,4,7,0]

```
L ::= x              |    the input
      single(N)      |    single(1) = [1]
      sort(L)        |    sort([6,9,2,5]) = [2,5,6,9]
      slice(L,N,N)   |    slice([6,9,2,5],0,2) = [6,9]
      concat(L,L)    |    concat([6,9],[2,5]) = [6,9,2,5]
N ::= find(L,N)      |    find([6,9],9) = 1
      0                   0
```

```
f(x) := concat( sort(slice(x,0,find(x,0))),  single(0))
```

# Regular tree grammars (RTGs)

starting nonterminal

```
L ::= x
    | single(N)
    | sort(L)
    | slice(L,N,N)
    | concat(L,L)
N ::= find(L,N)
    | 0
```

ranked alphabet (terminals)

nonterminals

productions

# Regular tree grammars (CFGs)

nonterminals   rules (productions)

alphabet   starting nonterminal

<Σ, N, R, S>

Terms: t ∈ $T_\Sigma(N)$ = all terms made from $N \cup \Sigma$

Rules are of the form: $A \to \sigma(A_1, \ldots, A_n)$

Derives in one step: $\mathcal{C}[A] \to \mathcal{C}[t]$ if $(A \to t) \in R$

(Incomplete) programs: $\{t \in T_\Sigma(N) | A \to^* t\}$

Ground programs: $\{t \in T_\Sigma | A \to^* t\}$

= programs without holes, complete programs

Whole programs: $\{t \in T_\Sigma | S \to^* t\}$

= roughly, programs of the right type

```
concat(L,0)
```

$L \to concat(L, L)$

```
concat(L,L) -> concat(x,L)

find(concat(L,L),N)

find(concat(x,x),0)

sort(concat(L,L))
```

# RTGs as structural constraints

Space of programs
= the *language* of an RTG
= all ground, whole programs

```
L ::= x             |
      single(N)     |
      sort(L)       |
      slice(L,N,N)  |
      concat(L,L)   |
N ::= find(L,N)     |
      0
```

```
x    sort(x)    concat(x, x)    slice(x,0,0)
...
slice(x,0,find(x,0))
...
concat(sort(slice(x,0,find(x,0))), single(0))
...
```

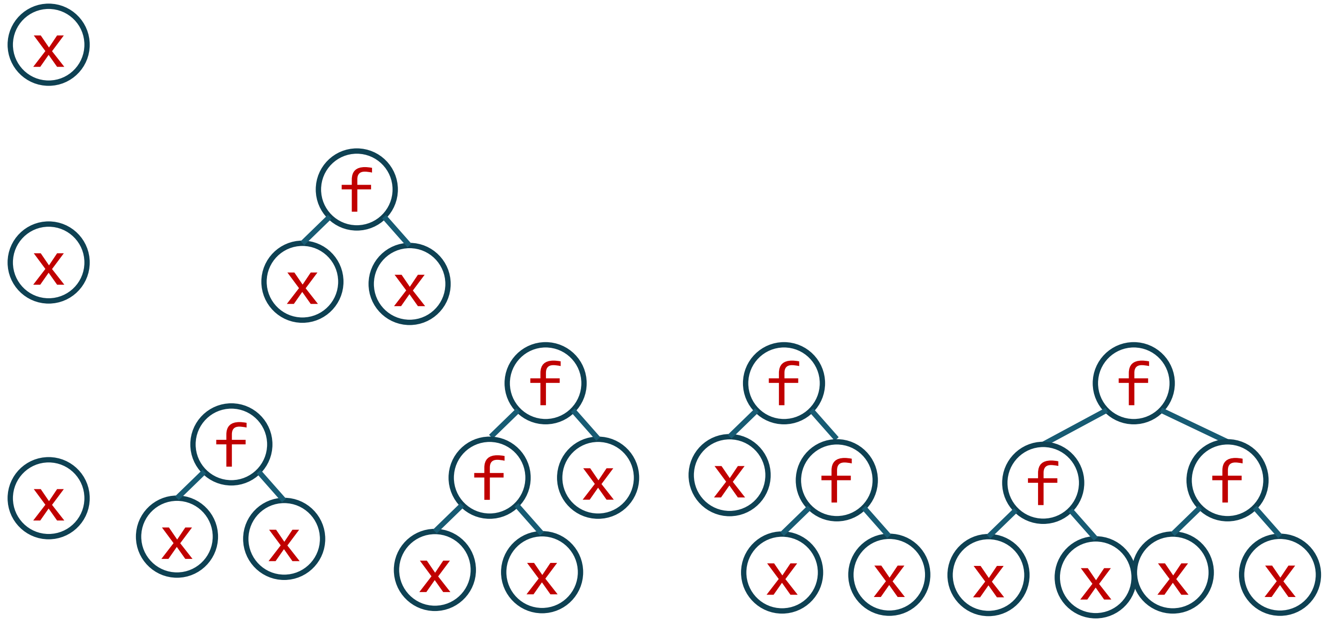
# How big is the space?

```
E ::= x | f(E,E)
```

depth <= 0 — N(0) = 1

depth <= 1 — N(1) = 2

depth <= 2 — N(2) = 5



$$N(d) = 1 + N(d - 1)^2$$

# How big is the space?

$$E ::= x \mid f(E,E)$$

$$N(d) = 1 + N(d - 1)^2 \qquad N(d) \sim c^{2^d} \qquad (c > 1)$$

N(1) = 1
N(2) = 2
N(3) = 5
N(4) = 26
N(5) = 677
N(6) = 458330
N(7) = 210066388901
N(8) = 44127887745906175987802
N(9) = 1947270476915296449559703445493848930452791205
N(10) = 3791862310265926082868235028027893277370233152247388584761734150717768254410341175325352026

# How big is the space?

$$E ::= x_1 \mid \dots \mid x_k \mid f_1(E,E) \mid \dots \mid f_m(E,E)$$

```
N(0) = k
```

$$N(d) = k + m * N(d - 1)^2$$

N(1) = 3
N(2) = 30
N(3) = 2703
N(4) = 21918630
N(5) = 1441279023230703
N(6) = 6231855668414547953818685622630
N(7) = 116508075215851596766492219468227024724121520304443212304350703

k = m = 3

# Syntactic sugar

Instead of this:

```
L ::= x              |
      single(N)      |
      sort(L)        |
      slice(L,N,N)   |
      concat(L,L)    |
N ::= find(L,N)      |
      0
```

We will often write this:

```
L ::= x           |
      [N]         |
      sort(L)     |
      L[N..N]     |
      L + L       |
N ::= find(L,N)   |
      0
```

- allow custom syntax for terminal symbols
- not an RTG strictly speaking, but you know what we mean…

# Syntactic sugar

```
L ::= sort(L)      |
      L[N..N]      |
      L + L        |
      [N]          |
      x
N ::= find(L,N)    |
      0
```

```
x    sort(x)    x + x    x[0..0]
...
x[0..find(x, 0)]
...
sort(x[0..find(x, 0)]) + [0]
...
```

# The SyGuS project

[Alur et al. 2013]

https://sygus.org/

Goal: Unify different syntax-guided approaches

Collection of synthesis benchmarks + yearly competition

- 6 competitions since 2013
- consider writing a SyGuS solver for your project!

Common input format + supporting tools

- parser, baseline synthesizers

# SyGuS problems

SyGuS problem = < theory, spec, grammar >

A “library” of types and function symbols

**Example:** Linear Integer Arithmetic (LIA)

```
True, False
0,1,2,...
∧, ∨, ¬, +, ≤, ite
```

RTG with terminals in the theory (+ input variables)

**Example:** Conditional LIA expressions w/o sums

```
E ::= x | ite C E E
C ::= E ≤ E | C ∧ C | ¬C
```

# SyGuS problems

SyGuS problem = < theory, spec, grammar >

A first-order logic formula over the theory

Examples:

```
f(0, 1) = 1 ∧
f(1, 0) = 1 ∧
f(1, 1) = 1 ∧
f(2, 0) = 2
```

# SyGuS demo

# SyGuS problems

SyGuS problem = < theory, spec, grammar >

can inductive synthesis handle these?

A first-order logic formula over the theory

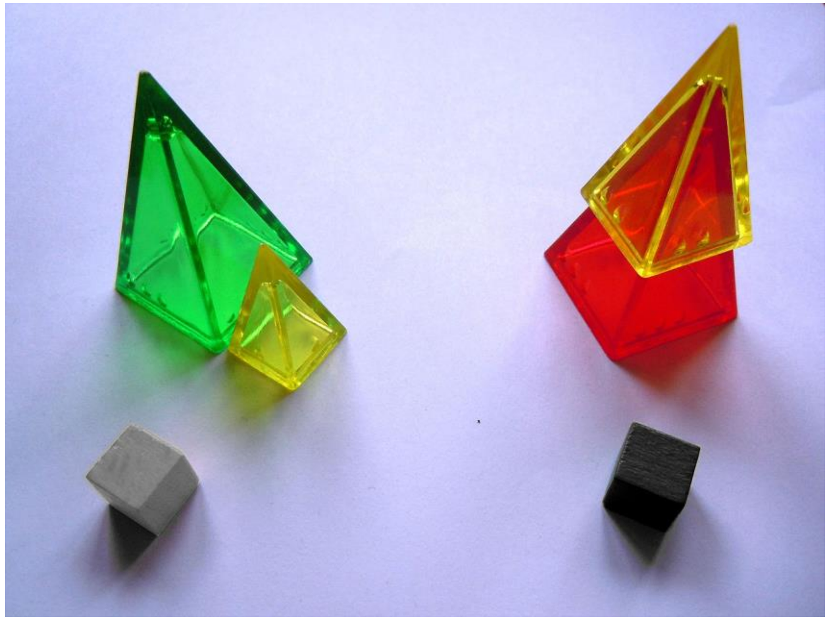
Examples:

$$
\begin{aligned}
&f(0, 1) = 1 \wedge \\
&f(1, 0) = 1 \wedge \\
&f(1, 1) = 1 \wedge \\
&f(2, 0) = 2
\end{aligned}
$$

Formula with free variables:

$$
\begin{aligned}
&x \leq f(x, y) \wedge \\
&y \leq f(x, y) \wedge \\
&(f(x, y) = x \vee f(x, y) = y)
\end{aligned}
$$

# The Zendo game



The teacher makes up a secret rule

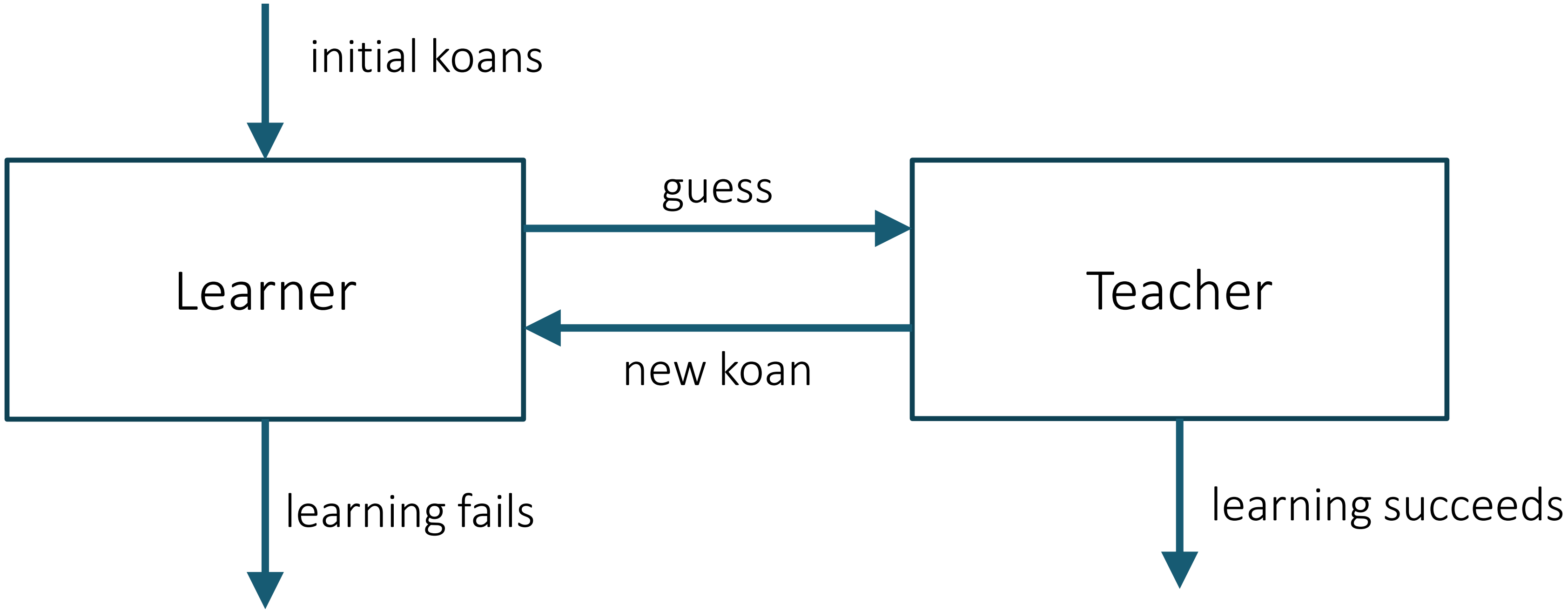
- e.g. all pieces must be grounded

The teacher builds two koans (a positive and a negative)

A student can try to guess the rule

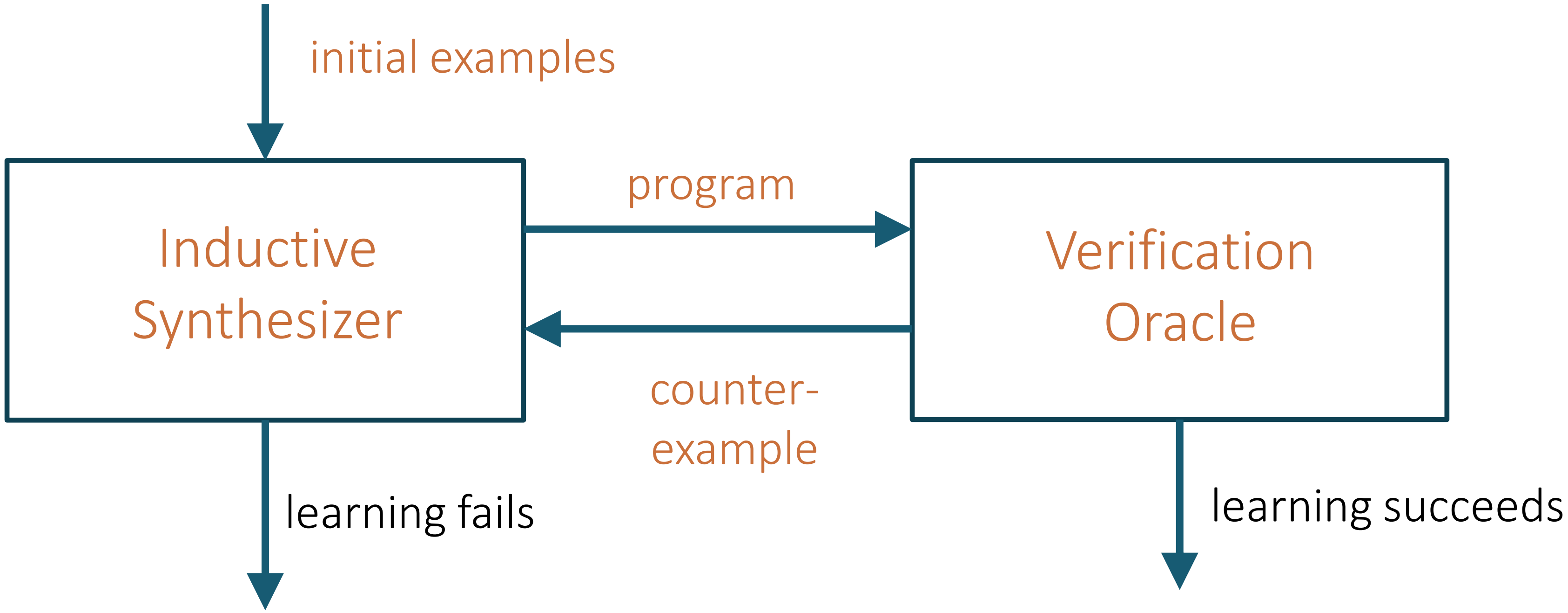
- if they are right, they win
- otherwise, the teacher builds a koan on which the two rules disagree

# The Zendo game



# Counter-example guided inductive synthesis (CEGIS)

The Zendo of program synthesis



# The problem statement

Behavioral constraints = examples

```
[1,4,7,2,0,6,9,2,5]  →  [1,2,4,7,0]
[0] → [0]
[5,1] → [1,5,0]
```

Search strategy?

Structural constraints = grammar

```
L ::= sort(L)  |  L[N..N]
   |  L + L  |  [N]  |  x
N ::= find(L,N)  |  0
```

# Enumerative search

# Enumerative search

=

Explicit / Exhaustive Search

Idea: Sample programs from the grammar one by one and test them on the examples

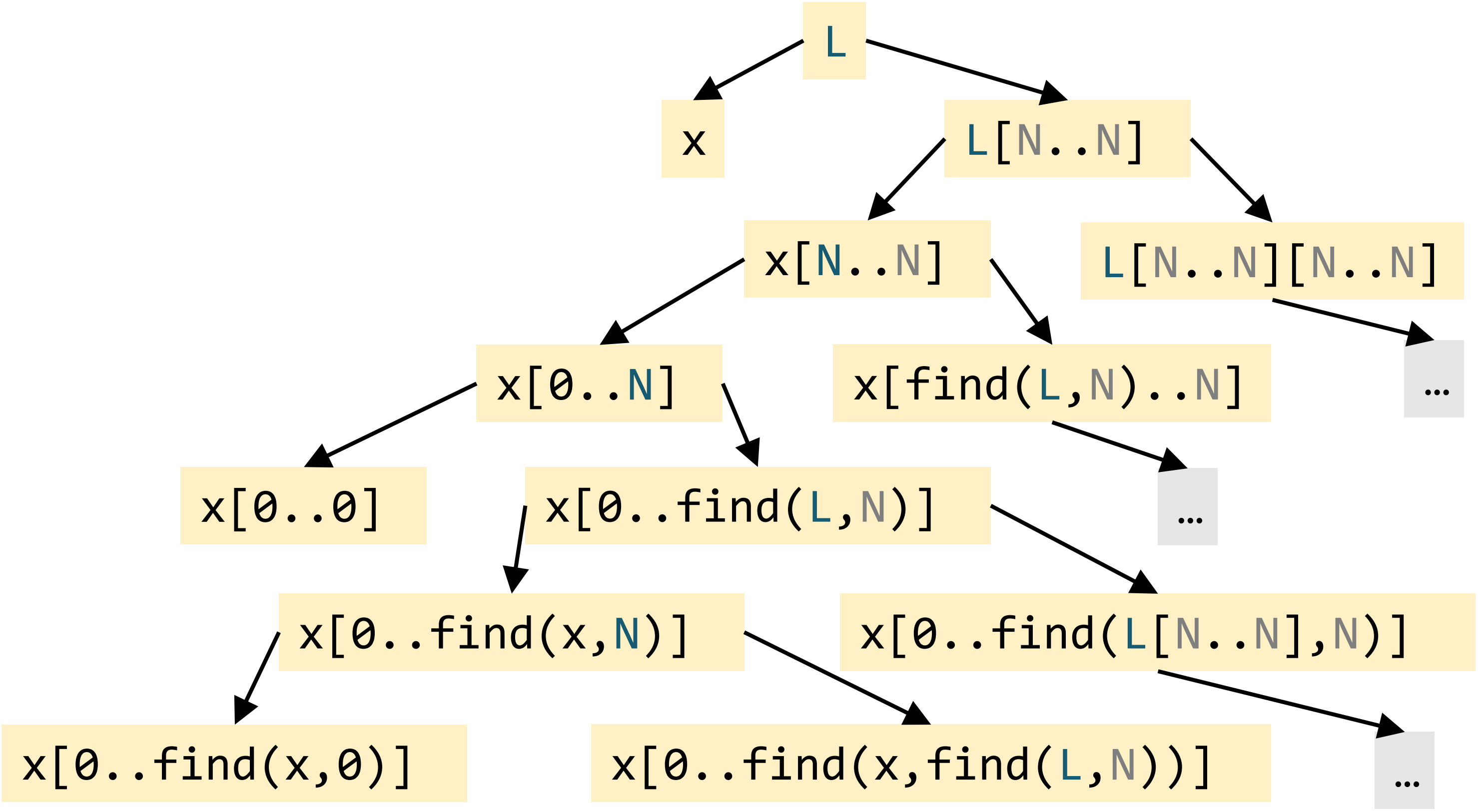
Challenge: How do we systematically enumerate all programs?

top-down vs bottom-up

# Top-down enumeration: search space

Search space is a tree where

- nodes are incomplete programs
- edges are left-most “rewrites to”




```
L ::= L[N..N]     |
      x
N ::= find(L,N)   |
      0
```

[[1,4,0,6] → [1,4]]

# Top-down enumeration = traversing the tree

Search tree can be traversed:

- depth-first (for fixed max depth)
- breadth-first
- best-first

General algorithm:

- Maintain a worklist of incomplete programs
- Initialize with the start non-terminal
- Expand left-most non-terminal using all productions

```
L ::= L[N..N]    |
      x
N ::= find(L,N)  |
      0

[[1,4,0,6]  →  [1,4]]
```

# Top-down: algorithm

nonterminals rules (productions)

alphabet starting nonterminal

```
top-down(<Σ, N, R, S>, [i → o]):
  wl := [S]
  while (wl != []):
    p := wl.dequeue()
    if (ground(p) ∧ p([i]) = [o]):
      return p
    wl.enqueue(unroll(p))

unroll(p):
  wl' := []
  A := left-most non-term in p
  forall (A → rhs) in R:
    p' = p[A -> rhs]
    if !exceeds_bound(p'): wl' += p'
  return wl';
```

can be smart about what to dequeue

depth- or breadth-first depending on where you enqueue

can impose bounds on depth/size

```
L ::= L[N..N]    |
      x
N ::= find(L,N)  |
      0
```

```
[[1,4,0,6] → [1,4]]
```

# Top-down: example (depth-first)

Worklist `wl`

iter 0: `L`

iter 1: `x` `L[N..N]`

iter 2: `L[N..N]`

iter 3: `x[N..N]` `L[N..N][N..N]`

iter 4: `x[0..N]` `x[find(L,N)..N]` `L[N..N][N..N]`

iter 5: `x[0..0]` `x[0.. find(L,N)]` `x[find(L,N)..N]` …

iter 6: `x[0.. find(L,N)]` `x[find(L,N)..N]` …

iter 7: `x[0.. find(x,N)]` `x[0.. find(L[N..N],N)]` …

iter 8: `x[0.. find(x,0)]` `x[0.. find(x,find(L,N))]` …

iter 9:

```
L ::= L[N..N]     |
      x
N ::= find(L,N)   |
      0
```

`[[1,4,0,6] → [1,4]]`

# Bottom-up enumeration

The dynamic programming approach

Maintain a bank of ground programs

Combine programs in the bank into larger programs using productions

```
L ::= sort(L)      |
      L[N..N]      |
      L + L        |
      [N]          |
      x
N ::= find(L,N)    |
      0
```

[[1,4,0,6] → [1,4]]

# Bottom-up: algorithm (take 1)

nonterminals rules (productions)

alphabet starting nonterminal

```
bottom-up (<Σ, N, R, S>, [i → o]):
  bank := {}
  for d in [0..]:
    forall (A → rhs) in R:
      forall p in new-terms(A→rhs, d, bank):
        if (A = S ∧ p([i]) = [o]):
          return p
        bank += p;

new-terms(A → σ(A_1…A_n), d, bank):
 if (d = 0 ∧ n = 0) yield σ
 else forall <p_1,…,p_n> in bank^n:
        if A_i ->* p_i: yield σ(p_1,…,p_n)
```

```
L ::= sort(L)    |
      L[N..N]    |
      L + L      |
      [N]        |
      x
N ::= find(L,N)  |
      0
```

[[1,4,0,6] → [1,4]]

# Bottom-up: algorithm (take 1)

nonterminals rules (productions)

alphabet starting nonterminal

```
bottom-up (<Σ, N, R, S>, [i → o]):
  bank := {}
  for d in [0..]:
    forall (A → rhs) in R:
      forall p in new-terms(A→rhs, d, bank):
        if (A = S ∧ p([i]) = [o]):
          return p
        bank += p;

new-terms(A → σ(A_1…A_n), d, bank):
 if (d = 0 ∧ n = 0) yield σ
 else forall <p_1,…,p_n> in bank^n:
        if A_i ->* p_i: yield σ(p_1,…,p_n)
```

```
L ::= sort(L)      |
      L[N..N]      |
      L + L        |
      [N]          |
      x
N ::= find(L,N)    |
      0
```

```
[[1,4,0,6] → [1,4]]
```

inefficient, better index bank by non-terminal!

# Bottom-up: algorithm (take 2)

```
bottom-up (<Σ, N, R, S>, [i → o]):
  bank[A] := {} forall A
  for d in [0..]:
    forall (A → rhs) in R:
      forall p in new-terms(A→rhs, d, bank):
        if (A = S ∧ p([i]) = [o]):
          return p
        bank += p;

new-terms(A → σ(A₁…Aₙ), d, bank):
 if (d = 0 ∧ n = 0) yield σ
 else forall <p₁,…,pₙ> in bank[A₁] × … × bank[Aₙ]:
           yield σ(p₁,…,pₙ)
```

```
L ::= sort(L)    |
      L[N..N]    |
      L + L      |
      [N]        |
      x
N ::= find(L,N)  |
      0
```

[[1,4,0,6] → [1,4]]

# Bottom-up: algorithm (take 2)

```
bottom-up (<Σ, N, R, S>, [i → o]):
  bank[A] := {} forall A
  for d in [0..]:
    forall (A → rhs) in R:
      forall p in new-terms(A→rhs, d, bank):
        if (A = S ∧ p([i]) = [o]):
          return p
        bank += p;

new-terms(A → σ(A₁…Aₙ), d, bank):
 if (d = 0 ∧ n = 0) yield σ
 else forall <p₁,…,pₙ> in bank[A₁] × … × bank[Aₙ]:
             yield σ(p₁,…,pₙ)
```

```
L ::= sort(L)     |
      L[N..N]     |
      L + L       |
      [N]         |
      x
N ::= find(L,N)   |
      0
```

[[1,4,0,6] → [1,4]]

inefficient, generating same terms again and again!
better index bank by depth

# Bottom-up enumeration

```
bottom-up (<Σ, N, R, S>, [i → o]):
  bank[A,d] := {} forall A, d
  for d in [0..]:
    forall (A → rhs) in R:
      forall p in new-terms(A→rhs, d, bank):
        if (A = S ∧ p([i]) = [o]):
          return p
        bank[A,d] += p;

new-terms(A → σ(A₁…Aₙ), d, bank):
 if (d = 0 ∧ n = 0) yield σ
 else forall <d₁,…,dₙ> in [0..d-1]ⁿ s.t. max(d₁,…,dₙ) = d-1:
      forall <p₁,…,pₙ> in bank[A₁,d₁] × … × bank[Aₙ,dₙ]:
        yield σ(p₁,…,pₙ)
```

```
L ::= sort(L)     |
      L[N..N]     |
      L + L       |
      [N]         |
      x
N ::= find(L,N)   |
      0
```

[[1,4,0,6] → [1,4]]

# Bottom-up: example

Program bank

d= 0: `x` `0`

d =1: `sort(x)` `x + x` `x[0..0]` `[0]` `find(x,0)`

d = 2: `sort(sort(x))` `sort(x[0..0])` `sort(x + x)` `sort([0])` `x + (x + x)` `x + [0]` `sort(x) + x` `x[0..0] + x` `(x + x) + x` `[0] + x` `x + x[0..0]` `x + sort(x)` `x[0..find(x,0)]`

```
L ::= sort(L)     |
      L + L       |
      L[N..N]     |
      [N]         |
      x
N ::= find(L,N)   |
      0
```

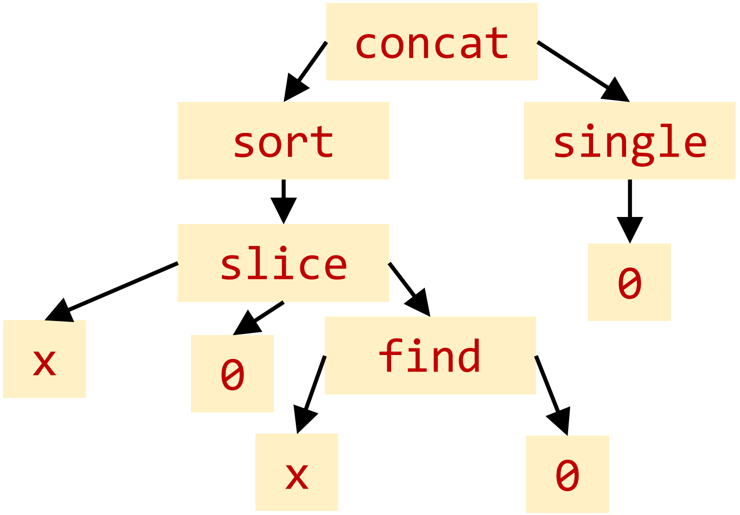
`[[1,4,0,6]` → `[1,4]]`

# Bottom-up: discussion

What are some optimizations that come to mind?

Instead of by depth, we can enumerate by size

- Why would we want that?




depth = 4, size = 10

programs of size <= 10: 8667

programs of depth <= 4: >1M

- Which parts of the algo would we need to change?

# Bottom-up vs top-down

**Top-down**

**Bottom-up**

Smaller to larger depth

- Has to explore between $3*10^9$ and $10^{23}$ programs to find `sort(x[0..find(x, 0)]) + [0]` (depth 6)

**Top-down:** Candidates are whole but might not be ground

- Cannot always run on inputs
- Can always relate to outputs (?)

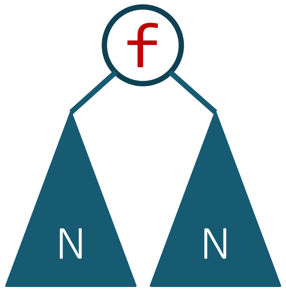
**Bottom-up:** Candidates are ground but might not be whole

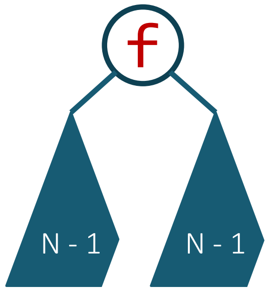
- Can always run on inputs
- Cannot always relate to outputs

# How to make it scale

## Prune

Discard useless subprograms




`m * N`$^2$ `m * (N - 1)`$^2$

## Prioritize

Explore more promising candidates first

```
P = { [0][N..N] ,
      x[N..N]  ,
      ... }
```

dequeue this first