Consider an object that is attached to a wall by a spring and forced to oscillate to and fro by a periodic driving function $f(t) = F_0 \cos(\omega t)$. Let $x(t)$ be the object's displacement from equilibrium at time t. Under the assumption of small displacements, the system is governed by the following second order, constant-coefficient, differential equation.

$$mx'' + bx' + kx = F_0 \cos(\omega t) \tag{1}$$

The letter m denotes the mass of the object, k is the spring constant, and the letter b denotes the damping constant. All three of these constants, and the constants F_0 and ω, are assumed to be positive; ω is called the *forcing frequency*. Recall that the *natural frequency* of the system is $\omega_0 = \sqrt{k/m}$. This is the frequency of the oscillations when the system is undamped and unforced: $mx'' + kx = 0$.

Steady-State Solutions

The general solution to Equation (1) is $x(t) = x_h(t) + x_p(t)$ where $x_h(t)$ is the general solution to the associated *homogeneous* equation and $x_p(t)$ is a *particular* solution to the forced equation. When the system is damped, $x_h(t)$ contains decaying exponentials and it approaches zero as $t \to \infty$. After sufficient time has passed, the motion will be governed by $x_p(t)$ and the system is said to have reached *steady-state*. The *steady-state solution*, also denoted $x_{ss}(t)$, is periodic with the same frequency as the driver. This is the motion that interests us here.

The steady-state solution for the simple driver $F_0 \cos(\omega t)$ in Equation (1) can be found by substituting $x = A\cos(\omega t) + B\sin(\omega t)$ into (1) and solving for A and B. When the driver is a general periodic function of frequency ω the steady-state solution can be found by a similar method involving the complex Fourier series for the driver.

Steady-State Solution for Any Periodic Driver

We wish to obtain the steady-state solution to

$$mx'' + bx' + kx = f(t) \tag{2}$$

where f is periodic of period P and (circluar) frequency $\omega = 2\pi/P$. Here is how.

1. Obtain the complex Fourier series representation for the driver.[1]

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

2. Substitute $x(t) = a_n e^{in\omega t}$ into the left side of (2) and solve for a_n, assuming that the driver is $c_n e^{in\omega t}$. That is, let $x = a_n e^{in\omega t}$ in

$$mx'' + bx' + kx = c_n e^{in\omega t} \tag{3}$$

and solve for a_n. This turns out to be easy because all of the exponential terms cancel. For the $n = 0$ case substitute $x = a_0$ and the constant a_0 will simply be c_0/k. However, for $n \geq 1$, a_n will be a complex number.

3. Once a_n is found, let $a_{-n} = \overline{a_n}$, and the steady-state solution to (2) is

$$x_{ss}(t) = \frac{c_0}{k} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} a_n e^{in\omega t} = \frac{c_0}{k} + \sum_{n=1}^{\infty} 2\,\Re(a_n e^{in\omega t})\,. \tag{4}$$

Here is an example.

Example 1. Obtain the steady-state solution to the mass-spring system $x'' + x' + 4x = f(t)$ where f is periodic of period $P = \pi$ defined on the interval $0 < t < \pi$ as follows.

$$f(t) = \begin{cases} 1 & , \quad 0 < t < \pi/2 \\ -1 & , \quad \pi/2 < t < \pi \end{cases}$$

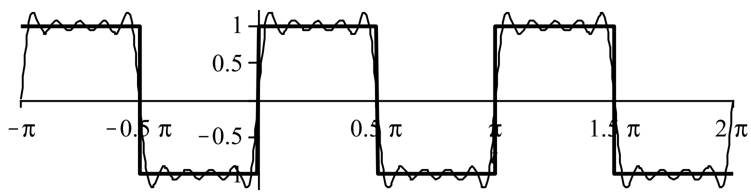
[1]Note the switch from k to n for the summation index in the Fourier series. This is to avoid confusion with the spring constant.

Plot the steady-state solution and the driver, suitably scaled to the dimensions of position vs time.

Solution. The driver is odd. See its graph below. Therefore, $c_0 = 0$. Since $P = \pi$, $\omega = 2\pi/P = 2$, so the Fourier series for the driver has the form $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2int}$ where

$$\begin{aligned} c_n = \frac{1}{\pi}\int_0^\pi f(t)e^{-2int}\,dt &= \frac{1}{\pi}\left(\int_0^{\pi/2} e^{-2int}\,dt + \int_{\pi/2}^{\pi}(-1)e^{-2int}\,dt\right) \\ &= \frac{1}{\pi}\left(\left.\frac{e^{-2int}}{-2in}\right|_{t=0}^{t=\pi/2} + \left.\frac{e^{-2int}}{2in}\right|_{t=\pi/2}^{t=\pi}\right) \\ &= \frac{1}{2\pi in}\left((1-e^{-\pi in}) + (e^{-2\pi in} - e^{-\pi in})\right) \\ &= \frac{1}{2\pi in}\left(2 - 2(-1)^n\right) \\ &= \frac{(-1)^n - 1}{\pi n}\, i\,. \end{aligned}$$

The coefficient c_n is pure imaginary. This is always the case when the waveform is odd. Observe also that when n is even, $c_n = 0$. The following plot confirms that these are the correct coefficients.



The driver f and its Fourier approximation $S_9(t)$.

Because there is no constant in the Fourier series for f, there is no constant in the steady-state solution. Therefore,

$$x_{ss}(t) = \sum_{n=1}^{\infty} 2\,\Re(a_n e^{2int})$$

where a_n is found by substituting $x = a_n e^{2int}$ into the differential equation

$$x'' + x' + 4x = c_n e^{2int}\,.$$

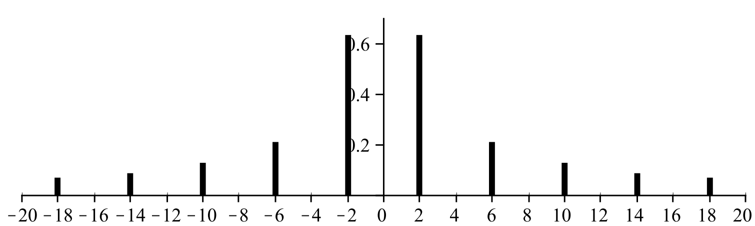
Doing so yields the following equation

$$-4n^2 a_n e^{2int} + 2ina_n e^{2int} + 4a_n e^{2int} = c_n e^{2int}\,,$$

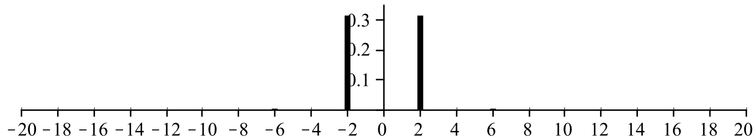
which simplifies to

$$(4 - 4n^2 + 2in)a_n = c_n\,.$$

Consequently, the n^{th} complex Fourier series coefficient for the steady-state solution is $a_n = c_n/(4 - 4n^2 + 2in)$. Observe that these coefficients approach 0 very quickly, implying that the oscillations in the steady-state solution will be completely determined by the first few non-zero harmonics. The amplitude spectrum for the driver is sketched below. The amplitude spectrum for the steady-state solution follows.

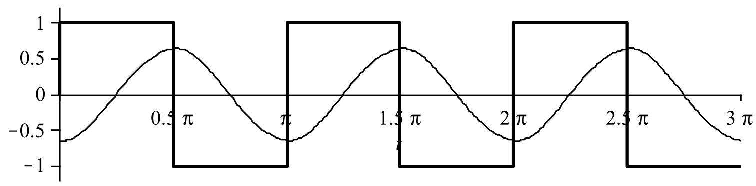


The amplitude spectrum for the periodic driver f.



The amplitude spectrum for the steady-state solution x_{ss} .

The following picture shows the driver and the steady-state solution. Five harmonics were used to plot x_{ss}. The picture obtained using just the *first* harmonic in the solution looks essentially the same.



The driver f and the steady-state solution x_{ss} .

A formula for the first harmonic in x_{ss}

First observe that c_n is zero when n is even. For odd n, $c_n = -2i/n\pi$. Therefore,

$$a_n = \frac{c_n}{4 - 4n^2 + 2in} = \frac{-2i}{n\pi(4 - 4n^2 + 2in)} \ , \ n \text{ odd} \, .$$

In particular, when $n = 1$, $a_1 = -1/\pi$. Therefore, using just the first harmonic,

$$x_{ss}(t) \approx 2\,\Re(a_1 e^{2it}) = -\frac{2}{\pi}\cos(2t) \, .$$

The amplitude of the first harmonic is 0.637. The next non-zero harmonic is $-\frac{2}{3\pi\sqrt{9+16^2}}\cos(6t - \delta_3)$ where $\delta_3 = \arctan(16^2/9)$. Its amplitude is 0.013.

A General Formula

The method illustrated in Example 1 is completely general, applying to any piecewise smooth driver of period P. In summary, it proceeds as follows. Begin by expressing the system in the form

$$mx'' + bx' + kx = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \ , \ \ \omega = 2\pi/P \, . \tag{5}$$

The steady-state solution will then have the form

$$x_{ss}(t) = \sum_{n=-\infty}^{\infty} a_n e^{in\omega t}$$

where $x = a_n e^{in\omega t}$ is a particular solution to

$$mx'' + bx' + kx = c_n e^{in\omega t} \, .$$

To find a_n simply substitute $x = a_n e^{in\omega t}$ into the last equation to obtain

$$-mn^2\omega^2 a_n e^{in\omega t} + ibn\omega a_n e^{in\omega t} + ka_n e^{in\omega t} = c_n e^{in\omega t}$$

which implies that

$$a_n = \frac{c_n}{k - mn^2\omega^2 + ibn\omega} \, .$$

Consequently,

$$x_{ss}(t) = \sum_{n=-\infty}^{\infty} \frac{c_n}{k - mn^2\omega^2 + ibn\omega} \, e^{in\omega t} \, . \tag{6}$$

Three comments:

1. The form of the n^{th} complex Fourier coefficient in Equation (6) makes it clear that the higher-order harmonics have almost no effect on the steady-state solution.

2. If information about a particular harmonic is needed, say the n^{th} one, its amplitude is $2\,|a_n|$ (why?). Therefore,

$$n^{\text{th}} \text{ Amplitude} = A_n = \frac{2|c_n|}{\sqrt{(k - mn^2\omega^2)^2 + b^2n^2\omega^2}}\,.$$

3. An explicit formula for the n^{th} steady-state harmonic is given by

$$n^{\text{th}} \text{ Harmonic} = A_n \cos(n\omega t - (\delta_n - \gamma_n))$$

   where δ_n is the argument of $k - mn^2\omega^2 + ibn\omega$ and γ_n is the argument of c_n.

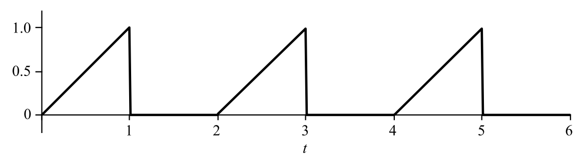
The following example illustrates these observations.

Example 2. Consider the mass-spring system

$$x'' + 0.1x' + 24x = f(t)$$

where f is the period 2 driver defined in Example 1 of the Complex Fourier Series, Part I handout.

$$f(t) = \begin{cases} t &, \quad 0 < t < 1 \\ 0 &, \quad 1 < t < 2 \end{cases}$$



The driver f has period 2.

Since $P = 2$, $\omega = 2\pi/P = \pi$, and $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\pi t}$ where

$$c_0 = \frac{1}{4} \quad \text{and} \quad c_n = \frac{1}{2}\left(\frac{(-1)^n}{n\pi} i + \frac{(-1)^n - 1}{n^2\pi^2}\right).$$

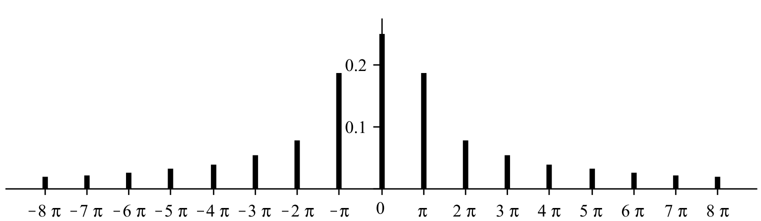
Therefore, the steady-state solution x_{ss} has the complex Fourier series

$$x_{ss}(t) = \frac{1/4}{k} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{c_n}{k - mn^2\pi^2 + ibn\pi}\, e^{in\pi t}$$

where $m = 1, b = 0.1$, and $k = 24$. That is,

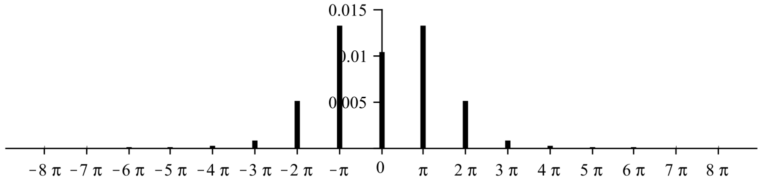
$$x_{ss}(t) = \frac{1}{96} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{c_n}{24 - n^2\pi^2 + 0.1in\pi}\, e^{in\pi t}\,.$$

The following picture shows the amplitude spectrum of the driver.



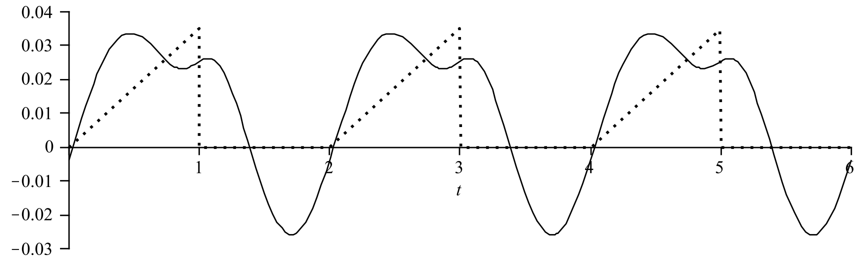
The driver's amplitude spectrum.

And here is the amplitude spectrum of the steady-state solution.



The amplitude spectrum of the steady-state solution.

The steady-state amplitude spectrum shows that the constant and the first three harmonics are enough to build an almost perfect picture of the steady-state motion. See the plot below, showing a scaled version of the driver and the steady-state approximation using 3 harmonics. The effects of the first two harmonics are especially evident. See Exercise 11 in Exercise Set 5.



The scaled driver (dotted) and the steady-state solution
using the constant term and the first three harmonics.

Interesting Observation. Using the constant and the first three harmonics yields only a mediocre approximation of the driver. See the plot in Example 1 of the Part I handout. However, the driver's power spectrum on page 4 of Part I clearly shows that its first three harmonics in the driver carry almost all of its power.