Response Spectrum Generation - FFT Method.

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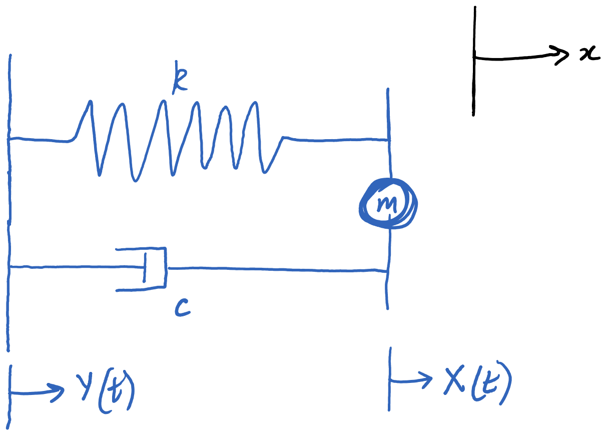
Response Spectra of a TH = find the peak response (acceleration / velocity / displacement) of an oscillator with natural frequency, w_n, and critical damping ratio, ζ. Repeat exercise for a range of frequencies to chart a spectrum.



$\ddot{y}(t) = a(t)$

$=$ Driving motion.

Want to find:

$S_a = \max |\ddot{X}(t)|$

$\Sigma F = m\ddot{X}$

$m\ddot{X} = -k(X(t) - Y(t)) - c(\dot{X}(t) - \dot{Y}(t))$

Define a new variable $\Longrightarrow$

$x(t) = X(t) - Y(t)$

then $\Longrightarrow$

$\dot{x}(t) = \dot{X}(t) - \dot{Y}(t)$

$\ddot{x}(t) = \ddot{X}(t) - \ddot{Y}(t)$
$= \ddot{X}(t) - a(t)$

$m(\ddot{x} + a(t)) = -kx - c\dot{x}$

$m\ddot{x} + c\dot{x} + kx = -ma(t)$

$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = -a(t)$

Let:

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2m\omega_n}$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = -a(t)$$

Where, $a(t)$ is the driving acceleration, defined over some time interval, $[0, T]$.

To solve for x, consider the fourier series representations of x and a:

$$a(t) = \sum_{n=-\infty}^{\infty} c_{a,n}\, e^{i.\frac{2n\pi}{T}t}, \quad \text{where}$$

$$c_{a,n} = \frac{1}{T}\int_0^T a(t)\, e^{-i.\frac{2n\pi}{T}t}\, dt.$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_{x,n}\, e^{i.\frac{2n\pi}{T}.t}, \quad \text{where}$$

$$c_{x,n} = \frac{1}{T}\int_0^T x(t)\, e^{-i.\frac{2n\pi}{T}.t}\, dt.$$

$$\dot{x}(t) = \frac{dx}{dt} = \frac{d}{dt}\left[\sum_{n=-\infty}^{\infty} c_{x,n}\, e^{i.\frac{2n\pi}{T}t}\right]$$

$$= \sum_{n=-\infty}^{\infty} c_{x,n}\, \frac{d}{dt}\left(e^{i.\frac{2n\pi}{T}.t}\right)$$

$$= \sum_{n=-\infty}^{\infty} c_{x,n}\left(i.\frac{2n\pi}{T}\right) e^{i.\frac{2n\pi}{T}.t}$$

$$\ddot{x}(t) = \frac{d}{dt}(\dot{x}(t)) = \sum_{n=-\infty}^{\infty} c_{x,n}\left(i.\frac{2n\pi}{T}\right)^2 . e^{i.\frac{2n\pi}{T}t}.$$

$$= -\sum_{n=-\infty}^{\infty} c_{x,n}\left(\frac{2n\pi}{T}\right)^2 . e^{i.\frac{2n\pi}{T}t}$$

Substitute these functions back into the main equation of motion $\Longrightarrow$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x + a(t) = 0$$

$$\sum_{n=-\infty}^{\infty}\left[-c_{x-n}\left(\frac{2n\pi}{T}\right)^2 + 2\zeta\omega_n\cdot c_{x-n}\cdot i\left(\frac{2n\pi}{T}\right) + c_{x-n}\omega_n^2 + c_{a-n}\right]e^{i\cdot\frac{2n\pi}{T}t} = 0$$

Take the inner product of both sides with $e^{-i\cdot\frac{2m\pi}{T}\cdot t}$ $\Rightarrow$

$$\int_0^T e^{i\frac{2n\pi}{T}t}\cdot e^{-i\frac{2m\pi}{T}t}\,dt \qquad n, m \in \mathbb{Z}$$

$$= \int_0^T e^{it\cdot\frac{2\pi}{T}(n-m)}\,dt$$

If $n \neq m \Longrightarrow$

$$= \int_0^T e^{it\cdot\frac{2\pi}{T}(n-m)}\,dt = \left[\frac{e^{it\cdot\frac{2\pi}{T}(n-m)}}{i(n-m)\cdot\frac{2\pi}{T}}\right]_0^T$$

$$= \frac{T}{i(n-m)\cdot 2\pi}\left[\underbrace{\cos[2\pi(n-m)]}_{=1} + i\,\underbrace{\sin[2\pi(n-m)]}_{=0} - 1\right]$$

$$= \frac{T}{2\pi i(n-m)}\left[1 - 1\right] = 0$$

If $n = m \Longrightarrow$

$$\int_0^T e^{i\frac{2n\pi}{T}t}\cdot e^{-i\frac{2n\pi}{T}t}\,dt$$

$$= \int_0^T e^{0}\,dt = T$$

So, $\forall n \Longrightarrow$

$$\left[-c_{x-n}\left(\frac{2n\pi}{T}\right)^2 + 2\zeta\omega_n c_{x-n} i\left(\frac{2\pi n}{T}\right) + c_{x-n}\omega_n^2 + c_{a-n}\right]T = 0$$

Let $\omega_f = \frac{2n\pi}{T} \Rightarrow$

$$c_{x\_n}\left[-\omega_f^2 + 2\zeta\omega_n\omega_f i + \omega_n^2\right] = -c_{a\_n}$$

$$\boxed{c_{x\_n} = \frac{-c_{a\_n}}{\left[-\omega_f^2 + 2\zeta\omega_n\omega_f i + \omega_n^2\right]}}$$

So, can calculate $\ddot{x}(t)$ by calculating c_{a_n} with an FFT of the input acceleration.

$$x(t) = \sum_{n=-\infty}^{\infty} c_{x\_n} e^{i\omega_f t}$$

$$\ddot{x}(t) = \sum_{n=-\infty}^{\infty} c_{\ddot{x}\_n} e^{i\omega_f t}$$

$$\boxed{c_{\ddot{x}\_n} = -\omega_f^2 c_{x\_n}}$$

$$\boxed{S_a = \max\left|\ddot{x}(t) + a(t)\right|}$$