Nicolino Primavera
FEM for Fluid Flow and FSI Interactions
Assignment 1
10/11/24

Problem 3

<u>Objective:</u>

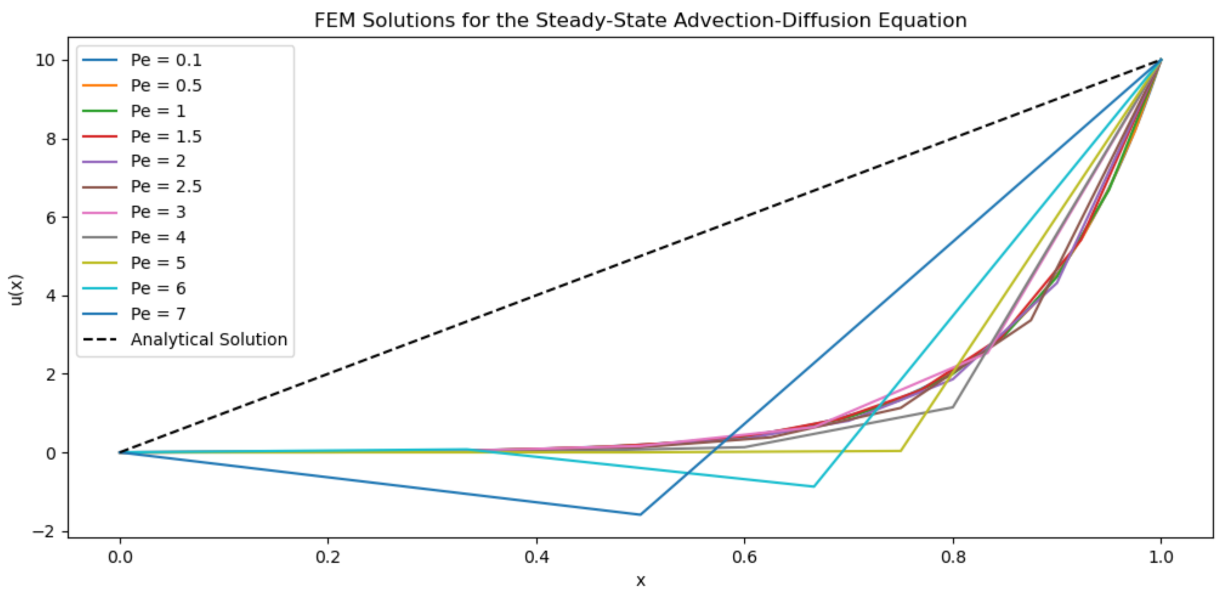
- Using the Finite Element Method (FEM) to analyze the Steady-State Advection-Diffusion Equation
- Steady-state Advection-Diffusion Equation: $\frac{du}{dx} = 0.05\frac{d^2u}{dx^2}\ ; 0 \leq x \leq 1,$ with boundary conditions u(0) = 0 & u(1) = 10
- Want to explore the effect of cell Peclet number on the computed solution when Galerkin FEM is applied to this problem
- Pe number represents the ratio of advection to diffusion
- Use numerical integration or Gauss quadrature-based integration (Gauss integration)
- Solve:
  - a. Choose at least four different grids for a range of cell Peclet numbers (Pe) and demonstrate that oscillatory solutions are obtained when Pe > 2
  - b. Compare the solutions for Pe < 2 against the analytical solution

<u>Methodology:</u>

- Utilized Gauss Quadrature points and weights for numerical integration
- Solved a 2-point Gauss Quadrature that was used to integrate the local stiffness matrix contributions within each element by evaluating the shape functions and their derivatives at the quadrature points
- Computed the local stiffness matrix for a finite element of size h and Peclet number Pe in a 1D advection-diffusion problem
  - Computed the diffusion and advection term for each matrix entry
  - Compiled the local stiffness matrix
- Calculated the global stiffness matrix
  - The Global Stiffness Matrix represents the entire system of equations for all nodes in the domain [0,1[- comprehensive assembly of contributions from each element's local stiffness matrix
  - Assembly: The local stiffness matrix for each element is computed and assembled into the global stiffness matrix by adding values to the corresponding rows and columns (nodes) of the global matrix
  - The Global Stiffness Matrix combines all the local element stiffness matrices, which creates a global system of equations that can be solved to approximate the solution of the advection-diffusion equation over the entire domain
  - Compute the Global Stiffness Matrix for the domain [0,1] with N nodes
  - Calculated by looping over the elements in the domain
- Calculated the grid size (h), number of nodes (N_no), and nodal positions (x) based on the selected cell peclet numbers
- Assembled the global stiffness matrices from the local element matrices and initialized a solution vector ‘u’ to store the unknowns at each node

- Applied the boundary conditions and modified the global stiffness matrix to ensure the proper boundary conditions were enforced.
- Solved the system and plotted against the analytical solutions to prove that for Pe < 2 there are stable solutions and for Pe > 2 there are unstable solutions

Results and Discussion:



The figure above displays my code's results. Peclet numbers below 2 have a smooth solution while peclet numbers above two have an oscillatory solution. This happens because convection and diffusion are not properly balanced in the numerical scheme (Galerkin Method). The oscillations in the solutions are due to numerical stability issues because convection and diffusion are not properly balanced in the numerical scheme. For convection dominated flows, the Galerkin method is unstable, but it is consistent and satisfies Galerkin orthogonality. The cell peclet number affects the accuracy of the Galerkin method. Large numbers (greater than two) are unstable and cause numerical oscillations.

Conclusion:

To conclude my findings, I can confidently say that oscillatory solutions are obtained when the peclet number is greater than two based on the plot my code generated. Once the peclet number was greater than two the solution transitioned from a smooth parabolic shape to an oscillatory triangular solution.

References:

Hughes, T. J. R. (2000). *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*. Dover Publications.

Code:

```
#!/usr/bin/env python3

"""
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Using the Finite Element Method (FEM) to analyze the Steady-State Advection-Diffusion
Equation
Steady-state Advection-Diffusion Equation: (du/dx) = (0.05)*(d^2u/dx^2) - strong form,
Domain: 0 <= x <= 1 , u(0) = 0 & u(1) = 10

Want to explore the effect of cell Peclet number on the computed solution when Galerkin FEM is
applied to this problem
Pe number represents the ratio of advection to diffusion
Use numerical inegration or Gauss quadrature-based integration (Gauss integration)

Solve:
a. Choose at least four different grids for a range of cell Peclet numbers (Pe) and demonstrate
that oscillatory solutions are obtained when Pe > 2
b. Compare the solutions for Pe < 2 against the analytical solution
"""

print("\nStarting Program...\n")

import math
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt

# Parameters
a = 1.0                                  # Advection speed
k = 0.05                                 # Diffusion coefficient
#Pe = (a*h)/k = 20*h                              # Cell peclet number
Pe_values = [0.1, 0.5, 1, 1.5, 2, 2.5, 3, 4, 5, 6, 7]  # Peclet numbers
#h = (Pe*k)/a                                 # Grid/cell spacing - use uniform spacing

# Domain
x_0 = 0
x_1 = 1

# Boundary Conditions
u_0 = 0
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u_1 = 10

# Gauss Quadrature / Numerical Integration
def gauss_quadrature(n_points):
    """
    Returns Gauss quadrature points and weights for numerical integration
    Gauss Quadrature Formula: Integral from -1 to +1 of (f(ξ)*dξ) = Summation from g=1 to # of int points (n) of f(ξ_g)*w_g
        f(ξ) - function being integrated
        ξ_g - quadrature points
        w_g - weights associated with each quadrature point , determine how much each function evaluation contributes to total integral
        n - number of quadrature points

    2 point Gauss Quadrature:
        Quadrature points: ξ_1 = -1/sqrt(3) , ξ_2 = 1/sqrt(3) - these points are extremely accurate for linear and quadratic basis functions
        Weights: w_1 = w_2 = 1
        Use to integrate polynomials up to 3rd degree

    Use quadrature points to integrate local stiffness matrix contributions within each element by evaluating the shape functions & their derivatives at the quadrature points
    Increase the number of quadrature points to improve accuracy of if integrating a more complex function
    """
    # Gauss Quadrature Rule for Numerical Integration
    if n_points == 2:                                           # 2-point Gauss Quadrature
        return np.array([-1/np.sqrt(3), 1/np.sqrt(3)]), np.array([1, 1])  # Returns 2-point quadrature points(2), weights
    else:
        raise ValueError("Only 2-point quadrature implemented.")         # Only allows for 2-point Gauss Quadrature

# Stiffness Matrix
def local_stiffness_matrix(h, Pe):
    """
    Compute the local stiffness matrix for a finite element of size h and Peclet number Pe in a 1D advection-diffusion problem

    Compute the Diffusion Term for each matrix entry:
        1/h - derivatives of the shape functions wrt physical coordinates
        diffusion term is proportional to the inverse of the element size
        when i == j the value is 1 (diagonal)
        when 1 /= j the value is -1 (off-diagonal)

    Compute the Advection Term for each matrix entry:
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        Pe - peclet number scales the contribution of advection relative to diffusion
        w[i] - weight associated w/ Gauss quadrature point i
        (0.5 * (1 - xi[i]) - shape function N_1(ξ) evaluated at quadrature point i
        (0.5 * (1 + xi[i]) - shape function N_2(ξ) evaluated at quadrature point i
    """
    K_local = np.zeros((2, 2))                                  # Initialize Local Stiffness Matrix [2 x 2]
    print(f"\nInitial Local Stiffness Matrix (should be 0): \n{K_local}\n") # Error handling

    xi, w = gauss_quadrature(2)                                  # Gauss quadrature points (xi) and
weights (w)
    print(f"Gauss Quadrature: {xi, w}")                          # Error handling
    print(f"Quadrature points: {xi}")                            # Error handling
    print(f"Weights: {w}\n")                                     # Error handling

    # Nested loop to fill the Local Stiffness Matrix [2 x 2]
    for i in range(2):
        for j in range(2):
            # Diffusion term (du/dx) (constant over the element for linear basis functions)
            diffusion = (1/h) * (1 if i == j else -1)            # Computes the Diffusion Term
            print(f"Diffusion Term: {diffusion}")                # Error handling

            # Advection term (Pe * u * v)
            advection = Pe * w[i] * (0.5 * (1 - xi[i]) if i == 0 else 0.5 * (1 + xi[i]))    # Computes the
Advection Term
            print(f"Advection Term: {advection}")                # Error handling

            # Local Stiffness Matrix
            K_local[i, j] += diffusion + advection               # Computes the Local Stiffness
Matrix
            print (f"Local Stiffness Matrix: \n{K_local}\n")     # Error handling

    print (f"Local Stiffness Matrix: {K_local}\n")               # Error handling
    return K_local

# Global Stiffness Matrix
def assemble_global_matrices(N, h, Pe):
    """
    The Global Stiffness Matrix represents the entire system of equations for all nodes in the
domain [0,1[ - comprehensive assembly of contributions from each element's local stiffness
matrix
    Assembly: The local stiffness matrix for each element is computed and assembled into the
global stiffness matrix by adding values to the corresponding rows and columns (nodes) of the
global matrix
    The Global Stiffness Matrix combines all the local element stiffness matrices, which creates a
global system of equations that can be solved to approx the soln of the advection-diffusion
equation over the entire domain
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```
    Compute the Global Stiffness Matrix for the domain [0,1] with N nodes

    N - number of nodes in the finite element mesh
    h - size of each element (distance b/w adjacent nodes), uniform for a structured grid
    Pe - peclet number
    """
    K_global = np.zeros((N, N))                              # Initialize an empty array of size [N x N]
(nodes)
    print(f"\nGlobal stiffness matrix (empty): \n{K_global}\n")      # Error handling - empty array
of size [N x N]

    # Loops over elements in the domain
    for e in range(N-1):                                   # Elements are iterated N-1 b/c there is one
element b/w every two nodes, element number (e) goes from 0 to N-2
        # Local stiffness matrix
        K_local = local_stiffness_matrix(h, Pe)                   # Assembling the local stiffness matrix
[2 x 2] for each element - represents the interactions b/w the two nodes that define the element
        print(f"Local Stiffness Matrix for {e}: \n{K_local}\n")      # Error handling

        # Global stiffness matrix - assembling the local stiffness matrices into the corresponding
part of the global stiffness matrix
        K_global[e:e+2, e:e+2] += K_local                         # Indexing: for element e, the nodes
involved are node e and node e+1 --> K_global[e:e+2, e:e+2] refers to the 2 x 2 submatrix of
K_global that corresponds to these two nodes
        print(f"Global Stiffness Matrix: \n{K_global}\n")           # Error handling

    print(f"Global Stiffness Matrix: \n{K_global}\n")              # Error handling
    return K_global                                        # Return the Global stiffness matrix

# Cell Peclet Number
for Pe in Pe_values:                                # Iterate over selected Peclet numbers from above
array
    # Compute grid size based on Pe and define nodes
    h = (Pe * k) / a                                  # Calculates grid size based on Pe
    N_no = int((x_1 - x_0) / h) + 1                     # Computes number of nodes based on total
domain length (x_1 - x_0)=1
    x = np.linspace(x_0, x_1, N_no)                      # Generates nodes positions , N_no evenly
spaced points b/w the domain boundaries --> positions of the nodes in the 1D domain
    print(f"Grid size: {h}\n")                       # Error handling
    print(f"Number of nodes: {N_no}\n")                 # Error handling
    print(f"Node positions:\n {x}\n")                 # Error handling

    # Initialize global matrices
    K_global = assemble_global_matrices(N_no, h, Pe)    # Assembles the Global Stiffness Matrix
from the local element matrices
```

```
    u = np.zeros(N_no)                          # Initialize solution vector 'u' - stores the unknowns at
each node
    print(f"Global Stiffness Matrix: \n{K_global}\n")   # Error handling
    print(f"Solution vector 'u': \n{u}\n")

    # Apply boundary conditions
    u[0] = u_0                                 # Sets first value of u at x=0 to the BC 0
    u[-1] = u_1                                # Sets final value of u at x=1 to th BC 10

    # Modify Global Stiffness Matrix to enforce BCs
    K_global[0, :] = 0                         # Clears the first row
    K_global[0, 0] = 1                         # Sets the first diagonal element to 1
    K_global[-1, :] = 0                        # Clears the last row
    K_global[-1, -1] = 1                       # Sets the last diagonal element to 1

    # Solve the system
    u = np.linalg.solve(K_global, u)           # Solves the system of linear eqns K_global(u)=g
    print(f"Solution vector 'u': \n{u}\n")     # Error handling

    # Plot
    plt.plot(x, u, label=f'Pe = {Pe}')

# Alalytical Solution for comparison - satisfies BCs u(0)=0 and u(1)=10 --> u(x)=10x
x_analytical = np.linspace(x_0, x_1, 100)                # X values of analytical soln
u_analytical = 10 * x_analytical                         # Linear solution from u(0) = 0 to u(1)
= 10 (u(x) values)
plt.plot(x_analytical, u_analytical, 'k--', label='Analytical Solution')  # Plot the analytical solution

# Plot
plt.xlabel('x')
plt.ylabel('u(x)')
plt.title('FEM Solutions for the Steady-State Advection-Diffusion Equation')
plt.legend()
plt.show()
```