Nicolino Primavera 10/11/24

① Consider the following boundary-value problem:

$$U_{,xx} - P(P-1)x^{(P-2)} = 0 \quad , \quad 0 < x < 1$$

$$\left.\begin{aligned} -U_{,x}(0) &= 0 \\ U(1) &= 1 \end{aligned}\right] \text{ constraints/BC's}$$

Where p is a given constant.

$\boxed{\text{Note:}}$

$$\frac{d^2U}{dx^2} = U_{,xx}$$

$$\frac{dU}{dx} = U_{,x}$$

ⓐ Obtain the exact solution to this problem for <u>p = 5.</u> Sketch.

$$U_{,xx} - P(P-1)x^{(P-2)} = 0$$

$$U_{,xx} - 5(5-1)x^{5-2} = 0 \rightarrow U_{,xx} - 20x^3 = 0 \rightarrow \frac{d^2U}{dx^2} - 20x^3 = 0$$

$$U_{,xx} = 20x^3$$

$$U_{,x} = \int 20x^3\,dx = \frac{20}{4}x^4 + C_1 = 5x^4 + C_1$$

$$U = \int 5x^4 + C_1 = \frac{5}{5}x^5 + \frac{C_1}{1}x + C_2 = x^5 + C_1x + C_2$$

$U = x^5 + C_1x + C_2 \rightarrow$ apply the BC's

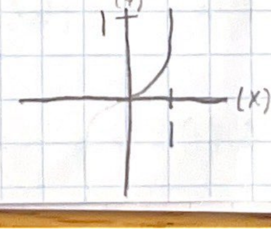
<u>$-U_{,x}(0) = 0$</u> $\rightarrow U_{,x}(0) = 0$

$U_{,x} = 5x^4 + C_1 \rightarrow U_{,x}(0) = (\cancel{5(0)^4} + C_1) \rightarrow U_{,x}(0) = C_1 \rightarrow \underline{C_1 = 0}$

<u>$U(1) = 1$</u>

$U = x^5 + C_1x + C_2 \rightarrow U(1) = (1)^5 + \cancel{(0)(1)} + C_2 = 1 + C_2 \rightarrow \underline{C_2 = 0}$

<u>exact solution:</u> $U(x) = x^5$




$0 < x < 1$

ⓑ State the Weak formulation of the problem and show its equivalence with the strong form.

*Note: (S) $u_{,xx} + f = 0$ in $\Omega$ & $\underbrace{u(1) = g}_{\text{Dirichlet BC}}$ & $\underbrace{u_{,xx}(0) = -h}_{\text{Neumann BC}}$

(W) $\int_0^1 w_{,x} u_{,x}\, dx = \int_0^1 wf\, dx + w(0)h$

$a(w,u) = (w,f) + w(0)h$

trial solution space

trial solutions

grid spacing

$\boxed{\text{Find } u \in S,\ w \in V:}$ $S: \{u \mid u \in H^1,\ u(1) = g$

test function space $V: \{w \mid w \in H^1,\ w(1) = 0$

test functions

$\boxed{\begin{aligned} u_{,xx} &= 20x^3 \\ u_{,x} &= 5x^4 \\ u &= x^5 \end{aligned}}$ w/ BC's $\begin{cases} -u_{,x}(0) = 0 \\ u(1) = 1 \end{cases}$

Equivalence b/w the Strong Form (S) and Weak Form (W)

↳ Assume that $u$ is a solution of $S$, then show that $u$ is a solution of (W) as well

(W) $a(w,u) = (w_{,x}, u_{,x}) = \int_0^1 w_{,x} u_{,x}\, dx = \overbrace{\int_0^1 wf\, dx}^{(w,f)} + w(0)h$

$\int_0^1 w_{,x} u_{,x}\, dx = \int_0^1 wf\, dx + w(0)h$

$0 = -\int_0^1 w_{,x} u_{,x}\, dx + \int_0^1 wf\, dx + w(0)h$ where $u_{,x} = 5x^4$

$0 = -a(w,u) + (w,f) + w(0)h$ where $h = -u_{,x}(0)$

$0 = -a(w,u) + (w,f) - w(0)u_{,x}(0)$

integrate this term from 0 to 1

Recall: $w(1) = 0$ ⟶ $[wu_{,x}]_0^1$

$$\left[wU_{,x}\right]_0^1 = \int_0^1 \left[wU_{,x}\right]_{,x} dx$$

$$0 = \underbrace{-\int_0^1 w_{,x} U_{,x}\, dx + \int_0^1 wf\, dx + \int_0^1 \left[wU_{,x}\right]_{,x} dx}_{\text{can be combined} \to \text{integration by parts}}$$

<u>Int by parts:</u>

$u = w \quad dv = U_{,xx}\, dx$

$du = w_{,x} \quad v = U_{,x}$

$$\int_0^1 wU_{,xx}\, dx = \int_0^1 (wU_{,x})_{,x}\, dx - \int_0^1 w_{,x}U_{,x}\, dx$$

$$\int u\, dv = uv - \int v\, du$$

$$\int_0^1 wU_{,xx}\, dx = wU_{,x} - \int_0^1 w_{,x}U_{,x}\, dx$$

$$\int_0^1 w(20x^3)\, dx = w5x^4 - \int_0^1 w_{,x} 5x^4\, dx$$

$$0 = \int_0^1 wU_{,xx}\, dx + \int_0^1 wf\, dx = \int_0^1 (U_{,xx} + f)\, w\, dx$$

divide by $w$ & differentiate

$$0 = \left(\cancel{\frac{d}{dx}} \cancel{\int_0^1} (U_{,xx} + f)\, \cancel{w}\, \cancel{dx}\right) \frac{1}{\cancel{w}}$$

(S) $\boxed{0 = U_{,xx} + f}$ — <u>Strong form</u>

$$0 = 20x^3 + f$$

(W) $\int_0^1 w_{,x} U_{,x}\, dx = \int_0^1 wf\, dx + w(0)h$

$\int_0^1 w_{,x} 5x^4\, dx = \int_0^1 wf\, dx + w(0)h$

Weak form

(c) State the Galerkin and Matrix formulations

*Note: (G) Introduce grid size: $h$

finite element spaces
- $S^h$: finite dimensional trial solution space
- $V^h$: finite dimensional test function space

If $u \in S$ & $u^h \in S^h$, then $u^h \in S$

If $w \in V$ & $w^h \in V^h$, then $w^h \in V$

Constraints / BC's: If $u^h \in S^h$, $w^h \in V^h$, then $\underbrace{u^h(1) = g, \; w^h(1) = 0}_{\text{Dirichlet BCs}}$

(G) Given $f, g, \hbar$, find $u^h = v^h + g^h$ where $v^h \in V$, $u^h \in S^h$ such that $\forall\, w^h \in V^h$, $a(w^h, v^h) = (w^h, f) + w^h(0)\hbar - a(w^h, g^h)$

Recall: $a(w^h, u^h) = (w^h, f) + w^h(0)\hbar$ (W)

$$u^h = v^h + g^h$$

Compare (G) against (W) where $u \in S$ & $w \in V$

$a(w, u) = \int_0^1 w_{,x} u_{,x}\, dx \rightarrow$ Galerkin

$u \in S \rightarrow \int_0^1 (u_{,x})^2 dx < \infty$, $w \in V \rightarrow \int_0^1 (w_{,x})^2 dx < \infty$

$\hookrightarrow$ $u \in S$ or $w \in V$ does not simplify $\int_0^1 w_{,x} u_{,x} < \infty$

but $v \in V$, $w \in V \rightarrow \int_0^1 (v_{,x})^2 dx < \infty$ & $\int_0^1 (w_{,x})^2 dx < \infty$

$\therefore\; a(w, v) = \int_0^1 (w_{,x} v_{,x})\, dx < \infty$

$\rightarrow a(w^h, v^h) = (w^h, f) + w^h(0)\hbar - a(w^h, g^h)$

$a(w^h, v^h) = \int_\Omega w^h_{,x} v^h_{,x}\, dx = a(w^h, v^h) = (w^h_{,x}, v^h_{,x})_\Omega$

Galerkin Solution: $a(w^h, v^h) = (w^h_{,x}, v^h_{,x})_\Omega$

$$\int_0^1 w^h_{,x} v^h_{,x} \, dx = \int_\Omega w^h_{,x} v^h_{,x} \, dx$$

arbitrary constant

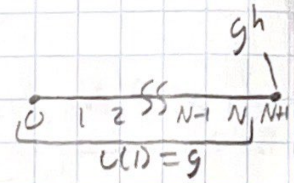
(M) $u^h = \sum_A d_A N_A(x)$ A: node number, $N_A(x) \subset \mathcal{V}^h / \mathcal{S}^h$

shape functions

For Galerkin method, $u^h$ and $w^h$ are obtained from the same function space

$w^A = \sum_A c_A N_A(x)$, $c_A$ - arbitrary constant

$N_A$: Shape functions / FE basis functions




$\underline{u^h = v^h + g^h}$, $v^h \subset \mathcal{V}^h$ test function space

$v^h = \sum_{A=0}^{N} d_A N_A$, $W = \sum_{A=0} c_A N_A$, $w^h = \sum_{A=0}^{N+1} c_A N_A$, $g^h = g N_{N+1}$

$N_A(1) = 0$ for $A = 0, 1, 2, \dots, N$

$N_{N+1}(1) = 1$

$N_{N+1} \begin{cases} = 1 & \text{at } x=1 \\ = 0 & \text{elsewhere} \end{cases}$

shape functions

$w^h = \sum_{A=1}^{N} c_A N_A + g N_{N+1} \longrightarrow$ plug into Galerkin FEM

$a(w^h, v^h) = (w^h, f) + w^h(0) h - a(w^h, g^h) \,/\, a(w^h, u^h) = (w^h, f) + w^h(0) h$

$$\sum_{A,B=0}^{N} c_A a(N_A, N_B) d_B = \sum_{A=0}^{N} c_A (N_A, f) + \sum_{A=0}^{N} c_A N_A(0) h - \left(\sum_{A=0}^{N} c_A N_A, g N_{N+1}\right)$$

$$\sum_{A=0}^{N} c_A \left(N_{A,x} \sum_{B=0}^{N} d_B N_{B,x} - N_A f - N_A(0) h + N_A g N_{N+1}\right) = 0$$

eqn. (A) $\sum_{A=0}^{N}(c_A R_A)=0$ where $R_A = a(N_A, \sum_{B=0}^{N} d_B N_B) - (N_A, f) \underbrace{- N_A(0)h + (N_A, g N_{N+1})}_{BC's}$

Residue at each node

☆ Eqn (A) is satisfied for all arbitrary $c_A$'s $\rightarrow$ $R_A = 0$

$$\underbrace{a(N_A \sum_{B=0}^{N} d_B N_B)}= (N_A, f) + N_A(0)h - (N_A, N_{N+1} g)$$

$$\hookrightarrow \sum_{B=0}^{N} N_{A,x} d_B N_{B,x} = \sum_{B=0}^{N} d_B (N_{A,x}, N_{B,x}) = \sum_{B=0}^{N} d_B\, a(N_A, N_B)$$

$$\sum_{B=0}^{N} \underbrace{d_B}_{\text{unknown / nodal displacement}} \underbrace{a(N_A, N_B)}_{\mathbb{K} - \text{stiffness matrix}} = \underbrace{(N_A, f) + N_A(0)h - (N_A, N_{N+1} g)}_{\text{load vector / forcing} \hookrightarrow \tilde{F}(N_A)}$$

$\mathbb{K}_{AB}\, d_B = F$ (G)

$$\mathbb{K}_{AB} = \int_0^1 N_{A,x} N_{B,x}\, dx = a(N_A, N_B)$$

$\hookrightarrow$ <u>stiffness matrix</u>

$U = \sum d_A N_A$ — given the $\overbrace{\text{coefficient matrix } (\mathbb{K})}^{\text{stiffness matrix}}$ and a force vector, $\tilde{F}(N_A)$ find $d_B$, nodal displacement

$d_B = F \cdot \mathbb{K}_{AB}^{-1}$ (M)

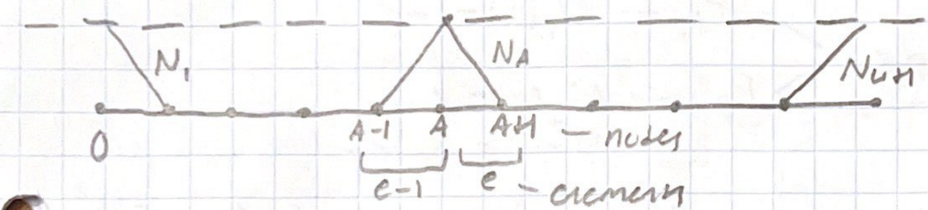
(d) Solve the matrix problem for $p=5$ using piecewise linear finite element space for,

i. Exactly one element

ii. Two equal-length elements

Shape functions: $N_A(x) = \begin{cases} \dfrac{x - x_{A-1}}{h_{A-1}} & ; x_{A-1} \le x \le x_A \\ \dfrac{x_{A+1} - x}{h_A} & ; x_A \le x \le x_{A+1} \end{cases}$

$h_A = h_e = x_{A+1} - x_A$




$$\mathbb{K}_{AB} = a(N_A, N_B) = \int_0^1 N_{A,x} N_{B,x} \, dx$$

we have $u_{,xx} - 20x^3 = 0$, $\underbrace{0 < x < 1}_{\text{domain}}$

BC's: $-u_{,x}(0) = 0$, $u(1) = 1$

Neumann BC    Dirichlet BC

$$K_{AB} = \int_0^1 N_{A,x} N_{B,x} \, dx$$

$N_1(x) = 1 - x \longrightarrow N_{1,x}(x) = -1$

$N_2(x) = x \longrightarrow N_{2,x}(x) = 1$

Calculate stiffness matrix

$$K_{11} = \int_0^1 (-1)(-1)\,dx = \int_0^1 dx = 1$$

$$K_{12} = \int_0^1 (-1)(1)\,dx = \int_0^1 -dx = -1$$

$$K_{21} = \int_0^1 (1)(-1)\,dx = \int_0^1 -dx = -1$$

$$K_{22} = \int_0^1 (1)(1)\,dx = \int_0^1 dx = 1$$

$$K = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Calculate load vector

$$F_i = \int_0^1 20x^3 N_A(x)\,dx$$

$F_1$ — $N_1(x) = 1 - x$

$F_2$ — $N_2(x) = x$

$$F_1 = \int_0^1 20x^3(1-x)\,dx = \int_0^1 (20x^3 - 20x^4)\,dx = 20\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1$$

$$= 20\left(\frac{1}{4} - \frac{1}{5}\right) = 5 - 4 = 1$$

$$F_2 = \int_0^1 20x^3(x)\,dx = \int_0^1 20x^4\,dx = 20 \cdot \frac{x^5}{5}\Big|_0^1 = 4x^5\Big|_0^1$$

$$= 4(1)^5 - 4(0)^5 = 4$$

$$F = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$K_{AB} d_B = F$ $\qquad$ $K \cdot u = F$

stiffness matrix — unknowns — load vector

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$u_1 - u_2 = 1$$
$$-u_1 + u_2 = 4$$

<u>BC's</u> $\quad -u_{,x}(0) = 0 \;,\; u(1) = 1$

Neumann $x = 0$ $\qquad$ Dirichlet $x = 1$

$\cancel{u_1 - u_2 = 1 \rightarrow u_1 = 1 + u_2}$

$\cancel{-u_1 + u_2 = 4 \rightarrow u_2 = 4 - u_1}$

$\cancel{u_1 = 1 + 4 - u_1}$

$u(1) = 1 \rightarrow$ fix $u_2 = 1$

$$u_x(1) = \frac{u_2 - u_1}{h} = \frac{1-2}{1} = -1$$

$u_1 - u_2 = 1 \rightarrow u_1 - 1 = 1 \rightarrow u_1 = 2$ & $u_2 = 1$

<u>Solve for two equal length elements</u>

Divide the domain $0 < x < 1$ into two elements

$e_1 \quad e_2$

$e_1 : [0, \frac{1}{2}]$

$e_2 : [\frac{1}{2}, 1]$

$\rightarrow$ 3 nodes

$N_1 : x_1 = 0$

$N_2 : x_2 = \frac{1}{2}$

$N_3 : x_3 = 1$

* two equal elements making up the domain

## Basis Functions — linear

$N_1(x) = 1-x$

$N_2(x) = x$

$e_1 : [0, \frac{1}{2}]$ , $e_2 : [\frac{1}{2}, 1]$

$x_1 = 0$

$x_2 = \frac{1}{2}$

$x_2 = \frac{1}{2}$

$x_3 = 1$

Node 1: $N_1(x_1) = 1-2x$

$= 1 \text{ @ } x=0$

$= 0 \text{ @ } x=\frac{1}{2}$

Node 2: $N_2(x_2) = 2x$

$= 1 \text{ @ } x=\frac{1}{2}$

$= 0 \text{ @ } x=0$

Node 2: $N_2(x_2) = 2(1-x)$

$= 1 \text{ @ } x=\frac{1}{2}$

$= 0 \text{ @ } x=1$

Node 3: $N_3(x_3) = 2x-1$

$= 1 \text{ @ } x=1$

$= 0 \text{ @ } x=\frac{1}{2}$

Compute local stiffness matrix

$$K_{AB} = \int_0^{1/2} N_{A,x} N_{B,x} \, dx$$

$$K_{11} = \int_0^{1/2} (-2)(-2)\,dx = 4 \cdot x \Big|_0^{\frac{1}{2}} = 2$$

$$K_{12} = \int_0^{1/2} (-2)(2)\,dx = -4 \cdot x \Big|_0^{1/2} = -2$$

$$K_{21} = \int_0^{1/2} (2)(-2)\,dx = -4 \cdot x \Big|_0^{1/2} = -2$$

$$K_{22} = \int_0^{1/2} (2)(2)\,dx = 4 \cdot x \Big|_0^{1/2} = 2$$

$$K = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$e_1$

$$K_{11} = \int_{1/2}^{1} (-2)(-2)\,dx = 4 \cdot x \Big|_{1/2}^{1} = 4(1-\tfrac{1}{2}) = 2$$

$$K_{12} = \int_{1/2}^{1} (-2)(2)\,dx = -4 \cdot x \Big|_{1/2}^{1} = -4(1-\tfrac{1}{2}) = -2$$

$$K_{21} = \int_{1/2}^{1} (2)(-2)\,dx = -4 \cdot x \Big|_{1/2}^{1} = -4(1-\tfrac{1}{2}) = -2$$

$$K_{22} = \int_{1/2}^{1} (2)(2)\,dx = 4(1-\tfrac{1}{2}) = 2$$

$$k = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$e_2$

Compute Global Stiffness Matrix

$$K = K_1 + K_2$$

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

Load Vector

$$F_i = \int_0^1 U_{,xx} N_A(x)\,dx = \int_0^1 20x^3 N_A(x)\,dx$$

$$F_1 = \int_0^{1/2} 20x^3(1-2x)\,dx = \int_0^{1/2} 20x^3 - 40x^4\,dx = \frac{20x^4}{4} - \frac{40x^5}{5} \Big|_0^{1/2}$$

$$= 5x^4 - 8x^5 \Big|_0^{1/2} = 5(\tfrac{1}{2})^4 - 8(\tfrac{1}{2})^5 = \underline{\underline{\tfrac{1}{16}}}$$

$$F_2 = \int_0^{1/2} 20x^3(2x)\,dx = \int_0^{1/2} 40x^4\,dx = 40\frac{x^5}{5}\Big|_0^{1/2} = 8x^5\Big|_0^{1/2}$$

$$= 8\left(\tfrac{1}{2}\right)^5 = \underline{\underline{\tfrac{1}{4}}}$$

$$F_2 = \int_{1/2}^{1} 20x^3(2-2x)\,dx = \int_{1/2}^{1} 40x^3 - 40x^4\,dx = 40\frac{x^4}{4} - 40\frac{x^5}{5}\Big|_{1/2}^{1}$$

$$= 10x^4 - 8x^5\Big|_{1/2}^{1} = 10 - 0.625 - 8 + 0.25 = \underline{\underline{1.625}}$$

$$F_3 = \int_{1/2}^{1} 20x^3(2x-1)\,dx = \int_{1/2}^{1} 40x^4 - 20x^3\,dx = 40\frac{x^5}{5} - 20\frac{x^4}{4}\Big|_{1/2}^{1}$$

$$= 8x^5 - 5x^4\Big|_{1/2}^{1} = 8 - 0.25 - 5 + 0.3125 = \underline{\underline{3.0625}}$$

$$F = \begin{pmatrix} \frac{1}{16} \\ \frac{1}{4} \\ \frac{13}{8} \end{pmatrix}$$

BC's — Neumann BC $- U_x(0) = 0 \rightarrow U_1$ is free

— Dirichlet BC $- U_3 = 1 \rightarrow$ fixed value at $N_3$

$$K \cdot dB = F$$

$$\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{pmatrix} \frac{1}{16} \\ \frac{1}{4} \end{pmatrix}$$

$$\left.\begin{aligned} 2U_1 - 2U_2 &= \frac{1}{16} \\ -2U_1 + 4U_2 &= \frac{1}{4} \end{aligned}\right]$$

$$U_1 = U_2 + \frac{1}{32}$$

$$-2\left(U_2 + \frac{1}{32}\right) + 4U_2 = \frac{1}{4} \rightarrow -2U_2 - \frac{1}{16} + 4U_2 = \frac{1}{4}$$

$$2U_2 = \frac{1}{4} + \frac{1}{16}$$

$$U_2 = \frac{5}{32}$$

$$2U_1 = 2U_2 + \frac{1}{16}$$

$$U_1 = U_2 + \frac{1}{32} = \frac{5}{32} + \frac{1}{32}$$

$$U_1 = \frac{6}{32} = \frac{3}{16}$$

$$U_3 = 1$$

$$U_x(1) \approx \frac{U_3 - U_2}{h} = \frac{1 - \frac{5}{32}}{0.5} = 1.6875$$

## ② Applying Galekin FEM for ADE

Class 4

- ↳ derived the global nodal equation for a generic internal node by assembling the local element stiffness matrix
- ↳ showed that for a uniform mesh, the nodal equation is the same as applying the central difference ($CD_2$) scheme to the original ADE

☆ in lecture we used the 'conservative' form of the weak form of the ADE to show the equivalence with $CD_2$

Demonstrate that this equivalence is retained even if we use the 'convective' form for the convection term in the weak form of the ADE.

the moment you add convection you lose the symmetric nature of the matrix

---

(W) $a(w,u) = \underbrace{(w, a u_{,x})_\Omega}_{\text{Convection term}} + \underbrace{(w_{,x}, K u_{,x})_\Omega}_{\text{Diffusion term}}$

$a(w,u) = (w, f) + BC's$

(G) $a(w^h, u^h) = (w^h, f) + BC's$

(M) $u^h(x) = \sum_A d_A N_A(x)$

$w^h(x) = \sum_A c_A N_A(x)$

plug into Galekin Form

↓

$\underbrace{a(N_A, N_B) d_B}_{\mathbb{K}_{AB}} = \underbrace{(N_A, f)}_{\text{load vector}} + BC's$

$\mathbb{K}_{AB}$ ↓ stiffness matrix

☆ A & B are node #'s

$\mathbb{K}_{AB} :=\!= \underbrace{(N_A, a N_{B,x})}_{\text{convection part}} + \underbrace{(N_{A,x}, K N_{B,x})}_{\text{Diffusion part}}$

$w_{,x}$

diffusion coefficient

$K_{AB} \neq K_{BA}$ ↳ Convection term: $K^{conv}_{AB} :=\!= (N_A, a N_{B,x})$

$K^{conv}_{BA} :=\!= (N_B, a N_{A,x})$ ] not equal

$\underbrace{\mathbb{K}^{conv}_{AB} = -\mathbb{K}^{conv}_{BA}}_{\text{skew-symmetric}}$ , $\mathbb{K}_{AB} = \mathbb{K}^{conv}_{AB} + \underbrace{\mathbb{K}^{diff}_{AB}}_{\text{symmetric}}$

# Weak Form for ADE

*convective form is the same as the advective form

Find $U \in \mathcal{S}$ such that $\forall\ w \in \mathcal{V}$

$$B(w,U) = F(w)$$

$$B(w,U) = \int_0^L (\underbrace{-w_{,x}\, aU}_{\text{conservative form}} + \underbrace{w_{,x} K U_{,x}}_{\text{diffusion}})\, dx$$

Convective or advective form: $waU_{,x}$

$$F(w) = \int_0^L wf\, dx$$

☆ Solve the weak form for ADE using the 'convective' form for the convection term → will show equivalence w/ $CD_2$ the same as the 'conservative' form did

## Galerkin Form

$v^h \in \mathcal{V}$ ; $U^h = v^h + g^h$

$$B(w^h, v^h) = F(w^h) - B(w^h, g^h)$$

$$B(w,U) = \int_0^L (\underbrace{waU_{,x}}_{\text{convective form}} + \underbrace{w_{,x} K U_{,x}}_{\text{diffusion form}})\, dx$$

(W) with convection term

## Introduce FE Basis Function → plug into Galerkin to get matrix form

$w^h = \sum c_A N_A$ ; $v^h = \sum_A d_A N_A$

test functions

linear combination of shape functions

(G) $B(w^h, v^h) = F(w^h) - B(w^h, g^h)$

$$\sum_{A,B} c_A B(N_A, N_B)\, d_B = \sum_A c_A [F(N_A) - B(N_A, g)]$$

$$\sum_A c_A \, \mathbb{K}_{AB} \, d_B = \sum_A c_A \tilde{F}(N_A) \longrightarrow \sum_A c_A \text{ cancel out}$$

$$\boxed{\mathbb{K}_{AB} \, d_B = \tilde{F}(N_A)} \longrightarrow \text{each 'A' node}$$

↳ applied to the sum of all A

Stiffness matrix ($\mathbb{K}_{AB}$) — load vector ($\tilde{F}(N_A)$)

$$\mathbb{K}_{AB} = \mathop{\mathrm{A}}_{e=1}^{N_{el}} \left[ \mathbb{k}^e_{ab} \right]$$

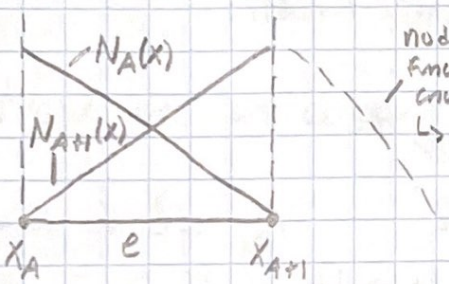
$\mathop{\mathrm{A}}$ — assembly operator over all the elements

$[\mathbb{k}^e_{ab}]$ — local element level stiffness matrix

assembly of local element wise stiffness matrix $\mathbb{k}^e$

## $\boxed{\text{How to construct } k^e}$

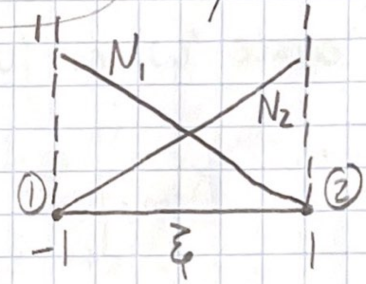
Assume $C^0$ continuity $\longrightarrow$ linear shape function




nodal shape function has another range ↳ next node

transformation $x(\xi)$ $\longrightarrow$

mapping to parent element




Known shape functions of the local element

coordinate psi

Shape functions: $N_A(x)$, $N_{A+1}(x)$

$$N_1(\xi) = \frac{1-\xi}{2} \quad N_1(-1) = 1, \quad N_1(1) = 0$$

$$N_2(\xi) = \frac{1+\xi}{2} \quad N_2(-1) = 0, \quad N_2(1) = 1$$

## Isoparametric Elements (or discretization)

*Locally

apply to local element: $U = \sum_A U_A N_A(x)$

position vector: $x = \sum_A x_A N_A(x)$ — nodal position vector

can compute the transformation

$$x_{,\xi} = \sum_A x_A \left[ N_{,\xi} \, \xi_{,x} \right]$$

our integrals:

$$\int dx$$

$$dx = x_{,\xi} \, d\xi$$

$$X = \sum_A x_A N_A(x)$$

$X = x_1 N_1 + x_2 N_2$ where $N_1 = \frac{1-\xi}{2}$

$N_2 = \frac{1+\xi}{2}$

$$X = x_1\left(\frac{1-\xi}{2}\right) + x_2\left(\frac{1+\xi}{2}\right)$$

$$\left[K^e_{ab}\right] = K^e_{convection} + K^e_{diffusion}$$

$$K_{conv} = \int_0^L \omega\, a\, U_{,x}\, dx$$

$$K_{diff} = \int_0^L \omega_{,x} K\, U_{,x}\, dx$$

☆ Replace 'ω' or 'U' with Na ☆ → Shape functions

$$\therefore\ K^e_{conv}\Big|_{ab} = \int_{\Omega^e} N_a\, a\, N_{b,x}\, d\Omega^e \quad \text{//}\, dx$$

, Ω is local x

Stiffness matrix for a & b

integrating over local element

↳ two summations → one is over A and the other B → U runs on B

$$K^e_{conv}\Big|_{ab} = \int_{\Omega^e} N_a\, a\, N_{b,x}\, dx = \int_{-1}^{1} N_a\, \overset{\text{speed}}{a} \underbrace{\left(N_{b,\xi}\, \cancel{\xi_{,x}}\right)}_{\text{chain rule}} \cancel{X_{,\xi}}\, d\xi$$

get canceled bc of convection

Write everything into change of coordinate so that we can integrate them → in the form of ξ (Psi)

☆ in diffusion there are two derivatives

☆ in convection there is only one derivative

$$\left[K^e_{conv}\right]_{ab} = \int_{-1}^{1} N_a\, a\, N_{b,\xi}\, d\xi$$ — local convection stiffness matrix for each node A & B

<u>ex: when a=1 & b=1</u>

$$\left[K^e_{conv}\right]_{11} = \int_{-1}^{1} N_1\, a\, N_{1,\xi}\, d\xi \qquad N_1 = \frac{1-\xi}{2} \qquad N_{1,\xi} = -\frac{1}{2}$$

$$\frac{d}{d\xi}\left(\frac{1-\xi}{2}\right) = \frac{1}{2}\frac{d}{d\xi}(1-\xi) = \frac{1}{2}\left(\frac{d}{d\xi}1 - \frac{d}{d\xi}\xi\right) = \frac{1}{2}(0-1) = -\frac{1}{2}$$

$$= \int_{-1}^{1} \left(\frac{1-\xi}{2}\right) a \left(-\frac{1}{2}\right) d\xi$$

$$= -\frac{1}{4}\int_{-1}^{1} (1-\xi)\, a\, d\xi = -\frac{1}{4}\int_{-1}^{1} (a - a\xi)\, d\xi$$

$$= -\frac{1}{4}\left(\int_{-1}^{1} a\, d\xi - \int_{-1}^{1} a\xi\, d\xi\right) = -\frac{1}{4}(2a - 0) = -\frac{a}{2}$$ — $\left[K^e_{conv}\right]_{11}$

Now we can compute $\left[K^e_{conv}\right]_{12}$ ; $\left[K^e_{conv}\right]_{21}$ ; $\left[K^e_{conv}\right]_{22}$

$$\left[K^e_{conv}\right]_{12} = \int_{-1}^{1} N_1\, a\, N_{2,\xi}\, d\xi \qquad \left|\; \begin{aligned} N_1 &= \frac{1-\xi}{2} \to N_{1,\xi} = -\frac{1}{2} \\ N_2 &= \frac{1+\xi}{2} \to N_{2,\xi} = \frac{1}{2} \end{aligned}\right.$$

$$= \int_{-1}^{1} \left(\frac{1-\xi}{2}\right) a \left(\frac{1}{2}\right) d\xi = \frac{a}{2}$$

$$\left[K^e_{conv}\right]_{21} = \int_{-1}^{1} N_2\, a\, N_{1,\xi}\, d\xi = \int_{-1}^{1} \left(\frac{1+\xi}{2}\right) a \left(-\frac{1}{2}\right) d\xi$$

$$= \left(-\frac{1}{4}\right)\int_{-1}^{1} (1+\xi)\, a\, d\xi = \left(-\frac{1}{4}\right)\left(\int_{-1}^{1} a\, d\xi + \cancelto{0}{\int_{-1}^{1} a\xi\, d\xi}\right)$$

$$= -\frac{a}{2}$$

$$\left[k^e_{conv}\right]_{22} = \int_{-1}^{1} N_2\, a\, N_{2,\xi}\, d\xi = \int_{-1}^{1} \left(\frac{1+\xi}{2}\right) a \left(\frac{1}{2}\right) d\xi$$

$$= \left(\frac{1}{4}\right)\int_{-1}^{1} (1+\xi)\, a\, d\xi = \left(\frac{1}{4}\right)\int_{-1}^{1} a\, d\xi + \cancel{\int_{-1}^{1} a\xi\, d\xi}^{\,0} = \frac{a}{2}$$

$$\left[k^e_{conv}\right] = \begin{bmatrix} -\frac{a}{2} & \frac{a}{2} \\ -\frac{a}{2} & \frac{a}{2} \end{bmatrix}$$

$dx = x_{,\xi}\, d\xi$

$$\left[k^e_{diff}\right]_{ab} = \int_{-1}^{1} N_{a,x}\, K\, N_{b,x}\, dx \qquad \Bigg| \qquad k_{diff} = \int_0^L \omega_{,x}\, K\, U_{,x}\, dx$$

$$= \int_{-1}^{1} \underbrace{\left(N_{a,\xi}\, \xi_{,x}\right)}_{\text{chain rule}} K \underbrace{\left(N_{b,\xi}\, \cancel{\xi_{,x}}\right)}_{\text{chain rule}} \underbrace{\cancel{x_{,\xi}}\, d\xi}_{dx} \qquad \Bigg| \qquad \begin{aligned} x_{,\xi} &= \textstyle\sum N_{a,\xi}\, x_a \\ \xi_{,x} &= (x_{,\xi})^{-1} \end{aligned}$$

$$= \int_{-1}^{1} N_{a,\xi}\, K\, N_{b,\xi}\, \xi_{,x}\, d\xi$$

$$x = \sum_A x_A N_A(x) = x_1 N_1 + x_2 N_2 \longrightarrow x_{,\xi} = x_1 N_{1,\xi} + x_2 N_{2,\xi} = x_1\left(-\tfrac{1}{2}\right) + x_2\left(\tfrac{1}{2}\right)$$

$$x_{,\xi} = \frac{h^e}{2}, \quad \xi_{,x} = \frac{2}{h^e}$$

<u>Solve when a=1 & b=1</u>

$$\left[k^e_{diff}\right]_{11} = \int_{-1}^{1} \underbrace{N_{1,\xi}\, K\, N_{1,\xi}}_{(-\frac{1}{2})(-\frac{1}{2}) = \frac{1}{4}} \frac{2}{h^e}\, d\xi = \frac{1}{4} \underbrace{\int_{-1}^{1} K \frac{2}{h^e}\, d\xi}$$

$$\int_{-1}^{1} d\xi = \xi\Big|_{-1}^{1} = 1-(-1) = 2$$

$$= \left(\frac{1}{4}\right) K \frac{2}{h^e}(2) = \frac{K}{h^e}$$

$$\left[K^e_{diff}\right] = \begin{bmatrix} \frac{K}{h^e} & -\frac{K}{h^e} \\ -\frac{K}{h^e} & \frac{K}{h^e} \end{bmatrix}, \qquad \left[K^e_{conv}\right] = \begin{bmatrix} -\frac{a}{2} & \frac{a}{2} \\ -\frac{a}{2} & \frac{a}{2} \end{bmatrix}$$

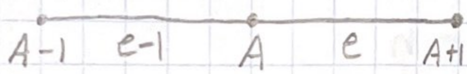
$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\left[K^e_{diff}\right]_A^{A+1} = \begin{bmatrix} \frac{K}{h^e} & -\frac{K}{h^e} \\ -\frac{K}{h^e} & \frac{K}{h^e} \end{bmatrix}, \qquad \left[K^e_{conv}\right]_A^{A+1} = \begin{bmatrix} -\frac{a}{2} & \frac{a}{2} \\ -\frac{a}{2} & \frac{a}{2} \end{bmatrix}$$

$$\left[K^e_{ab}\right] = K^e_{convection} + K^e_{diffusion} = \begin{array}{c c} & \begin{array}{cc} A & A+1 \end{array} \\ \begin{array}{c} A \\ A+1 \end{array} & \begin{bmatrix} \frac{K}{h^e} - \frac{a}{2} & \frac{a}{2} - \frac{K}{h^e} \\ -\frac{K}{h^e} - \frac{a}{2} & \frac{a}{2} + \frac{K}{h^e} \end{bmatrix} \end{array}$$

A · A+1

(A+1)(A+1)




$[K^{e-1}]$ $\quad$ $[K^e]$ — due to the interactions b/w A & A+1
— stiffness matrix

need to assemble both contributions to get the total contribution at Node A

$$\begin{array}{c c} & \begin{array}{ccc} A-1 & A & A+1 \end{array} \\ \begin{array}{c} A-1 \\ A \\ A+1 \end{array} & \begin{bmatrix} \frac{K}{h^{e-1}} - \frac{a}{2} & \frac{a}{2} - \frac{K}{h^{e-1}} & 0 \\ -\frac{K}{h^{e-1}} - \frac{a}{2} & \frac{a}{2} + \frac{K}{h^{e-1}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} + \begin{array}{c c} \begin{array}{ccc} A-1 & A & A+1 \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{K}{h^e} - \frac{a}{2} & \frac{a}{2} - \frac{K}{h^e} \\ 0 & -\frac{K}{h^e} - \frac{a}{2} & \frac{a}{2} + \frac{K}{h^e} \end{bmatrix} & \begin{array}{c} A-1 \\ A \\ A+1 \end{array} \end{array}$$

$$\left[K^{e-1}\right] + \left[K^e\right]$$

how you construct the whole stiffness matrix

Now that we have slightly expanded the matrix, we can add them
$[3 \times 3]$

Don't normally do this all the time → normally we write a nice assembly subroutine trying to identify all the node numbers and make sure that residue, stiffness matrix, gets added to its correct location

## $A^{th}$ Node:

$$\left[-\frac{k}{h^{e-1}}-\frac{a}{2}\right] \quad \left[\frac{k}{h^e}+\frac{k}{h^{e-1}}\right] \quad \left[\frac{a}{2}-\frac{k}{h^e}\right]$$ — Want to write this as an equation

$$\left[-\frac{k}{h^{e-1}}-\frac{a}{2}\right]U_{A-1}+\left[\frac{k}{h^e}+\frac{k}{h^{e-1}}\right]U_A+\left[\frac{a}{2}-\frac{k}{h^e}\right]U_{A+1}=\tilde{F}_A\cdot h \qquad [K]\{U\}$$

load vector

multiply K by the unknown vector

this term is the interaction b/w A-1 & A

☆ Assume uniform mesh, $h^{e-1}=h^e=h$

$$\left[-\frac{k}{h}-\frac{a}{2}\right]U_{A-1}+\left[\frac{k}{h}+\frac{k}{h}\right]U_A+\left[\frac{a}{2}-\frac{k}{h}\right]U_{A+1}=\tilde{F}_A\cdot h$$

☆ Rearrange and divide by h, assume $F=0$

$$-\frac{k}{h^2}U_{A-1}\underbrace{-\frac{a}{2h}U_{A-1}+\frac{2k}{h^2}U_A+\frac{a}{2h}U_{A+1}}_{a\frac{(U_{A+1}-U_{A-1})}{2h}}-\frac{k}{h^2}U_{A+1}=0$$

$$-\frac{k}{h^2}U_{A-1}+\frac{2k}{h^2}U_A-\frac{k}{h^2}U_{A+1}=-\frac{k(U_{A+1}-2U_A+U_{A-1})}{h^2}$$

$$a\frac{(U_{A+1}-U_{A-1})}{2h} - k\frac{(U_{A+1}-2U_A+U_{A-1})}{h^2} = 0$$

↳ Central Differencing (CD) scheme

Galerkin FEM is exactly same as CD for uniform mesh