MECE E6106 – Fall 2024
Homework # 1
(due Oct 11, 2024, 8pm)

1. Consider the following boundary-value problem: **[30 points]**

$$u_{,xx} - p(p-1)x^{(p-2)} = 0, \quad 0 < x < 1$$
$$-u_{,x}(0) = 0$$
$$u(1) = 1$$

   where p is a given constant.

   a. Obtain the exact solution to this problem for $p = 5$. Sketch.
   b. State the weak formulation of the problem and show its equivalence with the strong form.
   c. State the Galerkin and the matrix formulations.
   d. Solve the matrix problem for $p = 5$ using piecewise linear finite element space for,
      i. Exactly one element
      ii. Two equal-length elements
   e. Compare the exact value of $u_{,x}(1)$ against the corresponding values obtained from part (d). Is it possible for these results to agree favorably? Explain.

2. Let us focus on applying Galerkin finite element method (FEM) for ADE. In the class lecture, we derived the global nodal equation for a generic internal node by assembling the local element stiffness matrix and showed that for a uniform mesh, the nodal equation is the same as applying the central difference (CD_2) scheme to the original ADE. While in the lecture, we used the \`conservative' form of the weak form of the ADE to show the equivalence with $CD_2$, demonstrate that this equivalence is retained even if we use the \`convective' form for the convection term in the weak form of the ADE. **[10 points]**

3. We will use the finite element method (FEM) to analyze the steady-state advection-diffusion equation,

$$\frac{du}{dx} = 0.05\frac{d^2u}{dx^2}\ ; 0 \le x \le 1,$$

   with $u(0) = 0$ and $u(1) = 10$

   We want to explore the effect of cell Peclet number on the computed solution when Galerkin FEM is applied to this problem. **[60 points]**

   a. Choose at least four different grids for a range of cell Peclet numbers ($Pe_{\Delta x}$) and demonstrate that oscillatory solutions are obtained when $Pe_{\Delta x} > 2$.
   b. Compare the solutions for $Pe_{\Delta x} < 2$ against the analytical solution.

   Use element point-of-view as discussed in the class instead of assembling the stiffness matrix directly at the nodal level. You are welcome to use analytical integration at the local element level, but numerical integration is strongly recommended (e.g., Gauss quadrature-based integration).

For simplicity, you may choose uniform meshes, linear basis functions, and a direct solver to invert the assembled matrix. For the direct solver, you may use standard functions in Matlab or Python or implement your own Gauss Elimination method to compute the inverse.

Extra Credit (10%): Repeat the above analysis using quadratic basis functions and comment on the solution behavior. You may obtain the shape functions for quadratic basis online or from Chapter 3, Example 2 of the book by Hughes (Hughes, The Finite Element Method – Linear Static and Dynamic Finite Element Analysis, *Dover*)

General note for all programming-based assignments:

- Please submit only PDF files.
- Use carefully chosen plots to support your analysis and discussion. Plots should be only as big as they need to be and not any bigger.
- A thoughtful exploration of the problem beyond what is asked for is encouraged.
- Report style format to be used for all assignments with the following suggested sections:
    - Objective
    - Methodology
    - Results and Discussion
    - Conclusion
    - References
- While it is preferred that you type your report, you may write by hand as long as you do it neatly and legibly. For typed-up reports, equations and other expressions that are tedious to type can be handwritten.
- Please include a copy of your code. Please include comments in your code so that the workflow is easily understandable.