Chapter 13
Asian Options

Asian options are special cases of average value options, whose claim payoffs are determined by the difference between the average underlying asset price over a certain time interval and a strike price K. This chapter covers several probabilistic and PDE techniques for the pricing and hedging of Asian options. Due to their dependence on averaged asset prices. Asian options are less volatile than plain vanilla options whose claim payoffs depend only on the terminal value of the underlying asset.

13.1 Bounds on Asian Option Prices

Asian options were first traded in Tokyo in 1987, and have become particularly popular in commodities trading.

Arithmetic Asian options

Given an underlying asset price process $(S_t)_{t\in[0,T]}$, the payoff of the Asian call option on $(S_t)_{t\in[0,T]}$ with exercise date T and strike price K is given by

$$C = \left(\frac{1}{T}\int_0^T S_t dt - K\right)^+.$$

Similarly, the payoff of the Asian put option on $(S_t)_{t\in[0,T]}$ with exercise date T and strike price K is

$$C = \left(K - \frac{1}{T} \int_0^T S_t dt \right)^+ .$$

Due to their dependence on averaged asset prices, Asian options are less volatile than plain vanilla options whose payoffs depend only on the terminal value of the underlying asset. Examples of market price data for Asian option prices on Light Sweet Crude Oil Futures can be found **here**.

As an example, Figure 13.1 presents a graph of Brownian motion and its moving average process

$$X_t := \frac{1}{t} \int_0^t B_s ds, \qquad t > 0.$$

Fig. 13.1: Brownian motion B_t and its moving average X_t.[*]

Related exotic options include the Asian-American options, or Hawaiian options, that combine an Asian claim payoff with American style exercise, and can be priced by variational PDEs, cf. § 8.6.3.2 of Crépey (2013).

An option on average is an option whose payoff has the form

$$C = \phi(\Lambda_T, S_T),$$

where
$$\Lambda_T = S_0 \int_0^T e^{\sigma B_u + ru - \sigma^2 u/2} du = \int_0^T S_u du, \qquad T \geqslant 0.$$

- For example when $\phi(y, x) = (y/T - K)^+$ this yields the Asian call option with payoff

$$\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ = \left(\frac{\Lambda_T}{T} - K \right)^+, \tag{13.1}$$

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[*] The animation works in Acrobat Reader on the entire pdf file.

which is a path-dependent option whose price at time $t \in [0, T]$ is given by

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^* \left[\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \bigg| \mathcal{F}_t \right]. \tag{13.2}$$

- As another example, when $\phi(y, x) := \mathrm{e}^{-y}$ this yields the price

$$P(0, T) = \mathbb{E}^* \left[\mathrm{e}^{-\int_0^T S_u du} \right] = \mathbb{E}^* \left[\mathrm{e}^{-\Lambda_T} \right]$$

at time 0 of a bond with underlying short-term rate process S_t.

In the sequel, we assume that the underlying asset price process $(S_t)_{t \in \mathbb{R}_+}$ is a geometric Brownian motion satisfying the equation

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$.

Using the time homogeneity of the process $(S_t)_{t \in \mathbb{R}_+}$, the option with payoff $C = \phi(\Lambda_T, S_T)$ can be priced as

$$
\begin{aligned}
\mathrm{e}^{-(T-t)r}\mathbb{E}^*[\phi(\Lambda_T, S_T) \mid \mathcal{F}_t] &= \mathrm{e}^{-(T-t)r}\mathbb{E}^* \left[\phi \left(\Lambda_t + \int_t^T S_u du, S_T \right) \bigg| \mathcal{F}_t \right] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^* \left[\phi \left(y + x \int_t^T \frac{S_u}{S_t} du, x \frac{S_T}{S_t} \right) \right]_{y=\Lambda_t, x=S_t} \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^* \left[\phi \left(y + x \int_0^{T-t} \frac{S_u}{S_0} du, x \frac{S_{T-t}}{S_0} \right) \right]_{y=\Lambda_t, x=S_t}. \tag{13.3}
\end{aligned}
$$

Using the Markov property of the process $(S_t, \Lambda_t)_{t \in \mathbb{R}_+}$, we can write down the option price as a function

$$
\begin{aligned}
f(t, S_t, \Lambda_t) &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[\phi(\Lambda_T, S_T) \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[\phi(\Lambda_T, S_T) \mid S_t, \ \Lambda_t]
\end{aligned}
$$

of (t, S_t, Λ_t), where the function $f(t, x, y)$ is given by

$$f(t, x, y) = \mathrm{e}^{-(T-t)r}\mathbb{E}^* \left[\phi \left(y + x \int_0^{T-t} \frac{S_u}{S_0} du, x \frac{S_{T-t}}{S_0} \right) \right].$$

As we will see below there exists no easily tractable closed-form solution for the price of an arithmetically averaged Asian option.

Geometric Asian options

On the other hand, replacing the arithmetic average

⟳

$$\frac{1}{T}\sum_{k=1}^{n}S_{t_k}(t_k - t_{k-1}) \simeq \frac{1}{T}\int_0^T S_u du$$

with the geometric average

$$\begin{aligned}
\prod_{k=1}^{n}S_{t_k}^{(t_k - t_{k-1})/T} &= \exp\left(\log\prod_{k=1}^{n}S_{t_k}^{(t_k - t_{k-1})/T}\right) \\
&= \exp\left(\frac{1}{T}\sum_{k=1}^{n}\log S_{t_k}^{t_k - t_{k-1}}\right) \\
&= \exp\left(\frac{1}{T}\sum_{k=1}^{n}(t_k - t_{k-1})\log S_{t_k}\right) \\
&\simeq \exp\left(\frac{1}{T}\int_0^T \log S_u du\right)
\end{aligned}$$

leads to closed-form solutions using the Black Scholes formula, see Exercise 13.5.

Pricing by probability density functions

We note that the prices of option on averages can be estimated numerically using the joint probability density function $\psi_{\Lambda_{T-t}, B_{T-t}}$ of (Λ_{T-t}, B_{T-t}), as follows:

$$\begin{aligned}
f(t,x,y) &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\phi\left(y + x\int_0^{T-t}\frac{S_u}{S_0}du, x\frac{S_{T-t}}{S_0}\right)\right] \\
&= \mathrm{e}^{-(T-t)r}\int_0^\infty\int_{-\infty}^\infty \phi\left(y + xz, x\,\mathrm{e}^{\sigma u + (T-t)r - (T-t)\sigma^2/2}\right)\psi_{\Lambda_{T-t}, B_{T-t}}(z,u)dzdu,
\end{aligned}$$

see Section 13.2 for details.

Bounds on Asian option prices

As noted in the next proposition, arithmetic Asian call option prices can be lower bounded by geometric Asian call prices, as a consequence of the Jensen (1906) inequality. See Exercise 13.5 for the expression of the geometric Asian option call price.

Proposition 13.1. *Let ϕ be a non-decreasing payoff function. We have the bound*

$$\mathrm{e}^{-rT}\mathbb{E}^*\left[\phi\left(\exp\left(\frac{1}{T}\int_0^T\log S_u du\right) - K\right)\right] \leqslant \mathrm{e}^{-rT}\mathbb{E}^*\left[\phi\left(\frac{1}{T}\int_0^T S_u du - K\right)\right].$$

Proof. By Jensen's inequality applied to the (concave) log function and the uniform measure with probability density function $(1/T)\mathbb{1}_{[0,T]}$ on $[0,T]$, we have

$$\exp\left(\frac{1}{T}\int_0^T \log S_t dt\right) \leqslant \exp\left(\log\left(\frac{1}{T}\int_0^T S_t dt\right)\right) = \frac{1}{T}\int_0^T S_t dt. \quad (13.4)$$

$\square$

We also note (see Lemma 1 of Kemna and Vorst (1990) and Exercise 13.7 below for the discrete-time version of that result), that the Asian call option price can be upper bounded by the corresponding European call option price using convexity arguments.

Proposition 13.2. *Assume that $r \geqslant 0$, and let ϕ be a convex and non-decreasing payoff function. We have the bound*

$$e^{-rT}\mathbb{E}^*\left[\phi\left(\frac{1}{T}\int_0^T S_u du - K\right)\right] \leqslant e^{-rT}\mathbb{E}^*[\phi(S_T - K)].$$

Proof. By Jensen's inequality for the uniform measure with probability density function $(1/T)\mathbb{1}_{[0,T]}$ on $[0,T]$ and for the probability measure $\mathbb{P}^*$, we have

$$\begin{aligned}
e^{-rT}\mathbb{E}^*\left[\phi\left(\int_0^T S_u \frac{du}{T} - K\right)\right] &= e^{-rT}\mathbb{E}^*\left[\phi\left(\int_0^T (S_u - K)\frac{du}{T}\right)\right] \\
&\leqslant e^{-rT}\mathbb{E}^*\left[\int_0^T \phi(S_u - K)\frac{du}{T}\right] \\
&= e^{-rT}\mathbb{E}^*\left[\int_0^T \phi\left(e^{-(T-u)r}\mathbb{E}^*[S_T \mid \mathcal{F}_u] - K\right)\frac{du}{T}\right] \\
&= e^{-rT}\mathbb{E}^*\left[\int_0^T \phi\left(\mathbb{E}^*[e^{-(T-u)r}S_T - K \mid \mathcal{F}_u]\right)\frac{du}{T}\right] \\
&\leqslant e^{-rT}\mathbb{E}^*\left[\int_0^T \mathbb{E}^*\left[\phi\left(e^{-(T-u)r}S_T - K\right) \mid \mathcal{F}_u\right]\frac{du}{T}\right] \quad (13.5) \\
&\leqslant e^{-rT}\int_0^T \mathbb{E}^*\left[\mathbb{E}^*[\phi(S_T - K) \mid \mathcal{F}_u]\right]\frac{du}{T} \quad (13.6) \\
&= e^{-rT}\int_0^T \mathbb{E}^*[\phi(S_T - K)]\frac{du}{T} \\
&= e^{-rT}\mathbb{E}^*[\phi(S_T - K)],
\end{aligned}$$

where from (13.5) to (13.6) we used the facts that $r \geqslant 0$ and ϕ is non-decreasing. $\square$

In particular, taking $\phi(x) : (x - K)^+$, Proposition 13.2 shows that Asian option prices are upper bounded by European call option prices due to the lower volatility of arithmetic averages, as

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$$\mathrm{e}^{-rT}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+\right] \leqslant \mathrm{e}^{-rT}\mathbb{E}^*[(S_T - K)^+],$$

see Figure 13.2 for an illustration, as the averaging feature of Asian options reduces their underlying volatility.

```
1   nSim=99999;N=1000; t <- 1:N; dt <- 1.0/N; sigma=2;r=0.5; european=0;asian=0;K=1.5
2   dev.new(width=16,height=7); par(oma=c(0,5,0,0))
    for (j in 1:nSim){S<-exp(sigma*cumsum(rnorm( N,
        0,sqrt(dt)))+r*t/N-sigma**2*t/2/N);color="blue"
4   A<-sum(c(1,S))/(N+1);if (S[N]>=K) {european=european+S[N]-K}
    if (A>=K) {asian=asian+A-K};if (S[N]>A) {color="darkred"} else {color="darkgreen"}
6   plot(c(0,t/N),c(1,S), xlab = "Time", type='l', lwd = 3, ylab = "", ylim = c(0,exp(4*r)), col =
        color,main=paste("Asian Price=",format(round(asian,2)),"/",
        j,"=",format(round(asian/j,2)),"European Price=",format(round(european,2)),
        "/",j,"=",format(round(european/j,2))), xaxs='i',xaxt='n',yaxt='n', yaxs='i', yaxp =
        c(0,10,10), cex.lab=2, cex.main=2)
    text(0.3,6,paste("A-Payoff=",format(round(max(A-K,0),2)),", E-Payoff=",
        format(round(max(S[N]-K,0),2))), col=color,cex=2)
8   axis(1, las=1, cex.axis=2)
    axis(2, at=c(0,K,A,1,2,3,4,5,6,7,8,9,10), labels=c(0,"K","Average",1,2,3,4,5,6,7,8,9,10), las=2,
        cex.axis=2)
10  lines(c(0,t/N),rep(K,N+1),col = "red",lty = 1, lwd = 4);
    lines(c(0,t/N),rep(A,N+1),col = "darkgreen",lty = 2, lwd = 4); Sys.sleep(0.1)
12  if (S[N]>K || A>K) {readline(prompt = "Pause. Press <Enter> to continue...")}}
```

Fig. 13.2: Asian option price *vs.* European option price.[*]

In the case of Asian call options we have the following result.

Proposition 13.3. *Assume that $r \geqslant 0$. We have the conditional bound*

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \bigg| \mathcal{F}_t\right] \tag{13.7}$$

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[*] The animation works in Acrobat Reader on the entire pdf file.

$$\leqslant \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^t S_u du + \frac{T-t}{T}S_T - K\right)^+ \middle| \mathcal{F}_t\right]$$

on Asian option prices, $t \in [0, T]$.

Proof. Let the function $f(t, x, y)$ be defined as

$$f(t, S_t, \Lambda_t) = \mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \middle| \mathcal{F}_t\right],$$

i.e., from Proposition 13.2,

$$\begin{aligned}
f(t, x, y) &= \mathbb{E}^*\left[\left(\frac{1}{T}\left(y + x\int_0^{T-t}\frac{S_u}{S_0}du\right) - K\right)^+\right] \\
&= \mathbb{E}^*\left[\left(\frac{1}{T}\left(y + \frac{x}{S_0}\Lambda_{T-t}\right) - K\right)^+\right] \\
&= \mathbb{E}^*\left[\left(\frac{y}{T} - K + \frac{x}{TS_0}\Lambda_{T-t}\right)^+\right] \\
&= \frac{(T-t)x}{TS_0}\mathbb{E}^*\left[\left(\frac{yS_0}{(T-t)x} - \frac{KTS_0}{(T-t)x} + \frac{\Lambda_{T-t}}{T-t}\right)^+\right] \\
&\leqslant \frac{(T-t)x}{TS_0}\mathbb{E}^*\left[\left(\frac{yS_0}{(T-t)x} - \frac{KTS_0}{(T-t)x} + S_{T-t}\right)^+\right] \\
&= \mathbb{E}^*\left[\left(\frac{y}{T} - K + \frac{(T-t)xS_{T-t}}{TS_0}\right)^+\right], \qquad x, y > 0,
\end{aligned}$$

which yields (13.7). □

The right-hand side of the bound (13.7) can be computed from the Black-Scholes formula as

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^t S_u du + \frac{T-t}{T}S_T - K\right)^+ \middle| \mathcal{F}_t\right] \\
&= \frac{T-t}{T}\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(S_T + \frac{1}{T-t}\int_0^t S_u du - \frac{KT}{T-t}\right)^+ \middle| \mathcal{F}_t\right] \\
&= \frac{T-t}{T}\mathrm{Bl}\left(S_t, \frac{KT}{T-t} - \frac{1}{T-t}\int_0^t S_u du, \sigma, r, T-t\right), \quad 0 \leqslant t < T.
\end{aligned}$$

See also Proposition 3.2-(ii) of Geman and Yor (1993) for lower bounds when r takes negative values. We also have the following bound which yields the behavior of Asian call option prices in large time.

Proposition 13.4. *Assume that $r \geqslant 0$. The Asian call option price satisfies the bound*

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \bigg| \mathcal{F}_t\right] \leqslant \frac{\mathrm{e}^{-(T-t)r}}{T}\int_0^t S_u du + S_t \frac{1 - \mathrm{e}^{-(T-t)r}}{rT},$$

$t \in [0, T]$, *and tends to zero (almost surely) as time to maturity T tends to infinity:*

$$\lim_{T\to\infty}\left(\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \bigg| \mathcal{F}_t\right]\right) = 0, \quad t \geqslant 0.$$

Proof. Using the inequality $(x - K)^+ \leqslant x$ for $x \geqslant 0$, we have the bound

$$
\begin{aligned}
0 &\leqslant \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \bigg| \mathcal{F}_t\right] \\
&\leqslant \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\frac{1}{T}\int_0^T S_u du \bigg| \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\frac{1}{T}\int_0^t S_u du \bigg| \mathcal{F}_t\right] + \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\frac{1}{T}\int_t^T S_u du \bigg| \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_0^t S_u du + \frac{1}{T}\mathrm{e}^{-(T-t)r}\int_t^T \mathbb{E}^*[S_u \mid \mathcal{F}_t] du \\
&= \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_0^t S_u du + \frac{1}{T}\mathrm{e}^{-(T-t)r}\int_t^T \mathrm{e}^{(u-t)r} S_t du \\
&= \frac{1}{T}\mathrm{e}^{-(T-t)r}\int_0^t S_u du + \frac{S_t}{T}\int_t^T \mathrm{e}^{-(T-u)r} du \\
&= \frac{1}{T}\mathrm{e}^{-(T-t)r}\int_0^t S_u du + S_t \frac{1 - \mathrm{e}^{-(T-t)r}}{rT}.
\end{aligned}
$$

$\square$

Note that as T tends to infinity the Black-Scholes European call price tends to S_t, *i.e.*, we have

$$\lim_{T\to\infty}\left(\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t]\right) = S_t, \quad t \geqslant 0,$$

see Exercise 6.4-aiii).

13.2 Hartman-Watson Distribution

First, we note that the numerical computation of Asian call option prices can be done using the probability density function of

$$\Lambda_T = \int_0^T S_t dt.$$

In Yor (1992), Proposition 2, the joint probability density function of

$$\left(\Lambda_t, B_t \right) = \left(\int_0^t e^{\sigma B_s - p\sigma^2 s/2} ds, B_t - p\sigma t/2 \right), \qquad t > 0,$$

has been computed in the case $\sigma = 2$, cf. also Dufresne (2001) and Matsumoto and Yor (2005). In the next proposition, we restate this result for an arbitrary variance parameter σ after rescaling. Let $\theta(v, \tau)$ denote the function defined as

$$\theta(v, \tau) = \frac{v \, e^{\pi^2/(2\tau)}}{\sqrt{2\pi^3 \tau}} \int_0^\infty e^{-\xi^2/(2\tau)} e^{-v \cosh \xi} \sinh(\xi) \sin\left(\pi \xi / \tau\right) d\xi, \quad v, \tau > 0. \tag{13.8}$$

Proposition 13.5. *For all $t > 0$ we have*

$$\mathbb{P}\left(\int_0^t e^{\sigma B_s - p\sigma^2 s/2} ds \in dy, \ B_t - p\frac{\sigma t}{2} \in dz \right)$$

$$= \frac{\sigma}{2} e^{-p\sigma z/2 - p^2 \sigma^2 t/8} \exp\left(-2\frac{1 + e^{\sigma z}}{\sigma^2 y} \right) \theta\left(\frac{4 e^{\sigma z/2}}{\sigma^2 y}, \frac{\sigma^2 t}{4} \right) \frac{dy}{y} dz,$$

$y > 0$, $z \in \mathbb{R}$.

The expression of this probability density function can then be used for the pricing of options on average such as (13.3), as

$$f(t, x, y) = e^{-(T-t)r} \mathbb{E}^*\left[\phi\left(y + x \int_0^{T-t} \frac{S_v}{S_0} dv, x \frac{S_{T-t}}{S_0} \right) \right]$$

$$= e^{-(T-t)r}$$

$$\times \int_0^\infty \phi\left(y + xz, x \, e^{\sigma u + (T-t)r - (T-t)\sigma^2/2} \right) \mathbb{P}\left(\int_0^{T-t} \frac{S_v}{S_0} dv \in dz, B_{T-t} \in du \right)$$

$$= \frac{\sigma}{2} e^{-(T-t)r + (T-t)p^2\sigma^2/8} \int_0^\infty \int_{-\infty}^\infty \phi\left(y + xz, x \, e^{\sigma u + (T-t)r - (T-t)(1+p)\sigma^2/2} \right)$$

$$\times \exp\left(-2\frac{1 + e^{\sigma u - (T-t)p\sigma^2/2}}{\sigma^2 z} - \frac{p}{2}\sigma u \right) \theta\left(\frac{4 e^{\sigma u/2 - (T-t)p\sigma^2/4}}{\sigma^2 z}, \frac{(T-t)\sigma^2}{4} \right) du \frac{dz}{z}$$

$$= e^{-(T-t)r - (T-t)p^2\sigma^2/8} \int_0^\infty \int_0^\infty \phi\left(y + x/z, xv^2 \, e^{(T-t)r - (T-t)\sigma^2/2} \right)$$

$$\times v^{-1-p} \exp\left(-2z\frac{1 + v^2}{\sigma^2} \right) \theta\left(\frac{4vz}{\sigma^2}, \frac{(T-t)\sigma^2}{4} \right) dv \frac{dz}{z},$$

which actually stands as a triple integral due to the definition (13.8) of $\theta(v, \tau)$. Note that here the order of integration between du and dz cannot be exchanged without particular precautions, at the risk of wrong computations.

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By repeating the argument of (13.3) for $\phi(x, y) := (x - K)^+$, we have

$$
\mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \bigg| \mathcal{F}_t \right]
$$

$$
= \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\left(\frac{1}{T} \left(\Lambda_t + \int_t^T S_u du \right) - K \right)^+ \bigg| \mathcal{F}_t \right]
$$

$$
= \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\left(\frac{1}{T} \left(y + x \int_t^T \frac{S_u}{S_t} du \right) - K \right)^+ \bigg| \mathcal{F}_t \right]_{x=S_t,\ y=\Lambda_t}
$$

$$
= \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\left(\frac{1}{T} \left(y + x \int_0^{T-t} \frac{S_u}{S_0} du \right) - K \right)^+ \right]_{x=S_t,\ y=\Lambda_t}.
$$

Hence, the Asian call option can be priced as

$$
f(t, S_t, \Lambda_t) = \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \bigg| \mathcal{F}_t \right],
$$

where the function $f(t, x, y)$ is given by

$$
f(t, x, y) = \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\left(\frac{1}{T} \left(y + x \int_0^{T-t} \frac{S_u}{S_0} du \right) - K \right)^+ \right]
$$

$$
= \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\left(\frac{1}{T} \left(y + \frac{x}{S_0} \Lambda_{T-t} \right) - K \right)^+ \right], \quad x, y > 0. \quad (13.9)
$$

From Proposition 13.5, we deduce the marginal probability density function of Λ_T, also called the Hartman-Watson distribution see *e.g.* Barrieu et al. (2004).

Proposition 13.6. *The probability density function of*

$$
\Lambda_T := \int_0^T \mathrm{e}^{\sigma B_t - p\sigma^2 t/2} dt,
$$

is given by

$$
\mathbb{P} \left(\int_0^T \mathrm{e}^{\sigma B_t - p\sigma^2 t/2} dt \in du \right)
$$

$$
= \frac{\sigma}{2u} \mathrm{e}^{p^2 \sigma^2 T/8} \int_{-\infty}^\infty \exp \left(-2 \frac{1 + \mathrm{e}^{\sigma v - p\sigma^2 T/2}}{\sigma^2 u} - \frac{p}{2} \sigma v \right) \theta \left(\frac{4 \mathrm{e}^{\sigma v/2 - p\sigma^2 T/4}}{\sigma^2 u}, \frac{\sigma^2 T}{4} \right) dv du
$$

$$
= \mathrm{e}^{-p^2 \sigma^2 T/8} \int_0^\infty v^{-1-p} \exp \left(-2 \frac{1 + v^2}{\sigma^2 u} \right) \theta \left(\frac{4v}{\sigma^2 u}, \frac{\sigma^2 T}{4} \right) dv \frac{du}{u},
$$

$u > 0$.

From Proposition 13.6, we get

$$\mathbb{P}(\Lambda_T/S_0 \in du) = \mathbb{P}\left(\int_0^T S_t dt \in du\right) \tag{13.10}$$

$$= e^{-p^2\sigma^2 T/8} \int_0^\infty v^{-1-p} \exp\left(-2\frac{1+v^2}{\sigma^2 u}\right) \theta\left(\frac{4v}{\sigma^2 u}, \frac{\sigma^2 T}{4}\right) dv \frac{du}{u},$$

where $S_t = S_0 e^{\sigma B_t - p\sigma^2 t/2}$ and $p = 1 - 2r/\sigma^2$. By (13.9), this probability density function can then be used for the pricing of Asian options, as

$$f(t,x,y) = e^{-(T-t)r} \mathbb{E}^*\left[\left(\frac{1}{T}\left(y + \frac{x}{S_0}\Lambda_{T-t}\right) - K\right)^+\right] \tag{13.11}$$

$$= e^{-(T-t)r} \int_0^\infty \left(\frac{y+xz}{T} - K\right)^+ \mathbb{P}(\Lambda_{T-t}/S_0 \in dz)$$

$$= e^{-(T-t)r}\frac{\sigma}{2} e^{-(T-t)p^2\sigma^2/8} \int_0^\infty \int_0^\infty \left(\frac{y+xz}{T} - K\right)^+$$

$$\times v^{-1-p} \exp\left(-2\frac{1+v^2}{\sigma^2 z}\right) \theta\left(\frac{4v}{\sigma^2 z}, (T-t)\frac{\sigma^2}{4}\right) dv \frac{dz}{z}$$

$$= \frac{1}{T} e^{-(T-t)r - (T-t)p^2\sigma^2/8} \int_{0\vee(KT-y)/x}^\infty \int_0^\infty (xz + y - KT)$$

$$\times \exp\left(-2\frac{1+v^2}{\sigma^2 z}\right) \theta\left(\frac{4v}{\sigma^2 z}, (T-t)\frac{\sigma^2}{4}\right) dv \frac{dz}{z}$$

$$= \frac{4x}{\sigma^2 T} e^{-(T-t)r - (T-t)p^2\sigma^2/8} \int_0^\infty \int_0^\infty \left(\frac{1}{z} - \frac{(KT-y)\sigma^2}{4x}\right)^+$$

$$\times v^{-1-p} \exp\left(-z\frac{1+v^2}{2}\right) \theta\left(vz, (T-t)\frac{\sigma^2}{4}\right) dv \frac{dz}{z},$$

cf. Theorem in § 5 of Carr and Schröder (2004), which is actually a triple integral due to the definition (13.8) of $\theta(v,t)$. Note that since the integrals are not absolutely convergent, here the order of integration between dv and dz cannot be exchanged without particular precautions, at the risk of wrong computations.

13.3 Laplace Transform Method

The time Laplace transform of the rescaled option price

$$C(t) := \mathbb{E}^*\left[\left(\frac{1}{t}\int_0^t S_u du - K\right)^+\right], \qquad t > 0,$$

as

$$\int_0^\infty \mathrm{e}^{-\lambda t} C(t) dt = \frac{\int_0^{K/2} \mathrm{e}^{-x} x^{-2+(p+\sqrt{2\lambda+p^2})/2}(1-2Kx)^{2+(\sqrt{2\lambda+p^2}-p)/2} dx}{\lambda(\lambda-2+2p)\Gamma\big(-1+\big(p+\sqrt{2\lambda+p^2}\big)/2\big)},$$

with here $\sigma := 2$, and $\Gamma(z)$ denotes the gamma function, see Relation (3.10) in Geman and Yor (1993). This expression can be used for pricing by numerical inversion of the Laplace transform using *e.g.* the Widder method, the Gaver-Stehfest method, the Durbin-Crump method, or the Papoulis method. The following Figure 13.3 represents Asian call option prices computed by the Geman and Yor (1993) method.
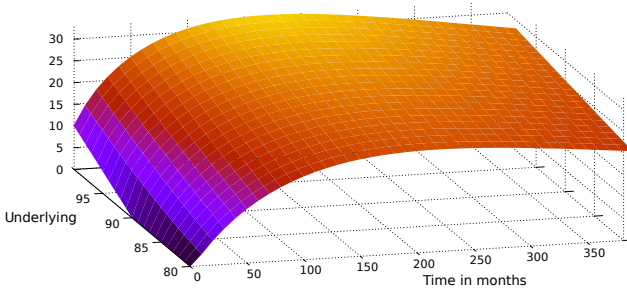

Fig. 13.3: Graph of Asian call option prices with $\sigma = 1$, $r = 0.1$ and $K = 90$.

We refer to *e.g.* Carr and Schröder (2004), Dufresne (2000), and references therein for more results on Asian option pricing using the probability density function of the averaged geometric Brownian motion.

Figure 9.5 presents a graph of implied volatility surface for Asian options on light sweet crude oil futures.

13.4 Moment Matching

Lognormal approximation

Other numerical approaches to the pricing of Asian options include Levy (1992), Turnbull and Wakeman (1992) which rely on approximations of the average price distribution based on the lognormal distribution. The lognormal distribution has the probability density function

$$g(x) = \frac{1}{\eta\sqrt{2\pi}} \mathrm{e}^{-(\mu-\log x)^2/(2\eta^2)} \frac{dx}{x}, \qquad x > 0,$$

where $\mu \in \mathbb{R}$, $\eta > 0$, with moments

$$\mathbb{E}[X] = e^{\mu + \eta^2/2} \quad \text{and} \quad \mathbb{E}[X^2] = e^{2\mu + 2\eta^2}. \tag{13.12}$$

The approximation is implemented by matching the above first two moments to those of time integral

$$\Lambda_T := \int_0^T S_t dt$$

of geometric Brownian motion

$$S_t = S_0 e^{\sigma B_t + (r - \sigma^2/2)t}, \qquad 0 \leqslant t \leqslant T,$$

as computed in the next proposition, cf. also (7) and (8) page 480 of Levy (1992), and Exercise 13.1.

Proposition 13.7. *The first and second moments of Λ_T are given by*

$$\mathbb{E}^*[\Lambda_T] = S_0 \frac{e^{rT} - 1}{r},$$

and

$$\mathbb{E}^*[(\Lambda_T)^2] = 2S_0^2 \frac{r e^{(\sigma^2 + 2r)T} - (\sigma^2 + 2r)e^{rT} + (\sigma^2 + r)}{(\sigma^2 + r)(\sigma^2 + 2r)r}.$$

Proof. The computation of the first moment is straightforward. We have

$$\begin{aligned}
\mathbb{E}^*[\Lambda_T] &= \mathbb{E}^*\Big[\int_0^T S_u du\Big] \\
&= \int_0^T \mathbb{E}^*[S_u] du \\
&= S_0 \int_0^T e^{ru} du \\
&= S_0 \frac{e^{rT} - 1}{r}.
\end{aligned}$$

For the second moment we have, letting $p := 1 - 2r/\sigma^2$,

$$\begin{aligned}
\mathbb{E}^*[(\Lambda_T)^2] &= S_0^2 \int_0^T \int_0^T e^{-p\sigma^2 a/2 - p\sigma^2 b/2} \mathbb{E}^*[e^{\sigma B_a} e^{\sigma B_b}] db\, da \\
&= 2S_0^2 \int_0^T \int_0^a e^{-p\sigma^2 a/2 - p\sigma^2 b/2} e^{(a+b)\sigma^2/2} e^{b\sigma^2} db\, da \\
&= 2S_0^2 \int_0^T e^{-(p-1)\sigma^2 a/2} \int_0^a e^{-(p-3)\sigma^2 b/2} db\, da \\
&= \frac{4S_0^2}{(p-3)\sigma^2} \int_0^T e^{-(p-1)\sigma^2 a/2} (1 - e^{-(p-3)\sigma^2 a/2}) da \\
&= \frac{4S_0^2}{(p-3)\sigma^2} \int_0^T e^{-(p-1)\sigma^2 a/2} da - \frac{4S_0^2}{(p-3)\sigma^2} \int_0^T e^{-(p-1)\sigma^2 a/2} e^{-(p-3)\sigma^2 a/2} da \\
&= \frac{8S_0^2}{(p-3)(p-1)\sigma^4} (1 - e^{-(p-1)\sigma^2 T/2}) - \frac{4S_0^2}{(p-3)\sigma^2} \int_0^T e^{-(2p-4)\sigma^2 a/2} da
\end{aligned}$$

↺

$$= \frac{8S_0^2}{(p-3)(p-1)\sigma^4}(1 - e^{-(p-1)\sigma^2 T/2}) - \frac{4S_0^2}{(p-3)(p-2)\sigma^4}(1 - e^{-(p-2)\sigma^2 T})$$

$$= 2S_0^2 \frac{r\, e^{(\sigma^2 + 2r)T} - (\sigma^2 + 2r)\, e^{rT} + (\sigma^2 + r)}{(\sigma^2 + r)(\sigma^2 + 2r)r},$$

since $r - \sigma^2/2 = -p\sigma^2/2$. □

By matching the first and second moments

$$\mathbb{E}[\Lambda_T] \simeq e^{\hat{\mu}_T + \hat{\eta}_T^2 T/2} \quad \text{and} \quad \mathbb{E}\big[\Lambda_T^2\big] \simeq e^{2(\hat{\mu}_T + \hat{\eta}_T^2 T)}$$

of the lognormal distribution with the moments of Proposition 13.7 we estimate $\widehat{\mu}_T$ and $\widehat{\eta}_T$ as

$$\widehat{\eta}_T^2 = \frac{1}{T} \log\left(\frac{E\big[\Lambda_T^2\big]}{(\mathbb{E}^*[\Lambda_T])^2}\right) \quad \text{and} \quad \widehat{\mu}_T = \frac{1}{T} \log \mathbb{E}^*[\Lambda_T] - \frac{1}{2}\widehat{\eta}_T^2.$$

Under this approximation, the probability density function φ_{Λ_T} of $\Lambda_T = \int_0^T S_t dt$ is approximated by the lognormal probability density function

$$\varphi_{\Lambda_T}(x) \approx \frac{1}{x\sigma_{t,T}\sqrt{2(T-t)\pi}} \exp\left(-\frac{(\widehat{\mu}_T - \log x)^2}{2(T-t)\widehat{\eta}_T^2}\right), \quad x > 0.$$
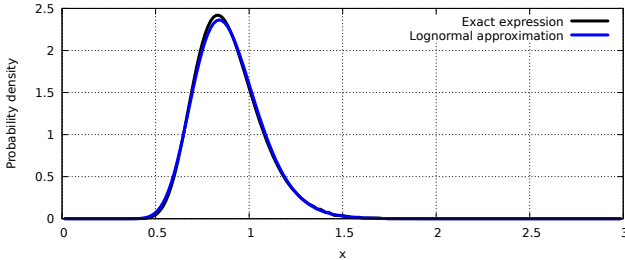

Fig. 13.4: Lognormal approximation for the probability density function of Λ_T.

As a consequence of Lemma 7.7 we find the approximation

$$e^{-rT}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] = e^{-rT}\int_0^\infty \left(\frac{x}{T} - K\right)^+ \varphi_{\Lambda_T}(x)dx$$

$$\simeq \frac{e^{-rT}}{\sigma_{t,T}\sqrt{2(T-t)\pi}} \int_0^\infty \left(\frac{x}{T} - K\right)^+ \exp\left(-\frac{(\widehat{\mu}_T - \log x)^2}{2(T-t)\widehat{\eta}_T^2}\right)\frac{dx}{x}$$

$$= \frac{1}{T}e^{\hat{\mu}_T + \hat{\eta}_T^2 T/2}\Phi(d_1) - K\Phi(d_2) \tag{13.13}$$

$$= \frac{\mathbb{E}[\Lambda_T]}{T}\Phi(d_1) - K\Phi(d_2)$$

$$= \mathrm{Bl}\left(\frac{\mathbb{E}[\Lambda_T]}{T}, K, 0, \widehat{\eta}_T, T - t\right),$$

where

$$d_1 = \frac{\log(\mathbb{E}^*[\Lambda_T]/(KT))}{\widehat{\eta}_T\sqrt{T}} + \widehat{\eta}_T\frac{\sqrt{T}}{2} = \frac{\widehat{\mu}_T T + \widehat{\eta}_T^2 T - \log(KT)}{\widehat{\eta}_T\sqrt{T}}$$

and

$$d_2 = d_1 - \widehat{\eta}_T\sqrt{T} = \frac{\log(\mathbb{E}^*[\Lambda_T]/(KT))}{\widehat{\eta}_T\sqrt{T}} - \widehat{\eta}_T\frac{\sqrt{T}}{2}.$$

The next Figure 13.5 compares the lognormal approximation to a Monte Carlo estimate of Asian call option prices with $\sigma = 1$, $r = 0.05$, and $S_t = 1.5$, and $K = 1.65$.
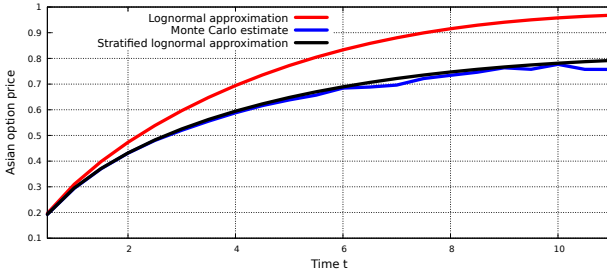

Fig. 13.5: Lognormal approximation to the Asian call option price.

```
1  library(devtools);
2  install_github("https://github.com/cran/fOptions")
   install_github("https://github.com/cran/fExoticOptions")
4  library(fExoticOptions);
   LevyAsianApproxOption(TypeFlag = "c", S = 1.5, SA = 1.5, X = 1.65, Time = 4, time = 4, r
       = 0.05, b = 0.05, sigma = 1.0)
```

Figure 13.5 also includes the stratified approximation

$$\mathrm{e}^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] \tag{13.14}$$

$$= \mathrm{e}^{-rT}\int_0^\infty \mathbb{E}\left[\left(\frac{x}{T} - K\right)^+ \,\Big|\, S_T = y\right]\varphi_{\Lambda_T|S_T=y}(x)\mathrm{d}\mathbb{P}(S_T \leqslant y)dx$$

$$\simeq \frac{\mathrm{e}^{-rT}}{T}\int_0^\infty \left(\mathrm{e}^{-p(y/x)\sigma^2(y/x)T/2 + \sigma^2(y/x)T/2}\Phi(d_+(K,y,x)) - KT\Phi(d_-(K,y,x))\right)$$

↺

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$$\times \, d\mathbb{P}(S_T \leqslant y) dx,$$

cf. Privault and Yu (2016), see the attached codes,[*] where

$$d_{\pm}(K, y, x) := \frac{1}{2\sigma(y/x)\sqrt{T}} \log\left(\frac{2x(b_T(y/x) - (1 + y/x)a_T(y/x))}{\sigma^2 K^2 T^2}\right) \pm \frac{\sigma(y/x)\sqrt{T}}{2}$$

and

$$
\begin{cases}
\sigma^2(z) := \dfrac{1}{T} \log\left(\dfrac{2}{\sigma^2 a_T(z)}\left(\dfrac{b_T(z)}{a_T(z)} - 1 - z\right)\right), \\[2mm]
a_T(z) := \dfrac{1}{\sigma^2 p(z)}\left(\Phi\left(\dfrac{\log z}{\sqrt{\sigma^2 T}} + \dfrac{1}{2}\sqrt{\sigma^2 T}\right) - \Phi\left(\dfrac{\log z}{\sqrt{\sigma^2 T}} - \dfrac{1}{2}\sqrt{\sigma^2 T}\right)\right), \\[2mm]
b_T(z) := \dfrac{1}{\sigma^2 q(z)}\left(\Phi\left(\dfrac{\log z}{\sqrt{\sigma^2 T}} + \sqrt{\sigma^2 T}\right) - \Phi\left(\dfrac{\log z}{\sqrt{\sigma^2 T}} - \sqrt{\sigma^2 T}\right)\right),
\end{cases}
$$

and

$$p(z) := \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-(\sigma^2 T/2 + \log z)^2/(2\sigma^2 T)}, \quad q(z) := \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-(\sigma^2 T + \log z)^2/(2\sigma^2 T)}.$$

Conditioning on the geometric mean price

Asian options on the arithmetic average

$$\frac{1}{T}\int_0^T S_t dt$$

have been priced by conditioning on the geometric mean underlying price

$$G := \exp\left(\frac{1}{T}\int_0^T \log S_t dt\right) \leqslant \exp\left(\log\left(\frac{1}{T}\int_0^T S_t dt\right)\right) = \frac{1}{T}\int_0^T S_t dt$$

in Curran (1994), as

$$
\begin{aligned}
e^{-rT}\mathbb{E}^*&\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+\right] \\
&= e^{-rT}\int_0^\infty \mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \Big| G = x\right] d\mathbb{P}(G \leqslant x) \\
&= e^{-rT}\int_0^K \mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \Big| G = x\right] d\mathbb{P}(G \leqslant x)
\end{aligned}
$$

---

[*] **C code** - **Matlab code** - **R code** (right-click to save as attachment - may not work on 🍎).

$$+ \, \mathrm{e}^{-rT} \int_K^\infty \mathbb{E}^* \left[\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \, \bigg| \, G = x \right] \mathrm{d}\mathbb{P}(G \leqslant x)$$
$$= C_1 + C_2,$$

where

$$C_1 := \, \mathrm{e}^{-rT} \int_0^K \mathbb{E}^* \left[\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \, \bigg| \, G = x \right] \mathrm{d}\mathbb{P}(G \leqslant x),$$

and

$$C_2 := \, \mathrm{e}^{-rT} \int_K^\infty \mathbb{E}^* \left[\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \, \bigg| \, G = x \right] \mathrm{d}\mathbb{P}(G \leqslant x)$$
$$= \, \mathrm{e}^{-rT} \int_K^\infty \mathbb{E}^* \left[\frac{1}{T} \int_0^T S_u du - K \, \bigg| \, G = x \right] \mathrm{d}\mathbb{P}(G \leqslant x)$$
$$= \frac{\mathrm{e}^{-rT}}{T} \int_K^\infty \mathbb{E}^* \left[\int_0^T S_u du \, \bigg| \, G = x \right] \mathrm{d}\mathbb{P}(G \leqslant x) - K \, \mathrm{e}^{-rT} \int_K^\infty \mathrm{d}\mathbb{P}(G \leqslant x)$$
$$= \frac{\mathrm{e}^{-rT}}{T} \mathbb{E}^* \left[\int_0^T S_u du \mathbb{1}_{\{G \geqslant K\}} \right] - K \, \mathrm{e}^{-rT} \mathbb{P}(G \geqslant K).$$

The term C_1 can be estimated by a lognormal approximation given that $G = x$. As for C_2, we note that

$$G = \exp \left(\frac{1}{T} \int_0^T \log S_t dt \right)$$
$$= \exp \left(\frac{1}{T} \int_0^T \left(\mu t + \sigma B_t - \frac{\sigma^2 t}{2} \right) dt \right)$$
$$= \exp \left(\frac{T}{2} \left(\mu - \frac{\sigma^2}{2} \right) + \frac{\sigma}{T} \int_0^T B_t dt \right),$$

hence

$$\log G = \frac{T}{2} (\mu - \sigma^2/2) + \frac{\sigma}{T} \int_0^T B_t dt$$

has the Gaussian distribution $\mathcal{N} \left((\mu - \sigma^2/2) T/2, \sigma^2 T/3 \right)$ with mean $(\mu - \sigma^2/2) T/2$, and variance

$$\mathbb{E} \left[\left(\int_0^T B_t dt \right)^2 \right] = \mathbb{E} \left[\int_0^T \int_0^T B_s B_t ds dt \right]$$
$$= \int_0^T \int_0^T \mathbb{E}[B_s B_t] ds dt$$
$$= 2 \int_0^T \int_0^t s ds dt$$

↻

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$$= \int_0^T t^2 dt$$
$$= \frac{T^3}{3}.$$

Hence, we have

$$
\begin{aligned}
\mathbb{P}(G \geqslant K) &= \mathbb{P}(\log G \geqslant \log K) \\
&= \mathbb{P}\left(\frac{T}{2}\left(\mu - \frac{\sigma^2}{2}\right) + \frac{\sigma}{T}\int_0^T B_t dt \geqslant \log K \right) \\
&= \mathbb{P}\left(\int_0^T B_t dt \geqslant \frac{T}{\sigma}\left(-\frac{T}{2}\left(\mu - \frac{\sigma^2}{2}\right) + \log K \right) \right) \\
&= \Phi\left(\frac{\sqrt{3}}{\sigma\sqrt{T}}\left(\frac{T}{2}\left(\mu - \frac{\sigma^2}{2}\right) - \log K \right) \right).
\end{aligned}
$$

Basket options

Basket options on the portfolio

$$A_T := \sum_{k=1}^N \alpha_k S_T^{(k)}$$

have also been priced in Milevsky (1998) by approximating A_T by a lognormal or a reciprocal gamma random variable, see also Deelstra et al. (2004) for additional conditioning on the geometric average of asset prices.

Asian basket options

Moment matching techniques combined with conditioning have been applied to Asian basket options in Deelstra et al. (2010). See also Dahl and Benth (2002) for the pricing of Asian basket options using quasi Monte Carlo simulation.

13.5 PDE Method

Two variables

The price at time t of the Asian call option with payoff (13.1) can be written as

$$
f(t, S_t, \Lambda_t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K \right)^+ \Bigg| \mathcal{F}_t \right], \quad 0 \leqslant t \leqslant T.
\tag{13.15}
$$

Next, we derive the Black-Scholes partial differential equation (PDE) for the value of a self-financing portfolio. Until the end of this chapter we model the asset price $(S_t)_{t\in[0,T]}$ as

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \qquad t \geqslant 0, \tag{13.16}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the historical probability measure $\mathbb{P}$.

Proposition 13.8. *Let $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ be a self-financing portfolio strategy whose value $V_t := \eta_t A_t + \xi_t S_t$, $t \geqslant 0$, takes the form*

$$V_t = f(t, S_t, \Lambda_t), \qquad t \geqslant 0,$$

where $f \in \mathcal{C}^{1,2,1}((0,T) \times (0,\infty)^2)$ is given by (13.15). *Then, the function $f(t,x,y)$ satisfies the PDE*

$$rf(t,x,y) = \frac{\partial f}{\partial t}(t,x,y) + x\frac{\partial f}{\partial y}(t,x,y) + rx\frac{\partial f}{\partial x}(t,x,y) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x,y), \tag{13.17}$$

$0 \leqslant t \leqslant T$, $x > 0$, *under the boundary conditions*

$$\begin{cases} f(t,0^+,y) = \lim_{x\searrow 0} f(t,x,y) = \mathrm{e}^{-(T-t)r}\left(\dfrac{y}{T} - K\right)^+, & \text{(13.18a)} \\[2mm] f(T,x,y) = \left(\dfrac{y}{T} - K\right)^+, & \text{(13.18b)} \end{cases}$$

$0 \leqslant t \leqslant T$, $x > 0$, $y \geqslant 0$,

and ξ_t is given by

$$\xi_t = \frac{\partial f}{\partial x}(t, S_t, \Lambda_t), \qquad 0 \leqslant t \leqslant T. \tag{13.19}$$

Proof. We note that the self-financing condition (5.9) implies

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t \\ &= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma\xi_t S_t dB_t, \qquad t \geqslant 0. \end{aligned} \tag{13.20}$$

Since $d\Lambda_t = S_t dt$, an application of Itô's formula to $f(t,x,y)$ leads to

$$dV_t = f(t, S_t, \Lambda_t) = \frac{\partial f}{\partial t}(t, S_t, \Lambda_t)dt + \frac{\partial f}{\partial y}(t, S_t, \Lambda_t)d\Lambda_t$$

↺

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$$+ \mu S_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t)dt + \frac{1}{2}S_t^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, \Lambda_t)dt + \sigma S_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t)dB_t$$

$$= \frac{\partial f}{\partial t}(t, S_t, \Lambda_t)dt + S_t \frac{\partial f}{\partial y}(t, S_t, \Lambda_t)dt$$

$$+ \mu S_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t)dt + \frac{1}{2}S_t^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, \Lambda_t)dt + \sigma S_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t)dB_t. \tag{13.21}$$

By respective identification of components in dB_t and dt in (13.20) and (13.21), we get

$$\begin{cases} r\eta_t A_t dt + \mu \xi_t S_t dt = \frac{\partial f}{\partial t}(t, S_t, \Lambda_t)dt + S_t \frac{\partial f}{\partial y}(t, S_t, \Lambda_t)dt + \mu S_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t)dt \\ \qquad\qquad + \frac{1}{2}S_t^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, \Lambda_t)dt, \\ \xi_t S_t \sigma dB_t = S_t \sigma \frac{\partial f}{\partial x}(t, S_t, \Lambda_t)dB_t, \end{cases}$$

hence

$$\begin{cases} rV_t - r\xi_t S_t = \frac{\partial f}{\partial t}(t, S_t, \Lambda_t) + S_t \frac{\partial f}{\partial y}(t, S_t, \Lambda_t)dt + \frac{1}{2}S_t^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, \Lambda_t), \\ \xi_t = \frac{\partial f}{\partial x}(t, S_t, \Lambda_t), \end{cases}$$

i.e.

$$\begin{cases} rf(t, S_t, \Lambda_t) = \frac{\partial f}{\partial t}(t, S_t, \Lambda_t) + S_t \frac{\partial f}{\partial y}(t, S_t, \Lambda_t) + rS_t \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) \\ \qquad\qquad + \frac{1}{2}S_t^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, \Lambda_t), \\ \xi_t = \frac{\partial f}{\partial x}(t, S_t, \Lambda_t). \end{cases}$$

$\square$

Remarks.

i) We have $\xi_T = 0$ at maturity from (13.18b) and (13.19), which is consistent with the fact that the Asian option is cash-settled at maturity and, close to maturity, its payoff $(\Lambda_T/T - K)^+$ becomes less dependent on the underlying asset price S_T.

ii) If $\Lambda_t/T \geqslant K$, by Exercise 13.8 we have

$$f(t, S_t, \Lambda_t) = e^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \bigg| \mathcal{F}_t\right]$$

$$= \mathrm{e}^{-(T-t)r}\left(\frac{\Lambda_t}{T} - K\right) + S_t \frac{1 - \mathrm{e}^{-(T-t)r}}{rT}, \quad (13.22)$$

$0 \leqslant t \leqslant T$. In particular, the function

$$f(t,x,y) = \mathrm{e}^{-(T-t)r}\left(\frac{y}{T} - K\right) + x\frac{1 - \mathrm{e}^{-(T-t)r}}{rT}, \qquad (13.23)$$

$0 \leqslant t \leqslant T$, $x > 0$, $y \geqslant KT$, solves the PDE (13.17), and (13.23) can be used to complete the boundary conditions (13.18a)-(13.18b).

iii) When $\Lambda_t/T \geqslant K$, the Delta ξ_t is given by

$$\xi_t = \frac{\partial f}{\partial x}(t, S_t, \Lambda_t) = \frac{1 - \mathrm{e}^{-(T-t)r}}{rT}, \qquad 0 \leqslant t \leqslant T. \qquad (13.24)$$

Next, we examine two methods which allow one to reduce the Asian option pricing PDE from three variables (t,x,y) to two variables (t,z). Reduction of dimensionality can be of crucial importance when applying discretization scheme whose complexity are of the form N^d where N is the number of discretization steps and d is the dimension of the problem (curse of dimensionality).

(1) One variable with time-independent coefficients

Following Lamberton and Lapeyre (1996), page 91, we define the auxiliary process

$$Z_t := \frac{1}{S_t}\left(\frac{1}{T}\int_0^t S_u du - K\right) = \frac{1}{S_t}\left(\frac{\Lambda_t}{T} - K\right), \qquad 0 \leqslant t \leqslant T.$$

With this notation, the price of the Asian call option at time t becomes

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \,\Big|\, \mathcal{F}_t\right] = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[S_T(Z_T)^+ \,\big|\, \mathcal{F}_t\right].$$

Lemma 13.9. *The price* (13.2) *at time t of the Asian call option with payoff* (13.1) *can be written as*

$$f(t, S_t, \Lambda_t) = S_t g(t, Z_t) \qquad\qquad\qquad (13.25)$$

$$= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \,\Big|\, \mathcal{F}_t\right], \qquad t \in [0,T],$$

with the relation

$$f(t,x,y) = xg\left(t, \frac{1}{x}\left(\frac{y}{T} - K\right)\right), \qquad x > 0,\ y \geqslant 0, \quad 0 \leqslant t \leqslant T,$$

where

$$g(t,z) = e^{-(T-t)r}\mathbb{E}^*\left[\left(z + \frac{1}{T}\int_0^{T-t}\frac{S_u}{S_0}du\right)^+\right] \qquad (13.26)$$

$$= e^{-(T-t)r}\mathbb{E}^*\left[\left(z + \frac{\Lambda_{T-t}}{S_0 T}\right)^+\right],$$

with the boundary condition

$$g(T,z) = z^+, \qquad z \in \mathbb{R}.$$

Proof. For $0 \leqslant s \leqslant t \leqslant T$, we have

$$d(S_t Z_t) = \frac{1}{T}d\left(\int_0^t S_u du - K\right) = \frac{S_t}{T}dt,$$

hence

$$S_t Z_t = S_s Z_s + \int_s^t d(S_u Z_u) = S_s Z_s + \int_s^t \frac{S_u}{T}du,$$

and therefore

$$\frac{S_t Z_t}{S_s} = Z_s + \frac{1}{T}\int_s^t \frac{S_u}{S_s}du, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Since for any $t \in [0,T]$, S_t is positive and $\mathcal{F}_t$-measurable, and S_u/S_t is independent of $\mathcal{F}_t$, $u \geqslant t$, we have:

$$e^{-(T-t)r}\mathbb{E}^*\left[S_T(Z_T)^+ \,\middle|\, \mathcal{F}_t\right] = e^{-(T-t)r}S_t\mathbb{E}^*\left[\left(\frac{S_T}{S_t}Z_T\right)^+ \,\middle|\, \mathcal{F}_t\right]$$

$$= e^{-(T-t)r}S_t\mathbb{E}^*\left[\left(Z_t + \frac{1}{T}\int_t^T \frac{S_u}{S_t}du\right)^+ \,\middle|\, \mathcal{F}_t\right]$$

$$= e^{-(T-t)r}S_t\mathbb{E}^*\left[\left(z + \frac{1}{T}\int_t^T \frac{S_u}{S_t}du\right)^+\right]_{z=Z_t}$$

$$= e^{-(T-t)r}S_t\mathbb{E}^*\left[\left(z + \frac{1}{T}\int_0^{T-t} \frac{S_u}{S_0}du\right)^+\right]_{z=Z_t}$$

$$= e^{-(T-t)r}S_t\mathbb{E}^*\left[\left(z + \frac{\Lambda_{T-t}}{S_0 T}\right)^+\right]_{z=Z_t}$$

$$= S_t g(t, Z_t),$$

which proves (13.26). □

When $\Lambda_t/T \geqslant K$ we have $Z_t \geqslant 0$, hence in this case by (13.22) and (13.25) we find

$$g(t, Z_t) = e^{-(T-t)r} Z_t + \frac{1 - e^{-(T-t)r}}{rT}, \qquad 0 \leqslant t \leqslant T. \qquad (13.27)$$

Note that as in (13.11), $g(t, z)$ can be computed from the probability density function (13.10) of Λ_{T-t}, as

$$g(t, z) = \mathbb{E}^* \left[\left(z + \frac{\Lambda_{T-t}}{S_0 T} \right)^+ \right]$$

$$= \int_0^\infty \left(z + \frac{u}{T} \right)^+ d\mathbb{P} \left(\frac{\Lambda_t}{S_0} \leqslant u \right)$$

$$= e^{-p^2 \sigma^2 t/8}$$

$$\times \int_0^\infty \left(z + \frac{u}{T} \right)^+ \int_0^\infty v^{-1-p} \exp \left(-2 \frac{1+v^2}{\sigma^2} \right) \theta \left(\frac{4v}{\sigma^2 u}, (T-t) \frac{\sigma^2}{4} \right) dv \frac{du}{u}$$

$$= e^{-p^2 \sigma^2 t/8}$$

$$\times \int_{(-zT)\vee 0}^\infty \left(z + \frac{u}{T} \right) \int_0^\infty v^{-1-p} \exp \left(-2 \frac{1+v^2}{\sigma^2} \right) \theta \left(\frac{4v}{\sigma^2 u}, (T-t) \frac{\sigma^2}{4} \right) dv \frac{du}{u}$$

$$= z \, e^{-p^2 \sigma^2 t/8} \int_{(-zT)\vee 0}^\infty \int_0^\infty v^{-1-p} \exp \left(-2 \frac{1+v^2}{\sigma^2} \right) \theta \left(\frac{4v}{\sigma^2 u}, (T-t) \frac{\sigma^2}{4} \right) dv \frac{du}{u}$$

$$+ \frac{1}{T} e^{-p^2 \sigma^2 t/8} \int_{(-zT)\vee 0}^\infty \int_0^\infty v^{-1-p} \exp \left(-2 \frac{1+v^2}{\sigma^2} \right) \theta \left(\frac{4v}{\sigma^2 u}, (T-t) \frac{\sigma^2}{4} \right) dv du.$$

The next proposition gives a replicating hedging strategy for Asian options.

Proposition 13.10. *(Rogers and Shi (1995)). Let $(\eta_t, \xi_t)_{t \in \mathbb{R}_+}$ be a self-financing portfolio strategy whose value $V_t := \eta_t A_t + \xi_t S_t$, $t \in [0, T]$, is given by*

$$V_t = S_t g(t, Z_t) = S_t g \left(t, \frac{1}{S_t} \left(\frac{\Lambda_t}{T} - K \right) \right), \qquad 0 \leqslant t \leqslant T,$$

where $g \in \mathcal{C}^{1,2}((0, T) \times (0, \infty))$ is given by (13.26). Then, the function $g(t, z)$ satisfies the PDE

$$\frac{\partial g}{\partial t}(t, z) + \left(\frac{1}{T} - rz \right) \frac{\partial g}{\partial z}(t, z) + \frac{1}{2} \sigma^2 z^2 \frac{\partial^2 g}{\partial z^2}(t, z) = 0, \qquad (13.28)$$

$0 \leqslant t \leqslant T$, *under the terminal condition*

$$g(T, z) = z^+, \qquad z \in \mathbb{R}, \qquad (13.29)$$

and the corresponding replicating portfolio Delta is given by

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$$\xi_t = g(t, Z_t) - Z_t \frac{\partial g}{\partial z}(t, Z_t), \qquad 0 \leqslant t \leqslant T. \qquad (13.30)$$

Proof. By (13.16) and the Itô formula applied to $1/S_t$, we have

$$
\begin{aligned}
d\left(\frac{1}{S_t}\right) &= -\frac{dS_t}{(S_t)^2} + \frac{2}{2}\frac{(dS_t)^2}{(S_t)^3} \\
&= \frac{1}{S_t}\left(\left(-\mu + \sigma^2\right) dt - \sigma dB_t\right),
\end{aligned}
$$

hence

$$
\begin{aligned}
dZ_t &= d\left(\frac{1}{S_t}\left(\frac{\Lambda_t}{T} - K\right)\right) \\
&= d\left(\frac{\Lambda_t}{T S_t} - \frac{K}{S_t}\right) \\
&= \frac{1}{T}d\left(\frac{\Lambda_t}{S_t}\right) - K d\left(\frac{1}{S_t}\right) \\
&= \frac{1}{T}\frac{d\Lambda_t}{S_t} + \left(\frac{\Lambda_t}{T} - K\right) d\left(\frac{1}{S_t}\right) \\
&= \frac{dt}{T} + S_t Z_t d\left(\frac{1}{S_t}\right) \\
&= \frac{dt}{T} + Z_t\left(-\mu + \sigma^2\right) dt - Z_t \sigma dB_t.
\end{aligned}
$$

By the self-financing condition (5.9) we have

$$
\begin{aligned}
dV_t &= \eta_t dA_t + \xi_t dS_t \\
&= r\eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t \\
&= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t, \qquad t \geqslant 0.
\end{aligned} \qquad (13.31)
$$

Another application of Itô's formula to $f(t, S_t, Z_t) = S_t g(t, Z_t)$ leads to

$$
\begin{aligned}
&d(S_t g(t, Z_t)) = g(t, Z_t) dS_t + S_t dg(t, Z_t) + dS_t \bullet dg(t, Z_t) \\
&= g(t, Z_t) dS_t + S_t \frac{\partial g}{\partial t}(t, Z_t) dt + S_t \frac{\partial g}{\partial z}(t, Z_t) dZ_t \\
&\quad + \frac{1}{2}S_t \frac{\partial^2 g}{\partial z^2}(t, Z_t)(dZ_t)^2 + dS_t \bullet dg(t, Z_t) \\
&= \mu S_t g(t, Z_t) dt + \sigma S_t g(t, Z_t) dB_t + S_t \frac{\partial g}{\partial t}(t, Z_t) dt \\
&\quad + S_t Z_t\left(-\mu + \sigma^2\right)\frac{\partial g}{\partial z}(t, Z_t) dt + \frac{1}{T}S_t \frac{\partial g}{\partial z}(t, Z_t) dt - \sigma S_t Z_t \frac{\partial g}{\partial z}(t, Z_t) dB_t \\
&\quad + \frac{1}{2}\sigma^2 Z_t^2 S_t \frac{\partial^2 g}{\partial z^2}(t, Z_t) dt - \sigma^2 S_t Z_t \frac{\partial g}{\partial z}(t, Z_t) dt
\end{aligned}
$$

$$= \mu S_t g(t, Z_t) dt + S_t \frac{\partial g}{\partial t}(t, Z_t) dt + S_t Z_t \left(-\mu + \sigma^2\right) \frac{\partial g}{\partial z}(t, Z_t) dt + \frac{1}{T} S_t \frac{\partial g}{\partial z}(t, Z_t) dt$$

$$+ \frac{1}{2} \sigma^2 Z_t^2 S_t \frac{\partial^2 g}{\partial z^2}(t, Z_t) dt - \sigma^2 S_t Z_t \frac{\partial g}{\partial z}(t, Z_t) dt$$

$$+ \sigma S_t g(t, Z_t) dB_t - \sigma S_t Z_t \frac{\partial g}{\partial z}(t, Z_t) dB_t. \tag{13.32}$$

By respective identification of components in dB_t and dt in (13.31) and (13.32), we get

$$\begin{cases} r\eta_t A_t + \mu \xi_t S_t = \mu S_t g(t, Z_t) + S_t \dfrac{\partial g}{\partial t}(t, Z_t) - \mu S_t Z_t \dfrac{\partial g}{\partial z}(t, Z_t) \\ \qquad\qquad\qquad + \dfrac{1}{T} S_t \dfrac{\partial g}{\partial z}(t, Z_t) + \dfrac{1}{2} \sigma^2 Z_t^2 S_t \dfrac{\partial^2 g}{\partial z^2}(t, Z_t), \\ \\ \xi_t S_t \sigma = \sigma S_t g(t, Z_t) - \sigma S_t Z_t \dfrac{\partial g}{\partial z}(t, Z_t), \end{cases}$$

hence

$$\begin{cases} rV_t - r\xi_t S_t = S_t \dfrac{\partial g}{\partial t}(t, Z_t) + \dfrac{1}{T} S_t \dfrac{\partial g}{\partial z}(t, Z_t) + \dfrac{1}{2} \sigma^2 Z_t^2 S_t \dfrac{\partial^2 g}{\partial z^2}(t, Z_t), \\ \xi_t = g(t, Z_t) - Z_t \dfrac{\partial g}{\partial z}(t, Z_t), \end{cases}$$

i.e.

$$\begin{cases} \dfrac{\partial g}{\partial t}(t, z) + \left(\dfrac{1}{T} - rz \right) \dfrac{\partial g}{\partial z}(t, z) + \dfrac{1}{2} \sigma^2 z^2 \dfrac{\partial^2 g}{\partial z^2}(t, z) = 0, \\ \xi_t = g(t, Z_t) - Z_t \dfrac{\partial g}{\partial z}(t, Z_t), \end{cases}$$

under the terminal condition $g(T, z) = z^+$, $z \in \mathbb{R}$, which follows from (13.26).
$\square$

When $\Lambda_t / T \geqslant K$ we have $Z_t \geqslant 0$ and (13.27) and (13.30) show that

$$\begin{aligned} \xi_t &= g(t, Z_t) - Z_t \frac{\partial g}{\partial z}(t, Z_t) \\ &= \mathrm{e}^{-(T-t)r} Z_t + \frac{1 - \mathrm{e}^{-(T-t)r}}{rT} - \mathrm{e}^{-(T-t)r} Z_t \\ &= \frac{1 - \mathrm{e}^{-(T-t)r}}{rT}, \qquad 0 \leqslant t \leqslant T, \end{aligned}$$

which recovers (13.24). Similarly, from (13.29) we recover

$$\xi_T = g(T, Z_T) - Z_T \frac{\partial g}{\partial z}(T, Z_T) = Z_T \mathbb{1}_{\{Z_T \geqslant 0\}} - Z_T \mathbb{1}_{\{Z_T \geqslant 0\}} = 0$$

$\circlearrowleft$

at maturity.

We also check that

$$\begin{aligned}
\xi_t &= \mathrm{e}^{-(T-t)r}\sigma S_t\frac{\partial f}{\partial x}f(t,S_t,Z_t) - \sigma Z_t\frac{\partial f}{\partial z}f(t,S_t,Z_t) \\
&= \mathrm{e}^{-(T-t)r}\left(-Z_t\frac{\partial g}{\partial z}(t,Z_t) + g(t,Z_t)\right) \\
&= \mathrm{e}^{-(T-t)r}\left(S_t\frac{\partial g}{\partial x}\left(t,\frac{1}{x}\left(\frac{1}{T}\int_0^t S_u du - K\right)\right)_{|x=S_t} + g(t,Z_t)\right) \\
&= \frac{\partial}{\partial x}\left(x\,\mathrm{e}^{-(T-t)r}g\left(t,\frac{1}{x}\left(\frac{1}{T}\int_0^t S_u du - K\right)\right)\right)_{|x=S_t}, \quad 0 \leqslant t \leqslant T.
\end{aligned}$$

We also find that the amount invested on the riskless asset is given by

$$\eta_t A_t = Z_t S_t\frac{\partial g}{\partial z}(t,Z_t).$$

Next we note that a PDE with no first-order derivative term can be obtained using time-dependent coefficients.

(2) One variable with time-dependent coefficients

Define now the auxiliary process

$$\begin{aligned}
U_t &:= \frac{1 - \mathrm{e}^{-(T-t)r}}{rT} + \mathrm{e}^{-(T-t)r}\frac{1}{S_t}\left(\frac{1}{T}\int_0^t S_u du - K\right) \\
&= \frac{1}{rT}\left(1 - \mathrm{e}^{-(T-t)r}\right) + \mathrm{e}^{-(T-t)r}Z_t, \quad 0 \leqslant t \leqslant T,
\end{aligned}$$

i.e.

$$Z_t = \mathrm{e}^{(T-t)r}U_t + \frac{\mathrm{e}^{(T-t)r} - 1}{rT}, \qquad 0 \leqslant t \leqslant T.$$

We have

$$\begin{aligned}
dU_t &= -\frac{1}{T}\mathrm{e}^{-(T-t)r}dt + r\,\mathrm{e}^{-(T-t)r}Z_t dt + \mathrm{e}^{-(T-t)r}dZ_t \\
&= \mathrm{e}^{-(T-t)r}\sigma^2 Z_t dt - \mathrm{e}^{-(T-t)r}\sigma Z_t dB_t - (\mu - r)\mathrm{e}^{-(T-t)r}Z_t dt \\
&= -\mathrm{e}^{-(T-t)r}\sigma Z_t d\widehat{B}_t, \qquad t \geqslant 0,
\end{aligned}$$

where

$$d\widehat{B}_t = dB_t - \sigma dt + \frac{\mu - r}{\sigma}dt = d\widetilde{B}_t - \sigma dt$$

is a standard Brownian motion under

$$d\widehat{\mathbb{P}} = e^{\sigma B_T - \sigma^2 t/2} d\mathbb{P}^* = e^{-rT} \frac{S_T}{S_0} d\mathbb{P}^*.$$

Lemma 13.11. *The Asian call option price can be written as*

$$S_t h(t, U_t) = e^{-(T-t)r} \mathbb{E}^* \left[\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \bigg| \mathcal{F}_t \right],$$

where the function $h(t, y)$ is given by

$$h(t, y) = \widehat{\mathbb{E}}\big[(U_T)^+ \,\big|\, U_t = y\big], \qquad 0 \leqslant t \leqslant T. \qquad (13.33)$$

Proof. We have

$$U_T = \frac{1}{S_T} \left(\frac{1}{T} \int_0^T S_u du - K \right) = Z_T,$$

and

$$\frac{d\widehat{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} = e^{(B_T - B_t)\sigma - (T-t)\sigma^2/2} = \frac{e^{-rT} S_T}{e^{-rt} S_t},$$

hence the price of the Asian call option is

$$\begin{aligned}
e^{-(T-t)r} \mathbb{E}^*[S_T(Z_T)^+ \mid \mathcal{F}_t] &= e^{-(T-t)r} \mathbb{E}^*[S_T(U_T)^+ \mid \mathcal{F}_t] \\
&= S_t \mathbb{E}^* \left[\frac{e^{-rT} S_T}{e^{-rt} S_t} (U_T)^+ \bigg| \mathcal{F}_t \right] \\
&= S_t \mathbb{E}^* \left[\frac{d\widehat{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} (U_T)^+ \bigg| \mathcal{F}_t \right] \\
&= S_t \widehat{\mathbb{E}}[(U_T)^+ \mid \mathcal{F}_t].
\end{aligned}$$

$\square$

The next proposition gives a replicating hedging strategy for Asian options. See § 7.5.3 of Shreve (2004) and references therein for a different derivation of the PDE (13.34).

Proposition 13.12. *(Večeř (2001)). Let $(\eta_t, \xi_t)_{t \in \mathbb{R}_+}$ be a self-financing portfolio strategy whose value $V_t := \eta_t A_t + \xi_t S_t$, $t \geqslant 0$, is given by*

$$V_t = S_t h(t, U_t), \qquad t \geqslant 0,$$

where $h \in \mathcal{C}^{1,2}((0, T) \times (0, \infty))$ is given by (13.33). Then, the function $h(t, z)$ satisfies the PDE

$$\frac{\partial h}{\partial t}(t,y) + \frac{\sigma^2}{2}\left(\frac{1-e^{-(T-t)r}}{rT} - y\right)^2 \frac{\partial^2 h}{\partial y^2}(t,y) = 0, \qquad (13.34)$$

under the terminal condition

$$h(T,z) = z^+,$$

and the corresponding replicating portfolio is given by

$$\xi_t = h(t,U_t) - Z_t \frac{\partial h}{\partial y}(t,U_t), \qquad 0 \leqslant t \leqslant T.$$

Proof. By the self-financing condition (13.20) we have

$$\begin{aligned}
dV_t &= r\eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t \\
&= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t,
\end{aligned} \qquad (13.35)$$

$t \geqslant 0$. By Itô's formula we get

$$\begin{aligned}
&d(S_t h(t,U_t)) = h(t,U_t)dS_t + S_t dh(t,U_t) + dS_t \bullet dh(t,U_t) \\
&= \mu S_t h(t,U_t)dt + \sigma S_t h(t,U_t)dB_t \\
&\quad + S_t \left(\frac{\partial h}{\partial t}(t,U_t)dt + \frac{\partial h}{\partial y}(t,U_t)dU_t + \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(t,U_t)(dU_t)^2\right) \\
&\quad + \frac{\partial h}{\partial y}(t,U_t)dS_t \bullet dU_t \\
&= \mu S_t h(t,U_t)dt + \sigma S_t h(t,U_t)dB_t - S_t(\mu - r)\frac{\partial h}{\partial y}(t,U_t)Z_t dt \\
&\quad + S_t \left(\frac{\partial h}{\partial t}(t,U_t)dt - \sigma \frac{\partial h}{\partial y}(t,U_t)Z_t d\widetilde{B}_t + \frac{\sigma^2}{2}Z_t^2 \frac{\partial^2 h}{\partial y^2}(t,U_t)dt\right) \\
&\quad - \sigma^2 S_t \frac{\partial h}{\partial y}(t,U_t)Z_t dt \\
&= \mu S_t h(t,U_t)dt + \sigma S_t h(t,U_t)dB_t - S_t(\mu - r)\frac{\partial h}{\partial y}(t,U_t)Z_t dt \\
&\quad + S_t \left(\frac{\partial h}{\partial t}(t,U_t)dt - \sigma \frac{\partial h}{\partial y}(t,U_t)Z_t(dB_t - \sigma dt) + \frac{\sigma^2}{2}Z_t^2 \frac{\partial^2 h}{\partial y^2}(t,U_t)dt\right) \\
&\quad - \sigma^2 S_t \frac{\partial h}{\partial y}(t,U_t)Z_t dt. \qquad (13.36)
\end{aligned}$$

By respective identification of components in dB_t and dt in (13.35) and (13.36), we get

$$\begin{cases} r\eta_t A_t + \mu\xi_t S_t = \mu S_t h(t, U_t) - (\mu - r)S_t Z_t \dfrac{\partial h}{\partial y}(t, U_t)dt + S_t \dfrac{\partial h}{\partial t}(t, U_t) \\ \qquad\qquad + \dfrac{\sigma^2}{2} S_t Z_t^2 \dfrac{\partial^2 h}{\partial y^2}(t, U_t), \\ \\ \xi_t = h(t, U_t) - Z_t \dfrac{\partial h}{\partial y}(t, U_t), \end{cases}$$

hence

$$\begin{cases} r\eta_t A_t = -rS_t(\xi_t - h(t, U_t)) + S_t \dfrac{\partial h}{\partial t}(t, U_t) + \dfrac{\sigma^2}{2} S_t Z_t^2 \dfrac{\partial^2 h}{\partial y^2}(t, U_t), \\ \xi_t = h(t, U_t) - Z_t \dfrac{\partial h}{\partial y}(t, U_t), \end{cases}$$

and

$$\begin{cases} \dfrac{\partial h}{\partial t}(t, y) + \dfrac{\sigma^2}{2} \left(\dfrac{1 - \mathrm{e}^{-(T-t)r}}{rT} - y \right)^2 \dfrac{\partial^2 h}{\partial y^2}(t, y) = 0, \\ \xi_t = h(t, U_t) + \left(\dfrac{1 - \mathrm{e}^{-(T-t)r}}{rT} - U_t \right) \dfrac{\partial h}{\partial y}(t, U_t), \end{cases}$$

under the terminal condition

$$h(T, z) = z^+.$$

$\square$

We also find the riskless portfolio allocation

$$\eta_t A_t = \mathrm{e}^{(T-t)r} S_t \left(U_t - \dfrac{1 - \mathrm{e}^{-(T-t)r}}{rT} \right) \dfrac{\partial h}{\partial y}(t, U_t) = S_t Z_t \dfrac{\partial h}{\partial y}(t, U_t).$$

Various implementations of Asian pricing methods can be found in this **IPython notebook** (right-click to save as attachment - may not work on 🍎). See also the **Premia** website.

Exercises

Exercise 13.1   Compute the first and second moments of the time integral $\int_\tau^T S_t dt$ for $\tau \in [0, T)$, where $(S_t)_{t \in \mathbb{R}_+}$ is the geometric Brownian motion $S_t := S_0 \mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}$, $t \geqslant 0$.

Exercise 13.2 Consider the short rate process $r_t = \sigma B_t$, where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion.

a) Find the probability distribution of the time integral $\int_0^T r_s ds$.

b) Compute the price

$$\mathrm{e}^{-rT} \mathbb{E}^* \left[\left(\int_0^T r_u du - K \right)^+ \right]$$

of a caplet on the forward rate $\int_0^T r_s ds$.

Exercise 13.3 Asian call option with a *negative* strike price. Consider the asset price process

$$S_t = S_0 \, \mathrm{e}^{rt + \sigma B_t - \sigma^2 t/2}, \qquad t \geqslant 0,$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion. Assuming that $K \leqslant 0$, compute the price

$$\mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \bigg| \mathcal{F}_t \right]$$

of the Asian option at time $t \in [0, T]$.

Exercise 13.4 Consider the Asian forward contract with payoff

$$\frac{1}{T} \int_0^T S_u du - K, \tag{13.37}$$

where $S_u = S_0 \, \mathrm{e}^{\sigma B_u + ru - \sigma^2 u/2}$, $u \geqslant 0$, and $(B_u)_{u \in \mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{P}^*$.

a) Price the Asian forward contract at any time $t \in [0, T]$.

b) Derive a call-put parity relation between the prices

$$C(t, K) := \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \bigg| \mathcal{F}_t \right]$$

and

$$P(t, K) := \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\left(K - \frac{1}{T} \int_0^T S_u du \right)^+ \bigg| \mathcal{F}_t \right]$$

of Asian call and put options.

c) Find the self-financing portfolio strategy $(\xi_t)_{t \in [0,T]}$ hedging the Asian forward contract with payoff (13.37), where ξ_t denotes the quantity invested at time $t \in [0, T]$ in the risky asset S_t.

d) Compute the numerical value of the price

$$e^{-(T-t)r}\mathbb{E}^*\left[\frac{1}{T}\int_0^T S_u du - K \,\bigg|\, \mathcal{F}_t\right]$$

of the long forward Asian contract on Light Sweet Crude Oil Futures (CLZ22.NYM) using the following market data:

Issue date: 2022-01-01,
Maturity $T =$2022-12-31,
Strike price $K =$\$80,
Interest rate $r =$2% per year,
$t =$2022-04-01,
Number of business days per month: 21, per year: 252.

```
1  library(quantmod)
   getSymbols("CLZ22.NYM",from="2022-01-01",to="2022-04-01",src="yahoo")
3  futures=Cl(`CLZ22.NYM`)
   chartSeries(futures,up.col="blue",theme="white")
5  n = length(!is.na(futures))
```

Exercise 13.5 Compute the price

$$e^{-(T-t)r}\mathbb{E}^*\left[\left(\exp\left(\frac{1}{T}\int_0^T \log S_u du\right) - K\right)^+ \,\bigg|\, \mathcal{F}_t\right], \qquad 0 \leqslant t \leqslant T.$$

at time t of the geometric Asian option with maturity T, where $S_t = S_0\,e^{rt+\sigma B_t - \sigma^2 t/2}$, $t \in [0, T]$.

Hint: When $X \simeq \mathcal{N}(0, v^2)$ is a centered Gaussian random variable with variance $v^2 > 0$, we have

$$\mathbb{E}^*[(e^{m+X} - K)^+] = e^{m+v^2/2}\Phi(v + (m - \log K)/v) - K\Phi((m - \log K)/v).$$

Exercise 13.6 Consider a CIR process $(r_t)_{t\in\mathbb{R}_+}$ given by

$$dr_t = -\lambda(r_t - m)dt + \sigma\sqrt{r_t}dB_t, \tag{13.38}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, and let

$$\Lambda_t := \frac{1}{T - \tau}\int_\tau^t r_s ds, \qquad t \in [\tau, T].$$

Compute the price at time $t \in [\tau, T]$ of the Asian option with payoff $(\Lambda_T - K)^+$, under the condition $\Lambda_t \geqslant K$.

↺

Exercise 13.7 Consider an asset price $(S_t)_{t\in\mathbb{R}_+}$ which is a *sub*martingale under the risk-neutral probability measure $\mathbb{P}^*$, in a market with risk-free interest rate $r > 0$, and let $\phi(x) = (x - K)^+$ be the (convex) payoff function of the European call option.

Show that, for any sequence $0 < T_1 < \cdots < T_n$, the price of the option on average with payoff

$$\phi\left(\frac{S_{T_1} + \cdots + S_{T_n}}{n}\right)$$

can be upper bounded by the price of the European call option with maturity T_n, *i.e.* show that

$$\mathbb{E}^*\left[\phi\left(\frac{S_{T_1} + \cdots + S_{T_n}}{n}\right)\right] \leqslant \mathbb{E}^*[\phi(S_{T_n})].$$

Exercise 13.8 Let $(S_t)_{t\in\mathbb{R}_+}$ denote a risky asset whose price S_t is given by

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$. Compute the price at time $t \in [\tau, T]$ of the Asian option with payoff

$$\left(\frac{1}{T-\tau}\int_\tau^T S_u du - K\right)^+,$$

under the condition that

$$A_t := \frac{1}{T-\tau}\int_\tau^t S_u du \geqslant K.$$

Exercise 13.9 Pricing Asian options by PDEs. Show that the functions $g(t, z)$ and $h(t, y)$ are linked by the relation

$$g(t, z) = h\left(t, \frac{1 - e^{-(T-t)r}}{rT} + e^{-(T-t)r}z\right), \qquad 0 \leqslant t \leqslant T, \quad z > 0,$$

and that the PDE (1.35) for $h(t, y)$ can be derived from the PDE (1.33) for $g(t, z)$ and the above relation.

Exercise 13.10 (Brown et al. (2016)) Given $S_t := S_0 e^{\sigma B_t + rt - \sigma^2 t/2}$ a geometric Brownian motion and letting

$$\widetilde{Z}_t := \frac{e^{-(T-t)r}}{S_t}\left(\frac{1}{T}\int_0^t S_u du - K\right) = \frac{e^{-(T-t)r}}{S_t}\left(\frac{\Lambda_t}{T} - K\right), \qquad 0 \leqslant t \leqslant T,$$

find the PDE satisfied by the pricing function $\widetilde{g}(t, z)$ such that

$$S_t \widetilde{g}(t, \widetilde{Z}_t) = \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \Bigg| \mathcal{F}_t \right].$$

Exercise 13.11 Hedging Asian options (Yang et al. (2011)).

a) Compute the Asian option price $f(t, S_t, \Lambda_t)$ when $\Lambda_t / T \geqslant K$.

b) Compute the hedging portfolio allocation (ξ_t, η_t) when $\Lambda_t / T \geqslant K$.

c) At maturity we have $f(T, S_T, \Lambda_T) = (\Lambda_T / T - K)^+$, hence $\xi_T = 0$ and

$$\eta_T A_T = A_T \frac{\mathrm{e}^{-rT}}{A_0} \left(\frac{\Lambda_T}{T} - K \right) \mathbb{1}_{\{\Lambda_T > KT\}} = \left(\frac{\Lambda_T}{T} - K \right)^+.$$

d) Show that the Asian option with payoff $(\Lambda_T - K)^+$ can be hedged by the self-financing portfolio

$$\xi_t = \frac{1}{S_t} \left(f(t, S_t, \Lambda_t) - \mathrm{e}^{-(T-t)r} \left(\frac{\Lambda_t}{T} - K \right) h \left(t, \frac{1}{S_t} \left(\frac{\Lambda_t}{T} - K \right) \right) \right)$$

in the asset S_t and

$$\eta_t = \frac{\mathrm{e}^{-rT}}{A_0} \left(\frac{\Lambda_t}{T} - K \right) h \left(t, \frac{1}{S_t} \left(\frac{\Lambda_t}{T} - K \right) \right), \qquad 0 \leqslant t \leqslant T,$$

in the riskless asset $A_t = A_0 \mathrm{e}^{rt}$, where $h(t, z)$ is solution to a partial differential equation to be written explicitly.

Exercise 13.12 Asian options with dividends. Consider an underlying asset price process $(S_t)_{t \in \mathbb{R}_+}$ modeled as $dS_t = (\mu - \delta) S_t dt + \sigma S_t dB_t$, where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion and $\delta > 0$ is a continuous-time dividend rate.

a) Write down the self-financing condition for the portfolio value $V_t = \xi_t S_t + \eta_t A_t$ with $A_t = A_0 \mathrm{e}^{rt}$, assuming that all dividends are reinvested.

b) Derive the Black-Scholes PDE for the function $g_\delta(t, x, y)$ such that $V_t = g_\delta(t, S_t, \Lambda_t)$ at time $t \in [0, T]$.

```r
1  install.packages("quantmod")
   library(quantmod)
3  getDividends("Z74.SI",from="2018-01-01",to="2018-12-31",src="yahoo")
   getSymbols("Z74.SI",from="2018-11-16",to="2018-12-19",src="yahoo")
5  T <- chart_theme(); T$col$line.col <- "black"
   chart_Series(Op(`Z74.SI`),name="Opening prices (black) - Closing prices
        (blue)",lty=4,theme=T)
7  add_TA(Cl(`Z74.SI`),lwd=2,lty=5,legend='Difference',col="blue",on = 1)
```

Z74.SI.div
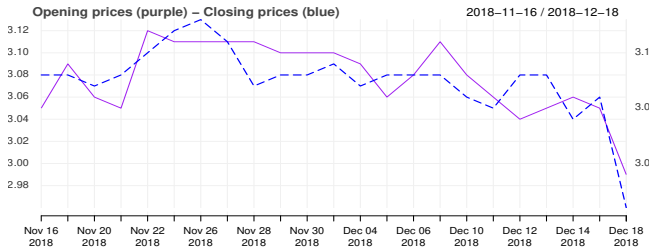2018-07-26 0.107
2018-12-17 0.068
2018-12-18 0.068

Fig. 13.6: SGD0.068 dividend detached on 18 Dec 2018 on Z74.SI.

The difference between the closing price on Dec 17 ($3.06) and the opening price on Dec 18 ($2.99) is $3.06 - $2.99 = 0.07. The adjusted price on Dec 17 ($2.992) is the closing price ($3.06) minus the dividend ($0.068).

Z74.SI	Open	High	Low	Close	Volume	Adjusted (ex-dividend)
2018-12-17	3.05	3.08	3.05	**3.06**	17441000	**2.992**
2018-12-18	**2.99**	2.99	2.96	2.96	28456400	2.960

The dividend rate α is given by $\alpha = 0.068/3.06 = 2.22\%$.