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Recursive computation of the Hawkes cumulants

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ABSTRACT

We propose a recursive method for the computation of the cumulants of self-exciting point processes of Hawkes type, based on standard combinatorial tools such as Bell polynomials. This closed-form approach is easier to implement on higher-order cumulants in comparison with existing methods based on differential equations, tree enumeration or martingale arguments. The results are corroborated by Monte Carlo simulations, and also apply to the computation of joint cumulants generated by multidimensional self-exciting processes.

1. Introduction

Hawkes processes were introduced in Hawkes (1971) as self-exciting point processes representing an alternative to doubly stochastic point processes. In recent years they have found applications in many fields, from neuroscience, see e.g. Cardanobile and Rotter (2010), to genomics analysis, see e.g. Reynaud-Bouret and Schbath (2010), as well as finance (Embrechts et al., 2011) and social media (Rizoiu et al., 2018). As noted in Jovanović et al. (2015), the analysis of statistical properties of Hawkes processes is still incomplete, in particular in terms of moments, cumulants and other statistical parameters such as skewness and kurtosis.

In Dassios and Zhao (2011) the moment and probability generating functions of (generalized) Hawkes processes and their intensity have been obtained by ODE methods, with the computation of first and second moments in the stationary case, see also Errais et al. (2010). In Bacry et al. (2012), a stochastic calculus and martingale approach has been applied to the computation of first and second moments, however it seems difficult to generalize to higher orders, see also Cui et al. (2020) and Daw and Pender (2020) for other methods based on differential equations. In Jovanović et al. (2015), a tree-based method for the computation of cumulants has been introduced, with an explicit computation of third order cumulants. However, this type of algorithm requires to perform tree enumerations, which can be computationally expensive.

Third-order cumulant expressions for Hawkes processes have been used in Achab et al. (2018) for the estimation of branching ratio matrices in the analysis of order books, and in Ocker et al. (2017) and Montangie et al. (2020) for the estimation of third order correlations in spiking neuronal networks. Higher order cumulants can also be useful in order to provide finer estimates of the evolution of time correlations and of the probability density functions of neuronal membrane potentials by Gram–Charlier density expansions, see e.g. Brigham and Destexhe (2015) and Privault (2020).

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In this paper, we derive a general recursion formula using the standard Bell polynomials for the computation of the cumulants of a self-exciting point process on $\mathbb{R}^d$, $d \geq 1$, with immigrant intensity $\nu(dx)$ and branching intensity $\gamma(dx)$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Our approach is based on a recursive relation for the Probability Generating Functional (PGFl) G_z of a self-exciting point process from a single point at $z \in \mathbb{R}^d$, derived in Proposition 3.1. Such an implicit relation has already been observed in Adamopoulos (1975), and applied in e.g. Bordenave and Torrisi (2007) to large deviations, however it does not seem to have been exploited for the computation of cumulants.

In Section 2 we start by reviewing the combinatorial approach of § 3.2 of Consul and Famoye (2006) to the computation of the cumulants of the integer-valued Borel distribution, and show that it can be extended as an explicit recursion using Bell polynomials. This provides an elementary model for subsequent computations, as the Borel distribution can be used to represent the cardinality of a self-exciting Poisson cluster point process.

Next, in Section 3 we extend this argument to the computation of the cumulants of self-exciting Hawkes Poisson cluster processes in Proposition 3.1, with an extension to the computation of joint cumulants. This provides a closed-form alternative, suitable for systematic higher-order computations, to the tree-based approach of Jovanović et al. (2015). Explicit computations for the time-dependent third and fourth cumulants and skewness and kurtosis of Hawkes processes with exponential kernels are presented in Section 4, and are confirmed by Monte Carlo estimates.

Cumulants, Faà di Bruno formula and Bell polynomials

We close this section with background results on combinatorics that will be needed in the sequel. Recall that if $f(t)$ admits the formal series expansion $f(t) = \sum_{n=1}^{\infty} \frac{a_n}{n!} t^n$, by the Faà di Bruno formula we have

$$e^{f(t)} - 1 = \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{k=1}^{n} \frac{n!}{k!} \sum_{\substack{l_1+\cdots+l_k=n \\ l_1,\ldots,l_k \geq 1}} \frac{a_{l_1}\cdots a_{l_k}}{l_1!\cdots l_k!} = \sum_{n=1}^{\infty} \frac{t^n}{n!} B_n(a_1,\ldots,a_n) \tag{1.1}$$

where the sum (1.1) holds on the integer compositions $(l_1, \ldots, l_k)$ of n, see e.g. Relation (2.5) in (Lukacs, 1955),

$$B_n(a_1,\ldots,a_n) = \sum_{k=1}^{n} B_{n,k}(a_1,\ldots,a_{n-k+1}) = \sum_{k=1}^{n} \frac{n!}{k!} \sum_{\substack{l_1+\cdots+l_k=n \\ l_1\geq 1,\ldots,l_k \geq 1}} \frac{a_{l_1}}{l_1!}\cdots\frac{a_{l_k}}{l_k!}$$

is the complete Bell polynomial of degree $n \geq 1$, and

$$B_{n,k}(a_1,\ldots,a_{n-k+1}) = \frac{n!}{k!} \sum_{\substack{l_1+\cdots+l_k=n \\ l_1\geq 1,\ldots,l_k \geq 1}} \frac{a_{l_1}}{l_1!}\cdots\frac{a_{l_k}}{l_k!}, \qquad 1 \leq k \leq n,$$

is the partial Bell polynomial of order (n, k). Given the Moment Generating Function (MGF)

$$M_X(t) := \mathbb{E}\big[e^{tX}\big] = 1 + \sum_{n\geq 1} \frac{t^n}{n!}\mathbb{E}[X^n]$$

of a random variable X, the cumulants of X are the coefficients $\big(\kappa^{(n)}\big)_{n\geq 1}$ appearing in the log-MGF series expansion

$$\log M_X(t) = \log\big(\mathbb{E}\big[e^{tX}\big]\big) = \sum_{n\geq 1} \kappa^{(n)} \frac{t^n}{n!}, \tag{1.2}$$

for t in a neighborhood of zero. The moments $\mathbb{E}[X^n]$ of a random variable X are linked to its cumulants $\big(\kappa^{(n)}\big)_{n\geq 1}$ through the relation

$$\mathbb{E}[X^n] = \sum_{k=1}^{n} \sum_{\pi_1\cup\cdots\cup\pi_k=\{1,\ldots,n\}} \kappa^{(|\pi_1|)}\cdots\kappa^{(|\pi_k|)},$$

which runs over the partitions $\pi_1, \ldots, \pi_k$ of the set $\{1, \ldots, n\}$, where $|\pi_i|$ denotes the cardinality of π_i. By the Faà di Bruno formula, (1.2) can be inverted as

$$\kappa^{(n)} = \sum_{k=1}^{n} (k-1)!(-1)^{k-1} \sum_{\pi_1\cup\cdots\cup\pi_k=\{1,\ldots,n\}} \mathbb{E}\big[X^{|\pi_1|}\big]\cdots\mathbb{E}\big[X^{|\pi_k|}\big], \quad n \geq 1,$$

see e.g. Theorem 1 of Lukacs (1955), and also Leonov and Shiryaev (1959), Relations (2.8)-(2.9) in McCullagh (1987), or Corollary 5.1.6 in Stanley (1999). The third and fourth cumulants can be used to define the skewness $\kappa^{(3)}/\big(\kappa^{(2)}\big)^{3/2}$ and the excess kurtosis $\kappa^{(4)}/\big(\kappa^{(2)}\big)^2$ of X.

2. Borel cumulants

In this section we consider the recursive computation of the cumulants of integer-valued Borel-distributed random variables using the Faà di Bruno formula. For this, we review the method of § 3.2 of Consul and Famoye (2006) which applies to Lagrangian distributions, and note that it admits an explicit formulation using Bell polynomials. Let $(X_n)_{n\geq 0}$ be a branching process started at $X_0 = 1$ with Poisson distributed offspring count N of parameter $\mu \in (0, 1)$. Denoting by X the total count of offsprings generated by $(X_n)_{n\geq 0}$ and letting $\big(X^{(l)}\big)_{l\geq 1}$ denote a sequence of independent copies of X, the Probability Generating Function (PGF) of X can be estimated by the standard branching recursion

$$G_X(s) = \mathbb{E}[s^X] = s\mathbb{E}\left[\prod_{l=1}^{N} s^{X^{(l)}}\right] = s\sum_{k\geq 0}\mathbb{E}\left[\prod_{l=1}^{k} s^{X^{(l)}}\right]\mathbb{P}(N=k) = s\mathrm{e}^{-\mu}\sum_{n\geq 0}\left(\mathbb{E}\left[s^{X^{(1)}}\right]\right)^n\frac{\mu^n}{n!} = sG_\mu(G_X(s)),$$

$-1 \leq s \leq 1$, where $G_\mu(s) := \mathrm{e}^{-\mu}\sum_{n=0}^{\infty}\frac{\mu^n}{n!}s^n = \mathrm{e}^{\mu(s-1)}$, $s \in \mathbb{R}$, is the PGF of the Poisson distribution with mean $\mu > 0$. The equation

$$G_X(s) = sG_\mu(G_X(s)), \qquad -1 \leq s \leq 1, \tag{2.1}$$

can be solved using Lagrange series, see page 145 of Pólya and Szegö (1998), showing that X has the Borel distribution $\mathbb{P}(X = n) = \mathrm{e}^{-\mu n}(\mu n)^{n-1}/n!$, $n \geq 1$, which belongs to the class of Lagrangian distributions, see § 8.4 of Consul and Famoye (2006). The following proposition then extends the relations (3.12) in Consul and Famoye (2006) for the computation of the cumulants of the Borel distribution, via a general expression based on the Bell polynomials. Another, less direct, recursion can be found in § 8.4.3 in Consul and Famoye (2006), based on the derivatives of moments of X with respect to μ.

Proposition 2.1. *Let X be a Borel distributed random variable with parameter $\mu \in (0, 1)$. We have $\kappa^{(1)} = 1/(1-\mu)$ and the induction relation*

$$\kappa^{(n)} = \frac{\mu}{1-\mu}\left(B_n\left(\kappa^{(1)},\ldots,\kappa^{(n)}\right) - \kappa^{(n)}\right) = \frac{\mu}{1-\mu}\sum_{k=2}^{n} B_{n,k}\left(\kappa^{(1)},\ldots,\kappa^{(n-k+1)}\right), \quad n \geq 2,$$

where B_n, resp. $B_{n,k}$, is the complete, resp. partial, Bell polynomial.

Proof. From (2.1), the moment generating function $M_X(t) = \mathbb{E}[\mathrm{e}^{tX}] = G(\mathrm{e}^t)$ satisfies

$$\log M_X(t) = t + \mu(M_X(t) - 1) = t + \mu\left(\mathrm{e}^{\log M_X(t)} - 1\right)$$

for t in a neighborhood of zero, see also Relation (19) in Haight and Breuer (1960) with $r = 1$. Based on the cumulant expansion (1.2) and the Faà di Bruno formula (1.1), we have

$$\sum_{n\geq 1}\kappa^{(n)}\frac{t^n}{n!} = \log M_X(t) = t + \mu\left(\mathrm{e}^{\log M_X(t)} - 1\right) = \left(1 + \mu\kappa^{(1)}\right)t + \mu\sum_{n=2}^{\infty}\frac{t^n}{n!}B_n\left(\kappa^{(1)},\ldots,\kappa^{(n)}\right),$$

which shows that $\kappa^{(1)} = 1 + \mu\kappa^{(1)}$ and

$$\kappa^{(n)} = \mu B_n\left(\kappa^{(1)},\ldots,\kappa^{(n)}\right) = \mu\kappa^{(n)} + \mu\sum_{k=2}^{n} B_{n,k}\left(\kappa^{(1)},\ldots,\kappa^{(n-k+1)}\right), \qquad n \geq 2,$$

since $B_{n,1}\left(\kappa^{(1)},\ldots,\kappa^{(n)}\right) = \kappa^{(n)}$. $\square$

In particular, since $B_2(x_1, x_2) = x_1^2 + x_2$ we have

$$\kappa^{(2)} = \frac{\mu}{1-\mu}\left(B_2\left(\frac{1}{1-\mu}, \kappa^{(2)}\right) - \kappa^{(2)}\right) = \frac{\mu}{(1-\mu)^3}.$$

Given that $B_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3$, we have

$$\kappa^{(3)} = \frac{\mu}{1-\mu}\left(B_3\left(\frac{1}{1-\mu}, \frac{\mu}{(1-\mu)^3}, \kappa^{(3)}\right) - \kappa^{(3)}\right) = \mu\frac{1+2\mu}{(1-\mu)^5}.$$

Since $B_4(x_1, x_2, x_3, x_4) = x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4$, we find

$$\kappa^{(4)} = \frac{\mu}{1-\mu}\left(B_4\left(\frac{1}{1-\mu}, \frac{\mu}{(1-\mu)^3}, \mu\frac{1+2\mu}{(1-\mu)^5}, \kappa^{(4)}\right) - \kappa^{(4)}\right) = \mu\frac{1+8\mu+6\mu^2}{(1-\mu)^7},$$

which recovers (8.85) page 159 of Consul and Famoye (2006).

3. Hawkes cumulants

In this section we work in the cluster process framework of Hawkes and Oakes (1974). We consider a self-exciting point process on $\mathbb{R}^d$, $d \geq 1$, with Poisson offspring intensity $\gamma(dx)$ and Poisson immigrant intensity $\nu(dx)$ on $\mathbb{R}^d$, built on the space

$$\Omega = \left\{\xi = \{x_i\}_{i\in I} \subset \mathbb{R}^d \ : \ \#(A \cap \xi) < \infty \text{ for all compact } A \in \mathcal{B}(\mathbb{R}^d)\right\}$$

of locally finite configurations on $\mathbb{R}^d$, whose elements $\xi \in \Omega$ are identified with the Radon point measures $\xi(dz) = \sum_{x\in\xi} \epsilon_x(dz)$, where ϵ_x denotes the Dirac measure at $x \in \mathbb{R}^d$. In particular, any initial immigrant point $y \in \mathbb{R}^d$ branches into a Poisson random sample denoted by $\xi_\gamma(y+dz) = \sum_{x\in\xi} \epsilon_{x+y}(dz)$ and centered at y, with intensity measure $\gamma(y+dz)$ on $\mathbb{R}^d$. We let $G_z(f) = f(z)\mathbb{E}\left[\prod_{x\in\xi} f(z+x)\right]$ denote the Probability Generating Functional (PGFl) of the branching process starting from a single point at $z \in \mathbb{R}^d$, for sufficiently integrable $f : \mathbb{R}^d \to \mathbb{R}$. The next proposition states a recursive property for the Probability Generating Functional $G_z(f)$, see also Theorem 1 in Adamopoulos (1975).

Proposition 3.1. *The Probability Generating Functional $G_z(f)$ satisfies*

$$G_z(f) = f(z)\exp\left(\int_{\mathbb{R}^d} (G_{z+x}(f) - 1)\gamma(dx)\right), \qquad z \in \mathbb{R}^d,$$

and the PGFl of the Hawkes process with immigrant intensity $\nu(dz)$ is given by

$$G_\nu(f) = \exp\left(\int_{\mathbb{R}^d} (G_z(f) - 1)\nu(dz)\right).$$

Proof. Viewing the self-exciting point process ξ as a marked point process we have, see e.g. Lemma 6.4.VI of Daley and Vere-Jones (2003),

$$\begin{aligned}
G_z(f) &= f(z)\mathbb{E}\left[\prod_{x\in\xi} f(z+x)\right] = f(z)\mathbb{E}\left[\prod_{x\in\xi_\gamma}\left(\prod_{y\in\xi} f(z+x+y)\right)\right] = f(z)\mathbb{E}\left[\prod_{x\in\xi_\gamma}\mathbb{E}\left[\prod_{y\in\xi} f(z+x+y)\right]\right] \\
&= f(z)\mathbb{E}\left[\prod_{x\in\xi_\gamma} G_{z+x}(f)\right] = \mathrm{e}^{-\gamma(\mathbb{R}^d)} f(z)\sum_{k=0}^\infty \frac{1}{k!}\int_{(\mathbb{R}^d)^k} G_{z+x_1}(f)\cdots G_{z+x_k}(f)\gamma(dx_1)\cdots\gamma(dx_k) \\
&= f(z)\exp\left(\int_{\mathbb{R}^d} (G_{z+x}(f) - 1)\gamma(dx)\right),
\end{aligned}$$

and

$$G_\nu(f) = \mathrm{e}^{-\nu(\mathbb{R}^d)}\sum_{n=0}^\infty \frac{1}{n!}\int_{(\mathbb{R}^d)^n} G_{z_1}(f)\cdots G_{z_n}(f)\nu(dz_1)\cdots\nu(dz_n) = \exp\left(\int_{\mathbb{R}^d} (G_z(f) - 1)\nu(dz)\right). \quad \square$$

Let $M_z(f) = G_z\left(e^f\right) = \mathbb{E}\left[\exp\left(f(z) + \sum_{x\in\xi} f(z+x)\right)\right]$ denote the Moment Generating Functional (MGFl) of the stochastic integral $\sum_{x\in\xi} f(x)$ given that the cluster process ξ starts from a single point at $z \in \mathbb{R}^d$. The following corollary is an immediate consequence of Proposition 3.1, see also Proposition 2.6 in Bogachev and Daletskii (2009) for Poisson cluster processes.

Corollary 3.2. *The Moment Generating Functional $M_z(f)$ satisfies the recursive relation*

$$M_z(f) = \exp\left(f(z) + \int_{\mathbb{R}^d} (M_{z+x}(f) - 1)\gamma(dx)\right), \qquad z \in \mathbb{R}^d. \tag{3.1}$$

The MGFl of the Hawkes process with immigrant intensity $\nu(dz)$ is given by

$$M_\nu(f) = \exp\left(\int_{\mathbb{R}^d} (M_z(f) - 1)\nu(dz)\right). \tag{3.2}$$

The next proposition provides a way to compute the cumulants $\kappa_z^{(n)}(f)$ of $\sum_{x\in\xi} f(x)$ by an induction relation based on the Bell polynomials. Note that the sum of coefficients in $B_n(x_1, \ldots, x_n)$ is the Bell number $B_n = \sum_{k=1}^n \frac{1}{k!}\sum_{\substack{l_1+\cdots+l_k=n \\ l_1,\ldots,l_k\geq 1}} \frac{n!}{l_1!\cdots l_k!}$ that represents the count of partitions of a set of n elements. In the sequel, we assume that $\gamma(\mathbb{R}^d) < 1$ and consider the integral operator Γ defined as $\Gamma f(z) = \int_{\mathbb{R}^d} f(z+y)\gamma(dy)$, $z \in \mathbb{R}^d$, and the inverse operator $(I_d - \Gamma)^{-1}$ given by

$$(I_d - \Gamma)^{-1} f(z) = f(z) + \sum_{m=1}^\infty \int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} f(z+x_1+\cdots+x_m)\gamma(dx_1)\cdots\gamma(dx_m), \quad z \in \mathbb{R}^d.$$

Proposition 3.3. *The first cumulant $\kappa_z^{(1)}(f)$ of $\sum_{x\in\xi} f(x)$ given that ξ is started from a single point at $z \in \mathbb{R}^d$ is given by $\kappa_z^{(1)}(f) = (I_d - \Gamma)^{-1} f(z)$ for $n = 1$, and $\kappa_z^{(n)}(f)$ is given for $n \geq 2$ by the induction relation*

$$\kappa_z^{(n)}(f) = (I_d - \Gamma)^{-1}\Gamma\Big(B_n\big(\kappa_{z+\cdot}^{(1)}, \ldots, \kappa_{z+\cdot}^{(n)}\big) - \kappa_{z+\cdot}^{(n)}\Big) = \sum_{k=2}^{n} (I_d - \Gamma)^{-1}\Gamma B_{n,k}\big(\kappa_{z+\cdot}^{(1)}, \ldots, \kappa_{z+\cdot}^{(n-k+1)}\big).$$

Proof. By (1.2), (3.1) and the Faà di Bruno formula (1.1), we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{t^n}{n!}\kappa_z^{(n)}(f) &= \log M_z(tf) = tf(z) + \int_{\mathbb{R}^d} \big(e^{\log M_{z+x}(tf)} - 1\big)\gamma(dx) \\ &= tf(z) + t\int_{\mathbb{R}^d} \kappa_{z+x}^{(1)}\gamma(dx) + \sum_{n=2}^{\infty}\frac{t^n}{n!}\int_{\mathbb{R}^d} B_n\big(\kappa_{z+x}^{(1)}, \ldots, \kappa_{z+x}^{(n)}\big)\gamma(dx), \qquad (3.3)\end{aligned}$$

hence

$$\kappa_z^{(1)}(f) = f(z) + \int_{\mathbb{R}^d} \kappa_{z+x}^{(1)}(f)\gamma(dx) = f(z) + \sum_{m=1}^{\infty}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} f(z + x_1 + \cdots + x_m)\gamma(dx_1)\cdots\gamma(dx_m),$$

as solution of the renewal equation

$$\kappa_z^{(1)}(f) = h(z) + \int_{\mathbb{R}^d} \kappa_{z+x}^{(1)}(f)\gamma(dx), \quad z \in \mathbb{R}^d.$$

For $n \geq 2$, (3.3) yields

$$\kappa_z^{(n)}(f) = \int_{\mathbb{R}^d} B_n\big(\kappa_{z+x}^{(1)}, \ldots, \kappa_{z+x}^{(n)}\big)\gamma(dx) = \Gamma\kappa_{z+\cdot}^{(n)}(f) + \Gamma\Big(B_n\big(\kappa_{z+\cdot}^{(1)}, \ldots, \kappa_{z+\cdot}^{(n)}\big) - \kappa_{z+\cdot}^{(n)}\Big),$$

yet $(I_d - \Gamma)\kappa_z^{(n)}(f) = \Gamma\Big(B_n\big(\kappa_{z+\cdot}^{(1)}, \ldots, \kappa_{z+\cdot}^{(n)}\big) - \kappa_{z+\cdot}^{(n)}\Big)$, which yields

$$\begin{aligned}&\kappa_z^{(n)}(f) = (I_d - \Gamma)^{-1}\Gamma\Big(B_n\big(\kappa_{z+\cdot}^{(1)}, \ldots, \kappa_{z+\cdot}^{(n)}\big) - \kappa_{z+\cdot}^{(n)}\Big) \\ &= \sum_{m=1}^{\infty}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\Big(B_n\big(\kappa_{z+x_1+\cdots+x_m}^{(1)}, \ldots, \kappa_{z+x_1+\cdots+x_m}^{(n)}\big) - \kappa_{z+x_1+\cdots+x_m}^{(n)}\Big)\gamma(dx_1)\cdots\gamma(dx_m) \\ &= \sum_{m=1}^{\infty}\sum_{k=2}^{n}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} B_{n,k}\big(\kappa_{z+x_1+\cdots+x_m}^{(1)}, \ldots, \kappa_{z+x_1+\cdots+x_m}^{(n-k+1)}\big)\gamma(dx_1)\cdots\gamma(dx_m), \qquad n \geq 2. \quad \square\end{aligned}$$

Unconditional cumulants can be obtained in the next corollary as a consequence of Proposition 3.3.

Corollary 3.4. *The cumulant of order $n \geq 1$ of $\sum_{x\in\xi} f(x)$ is given by*

$$\kappa^{(n)}(f) = \int_{\mathbb{R}^d} B_n\big(\kappa_z^{(1)}(f), \ldots, \kappa_z^{(n)}(f)\big)\nu(dz),$$

and the recursion

$$\begin{aligned}B_n\big(\kappa_z^{(1)}(f), \ldots, \kappa_z^{(n)}(f)\big) &= (I_d - \Gamma)^{-1}\Big(B_n\big(\kappa_{z+\cdot}^{(1)}, \ldots, \kappa_{z+\cdot}^{(n)}\big) - \kappa_{z+\cdot}^{(n)}\Big) \\ &= \sum_{k=2}^{n}(I_d - \Gamma)^{-1}B_{n,k}\big(\kappa_{z+\cdot}^{(1)}, \ldots, \kappa_{z+\cdot}^{(n-k+1)}\big), \quad z \in \mathbb{R}^d.\end{aligned}$$

Proof. By (1.2), (3.2) and the Faà di Bruno formula (1.1), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{t^n}{n!}\kappa^{(n)}(f) &= \log M_\nu(f) = \int_{\mathbb{R}^d}(M_z(f) - 1)\nu(dz) = \int_{\mathbb{R}^d}(e^{\log M_z(f)} - 1)\nu(dz) \\ &= \sum_{n=1}^{\infty}\frac{t^n}{n!}B_n\big(\kappa_z^{(1)}(f), \ldots, \kappa_z^{(n)}(f)\big)\nu(dz),\end{aligned}$$

and therefore

$$\kappa^{(n)}(f) = \int_{\mathbb{R}^d} B_n\big(\kappa_z^{(1)}(f), \ldots, \kappa_z^{(n)}(f)\big)\nu(dz), \qquad n \geq 2.$$

We conclude from the equalities

$$
\begin{aligned}
& B_n\big(\kappa_z^{(1)}(f),\ldots,\kappa_z^{(n)}(f)\big) = \kappa_z^{(n)}(f) + \big(B_n\big(\kappa_z^{(1)}(f),\ldots,\kappa_z^{(n)}(f)\big) - \kappa_z^{(n)}(f)\big) \\
= & \sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \big(B_n\big(\kappa^{(1)}_{z+x_1+\cdots+x_m},\ldots,\kappa^{(n)}_{z+x_1+\cdots+x_m}\big) - \kappa^{(n)}_{z+x_1+\cdots+x_m}\big)\gamma(dx_1)\cdots\gamma(dx_m) \\
& + \big(B_n\big(\kappa_z^{(1)}(f),\ldots,\kappa_z^{(n)}(f)\big) - \kappa_z^{(n)}(f)\big) \\
= & \sum_{m=0}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \big(B_n\big(\kappa^{(1)}_{z+x_1+\cdots+x_m},\ldots,\kappa^{(n)}_{z+x_1+\cdots+x_m}\big) - \kappa^{(n)}_{z+x_1+\cdots+x_m}\big)\gamma(dx_1)\cdots\gamma(dx_m),
\end{aligned}
$$

that follow from Proposition 3.3. □

Second cumulant. For $n = 2$, Proposition 3.3 shows that

$$
\kappa_z^{(2)}(f) = \sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \big(\kappa^{(1)}_{z+x_1+\cdots+x_m}(f)\big)^2 \gamma(dx_1)\cdots\gamma(dx_m),
$$

and by Corollary 3.4 we have

$$
\kappa^{(2)}(f) = \int_{\mathbb{R}^d} \kappa_z^{(2)}(f)\nu(dz) + \int_{\mathbb{R}^d} \big(\kappa_z^{(1)}(f)\big)^2 \nu(dz) = \sum_{m=0}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \big(\kappa^{(1)}_{z+x_1+\cdots+x_m}(f)\big)^2 \gamma(dx_1)\cdots\gamma(dx_m)\nu(dz),
$$

see e.g. Proposition 2 in Bacry et al. (2012) and Eq. (37) in Jovanović et al. (2015).

Third cumulant. For $n = 3$, we have

$$
\begin{aligned}
\kappa_z^{(3)}(f) = & \ 3\sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \kappa^{(1)}_{z+x_1+\cdots+x_m}(f)\kappa^{(2)}_{z+x_1+\cdots+x_m}(f)\gamma(dx_1)\cdots\gamma(dx_m) \\
& + \sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \big(\kappa^{(1)}_{z+x_1+\cdots+x_m}(f)\big)^3 \gamma(dx_1)\cdots\gamma(dx_m), \qquad (3.4)
\end{aligned}
$$

and

$$
\kappa^{(3)}(f) = \int_{\mathbb{R}^d} B_3\big(\kappa_z^{(1)},\kappa_z^{(2)},\kappa_z^{(3)}\big)\nu(dz) = \int_{\mathbb{R}^d} \big(\kappa_z^{(1)}\big)^3 \nu(dz) + 3\int_{\mathbb{R}^d} \kappa_z^{(1)}\kappa_z^{(2)}\nu(dz) + \int_{\mathbb{R}^d} \kappa_z^{(3)}\nu(dz) \qquad (3.5)
$$

$$
= 3\sum_{m=0}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \kappa^{(1)}_{z+x_1+\cdots+x_m}(f)\kappa^{(2)}_{z+x_1+\cdots+x_m}(f)\gamma(dx_1)\cdots\gamma(dx_m)\nu(dz) \qquad (3.6)
$$

$$
+ \sum_{m=0}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \big(\kappa^{(1)}_{z+x_1+\cdots+x_m}(f)\big)^3 \gamma(dx_1)\cdots\gamma(dx_m)\nu(dz), \qquad (3.7)
$$

which corresponds to Eq. (39) in Jovanović et al. (2015).

Fourth cumulant. For $n = 4$, we have

$$
\begin{aligned}
\kappa_z^{(4)}(f) = & \ 6\sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \big(\kappa^{(1)}_{z+x_1+\cdots+x_m}(f)\big)^2 \kappa^{(2)}_{z+x_1+\cdots+x_m}(f)\gamma(dx_1)\cdots\gamma(dx_m) \\
& + 4\sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \kappa^{(1)}_{z+x_1+\cdots+x_m}(f)\kappa^{(3)}_{z+x_1+\cdots+x_m}(f)\gamma(dx_1)\cdots\gamma(dx_m) \\
& + 3\sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \big(\kappa^{(2)}_{z+x_1+\cdots+x_m}(f)\big)^2 \gamma(dx_1)\cdots\gamma(dx_m) \\
& + \sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \big(\kappa^{(1)}_{z+x_1+\cdots+x_m}(f)\big)^4 \gamma(dx_1)\cdots\gamma(dx_m),
\end{aligned}
$$

and

$$
\kappa^{(4)}(f) = \int_{\mathbb{R}^d} B_4\big(\kappa_z^{(1)},\kappa_z^{(2)},\kappa_z^{(3)},\kappa_z^{(4)}\big)\nu(dz) \qquad (3.8)
$$

$$
= \int_{\mathbb{R}^d} \big(\kappa_z^{(1)}\big)^4 \nu(dz) + 6\int_{\mathbb{R}^d} \big(\kappa_z^{(1)}\big)^2 \kappa_z^{(2)}\nu(dz) + 4\int_{\mathbb{R}^d} \kappa_z^{(1)}\kappa_z^{(3)}\nu(dz) + 3\int_{\mathbb{R}^d} \big(\kappa_z^{(2)}\big)^2 \nu(dz) + \int_{\mathbb{R}^d} \kappa_z^{(4)}\nu(dz)
$$

$$= 6 \sum_{m=0}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \left(\kappa^{(1)}_{z+x_1+\cdots+x_m}(f)\right)^2 \kappa^{(2)}_{z+x_1+\cdots+x_m}(f) \gamma(dx_1) \cdots \gamma(dx_m) \tag{3.9}$$

$$+ 4 \sum_{m=0}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \kappa^{(1)}_{z+x_1+\cdots+x_m}(f) \kappa^{(3)}_{z+x_1+\cdots+x_m}(f) \gamma(dx_1) \cdots \gamma(dx_m)$$

$$+ 3 \sum_{m=0}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \left(\kappa^{(2)}_{z+x_1+\cdots+x_m}(f)\right)^2 \gamma(dx_1) \cdots \gamma(dx_m) + \sum_{m=0}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \left(\kappa^{(1)}_{z+x_1+\cdots+x_m}(f)\right)^4 \gamma(dx_1) \cdots \gamma(dx_m). \tag{3.10}$$

We note that the count of 4 terms in (3.6)–(3.7) and the total count of $6 + 4 \times 4 + 3 \times 1 + 1 \times 1 = 26$ terms in (3.9)–(3.10), due to 4 terms in $\kappa^{(3)}_{z+x_1+\cdots+x_m}(f)$, is matching the 26 terms obtained in Figure 4 of Jovanović et al. (2015) using tree enumeration.

Joint cumulants. The expression of Proposition 3.3 can be extended to joint cumulants by standard combinatorial arguments.

Proposition 3.5. *For $n \geq 2$, the joint cumulants $\kappa^{(n)}_z(f_1, \ldots, f_n)$ of $\sum_{x\in\xi} f_1(x), \ldots, \sum_{x\in\xi} f_n(x)$ given that ξ is started from a single point at $z \in \mathbb{R}^d$ are given by the induction relation*

$$\kappa^{(n)}_z(f_1, \ldots, f_n) = \sum_{k=2}^{n} \sum_{\pi_1\cup\cdots\cup\pi_k=\{1,\ldots,n\}} \sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{j=1}^{k} \kappa^{(|\pi_j|)}_{z+x_1+\cdots+x_m}((f_i)_{i\in\pi_j}) \gamma(dx_1) \cdots \gamma(dx_m),$$

$n \geq 2$, where the above sum is over set partitions $\pi_1 \cup \cdots \cup \pi_k = \{1, \ldots, n\}$, $k = 2, \ldots, n$.

As in Corollary 3.4, we obtain the expressions

$$\kappa^{(n)}(f_1, \ldots, f_n) = \sum_{k=1}^{n} \sum_{\pi_1\cup\cdots\cup\pi_k=\{1,\ldots,n\}} \int_{\mathbb{R}^d} \prod_{j=1}^{k} \kappa^{(|\pi_j|)}_z((f_i)_{i\in\pi_j}) \nu(dz)$$

$$= \sum_{k=2}^{n} \sum_{\pi_1\cup\cdots\cup\pi_k=\{1,\ldots,n\}} \sum_{m=0}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{j=1}^{k} \kappa^{(|\pi_j|)}_{z+x_1+\cdots+x_m}((f_i)_{i\in\pi_j}) \gamma(dx_1) \cdots \gamma(dx_m) \nu(dz),$$

as a consequence of Proposition 3.5.

Second joint cumulant. We have

$$\kappa^{(2)}_z(f_1, f_2) = \sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \kappa^{(1)}_{z+x_1+\cdots+x_m}(f_1) \kappa^{(1)}_{z+x_1+\cdots+x_m}(f_2) \gamma(dx_1) \cdots \gamma(dx_m),$$

and

$$\kappa^{(2)}(f_1, f_2) = \int_{\mathbb{R}^d} \kappa^{(2)}_z(f_1, f_2) \nu(dz) + \int_{\mathbb{R}^d} \kappa^{(1)}_z(f_1) \kappa^{(1)}_z(f_2) \nu(dz).$$

Third joint cumulant. For $n = 3$, we have

$$\kappa^{(3)}_z(f_1, f_2, f_3) = \sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \kappa^{(1)}_{z+x_1+\cdots+x_m}(f_1) \kappa^{(2)}_{z+x_1+\cdots+x_m}(f_2, f_3) \gamma(dx_1) \cdots \gamma(dx_m)$$

$$+ \sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \kappa^{(1)}_{z+x_1+\cdots+x_m}(f_2) \kappa^{(2)}_{z+x_1+\cdots+x_m}(f_1, f_3) \gamma(dx_1) \cdots \gamma(dx_m)$$

$$+ \sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \kappa^{(1)}_{z+x_1+\cdots+x_m}(f_3) \kappa^{(2)}_{z+x_1+\cdots+x_m}(f_1, f_2) \gamma(dx_1) \cdots \gamma(dx_m)$$

$$+ \sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \kappa^{(1)}_{z+x_1+\cdots+x_m}(f_1) \kappa^{(1)}_{z+x_1+\cdots+x_m}(f_2) \kappa^{(1)}_{z+x_1+\cdots+x_m}(f_3) \gamma(dx_1) \cdots \gamma(dx_m),$$

and

$$\kappa^{(3)}(f_1, f_2, f_3) = \int_{\mathbb{R}^d} \kappa^{(1)}_z(f_1) \kappa^{(1)}_z(f_2) \kappa^{(1)}_z(f_3) \nu(dz) + \int_{\mathbb{R}^d} \kappa^{(1)}_z(f_1) \kappa^{(2)}_z(f_2, f_3) \nu(dz)$$

$$+ \int_{\mathbb{R}^d} \kappa^{(1)}_z(f_2) \kappa^{(2)}_z(f_1, f_3) \nu(dz) + \int_{\mathbb{R}^d} \kappa^{(1)}_z(f_3) \kappa^{(2)}_z(f_1, f_2) \nu(dz) + \int_{\mathbb{R}^d} \kappa^{(3)}_z(f_1, f_2, f_3) \nu(dz).$$

Similar expressions for $\kappa^{(4)}(f)$ can be obtained from (3.8).

4. Example - exponential kernel

In this section we take $d = 1$ and consider the exponential kernel $\gamma(dx) = a\mathbf{1}_{[0,\infty)}(x)\mathrm{e}^{-bx}dx$, $0 < a < b$, and constant Poisson intensity $\nu(dz) = \nu dz$, $\nu > 0$. In this case, $N_t(\xi) := \xi([0,t]) = \sum_{x\in\xi}\mathbf{1}_{[0,t]}(x)$ defines the self-exciting Hawkes process with stochastic intensity $\lambda_t := \nu + a\int_0^t \mathrm{e}^{-b(t-s)}dN_s$, $t \in \mathbb{R}_+$. The recursive calculation of the cumulants $\kappa_z^{(n)}(t) := \kappa_z^{(n)}(\mathbf{1}_{[0,t]})$ will be performed using the family of functions $e_{p,\eta}(x) := x^p\mathrm{e}^{\eta x}\mathbf{1}_{[0,t]}(x)$, $\eta < b$, $p \geq 0$, which satisfy the relation

$$(I_d - \Gamma)^{-1}\Gamma e_{p,\eta}(z) = \sum_{n=1}^{\infty}\int_0^t\cdots\int_0^t e_{p,\eta}(z + x_1 + \cdots + x_n)\gamma(dx_1)\cdots\gamma(dx_n) = a\int_0^{t-z} y^p\mathrm{e}^{(\eta+a-b)y}dy, \quad z \in [0,t],$$

with

$$(I_d - \Gamma)^{-1}\Gamma e_{0,\eta}(z) = a\mathrm{e}^{\eta z}\mathbf{1}_{(-\infty,t]}(z)\frac{\mathrm{e}^{(\eta+a-b)(t-z)} - 1}{\eta + a - b} = a\mathrm{e}^{\eta z}\frac{\mathrm{e}^{(\eta+a-b)t}e_{0,-\eta+b-a}(z) - e_{0,0}(z)}{\eta + a - b}, \tag{4.1}$$

and

$$(I_d - \Gamma)^{-1}\Gamma e_{1,\eta}(z) = a\mathrm{e}^{\eta z}\frac{e_{0,0}(z) + \mathrm{e}^{(\eta+a-b)t}e_{0,-\eta+b-a}(z)((\eta + a - b)(t - z) - 1)}{(\eta + a - b)^2}, \tag{4.2}$$

where $(I_d - \Gamma)^{-1}\Gamma e_{p,\eta}(z)$ can be similarly evaluated for $p \geq 2$.

First cumulant. We have

$$\begin{aligned}\kappa_z^{(1)}(t) &= 1 + \sum_{n=1}^{\infty}\int_0^t\cdots\int_0^t e_{0,0}(z + x_1 + \cdots + x_n)\gamma(dx_1)\cdots\gamma(dx_n)\\ &= 1 + \frac{a}{a-b}\mathrm{e}^{(a-b)t}e_{0,b-a}(z) - \frac{a}{a-b}e_{0,0}(z), \qquad z \in \mathbb{R}_+,\end{aligned} \tag{4.3}$$

which recovers

$$\mathbb{E}[N_t] = \kappa^{(1)}(t) = \int_0^t \kappa_z^{(1)}(t)\nu(dz) = \frac{\nu}{(b-a)^2}\big(-a + b(b-a)t + a\mathrm{e}^{(a-b)t}\big).$$

Second cumulant. Using (4.1), we have

$$\begin{aligned}\kappa_z^{(2)}(t) &= \sum_{m=1}^{\infty}\int_0^t\cdots\int_0^t \big(\kappa_{z+x_1+\cdots+x_m}^{(1)}(t)\big)^2\gamma(dx_1)\cdots\gamma(dx_m)\\ &= \frac{b^2}{(b-a)^2}\sum_{m=1}^{\infty}\int_0^t\cdots\int_0^t e_{0,0}(z)\gamma(dx_1)\cdots\gamma(dx_m)\\ &\quad -\frac{2ab}{(b-a)^2}\mathrm{e}^{(b-a)t}\sum_{m=1}^{\infty}\int_0^t\cdots\int_0^t e_{0,b-a}(z + x_1 + \cdots + x_m)\gamma(dx_1)\cdots\gamma(dx_m)\\ &\quad +\frac{a^2}{(b-a)^2}\mathrm{e}^{2(b-a)t}\sum_{m=1}^{\infty}\int_0^t\cdots\int_0^t e_{0,2(b-a)}(z + x_1 + \cdots + x_m)\gamma(dx_1)\cdots\gamma(dx_m)\\ &= \frac{ab^2}{(b-a)^3}(1 - \mathrm{e}^{-(b-a)(t-z)}) - \frac{2a^2b}{(b-a)^2}(t-z)\mathrm{e}^{-(b-a)(t-z)} - a^3\frac{\mathrm{e}^{-2(b-a)(t-z)} - \mathrm{e}^{-(b-a)(t-z)}}{(b-a)^3},\end{aligned} \tag{4.4}$$

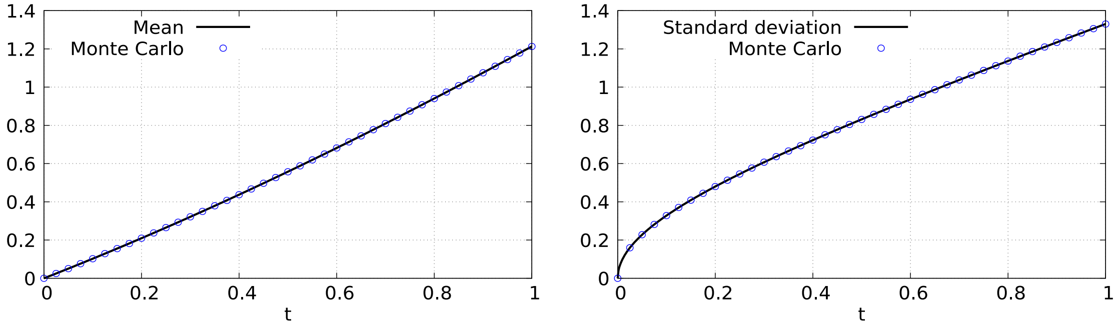
and

$$\begin{aligned}\int_0^t \big(\kappa_z^{(1)}(t)\big)^2\nu(dz) &= \int_0^t\left(\frac{b}{b-a} - \frac{a}{b-a}\mathrm{e}^{-(b-a)(t-z)}\right)^2\nu(dz)\\ &= \frac{\nu b^2 t}{(b-a)^2} - \frac{2\nu ba}{(b-a)^3}(1 - \mathrm{e}^{-(b-a)t}) + \frac{\nu a^2}{2(b-a)^3}(1 - \mathrm{e}^{-2(b-a)t}),\end{aligned}$$

hence

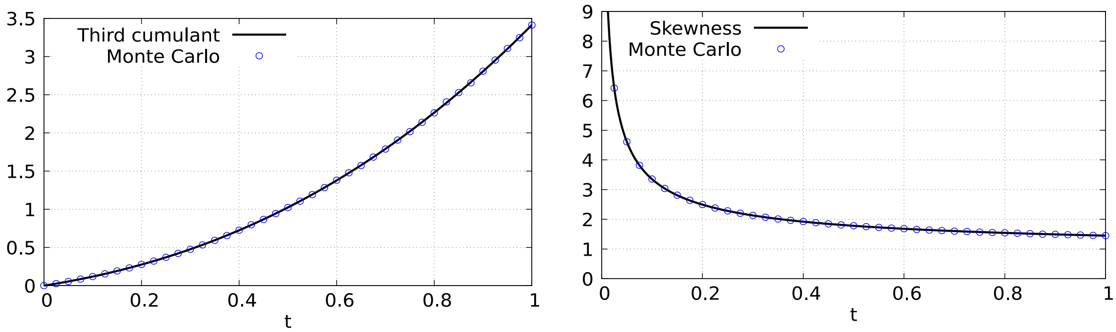
$$\mathrm{Var}[N_t] = \kappa^{(2)}(t) = -\frac{\nu}{2(a-b)^4}\Big(6ab^2 - a^2b + 2b^3(a-b)t + 2a\big(a^2 - 3b^2 + 2ab(a-b)t\big)\mathrm{e}^{(a-b)t} + a^2(b-2a)\mathrm{e}^{2(a-b)t}\Big).$$

Figs. 1–3 are plotted with $\nu = 1$, $a = 0.5$, $b = 1$, and 10^7 Monte Carlo samples.



(a) First cumulant $\kappa^{(1)}(t)$. (b) Square root of second cumulant $\kappa^{(2)}(t)$.

Fig. 1. Mean and standard deviation with exponential kernel.



(a) Third cumulant $\kappa^{(3)}(t)$. (b) Skewness $\kappa^{(3)}(t)/\left(\kappa^{(2)}(t)\right)^{3/2}$.

Fig. 2. Third cumulant and skewness with exponential kernel.

Third cumulant. The recursive computation of $\kappa^{(3)}(t)$ can be carried out from (3.4)–(3.5) and (4.1)–(4.2) using Mathematica, based on the expressions of $\kappa^{(1)}_{z+x_1+\cdots+x_m}(f)$ and $\kappa^{(2)}_{z+x_1+\cdots+x_m}(f)$ given in (4.3)–(4.4), which yields

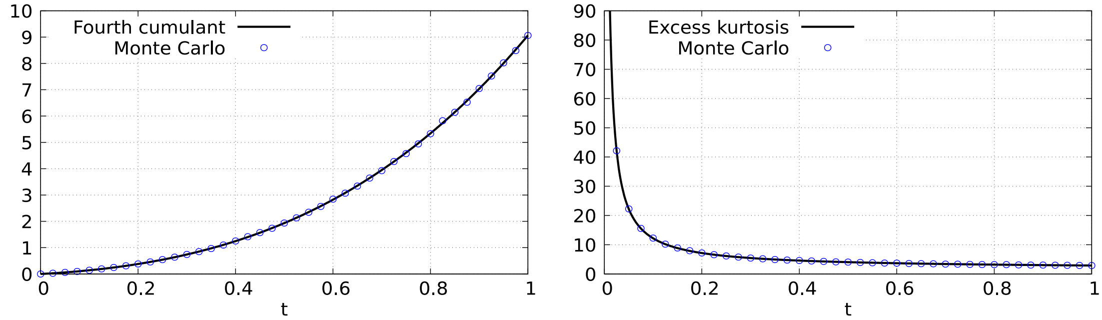
$$\begin{aligned}
\kappa^{(3)}(t) &= -\frac{\nu}{6(a-b)^6}\Big(42ab^4+30a^2b^3-7a^3b^2+a^4b+6b^4(2a^2-ab-b^2)t \\
&+3\big(18a^3b^2-16a^2b^3-a^4b-14ab^4-2a^5+6a^2b(4ab^2-4b^3+a^2b-a^3)t-6a^3b^2(a-b)^2t^2\big)\mathrm{e}^{(a-b)t} \\
&+9\big(2a^2b^3-5a^3b^2-a^4b+2a^5+2a^3b(b^2-3ab+2a^2)t\big)\mathrm{e}^{2(a-b)t}-a^3\big(2b^2-11ab+12a^2\big)\mathrm{e}^{3(a-b)t}\Big).
\end{aligned}$$

Fig. 2 shows the numerical evaluation of $\kappa^{(3)}(t)$ and of the associated skewness $\kappa^{(3)}(t)/(\kappa^{(2)}(t))^{3/2}$.

Fourth cumulant. The recursive computation of $\kappa^{(4)}(t)$ can be similarly carried out from (3.8) and (4.1)–(4.2) using Mathematica, which yields

$$\begin{aligned}
\kappa^{(4)}(t) &= -\frac{\nu}{12(a-b)^8}\Big(180ab^6+570a^2b^5+100a^3b^4-15a^4b^3+2a^5b^2+12b^5(6a^3+2a^2b-b^3-7ab^2)t \\
&+4\big(5a^6b-45ab^6+3a^7-59a^5b^2-180a^2b^5+75a^3b^4+75a^4b^3 \\
&+6a^2b(5ab^4-25b^5+41a^2b^3-22a^3b^2-a^4b+2a^5)t+18a^3b^2(10ab^3-5b^4-4a^2b^2-2a^3b+a^4)t^2 \\
&+12a^4b^3(a-b)^3t^3\big)\mathrm{e}^{(a-b)t}+\big(150a^2b^5-360a^3b^4-564a^4b^3+588a^5b^2+18a^6b-84a^7 \\
&+4a^3b(90b^4-306ab^3+180a^2b^2+108a^3b-72a^4)t+144a^4b^2(-4ab^2+b^3+5a^2b-2a^3)t^2\big)\mathrm{e}^{2(a-b)t} \\
&+\big(276a^4b^3-40a^3b^4-132a^6b-320a^5b^2+144a^7+24a^4b(13ab^2-23a^2b-2b^3+12a^3)t\big)\mathrm{e}^{3(a-b)t} \\
&+a^4\big(3b^3-34ab^2+94a^2b-72a^3\big)\mathrm{e}^{4(a-b)t}\Big).
\end{aligned}$$

Fig. 2 shows the numerical evaluation of $\kappa^{(4)}(t)$ and of the associated excess kurtosis $\kappa^{(4)}(t)/(\kappa^{(2)}(t))^2$.



(a) Fourth cumulant $\kappa^{(4)}(t)$. (b) Excess kurtosis $\kappa^{(4)}(t)/\big(\kappa^{(2)}(t)\big)$.

Fig. 3. Fourth cumulant and excess kurtosis with exponential kernel.

Intensity cumulants. Intensity cumulants can also be computed recursively from Corollary 3.4 and the expression of λ_t. We have

$$\mathbb{E}[\lambda_t] = \nu + a\mathbb{E}\left[\int_0^t \mathrm{e}^{-b(t-s)} dN_s\right] = \nu + a\mathrm{e}^{-bt}\kappa^{(1)}(e_{0,b}) = \nu + a\mathrm{e}^{-bt}\int_0^t \kappa_z^{(1)}(e_{0,b})\nu(dz),$$

with

$$\begin{aligned}\kappa_z^{(1)}(e_{0,b}) &= e_{0,b}(z) + \sum_{n=1}^{\infty}\int_0^t \cdots \int_0^t e_{0,b}(z + x_1 + \cdots + x_n)\gamma(dx_1)\cdots\gamma(dx_n)\\ &= \mathrm{e}^{bz} e_{0,a}(t-z) = \mathrm{e}^{bt} e_{0,a-b}(t-z) = \mathrm{e}^{at} e_{0,b-a}(z),\end{aligned}$$

hence

$$\mathbb{E}[\lambda_t] = \nu + a\mathrm{e}^{-bt}\kappa^{(1)}(e_{0,b}) = \nu + a\mathrm{e}^{-bt}\int_0^t \kappa_z^{(1)}(e_{0,b})\nu(dz) = \nu + \frac{\nu a}{b-a}(1 - \mathrm{e}^{(a-b)t}),$$

see e.g. Theorem 3.6 in Dassios and Zhao (2011). Next, we compute the joint moment $\mathbb{E}[\lambda_t N_t]$. Using (4.1), we have

$$\begin{aligned}\kappa_z^{(2)}(e_{0,0}, e_{0,b}) &= \sum_{m=1}^{\infty}\int_0^t \cdots \int_0^t \kappa^{(1)}_{z+x_1+\cdots+x_m}(e_{0,0})\kappa^{(1)}_{z+x_1+\cdots+x_m}(e_{0,b})\gamma(dx_1)\cdots\gamma(dx_m)\\ &= \mathrm{e}^{at}\frac{b}{b-a}\sum_{m=1}^{\infty}\int_0^t \cdots \int_0^t e_{0,b-a}(z + x_1 + \cdots + x_m)\gamma(dx_1)\cdots\gamma(dx_m)\\ &\quad - \frac{a}{b-a}\mathrm{e}^{(2a-b)t}\sum_{m=1}^{\infty}\int_0^t \cdots \int_0^t e_{0,2(b-a)}(z + x_1 + \cdots + x_m)\gamma(dx_1)\cdots\gamma(dx_m)\\ &= \frac{ab}{b-a}\mathrm{e}^{at}z + a^2\mathrm{e}^{(2a-b)t}\frac{1-\mathrm{e}^{(b-a)(t-z)}}{(b-a)^2}, \qquad z \in [0,t].\end{aligned}$$

Hence we have

$$\begin{aligned}a\mathbb{E}\left[N_t\int_0^t \mathrm{e}^{-b(t-s)} dN_s\right] &= a\mathrm{e}^{-bt}\kappa^{(2)}(e_{0,0}, e_{0,b}) + a\mathrm{e}^{-bt}\kappa^{(1)}(e_{0,b})\kappa^{(1)}(e_{0,0})\\ &= a\mathrm{e}^{-bt}\int_0^t \kappa_z^{(2)}(e_{0,0}, e_{0,b})\nu(dz) + a\mathrm{e}^{-bt}\int_0^t \kappa_z^{(1)}(e_{0,0})\kappa_z^{(1)}(e_{0,b})\nu(dz) + a\mathrm{e}^{-bt}\kappa^{(1)}(e_{0,b})\kappa^{(1)}(e_{0,0})\\ &= -\frac{\nu a}{(a-b)^3}\Big(b^2 - \nu a - ab + \nu b(b-a)t + a\big(a - \nu + a(b-a)t\big)\mathrm{e}^{2(a-b)t}\\ &\quad + \big(2\nu a - a^2 + ab - b^2 + (\nu ab - \nu b^2 - a(a-b)^2)t + ab(a-b)^2t^2/2\big)\mathrm{e}^{(a-b)t}\Big), \quad t \geq 0.\end{aligned}$$

Fig. 4 are plotted with $\nu = 2$, $a = 0.5$, $b = 1$, and 10^6 Monte Carlo samples.

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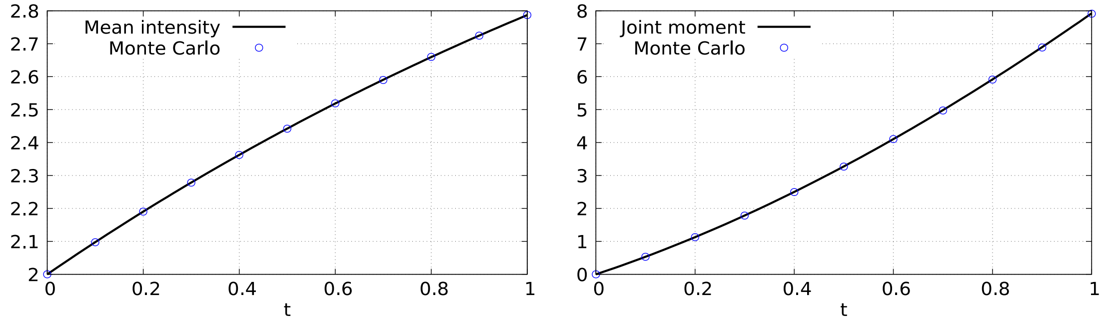

(a) Mean intensity $\mathbb{E}[\lambda_t]$. (b) Joint moment $\mathbb{E}[\lambda_t N_t]$.

Fig. 4. Mean intensity and joint moment with exponential kernel.

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