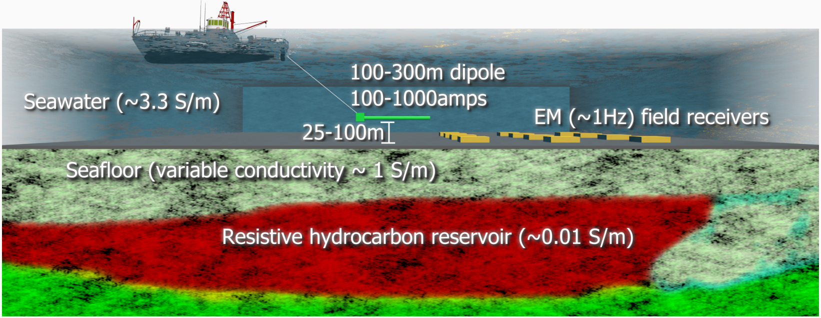
UNCERTAINTY QUANTIFICATION IN THE CONTROLLED SOURCE ELECTROMAGNETIC MEASUREMENTS

Dimitris Kamilis and Nick Polydorides

June 26, 2017

Institute for Digital Communications
School of Engineering
University of Edinburgh

Controlled-Source Electromagnetic Imaging



Modelled by time-harmonic quasi-magnetostatic Maxwell's equations

$$\begin{aligned} \Lambda_\sigma \mathsf{E} = \nabla \times \mu^{-1} \nabla \times \mathsf{E} - \imath\omega\sigma \mathsf{E} &= \imath\omega \mathbf{j}_{\text{ext}} && \text{in } D \subset \mathbb{R}^3 \\ \mathsf{E} \times \mathbf{n} &= 0 && \text{on } \partial D \end{aligned}$$

Variational formulation

Find $\boldsymbol{u} \in X = H_0(\text{curl}; D)$ such that

$$\begin{aligned} \alpha(\boldsymbol{u}, \boldsymbol{v}) &= \frac{1}{\mu}(\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) - \imath\omega(\sigma \boldsymbol{u}, \boldsymbol{v}) \\ &= \imath\omega(\boldsymbol{j}_{\text{ext}}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in X \end{aligned}$$

Properties: well-posed provided if $\sigma > 0$.

MATHEMATICAL MODEL

- Deterministic models yield one-to-one ‘model to measurement’ maps
- Why do we need the stochastic models?
- Encode prior information on electrical properties into the model to get the measurement likelihood
- Obtain more information on the measurements
- Design experiments to maximise information content in the data
- Construct explicit, deterministic expressions of the posterior density

- **Uncertainty** leads to probabilistic description and Bayesian framework
- Quantify rational degrees of belief
- Model conductivity as a **random field** $\sigma(x, \theta)$ with infinite degrees of freedom.
- **Constraints** through prior information.
- **Objective**: Given a prior model, estimate a quantity of interest and quantify the uncertainty in the system response (forward UQ).

Parametric uncertainty

Represent the uncertainty in terms of an infinite sequence of random parameters $\mathbf{y} = (y_j)_{j\geq 1} \in Y \subset \mathbb{R}^{\mathbb{N}}$ with law defined on the product measure space $(Y, \mathcal{B}(Y), \pi_0)$.

Assume $a \in X$ and there is a basis ψ_j such that

$$a(\mathbf{x}, \mathbf{y}) = \sum_{j\geq 1} \phi_j(\mathbf{y})\psi_j(\mathbf{x}) \qquad \text{Affine expansion}$$

$$\sigma(\mathbf{x}, \mathbf{y}) = \sigma_0(\mathbf{x}) + \sigma_*(\mathbf{x}) \exp\left[a(\mathbf{x}, \mathbf{y})\right] \quad \sigma_0(\mathbf{x}) \geq 0, \sigma_*(\mathbf{x}) > 0 \quad \forall x \in D$$

Parametric uncertainty

Represent the uncertainty in terms of an infinite sequence of random parameters $\mathbf{y} = (y_j)_{j\geq 1} \in Y \subset \mathbb{R}^{\mathbb{N}}$ with law defined on the product measure space $(Y, \mathcal{B}(Y), \pi_0)$.

Assume $a \in X$ and there is a basis ψ_j such that

$$a(\mathbf{x}, \mathbf{y}) = \sum_{j\geq 1} \phi_j(\mathbf{y})\psi_j(\mathbf{x}) \qquad \text{Affine expansion}$$

$$\sigma(\mathbf{x}, \mathbf{y}) = \sigma_0(\mathbf{x}) + \sigma_*(\mathbf{x}) \exp\left[a(\mathbf{x}, \mathbf{y})\right] \quad \sigma_0(\mathbf{x}) \geq 0, \sigma_*(\mathbf{x}) > 0 \quad \forall x \in D$$

Log-normal random field σ

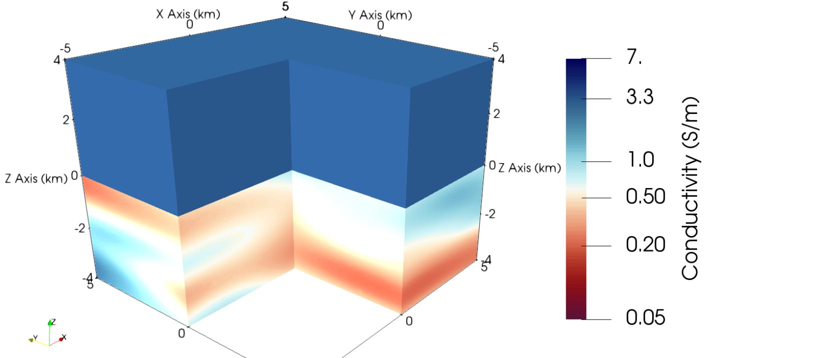
Karhunen–Loève expansion for Gaussian random field a

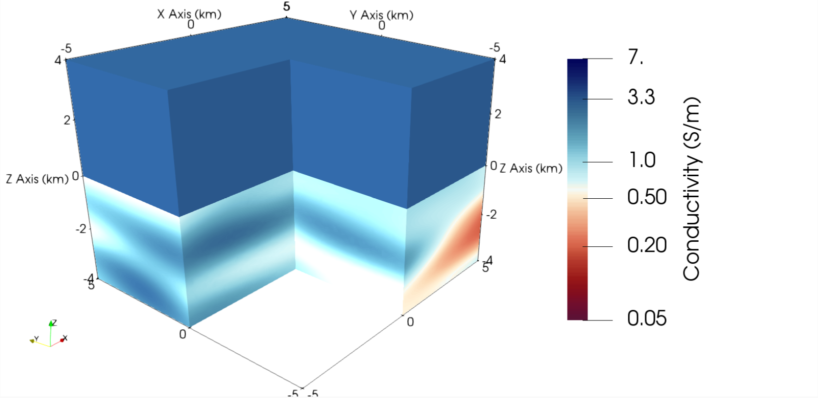
$$\sigma(\mathbf{x}, \mathbf{y}) = \sigma_0(\mathbf{x}) + \sigma_*(\mathbf{x}) \exp\left(\sum_{j=1}^{\infty} \sqrt{\lambda_j}\xi_j(\mathbf{x})y_j\right), \quad y_j \sim \mathcal{N}(0, 1)$$

Assume covariance function belongs to Matérn-Whittle class

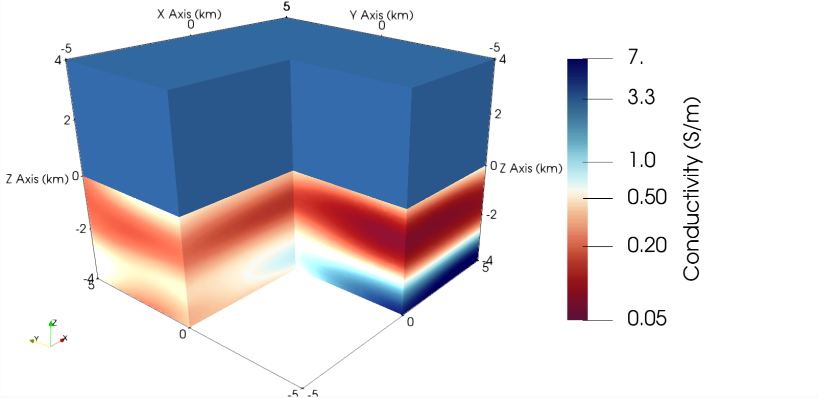
$$C_a(x, y) = \frac{\mathrm{Var}(a)}{2^{\nu-1}\Gamma(\nu)} \left(\|x_1 - x_2\|_M\right)^{\nu} K_{\nu}\left(\|x_1 - x_2\|_M\right)$$

For $\nu = 15/2$, $M^{1/2} = \mathrm{diag}(1250, 1250, 300)$
$\sigma_+ = 3.3$, $\mathrm{Var}[a] = 1$, $\sigma_*(\mathbf{x}) = 1/2$ and $\sigma_0(\mathbf{x}) = 0$
$D = (-5000, 5000) \times (-5000, 5000) \times (-4000, 4000)$, separated into D_+ and D_- by $z = 0$

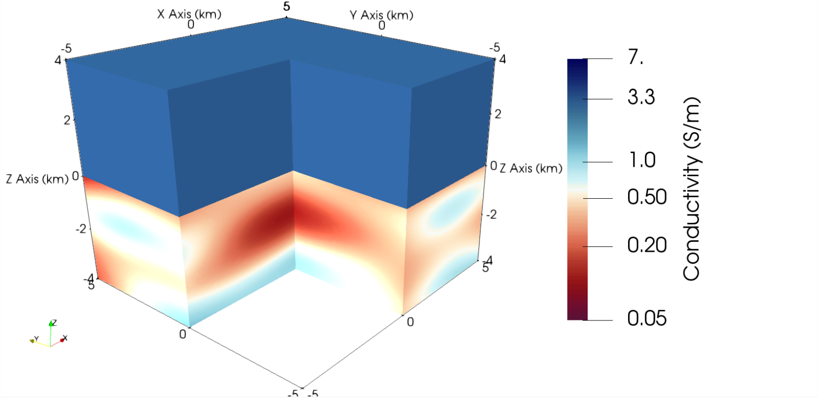



X Axis (km)
Y Axis (km)
Z Axis (km)
Z Axis (km)
7.
3.3
1.0
0.50
0.20
0.05
Conductivity (S/m)

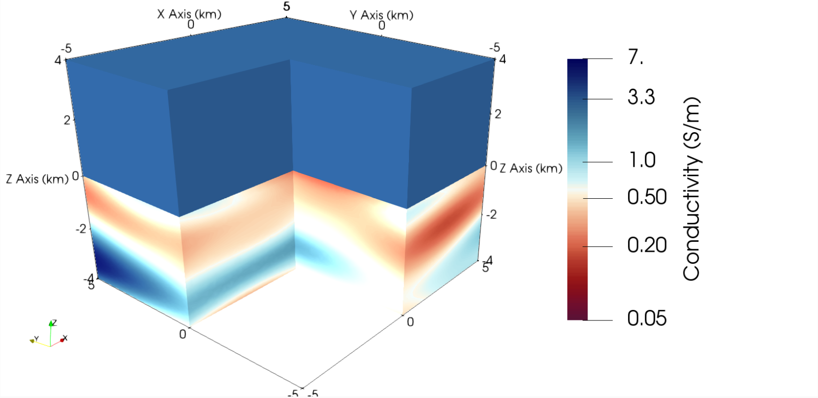
EXAMPLE RANDOM FIELD



EXAMPLE RANDOM FIELD



EXAMPLE RANDOM FIELD



Input-output map $\mathcal{G}$

Assume that for each parametric input $\mathbf{y}$ we have the solution $\mathbf{E}(\mathbf{y})$ given as

$$\mathcal{F} : \mathbf{y} \to \Lambda_\sigma^{-1} f \to \mathbf{E}(\mathbf{y}) \in H(\mathrm{curl}, D), \quad \text{FEM with Nedèlec edge elements}$$

Measurements are also modelled through through an observation operator $\mathcal{O}(\mathbf{E}) \in \mathbb{C}$

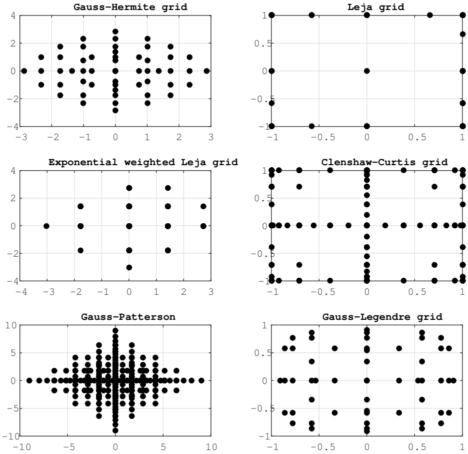
Estimation of uncertainty

We can calculate a quantity of interest Q (mean, covariance etc.) by approximating the integral

$$\mathbb{E}[Q] = \int_Y Q(\mathbf{y})\rho(\mathbf{y})\, d\mathbf{y}$$

Sparse-grids

Sparse grids $\mathcal{S}_{\Lambda_M}$ are constructed by union of tensorizations of one-dimensional ensemble grids, based on multi-index sets Λ_M



Calculation of moments

Approximation by sparse-grid quadrature $I_{\Lambda_M}[Q]$.

$$\mathbb{E}[Q] \approx I_{\Lambda_M}[Q] = \sum_m Q(y_m) w_m$$

where y_m and w_m are precomputed nodes and weights according to a quadrature rule.

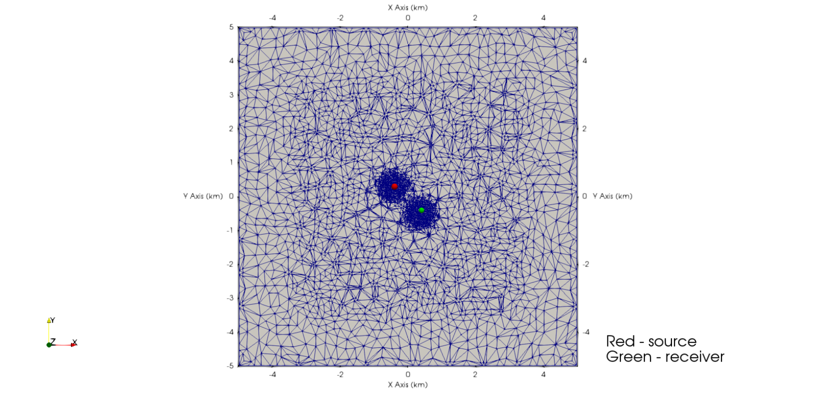
Convergence of approximation

Generalized sparse quadrature exploits the properties of the forward model to derive an approximation of the integral using fewer forward solves than typical Monte Carlo methods for the same accuracy, while being also immune to the curse of high dimensionality.

$\rightarrow$*Adaptive algorithm* to exploit anisotropy

$\mu = \mu_0$, $\omega = 2\pi$, $\|\mathbf{p}_s\|_2 = 50000$

Mesh a priori refined at the source $\mathbf{x}_s = (-350, 400, 250)$ and sensor $\mathbf{x}_r = (200, -300, 150)$ positions (x-oriented dipole source and receiver) $\rightarrow$ 81287 tetrahedra, 98030 dofs



numerical experiments (measurements uncertainty)

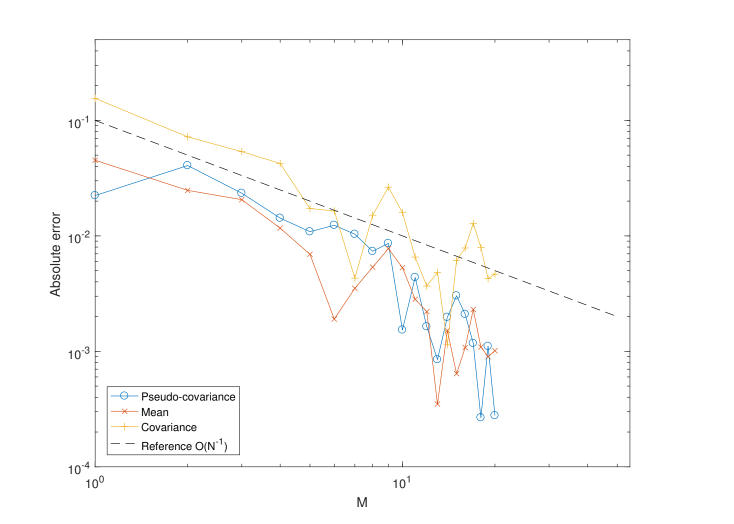
Statistical characterization of measurement at x_r with $Q = \mathcal{O}(\mathsf{E}) = E_{x_r}$

	SGQ approximation
$\mathbb{E}[Q]$	$(-4.66 - 28.02\mathrm{i}) \times 10^{-6}$
$\mathrm{Cov}[Q, Q]$	$(-8.26 + 30.12\mathrm{i}) \times 10^{-12}$
$\mathrm{Cov}[Q, \overline{Q}]$	3.29×10^{-11}

Table 1: Values in SI units of the mean $\mathbb{E}[Q]$, covariance $\mathrm{Cov}[Q, \overline{Q}]$ and pseudo-covariance $\mathrm{Cov}[Q, Q]$ as approximated by the SGQ (using $I_{\Lambda_{100}}[Q]$) method at $M = 100$.

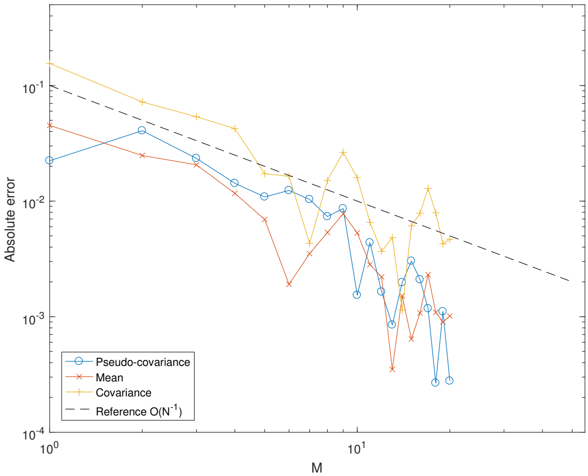
Number of forward solves ≈ 1000 to achieve accuracy $\epsilon = 0.01$ (including cost of exploration in adaptive algorithm). Compare with Monte Carlo which requires ≈ 10000 evaluations.

Convergence of sparse grid approximation with respect to number of indices M

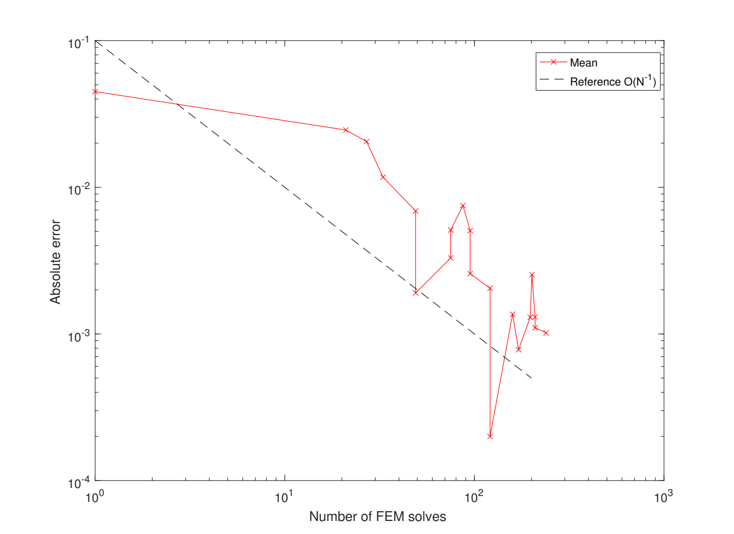


NUMERICAL EXPERIMENTS (MEASUREMENTS UNCERTAINTY)

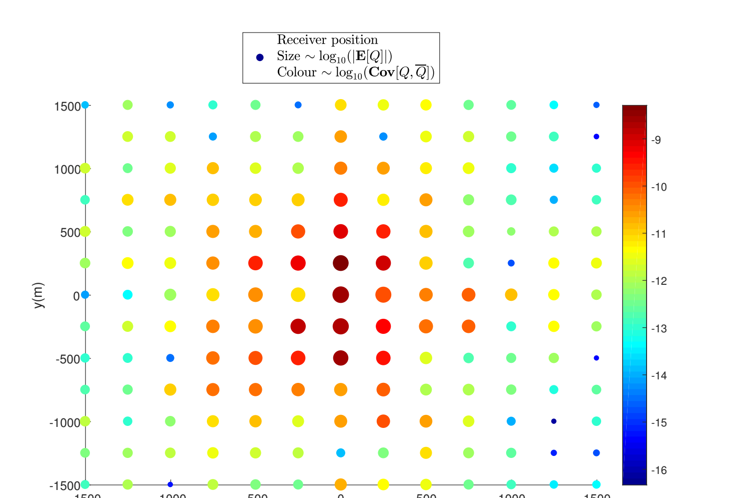
Convergence of sparse grid approximation with respect to number of indices M



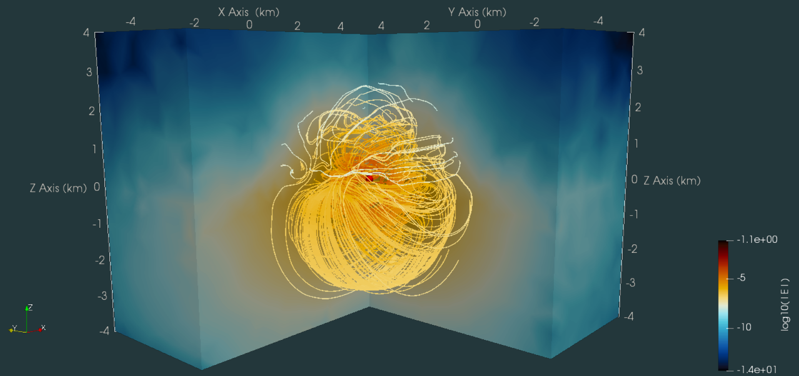
Convergence of sparse grid approximation with respect to number of forward solves



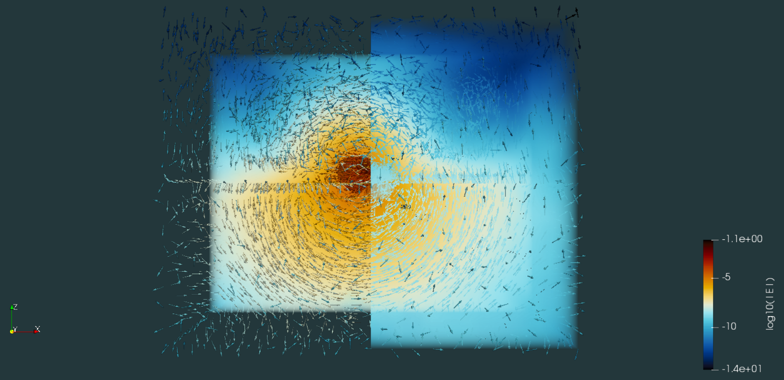
Source at $x_s = (0, 0, 250)$ (x-oriented), grid of 169 receivers at $z = 150$ (x-oriented for $x > 0$ and y-oriented for $x < 0$). 86397 tets, 102719 dofs



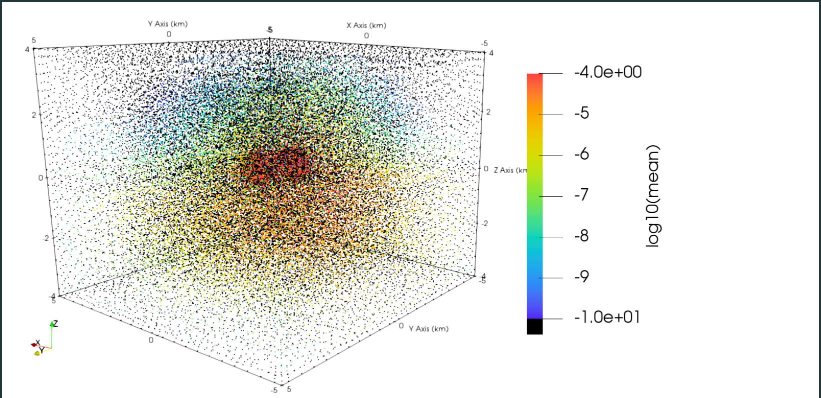
Streamlines for the mean FEM solution



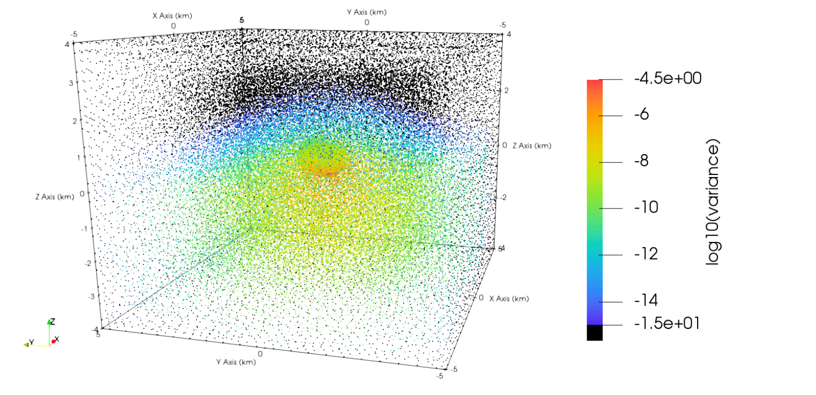
Glyph vectors for the mean FEM solution, 62786 tets, 76431 dofs



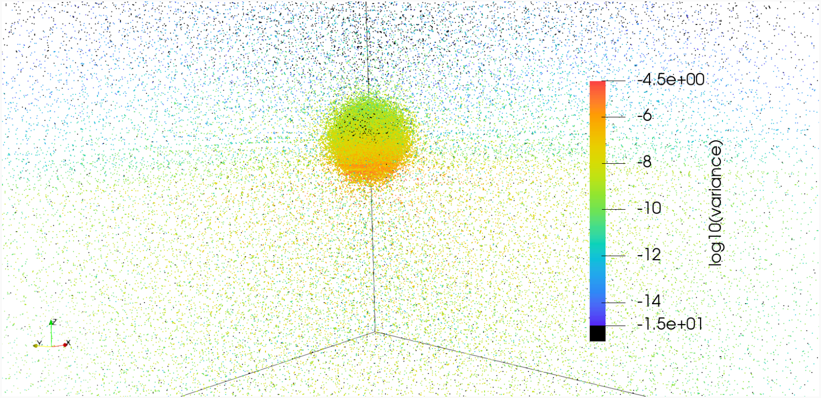
Mean FEM solution at dofs (edges)



Points represent the values of the diagonal of the FEM solution covariance matrix at the dofs (edges)



Zoom at source location



Model order reduction

Instead of solving the FEM high-fidelity problem with N_0 d.o.f for each y_m, solve a reduced problem.

Construct reduced basis of dimension $N \ll N_0$, based on *snapshots* from a training set and a *greedy* procedure.

Offline-online approach

Offline: project FEM matrices on reduced space and save them
Online: solve the reduced problem for each y_m

Remark

Requires affine expansion for σ
Solution: Empirical Interpolation Method - greedy interpolation

CONCLUSION

- Infinite dimensional Bayesian inversion becomes a parametric, deterministic, high-dimensional quadrature problem
- To tackle high-dimensionality we exploit sparsity and smoothness.
- We perform the integration using adaptive sparse-grid quadrature
- We achieve dimension-independent rates for the error in the estimation of the quantity of interest.
- To further reduce the complexity, we employ the reduced basis method with training sets chosen adaptively.

Questions?

P. Chen and C. Schwab.
Sparse-grid, reduced-basis bayesian inversion.
Computer Methods in Applied Mechanics and Engineering, 297:84 – 115, 2015.

A. Cohen and R. DeVore.
Approximation of high-dimensional parametric pdes.
Acta Numerica, 24:1–159, 5 2015.

A. M. Stuart.
Inverse problems: a bayesian perspective.
Acta Numerica, 19:451–559, 2010.