

Chapter 17

Short Rates and Bond Pricing

Short-term rates, typically daily rates, are the interest rates applied to short-term lending between financial institutions. The stochastic modeling of short-term interest rate processes is based on the mean reversion property, as in the Vasicek, CIR, CEV, and affine-type models studied in this chapter. The pricing of fixed income products, such as bonds, is considered in this framework using probabilistic and PDE arguments.

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17.1 Short-Term Mean-Reverting Models

Money market accounts with price $(A_t)_{t \in \mathbb{R}_+}$ can be defined from a short-term interest rate process $(r_t)_{t \in \mathbb{R}_+}$ as

$$\frac{A_{t+dt} - A_t}{A_t} = r_t dt, \quad \frac{dA_t}{A_t} = r_t dt, \quad \frac{dA_t}{dt} = r_t A_t, \quad t \geq 0,$$

with

$$A_t = A_0 \exp \left(\int_0^t r_s ds \right), \quad t \geq 0.$$

As short-term interest rates behave differently from stock prices, they require the development of specific models to account for properties such as positivity, boundedness, and return to equilibrium.

Vašíček (1977) model

The first model to capture the mean reversion property of interest rates, a property not possessed by geometric Brownian motion, is the Vašíček (1977) model, which is based on the Ornstein-Uhlenbeck process. Here, the short-term interest rate process $(r_t)_{t \in \mathbb{R}_+}$ solves the equation

$$dr_t = (a - br_t)dt + \sigma dB_t, \quad (17.1)$$

where $a, \sigma \in \mathbb{R}$, $b > 0$, and $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion, with solution

$$r_t = r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt}) + \sigma \int_0^t e^{-(t-s)b} dB_s, \quad t \geq 0, \quad (17.2)$$

see Exercise 17.1. The probability distribution of r_t is Gaussian at all times t , with mean

$$\mathbb{E}[r_t] = r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt}),$$

and variance given from the Itô isometry (4.16) as

$$\begin{aligned} \text{Var}[r_t] &= \text{Var}\left[\sigma \int_0^t e^{-(t-s)b} dB_s\right] \\ &= \sigma^2 \int_0^t (e^{-(t-s)b})^2 ds \\ &= \sigma^2 \int_0^t e^{-2bs} ds \\ &= \frac{\sigma^2}{2b}(1 - e^{-2bt}), \quad t \geq 0, \end{aligned}$$

i.e.

$$r_t \simeq \mathcal{N}\left(r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt}), \frac{\sigma^2}{2b}(1 - e^{-2bt})\right), \quad t > 0.$$

In particular, the probability density function $f_t(x)$ of r_t at time $t > 0$ is given by

$$f_t(x) = \frac{\sqrt{b/\pi}}{\sigma\sqrt{1 - e^{-2bt}}} \exp\left(-\frac{(r_0 e^{-bt} + a(1 - e^{-bt})/b - x)^2}{\sigma^2(1 - e^{-2bt})/b}\right), \quad x \in \mathbb{R}.$$

In the long run,* i.e. as time t becomes large we have, assuming $b > 0$,

$$\lim_{t \rightarrow \infty} \mathbb{E}[r_t] = \frac{a}{b} \quad \text{and} \quad \lim_{t \rightarrow \infty} \text{Var}[r_t] = \frac{\sigma^2}{2b}, \quad (17.3)$$

* "But this *long run* is a misleading guide to current affairs. *In the long run* we are all dead." Keynes (1924), Ch. 3, p. 80.



and this distribution converges to the Gaussian $\mathcal{N}(a/b, \sigma^2/(2b))$ distribution, which is also the *invariant* (or stationary) distribution of $(r_t)_{t \in \mathbb{R}_+}$, see Exercise 17.1. In addition, the process tends to revert to its long term mean $a/b = \lim_{t \rightarrow \infty} \mathbb{E}[r_t]$ which makes the average drift vanish, *i.e.*:

$$\lim_{t \rightarrow \infty} \mathbb{E}[a - br_t] = a - b \lim_{t \rightarrow \infty} \mathbb{E}[r_t] = 0.$$

Figure 17.1 presents a random simulation of $t \mapsto r_t$ in the Vasicek model with $r_0 = 3\%$, and shows the mean-reverting property of the process with respect to $a/b = 2.5\%$.

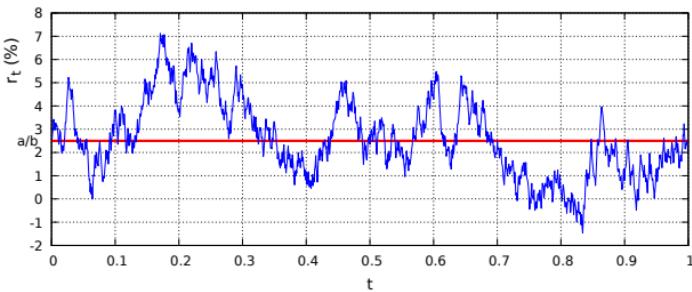


Fig. 17.1: Graph of the Vasicek short rate $t \mapsto r_t$ with $a = 0.025$, $b = 1$, and $\sigma = 0.1$.

As can be checked from the simulation of Figure 17.1 the value of r_t in the Vasicek model may become negative due to its Gaussian distribution. Although real interest rates may sometimes fall below zero,* this can be regarded as a potential drawback of the Vasicek model.

The next code provides a numerical solution of the Vasicek stochastic differential equation (17.1) using the Euler method, see Figure 17.1. For this, we discretize (17.1) according to a discrete-time sequence $(t_k)_{k \geq 0} = (t_0, t_1, t_2, \dots)$ of time instants, as

$$r_{t_{k+1}} - r_{t_k} = (a - br_{t_k})\Delta t + \sigma Z_k, \quad k \geq 0,$$

where $\Delta t := t_{k+1} - t_k$ and $(Z_k)_{k \geq 0}$ is a Gaussian white noise with variance Δt , *i.e.* a sequence of independent, centered and identically distributed $\mathcal{N}(0, \Delta t)$ Gaussian random variables, which yields

$$r_{t_{k+1}} = r_{t_k} + (a - br_{t_k})\Delta t + \sigma Z_k = a\Delta t + (1 - b\Delta t)r_{t_k} + \sigma Z_k, \quad k \geq 0. \quad (17.4)$$

* Eurozone interest rates turned negative in 2014.

```

1 N=10000;t<-0:(N-1);dt<-1.0/N;nsim<-2; a=0.025;b=1;sigma=0.1;
2 dB <- matrix(rnorm(nsim*N,mean=0,sd=sqrt(dt)), nsim, N)
R <- matrix(0,nsim,N);R[,1]=0.03
4 dev.new(width=10,height=7);
for (i in 1:nsim){for (j in 2:N){R[i,j]=R[i,j-1]+(a-b*R[i,j-1])*dt+sigma*dB[i,j]}}
6 par(mar=c(0,1,1,1));par(oma=c(0,1,1,1));par(mgp=c(-5,1,1))
plot(t,R[,1],xlab = "Time",ylab = "",type = "l",ylim = c(R[1,1]-0.2,R[1,1]+0.2),col =
0,axes=FALSE)
axis(2, pos=0,las=1);for (i in 1:nsim){lines(t, R[i, ], xlab = "time", type = "l", col = i+0)}
8 abline(h=a/b,col="blue",lwd=3);abline(h=0)

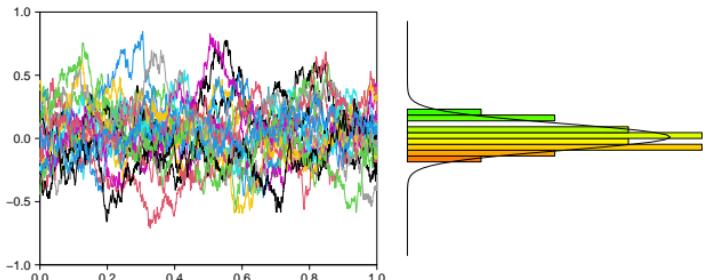
```

We note that the process remains in the $\mathcal{N}(a/b, \sigma^2/2/b)$ Gaussian distribution if it is started from this distribution, see Exercise 17.1-(b).

```

1 a=0.25;b=10;sigma=1; N=1000; t <- 0:N; dt <- 1.0/N; nsim=20;
dB <- matrix(rnorm(nsim * N, 0, sqrt(dt)), nsim, N); Y <- matrix(0, nsim, N+1)
3 for (i in 1:nsim){Y[i,1]=rnorm(1,a/b,sqrt(sigma**2/2/b));
for (j in 2:N){Y[i,j] = Y[i,j-1] +(a-b*Y[i,j-1])*dt + sigma*dB[i,j];}}
5 H<-hist(Y[,N],plot=FALSE,breaks=10); dev.new(width=16,height=7);
layout(matrix(c(1,2), nrow = 1, byrow = TRUE));par(mar=c(2,2,2,2), oma = c(2, 2, 2, 2))
7 plot(t*dt, Y[, 1], xlab = "", ylab = "", type = "l", ylim = c(-1, 1), col = 0,
xaxs="T",yaxs="T",las=1, cex.axis=1.6)
for (i in 1:nsim){lines(t*dt, Y[i, ], type = "l", ylim = c(-1, 1), col = i,lwd=2)}
9 for (i in 1:nsim){points(0.999, Y[i,N], pch=1, lwd = 5, col = i)}
x <- seq(-2,2, length=100); px <- dnorm(x,a/b,sqrt(sigma**2/2/b));par(mar = c(2,2,2,2))
11 plot(NULL , xlab="", ylab="", xlim = c(0, max(px,H$density)), ylim = c(-2,2),axes=F)
rect(0, H$breaks[1:(length(H$breaks) - 1)], col=rainbow(20,start=0.08,end=0.6), H$density,
H$breaks[2:length(H$breaks)]); lines(px,x, lty=1, col="black",lwd=2)

```



Example - TNX yield

We consider the yield of the 10 Year Treasury Note on the Chicago Board Options Exchange (CBOE), for later use in the calibration of the Vasicek model. Treasury notes usually have a maturity between one and 10 years, whereas treasury bonds have maturities beyond 10 years).



```

1 library(quantmod)
2 getSymbols("TNX",from="2012-01-01",to="2016-01-01",src="yahoo")
3 rate=Ad(`TNX`);rate<-rate[is.na(rate)]
4 dev.new(width=10,height=7);chartSeries(rate,up.col="blue",theme="white")
n = length(is.na(rate))

```

The next Figure 17.2 displays the yield of the 10 Year Treasury Note.

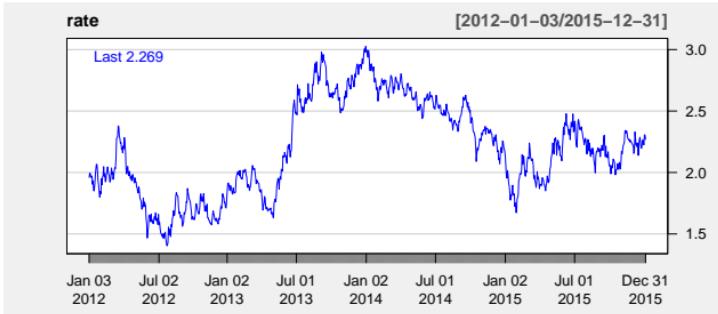


Fig. 17.2: CBOE 10 Year Treasury Note (TNX) yield.

Cox-Ingersoll-Ross (CIR) model

The Cox et al. (1985) (CIR) model brings a solution to the positivity problem encountered with the Vasicek model, by the use the nonlinear stochastic differential equation

$$dr_t = \beta(\alpha - r_t)dt + \sigma\sqrt{r_t}dB_t, \quad (17.5)$$

with $\alpha > 0$, $\beta > 0$, $\sigma > 0$. The probability distribution of r_t at time $t > 0$ admits the noncentral Chi square probability density function given by

$$f_t(x) = \frac{2\beta}{\sigma^2(1 - e^{-\beta t})} \exp\left(-\frac{2\beta(x + r_0 e^{-\beta t})}{\sigma^2(1 - e^{-\beta t})}\right) \left(\frac{x}{r_0 e^{-\beta t}}\right)^{\alpha\beta/\sigma^2 - 1/2} I_{2\alpha\beta/\sigma^2 - 1}\left(\frac{4\beta\sqrt{r_0 x e^{-\beta t}}}{\sigma^2(1 - e^{-\beta t})}\right), \quad (17.6)$$

$x > 0$, where

$$I_\lambda(z) := \left(\frac{z}{2}\right)^\lambda \sum_{k \geq 0} \frac{(z^2/4)^k}{k!\Gamma(\lambda + k + 1)}, \quad z \in \mathbb{R},$$

is the modified Bessel function of the first kind, see Lemma 9 in Feller (1951) and Corollary 24 in Albanese and Lawi (2005). Note that $f_t(x)$ is not defined at $x = 0$ if $\alpha\beta/\sigma^2 - 1/2 < 0$, i.e. $\sigma^2 > 2\alpha\beta$, in which case the probability

distribution of r_t admits a point mass at $x = 0$. On the other hand, r_t remains almost surely strictly positive under the Feller condition $2\alpha\beta \geq \sigma^2$, cf. the study of the associated probability density function in Lemma 4 of [Feller \(1951\)](#) for $\alpha, \beta \in \mathbb{R}$.

Figure 17.3 presents a random simulation of $t \mapsto r_t$ in the [Cox et al. \(1985\)](#) (CIR) model in the case $\sigma^2 > 2\alpha\beta$, in which the process is mean reverting with respect to $\alpha = 2.5\%$ and has a nonzero probability of hitting 0.

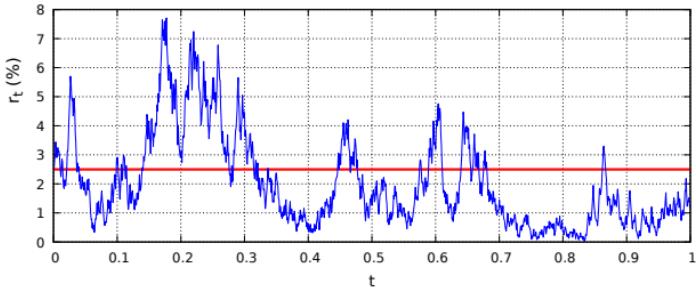


Fig. 17.3: Graph of the CIR short rate $t \mapsto r_t$ with $\alpha = 2.5\%$, $\beta = 1$, and $\sigma = 1.3$.

The next [R](#) code provides a numerical solution of the stochastic differential equation (17.5) using the Euler method, see Figure 17.3.

```

1 N=10000; t <- 0:(N-1); dt <- 1.0/N; nsim <- 2; a=0.025; b=1; sigma=0.1;
2 sd=sqrt(sigma^2/2/b); R <- matrix(0,nsim,N);R[,1]=0.03
3 x <- matrix(rnorm(nsim*N,mean=0,sd=sqrt(dt)), nsim, N)
4 for (i in 1:nsim){for (j in
5   2:N){R[i,j]=max(0,R[i,j-1]+(a+b*R[i,j-1])*dt+sigma*sqrt(R[i,j-1])*X[i,j])}
6 plot(t,R[,1],xlab="time",ylab="",type="l",ylim=c(0,R[1,1]+sd/5),col=0,axes=FALSE)
7 axis(2, pos=0)
8 for (i in 1:nsim){lines(t, R[i, ], xlab = "time", type = "l", col = i+8)}
9 abline(h=a/b,col="blue",lwd=3);abline(h=0)

```

In large time $t \rightarrow \infty$, using the asymptotics

$$I_\lambda(z) \simeq_{z \rightarrow 0} \frac{1}{\Gamma(\lambda + 1)} \left(\frac{z}{2}\right)^\lambda,$$

the probability density function (17.6) becomes the gamma density function

$$f(x) = \lim_{t \rightarrow \infty} f_t(x) = \frac{1}{\Gamma(2\alpha\beta/\sigma^2)} \left(\frac{2\beta}{\sigma^2}\right)^{2\alpha\beta/\sigma^2} x^{-1+2\alpha\beta/\sigma^2} e^{-2\beta x/\sigma^2}, \quad x > 0. \quad (17.7)$$

with shape parameter $2\alpha\beta/\sigma^2$ and scale parameter $\sigma^2/(2\beta)$, which is also the *invariant distribution* of r_t .

The family of classical mean-reverting models also includes the [Courtadon \(1982\)](#) model



$$dr_t = \beta(\alpha - r_t)dt + \sigma r_t dB_t,$$

where α, β, σ are nonnegative, cf. Exercise 17.6, and the exponential Vasicek model

$$dr_t = r_t(\eta - a \log r_t)dt + \sigma r_t dB_t,$$

where $a, \eta, \sigma > 0$, cf. Exercises 4.18 and 4.19.

Constant Elasticity of Variance (CEV) model

Constant Elasticity of Variance models are designed to take into account nonconstant volatilities that can vary as a power of the underlying asset price. The [Marsh and Rosenfeld \(1983\)](#) short-term interest rate model

$$dr_t = (\beta r_t^{\gamma-1} + \alpha r_t)dt + \sigma r_t^{\gamma/2} dB_t, \quad (17.8)$$

where $\alpha \in \mathbb{R}$, $\beta, \sigma > 0$ are constants and $\gamma > 0$ is the variance (or diffusion) elasticity coefficient, covers most of the CEV models. Here, the elasticity coefficient is defined as ratio

$$\frac{dv^2(r)/v^2(r)}{dr/r}$$

between the relative change $dv(r)/v(r)$ in the variance $v(r)$ and the relative change dr/r in r . Denoting by $v^2(r) := \sigma^2 r^\gamma$ the variance coefficient in (17.8), constant elasticity refers to the constant ratio

$$\frac{dv^2(r)/v^2(r)}{dr/r} = 2 \frac{r}{v(r)} \frac{dv(r)}{dr} = 2 \frac{d \log v(r)}{d \log r} = 2 \frac{d \log r^{\gamma/2}}{d \log r} = \gamma.$$

For $\gamma = 1$, (17.8) yields the [Cox et al. \(1985\)](#) (CIR) equation

$$dr_t = (\beta + \alpha r_t)dt + \sigma \sqrt{r_t} dB_t.$$

For $\beta = 0$ we get the standard CEV model

$$dr_t = \alpha r_t dt + \sigma r_t^{\gamma/2} dB_t,$$

and for $\gamma = 2$ and $\beta = 0$ this yields the [Dothan \(1978\)](#) model

$$dr_t = \alpha r_t dt + \sigma r_t dB_t,$$

which is a version of geometric Brownian motion used for short-term interest rate modeling.

Time-dependent affine models

The class of short rate interest rate models admits a number of generalizations (see the references quoted in the introduction of this chapter), including the class of affine models of the form

$$dr_t = (\eta(t) + \lambda(t)r_t)dt + \sqrt{\delta(t) + \gamma(t)r_t}dB_t. \quad (17.9)$$

Such models are called *affine* because the associated bonds can be priced using an *affine* PDE of the type (17.27) below with solution of the form (17.28), as will be seen after Proposition 17.2.

The family of affine models also includes:

- i) the [Ho and Lee \(1986\)](#) model

$$dr_t = \theta(t)dt + \sigma dB_t,$$

where $\theta(t)$ is a deterministic function of time, as an extension of the Merton model $dr_t = \theta dt + \sigma dB_t$,

- ii) the [Hull and White \(1990\)](#) model

$$dr_t = (\theta(t) - \alpha(t)r_t)dt + \sigma(t)dB_t$$

which is a time-dependent extension of the Vasicek model (17.1), with the explicit solution

$$r_t = r_0 e^{-\int_0^t \alpha(\tau)d\tau} + \int_0^t e^{-\int_u^t \alpha(\tau)d\tau} \theta(u)du + \int_0^t \sigma(u) e^{-\int_u^t \alpha(\tau)d\tau} dB_u,$$

$$t \geq 0.$$

17.2 Calibration of the Vasicek Model

Using the discretization (17.4) of the Vasicek equation (17.1) and a set $(\tilde{r}_{t_k})_{k=0,1,\dots,n}$ of *market data*, we consider the quadratic residual

$$\sum_{k=0}^{n-1} (\tilde{r}_{t_{k+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_k})^2 \quad (17.10)$$

which represents the (squared) quadratic distance between the observed data sequence $(\tilde{r}_{t_k})_{k=1,2,\dots,n}$ and its predictions $(a\Delta t + (1 - b\Delta t)\tilde{r}_{t_k})_{k=0,1,\dots,n-1}$.

In order to minimize the residual (17.10) over a and b we use Ordinary Least Square (OLS) regression, and equate the following derivatives to zero. Namely, we have



$$\begin{aligned}
& \frac{\partial}{\partial a} \sum_{l=0}^{n-1} (\tilde{r}_{t_{l+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_l})^2 \\
&= -2\Delta t \left(-an\Delta t + \sum_{l=0}^{n-1} (\tilde{r}_{t_{l+1}} - (1 - b\Delta t)\tilde{r}_{t_l}) \right) \\
&= 0,
\end{aligned}$$

hence

$$a\Delta t = \frac{1}{n} \sum_{l=0}^{n-1} (\tilde{r}_{t_{l+1}} - (1 - b\Delta t)\tilde{r}_{t_l}),$$

and

$$\begin{aligned}
& \frac{\partial}{\partial b} \sum_{k=0}^{n-1} (\tilde{r}_{t_{k+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_k})^2 \\
&= 2\Delta t \sum_{k=0}^{n-1} \tilde{r}_{t_k} (-a\Delta t + \tilde{r}_{t_{k+1}} - (1 - b\Delta t)\tilde{r}_{t_k}) \\
&= 2\Delta t \sum_{k=0}^{n-1} \tilde{r}_{t_k} \left(\tilde{r}_{t_{k+1}} - (1 - b\Delta t)\tilde{r}_{t_k} - \frac{1}{n} \sum_{l=0}^{n-1} (\tilde{r}_{t_{l+1}} - (1 - b\Delta t)\tilde{r}_{t_l}) \right) \\
&= 2\Delta t \sum_{k=0}^{n-1} \tilde{r}_{t_k} \tilde{r}_{t_{k+1}} - \frac{\Delta t}{n} \sum_{k,l=0}^{n-1} \tilde{r}_{t_k} \tilde{r}_{t_{l+1}} - \Delta t(1 - b\Delta t) \left(\sum_{k=0}^{n-1} (\tilde{r}_{t_k})^2 - \frac{1}{n} \sum_{k,l=0}^{n-1} \tilde{r}_{t_k} \tilde{r}_{t_l} \right) \\
&= 0.
\end{aligned}$$

This leads to estimators for the parameters a and b , respectively as the empirical mean and covariance of $(\tilde{r}_{t_k})_{k=0,1,\dots,n}$, i.e.

$$\left\{ \begin{array}{l} \hat{a}\Delta t = \frac{1}{n} \sum_{k=0}^{n-1} \left(\tilde{r}_{t_{k+1}} - (1 - \hat{b}\Delta t) \tilde{r}_{t_k} \right), \\ \text{and} \\ 1 - \hat{b}\Delta t = \frac{\sum_{k=0}^{n-1} \tilde{r}_{t_k} \tilde{r}_{t_{k+1}} - \frac{1}{n} \sum_{k,l=0}^{n-1} \tilde{r}_{t_k} \tilde{r}_{t_{l+1}}}{\sum_{k=0}^{n-1} (\tilde{r}_{t_k})^2 - \frac{1}{n} \sum_{k,l=0}^{n-1} \tilde{r}_{t_k} \tilde{r}_{t_l}} \\ = \frac{\sum_{k=0}^{n-1} \left(\tilde{r}_{t_k} - \frac{1}{n} \sum_{l=0}^{n-1} \tilde{r}_{t_l} \right) \left(\tilde{r}_{t_{k+1}} - \frac{1}{n} \sum_{l=0}^{n-1} \tilde{r}_{t_{l+1}} \right)}{\sum_{k=0}^{n-1} \left(\tilde{r}_{t_k} - \frac{1}{n} \sum_{k=0}^{n-1} \tilde{r}_{t_k} \right)^2}. \end{array} \right. \quad (17.11)$$

This also yields

$$\begin{aligned} \sigma^2 \Delta t &= \text{Var}[\sigma Z_k] \\ &\simeq \mathbb{E}[(\tilde{r}_{t_{k+1}} - (1 - b\Delta t)\tilde{r}_{t_k} - a\Delta t)^2], \quad k \geq 0, \end{aligned}$$

hence σ can be estimated as

$$\hat{\sigma}^2 \Delta t = \frac{1}{n-1} \sum_{k=0}^{n-1} (\tilde{r}_{t_{k+1}} - \tilde{r}_{t_k} (1 - \hat{b}\Delta t) - \hat{a}\Delta t)^2, \quad (17.12)$$

see also Relation (4.7) page 49 of [Greene \(2003\)](#).

Exercise. Show that (17.12) can be recovered by minimizing the residual

$$\eta \mapsto \sum_{k=0}^{n-1} ((\tilde{r}_{t_{k+1}} - \tilde{r}_{t_k} (1 - \hat{b}\Delta t) - \hat{a}\Delta t)^2 - \eta \Delta t)^2$$

as a function of $\eta > 0$, see also Exercise 17.3.

Time series modeling

Defining $\hat{r}_{t_n} := r_{t_n} - a/b$, $n \geq 0$, we have

$$\begin{aligned} \hat{r}_{t_{n+1}} &= r_{t_{n+1}} - \frac{a}{b} \\ &= r_{t_n} - \frac{a}{b} + (a - br_{t_n})\Delta t + \sigma Z_n \end{aligned}$$



$$\begin{aligned}
&= r_{t_n} - \frac{a}{b} - b \left(r_{t_n} - \frac{a}{b} \right) \Delta t + \sigma Z_n \\
&= \hat{r}_{t_n} - b \hat{r}_{t_n} \Delta t + \sigma Z_n \\
&= (1 - b \Delta t) \hat{r}_{t_n} + \sigma Z_n, \quad n \geq 0.
\end{aligned}$$

In other words, the sequence $(\hat{r}_{t_n})_{n \geq 0}$ is modeled according to an autoregressive AR(1) time series $(X_n)_{n \geq 0}$ with parameter $\alpha = 1 - b \Delta t$, in which the current state X_n of the system is expressed as the linear combination

$$X_n := \sigma Z_n + \alpha X_{n-1}, \quad n \geq 1, \quad (17.13)$$

where $(Z_n)_{n \geq 1}$ another Gaussian white noise sequence with variance Δt . This equation can be solved recursively as the causal series

$$X_n = \sigma Z_n + \alpha(\sigma Z_{n-1} + \alpha X_{n-2}) = \dots = \sigma \sum_{k \geq 0} \alpha^k Z_{n-k},$$

which converges when $|\alpha| < 1$, i.e. $|1 - b \Delta t| < 1$, in which case the time series $(X_n)_{n \geq 0}$ is weakly stationary, with

$$\begin{aligned}
\mathbb{E}[X_n] &= \sigma \sum_{k \geq 0} \alpha^k \mathbb{E}[Z_{n-k}] \\
&= \sigma \mathbb{E}[Z_0] \sum_{k \geq 0} \alpha^k \\
&= \frac{\sigma}{1 - \alpha} \mathbb{E}[Z_0] \\
&= 0, \quad n \geq 0.
\end{aligned}$$

The variance of X_n is given by

$$\begin{aligned}
\text{Var}[X_n] &= \sigma^2 \text{Var} \left[\sum_{k \geq 0} \alpha^k Z_{n-k} \right] \\
&= \sigma^2 \Delta t \sum_{k \geq 0} \alpha^{2k} \\
&= \sigma^2 \Delta t \sum_{k \geq 0} (1 - b \Delta t)^{2k} \\
&= \frac{\sigma^2 \Delta t}{1 - (1 - b \Delta t)^2} \\
&= \frac{\sigma^2 \Delta t}{2b \Delta t - b^2 (\Delta t)^2} \\
&\simeq \frac{\sigma^2}{2b}, \quad [\Delta t \simeq 0],
\end{aligned}$$

which coincides with the variance (17.3) of the Vasicek process in the stationary regime.

Example - TNX yield calibration

The next  code is estimating the parameters of the Vasicek model using the 10 Year Treasury Note yield data of Figure 17.2, by implementing the formulas (17.11).

```

1 ratek<-as.vector(ratek);ratekplus1 <- c(ratek[-1],0)
2 oneminusbdt <- (sum(ratek*ratekplus1) - sum(ratek)*sum(ratekplus1)/n)/(sum(ratek*ratek) -
   sum(ratek)*sum(ratek)/n)
3 adt <- sum(ratekplus1)/n-oneminusbdt*sum(ratek)/n;
4 sigmadt <- sqrt(sum((ratekplus1-oneminusbdt*ratek-adt)^2)/n)

```

Parameter estimation can also be implemented using the linear regression command

$$\ln(c(\text{diff}(\text{ratek})) \sim \text{ratek}[1:\text{length}(\text{ratek})-1])$$

in , which estimates the values of $a\Delta t \simeq 0.017110$ and $-b\Delta t \simeq -0.007648$ in the regression

$$r_{t+1} - r_t = (a - br_t)\Delta t + \sigma Z_k, \quad k \geq 0,$$

Coefficients:

(Intercept)	ratek[1:length(ratek) - 1]
0.017110	-0.007648

```

1 for (i in 1:100) {ar.sim<-arima.sim(model=list(ar=c(oneminusbdt)),n,start=100,n)
2 y=adt/oneminusbdt+sigmadt*ar.sim;y=y+ratek[1]-y[1]
3 time <- as.POSIXct(time(rate), format = "%Y-%m-%d")
4 sim_yield <- xts(x = y, order.by = time);
5 myPars <- chart_pars();myPars$ex<-1.4
6 myTheme <- chart_theme();myTheme$col$line.col <- "blue"
7 myTheme$rylab <- FALSE; dev.new(width=24,height=8)
8 par(mfrow=c(1,2));print(chart_Series(rate,theme=myTheme,pars=myPars))
9 graph <-chart_Series(sim_yield,theme=myTheme,pars=myPars); myylim <- graph$get ylim()
10 myylim([2]) <- structure(c(min(rate),max(rate)), fixed=TRUE)
graph$set ylim(myylim); print(graph); Sys.sleep(1);dev.off()

```

The above  code is generating Vasicek random samples according to the AR(1) time series (17.13), see Figure 17.4.





(a) CBOE TNX market yield.

(b) Calibrated Vasicek sample path.

Fig. 17.4: Calibrated Vasicek simulation *vs.* market data.

The **R** package `Sim.DiffProc` can also be used to estimate the coefficients $a\Delta t$ and $b\Delta t$.

```
1 install.packages("Sim.DiffProc");library(Sim.DiffProc)
fx <- expression( theta[1]-theta[2]*x ); gx <- expression( theta[3] )
3 fitsde(data = as.ts(ratek), drift = fx, diffusion = gx, start = list(theta1=0.1, theta2=0.1,
theta3=0.1),pmle="euler")
```

17.3 Zero-Coupon and Coupon Bonds

A zero-coupon bond is a contract priced $P(t, T)$ at time $t < T$ to deliver the *face value* (or *par value*) $P(T, T) = \$1$ at time T . In addition to its value at maturity, a bond may yield a periodic *coupon* payment at regular time intervals until the maturity date.



Fig. 17.5: Five-dollar 1875 Louisiana bond with 7.5% biannual coupons and maturity $T = 1/1/1886$.

The computation of the arbitrage-free price $P_0(t, T)$ of a zero-coupon bond based on an underlying short-term interest rate process $(r_t)_{t \in \mathbb{R}_+}$ is a basic and important issue in interest rate modeling.

Constant short rate

In case the short-term interest rate is a constant $r_t = r$, $t \geq 0$, a standard arbitrage argument shows that the price $P(t, T)$ of the bond is given by

$$P(t, T) = e^{-(T-t)r}, \quad 0 \leq t \leq T.$$

Indeed, if $P(t, T) > e^{-(T-t)r}$ we could issue a bond at the price $P(t, T)$ and invest this amount at the compounded risk-free interest rate r , which would yield $P(t, T) e^{(T-t)r} > 1$ at time T .

On the other hand, if $P(t, T) < e^{-(T-t)r}$ we could borrow $P(t, T)$ at the rate r to buy a bond priced $P(t, T)$. At maturity time T we would receive \$1 and refund only $P(t, T) e^{(T-t)r} < 1$.

The price $P(t, T) = e^{-(T-t)r}$ of the bond is the value of $P(t, T)$ that makes the potential profit $P(t, T) e^{(T-t)r} - 1$ vanish for both traders.

Time-dependent deterministic short rates

Similarly to the above, when the short-term interest rate process $(r(t))_{t \in \mathbb{R}_+}$ is a deterministic function of time, a similar argument shows that

$$P(t, T) = e^{-\int_t^T r(s)ds}, \quad 0 \leq t \leq T. \quad (17.14)$$

Stochastic short rates

In case $(r_t)_{t \in \mathbb{R}_+}$ is an $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ -adapted random process the formula (17.14) is no longer valid as it relies on future information, and we replace it with the averaged discounted payoff

$$P(t, T) = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T, \quad (17.15)$$

under a risk-neutral probability measure \mathbb{P}^* . It is natural to write $P(t, T)$ as a conditional expectation under a martingale measure, as the use of conditional expectation helps to “filter out” the (random/unknown) future information past time t contained in $\int_t^T r_s ds$. The expression (17.15) makes sense as the “best possible estimate” of the future quantity $e^{-\int_t^T r_s ds}$ in mean-square sense, given the information known up to time t .



Coupon bonds

Pricing bonds with nonzero coupon is not difficult since in general the amount and periodicity of coupons are deterministic.* In the case of a succession of coupon payments c_1, c_2, \dots, c_n at times $T_1, T_2, \dots, T_n \in (t, T]$, another application of the above absence of arbitrage argument shows that the price $P_c(t, T)$ of the coupon bond with discounted (deterministic) coupon payments is given by the linear combination of zero-coupon bond prices

$$\begin{aligned} P_c(t, T) &:= \mathbb{E}^* \left[\sum_{k=1}^n c_k e^{-\int_t^{T_k} r_s ds} \middle| \mathcal{F}_t \right] + \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right] \\ &= \sum_{k=1}^n c_k \mathbb{E}^* \left[e^{-\int_t^{T_k} r_s ds} \middle| \mathcal{F}_t \right] + P_0(t, T) \\ &= P_0(t, T) + \sum_{k=1}^n c_k P_0(t, T_k), \quad 0 \leq t \leq T_1, \end{aligned} \quad (17.16)$$

which represents the present value at time t of future $\$c_1, \$c_2, \dots, \$c_n$ receipts respectively at times T_1, T_2, \dots, T_n , in addition to a terminal $\$1$ principal payment.

In the case of a constant coupon rate c paid at regular time intervals $\tau = T_{k+1} - T_k$, $k = 0, 1, \dots, n-1$, with $T_0 = t$, $T_n = T$, and a constant deterministic short rate r , we find

$$\begin{aligned} P_c(T_0, T_n) &= e^{-rn\tau} + c \sum_{k=1}^n e^{-(T_k - T_0)r} \\ &= e^{-rn\tau} + c \sum_{k=1}^n e^{-kr\tau} \\ &= e^{-rn\tau} + c \frac{e^{-r\tau} - e^{-r(n+1)\tau}}{1 - e^{-r\tau}}. \end{aligned}$$

In terms of the discrete-time interest rate $\tilde{r} := e^{r\tau} - 1$, we also have

$$P_c(T_0, T_n) = \frac{c}{\tilde{r}} + \frac{\tilde{r} - c}{(1 + \tilde{r})^n \tilde{r}}.$$

* However, coupon default cannot be excluded.

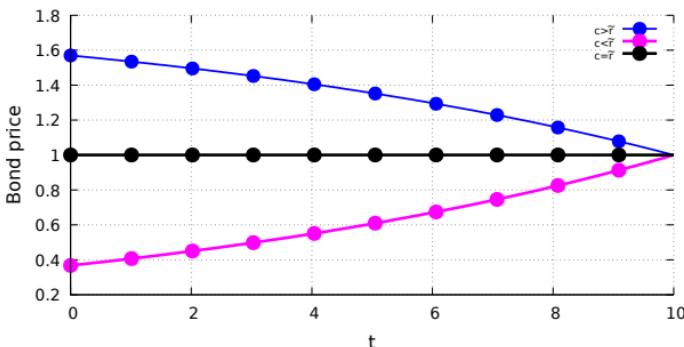


Fig. 17.6: Discrete-time coupon bond pricing.

In the case of a continuous-time coupon rate $c > 0$, the above discrete-time calculation (17.16) can be reinterpreted as follows:

$$P_c(t, T) = P_0(t, T) + c \int_t^T P_0(t, u) du \quad (17.17)$$

$$= P_0(t, T) + c \int_t^T e^{-(u-t)r} du$$

$$= e^{-(T-t)r} + c \int_0^{T-t} e^{-ru} du$$

$$= e^{-(T-t)r} + \frac{c}{r} (1 - e^{-(T-t)r}),$$

$$= \frac{c}{r} + \frac{r-c}{r} e^{-(T-t)r}, \quad 0 \leq t \leq T, \quad (17.18)$$

where the coupon bond price $P_c(t, T)$ solves the ordinary differential equation

$$dP_c(t, T) = (r - c) e^{-(T-t)r} dt = -cdt + rP_c(t, T)dt, \quad 0 \leq t \leq T,$$

see also Figures 17.7 and 17.11 below.



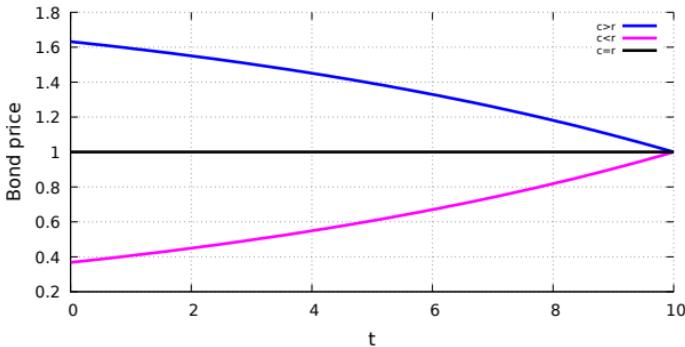


Fig. 17.7: Continuous-time coupon bond pricing.

In what follows, we will mostly consider zero-coupon bonds priced as $P(t, T) = P_0(t, T)$, $0 \leq t \leq T$, in the setting of stochastic short rates.

Martingale property of discounted bond prices

The following proposition shows that Assumption (A) of Chapter 16 is satisfied, in other words, the bond price process $t \mapsto P(t, T)$ can be used as a numéraire.

Proposition 17.1. *The discounted bond price process*

$$t \mapsto \tilde{P}(t, T) := e^{-\int_0^t r_s ds} P(t, T)$$

is a martingale under \mathbb{P}^* .

Proof. By (17.15) we have

$$\begin{aligned} \tilde{P}(t, T) &= e^{-\int_0^t r_s ds} P(t, T) \\ &= e^{-\int_0^t r_s ds} \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right] \\ &= \mathbb{E}^* \left[e^{-\int_0^t r_s ds} e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right] \\ &= \mathbb{E}^* \left[e^{-\int_0^T r_s ds} \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T, \end{aligned}$$

and this suffices in order to conclude, since by the tower property (A.33) of conditional expectations, see, *e.g.*, Example 1 page 271, any process $(X_t)_{t \in \mathbb{R}_+}$ of the form $t \mapsto X_t := \mathbb{E}^*[F \mid \mathcal{F}_t]$, $F \in L^1(\Omega)$, is a martingale, see also Relation (7.1). In other words, we have

$$\mathbb{E}^* [\tilde{P}(t, T) \mid \mathcal{F}_u] = \mathbb{E}^* \left[\mathbb{E}^* \left[e^{-\int_0^T r_s ds} \mid \mathcal{F}_t \right] \mid \mathcal{F}_u \right]$$

$$\begin{aligned}
&= \mathbb{E}^* \left[e^{-\int_0^T r_s ds} \mid \mathcal{F}_u \right] \\
&= \tilde{P}(u, T), \quad 0 \leq u \leq t.
\end{aligned}$$

□

17.4 Bond Pricing PDE

We assume from now on that the underlying short rate process solves the stochastic differential equation

$$dr_t = \mu(t, r_t) dt + \sigma(t, r_t) dB_t \quad (17.19)$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under \mathbb{P}^* . Note that specifying the dynamics of $(r_t)_{t \in \mathbb{R}_+}$ under the historical probability measure \mathbb{P} will also lead to a notion of market price of risk (MPoR) for the modeling of short rates.

As all solutions of stochastic differential equations such as (17.19) have the *Markov property*, cf. e.g. Theorem V-32 of [Protter \(2004\)](#), the arbitrage-free price $P(t, T)$ can be rewritten as a function $F(t, r_t)$ of r_t , i.e.

$$\begin{aligned}
P(t, T) &= \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right] \\
&= \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \mid r_t \right] \\
&= F(t, r_t),
\end{aligned} \quad (17.20)$$

and depends on (t, r_t) only, instead of depending on the whole information available in \mathcal{F}_t up to time t , meaning that the pricing problem can now be formulated as a search for the function $F(t, x)$.

Proposition 17.2. (*Bond pricing PDE*). Consider a short rate $(r_t)_{t \in \mathbb{R}_+}$ modeled by a diffusion equation of the form

$$dr_t = \mu(t, r_t) dt + \sigma(t, r_t) dB_t.$$

The bond pricing PDE for $P(t, T) = F(t, r_t)$ as in (17.20) is written as

$$xF(t, x) = \frac{\partial F}{\partial t}(t, x) + \mu(t, x) \frac{\partial F}{\partial x}(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 F}{\partial x^2}(t, x), \quad (17.21)$$

$t \in [0, T]$, $x \in \mathbb{R}$, subject to the terminal condition

$$F(T, x) = 1, \quad x \in \mathbb{R}. \quad (17.22)$$



In addition, the bond price dynamics is given by

$$dP(t, T) = r_t P(t, T) dt + \sigma(t, r_t) \frac{\partial F}{\partial x}(t, r_t) dB_t. \quad (17.23)$$

Proof. By Itô's formula, we have

$$\begin{aligned} d \left(e^{-\int_0^t r_s ds} P(t, T) \right) &= -r_t e^{-\int_0^t r_s ds} P(t, T) dt + e^{-\int_0^t r_s ds} dP(t, T) \\ &= -r_t e^{-\int_0^t r_s ds} F(t, r_t) dt + e^{-\int_0^t r_s ds} dF(t, r_t) \\ &= -r_t e^{-\int_0^t r_s ds} F(t, r_t) dt + e^{-\int_0^t r_s ds} \frac{\partial F}{\partial x}(t, r_t) dr_t \\ &\quad + \frac{1}{2} e^{-\int_0^t r_s ds} \frac{\partial^2 F}{\partial x^2}(t, r_t) (dr_t)^2 + e^{-\int_0^t r_s ds} \frac{\partial F}{\partial t}(t, r_t) dt \\ &= -r_t e^{-\int_0^t r_s ds} F(t, r_t) dt + e^{-\int_0^t r_s ds} \frac{\partial F}{\partial x}(t, r_t) (\mu(t, r_t) dt + \sigma(t, r_t) dB_t) \\ &\quad + e^{-\int_0^t r_s ds} \left(\frac{1}{2} \sigma^2(t, r_t) \frac{\partial^2 F}{\partial x^2}(t, r_t) + \frac{\partial F}{\partial t}(t, r_t) \right) dt \\ &= e^{-\int_0^t r_s ds} \sigma(t, r_t) \frac{\partial F}{\partial x}(t, r_t) dB_t \\ &\quad + e^{-\int_0^t r_s ds} \left(-r_t F(t, r_t) + \mu(t, r_t) \frac{\partial F}{\partial x}(t, r_t) + \frac{1}{2} \sigma^2(t, r_t) \frac{\partial^2 F}{\partial x^2}(t, r_t) + \frac{\partial F}{\partial t}(t, r_t) \right) dt. \end{aligned} \quad (17.24)$$

Given that $t \mapsto e^{-\int_0^t r_s ds} P(t, T)$ is a martingale, the above expression (17.24) should only contain terms in dB_t (cf. Corollary II-6-1, page 72 of [Protter \(2004\)](#)), and all terms in dt should vanish inside (17.24). This leads to the identities

$$\begin{cases} r_t F(t, r_t) = \mu(t, r_t) \frac{\partial F}{\partial x}(t, r_t) + \frac{1}{2} \sigma^2(t, r_t) \frac{\partial^2 F}{\partial x^2}(t, r_t) + \frac{\partial F}{\partial t}(t, r_t) \\ d \left(e^{-\int_0^t r_s ds} P(t, T) \right) = e^{-\int_0^t r_s ds} \sigma(t, r_t) \frac{\partial F}{\partial x}(t, r_t) dB_t, \end{cases} \quad (17.25a)$$

which lead to (17.21) and (17.23). Condition (17.22) is due to the fact that $P(T, T) = \$1$. \square

By (17.25a), the proof of Proposition 17.2 also shows that

$$\begin{aligned} \frac{dP(t, T)}{P(t, T)} &= \frac{1}{P(t, T)} d \left(e^{\int_0^t r_s ds} e^{-\int_0^t r_s ds} P(t, T) \right) \\ &= \frac{1}{P(t, T)} \left(r_t P(t, T) dt + e^{\int_0^t r_s ds} d \left(e^{-\int_0^t r_s ds} P(t, T) \right) \right) \\ &= r_t dt + \frac{1}{P(t, T)} e^{\int_0^t r_s ds} d \left(e^{-\int_0^t r_s ds} P(t, T) \right) \end{aligned}$$

$$\begin{aligned}
&= r_t dt + \frac{1}{F(t, r_t)} \frac{\partial F}{\partial x}(t, r_t) \sigma(t, r_t) dB_t \\
&= r_t dt + \sigma(t, r_t) \frac{\partial}{\partial x} \log F(t, r_t) dB_t.
\end{aligned} \tag{17.26}$$

In the case of an interest rate process modeled by (17.9), we have

$$\mu(t, x) = \eta(t) + \lambda(t)x, \quad \text{and} \quad \sigma(t, x) = \sqrt{\delta(t) + \gamma(t)x},$$

hence (17.21) yields the *affine* PDE

$$xF(t, x) = \frac{\partial F}{\partial t}(t, x) + (\eta(t) + \lambda(t)x) \frac{\partial F}{\partial x}(t, x) + \frac{1}{2}(\delta(t) + \gamma(t)x) \frac{\partial^2 F}{\partial x^2}(t, x) \tag{17.27}$$

with time-dependent coefficients, $t \geq 0$, $x \in \mathbb{R}$.

Note that more generally, all affine short rate models as defined in Relation (17.9), including the Vasicek model, will yield a bond pricing formula of the form

$$P(t, T) = e^{A(T-t)+r_t C(T-t)}, \tag{17.28}$$

cf. e.g. § 3.2.4. in [Brigo and Mercurio \(2006\)](#).

Vašíček (1977) bond pricing

In the Vasicek case

$$dr_t = (a - br_t)dt + \sigma dB_t,$$

the bond price takes the form

$$F(t, r_t) = P(t, T) = e^{A(T-t)+r_t C(T-t)},$$

where $A(\cdot)$ and $C(\cdot)$ are deterministic functions of time, see (17.34)-(17.35) below, and (17.26) yields

$$\begin{aligned}
\frac{dP(t, T)}{P(t, T)} &= r_t dt + \sigma C(T-t) dB_t \\
&= r_t dt - \frac{\sigma}{b} (1 - e^{-(T-t)b}) dB_t,
\end{aligned} \tag{17.29}$$

since $F(t, x) = e^{A(T-t)+xC(T-t)}$.

Probabilistic solution of the Vasicek PDE

Next, we solve the PDE (17.21), written with $\mu(t, x) = a - bx$ and $\sigma(t, x) = \sigma$ in the [Vašíček \(1977\)](#) model



$$dr_t = (a - br_t)dt + \sigma dB_t \quad (17.30)$$

as

$$\begin{cases} xF(t, x) = \frac{\partial F}{\partial t}(t, x) + (a - bx)\frac{\partial F}{\partial x}(t, x) + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial x^2}(t, x), \\ F(T, x) = 1. \end{cases} \quad (17.31)$$

For this, Proposition 17.3 relies on a direct computation of the conditional expectation

$$F(t, r_t) = P(t, T) = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right]. \quad (17.32)$$

See also Exercise 17.8 for a closed-form bond pricing formula in the Cox et al. (1985) (CIR) model.

Proposition 17.3. *The zero-coupon bond price in the Vasicek model (17.30) can be expressed as*

$$P(t, T) = e^{A(T-t)+r_tC(T-t)}, \quad 0 \leq t \leq T, \quad (17.33)$$

where $A(x)$ and $C(x)$ are functions of time to maturity given by

$$C(x) := -\frac{1}{b}(1 - e^{-bx}), \quad (17.34)$$

and

$$\begin{aligned} A(x) &:= \frac{4ab - 3\sigma^2}{4b^3} + \frac{\sigma^2 - 2ab}{2b^2}x + \frac{\sigma^2 - ab}{b^3}e^{-bx} - \frac{\sigma^2}{4b^3}e^{-2bx} \\ &= -\left(\frac{a}{b} - \frac{\sigma^2}{2b^2}\right)(x + C(x)) - \frac{\sigma^2}{4b}C^2(x), \quad x \geq 0. \end{aligned} \quad (17.35)$$

Proof. Recall that in the Vasicek model (17.30), the short rate process $(r_t)_{t \in \mathbb{R}_+}$ solution of (17.30) has the expression

$$r_t = g(t) + \int_0^t h(t, s)dB_s = r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt}) + \sigma \int_0^t e^{-(t-s)b}dB_s,$$

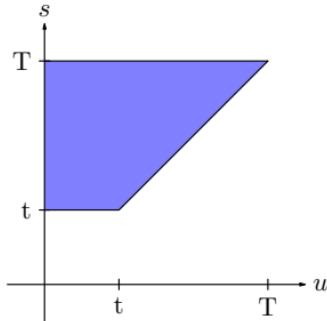
see Exercise 17.1, where g and h are the deterministic functions

$$g(t) := r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt}), \quad t \geq 0,$$

and

$$h(t, s) := \sigma e^{-(t-s)b}, \quad 0 \leq s \leq t.$$

Using the fact that Wiener integrals are Gaussian random variables and the Gaussian moment generating function, and exchanging the order of integration between ds and du over $[t, T]$ according to the following picture:



we have

$$\begin{aligned}
 P(t, T) &= \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right] \\
 &= \mathbb{E}^* \left[e^{-\int_t^T (g(s) + \int_0^s h(s, u) dB_u) ds} \mid \mathcal{F}_t \right] \\
 &= \exp \left(- \int_t^T g(s) ds \right) \mathbb{E}^* \left[e^{-\int_t^T \int_0^s h(s, u) dB_u ds} \mid \mathcal{F}_t \right] \\
 &= \exp \left(- \int_t^T g(s) ds \right) \mathbb{E}^* \left[e^{-\int_0^T \int_{\max(u,t)}^T h(s, u) ds dB_u} \mid \mathcal{F}_t \right] \\
 &= \exp \left(- \int_t^T g(s) ds - \int_0^t \int_{\max(u,t)}^T h(s, u) ds dB_u \right) \mathbb{E}^* \left[e^{-\int_t^T \int_{\max(u,t)}^T h(s, u) ds dB_u} \mid \mathcal{F}_t \right] \\
 &= \exp \left(- \int_t^T g(s) ds - \int_0^t \int_t^T h(s, u) ds dB_u \right) \mathbb{E}^* \left[e^{-\int_t^T \int_u^T h(s, u) ds dB_u} \mid \mathcal{F}_t \right] \\
 &= \exp \left(- \int_t^T g(s) ds - \int_0^t \int_t^T h(s, u) ds dB_u \right) \mathbb{E}^* \left[e^{-\int_t^T \int_u^T h(s, u) ds dB_u} \right] \\
 &= \exp \left(- \int_t^T g(s) ds - \int_0^t \int_t^T h(s, u) ds dB_u + \frac{1}{2} \int_t^T \left(\int_u^T h(s, u) ds \right)^2 du \right) \\
 &= \exp \left(- \int_t^T (r_0 e^{-bs} + \frac{a}{b} (1 - e^{-bs})) ds - \sigma \int_0^t \int_t^T e^{-(s-u)b} ds dB_u \right) \\
 &\quad \times \exp \left(\frac{\sigma^2}{2} \int_t^T \left(\int_u^T e^{-(s-u)b} ds \right)^2 du \right) \\
 &= \exp \left(- \int_t^T (r_0 e^{-bs} + \frac{a}{b} (1 - e^{-bs})) ds - \frac{\sigma}{b} (1 - e^{-(T-t)b}) \int_0^t e^{-(t-u)b} dB_u \right)
 \end{aligned}$$



$$\begin{aligned}
& \times \exp \left(\frac{\sigma^2}{2} \int_t^T e^{2bu} \left(\frac{e^{-bu} - e^{-bT}}{b} \right)^2 du \right) \\
& = \exp \left(-\frac{rt}{b} (1 - e^{-(T-t)b}) + \frac{1}{b} (1 - e^{-(T-t)b}) \left(r_0 e^{-bt} + \frac{a}{b} (1 - e^{-bt}) \right) \right) \\
& \quad \times \exp \left(- \int_t^T \left(r_0 e^{-bs} + \frac{a}{b} (1 - e^{-bs}) \right) ds + \frac{\sigma^2}{2} \int_t^T e^{2bu} \left(\frac{e^{-bu} - e^{-bT}}{b} \right)^2 du \right) \\
& = e^{A(T-t)+r_t C(T-t)}, \tag{17.36}
\end{aligned}$$

where $A(x)$ and $C(x)$ are the functions given by (17.34) and (17.35). \square

Analytical solution of the Vasicek PDE

In order to solve the PDE (17.31) analytically, we may start by looking for a solution of the form

$$F(t, x) = e^{A(T-t)+xC(T-t)}, \tag{17.37}$$

where $A(\cdot)$ and $C(\cdot)$ are functions to be determined under the conditions $A(0) = 0$ and $C(0) = 0$. Substituting (17.37) into the PDE (17.21) with the Vasicek coefficients $\mu(t, x) = (a - bx)$ and $\sigma(t, x) = \sigma$ shows that

$$\begin{aligned}
x e^{A(T-t)+xC(T-t)} &= -(A'(T-t) + xC'(T-t)) e^{A(T-t)+xC(T-t)} \\
&\quad + (a - bx) C(T-t) e^{A(T-t)+xC(T-t)} \\
&\quad + \frac{1}{2} \sigma^2 C^2(T-t) e^{A(T-t)+xC(T-t)},
\end{aligned}$$

i.e.

$$x = -A'(T-t) - xC'(T-t) + (a - bx) C(T-t) + \frac{1}{2} \sigma^2 C^2(T-t).$$

By identification of terms for $x = 0$ and $x \neq 0$, this yields the system of Riccati and linear differential equations

$$\begin{cases} A'(s) = aC(s) + \frac{\sigma^2}{2} C^2(s) \\ C'(s) = -1 - bC(s), \end{cases}$$

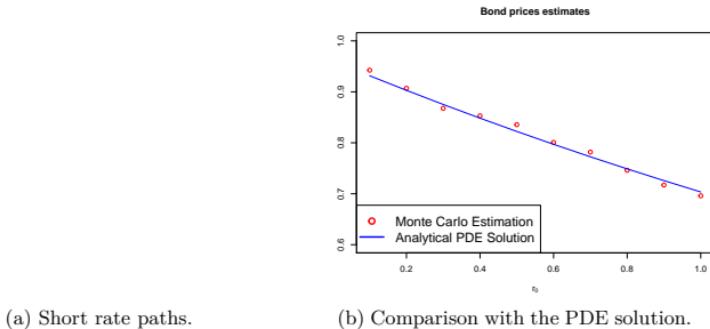
which can be solved to recover the above value of $P(t, T) = F(t, r_t)$ via

$$C(s) = -\frac{1}{b} (1 - e^{-bs})$$

and

$$\begin{aligned}
A(t) &= A(0) + \int_0^t A'(s) ds \\
&= \int_0^t \left(aC(s) + \frac{\sigma^2}{2} C^2(s) \right) ds \\
&= \int_0^t \left(\frac{a}{b} (1 - e^{-bs}) + \frac{\sigma^2}{2b^2} (1 - e^{-bs})^2 \right) ds \\
&= \frac{a}{b} \int_0^t (1 - e^{-bs}) ds + \frac{\sigma^2}{2b^2} \int_0^t (1 - e^{-bs})^2 ds \\
&= \frac{4ab - 3\sigma^2}{4b^3} + \frac{\sigma^2 - 2ab}{2b^2} t + \frac{\sigma^2 - ab}{b^3} e^{-bt} - \frac{\sigma^2}{4b^3} e^{-2bt}, \quad t \geq 0.
\end{aligned}$$

The next Figure 17.8 shows the output of the attached [R code](#) for the Monte Carlo and analytical estimation of Vasicek bond prices.



(a) Short rate paths.

(b) Comparison with the PDE solution.

Fig. 17.8: Comparison of Monte Carlo and analytical PDE solutions.

Vasicek bond price simulations

In this section we consider again the Vasicek model, in which the short rate $(r_t)_{t \in \mathbb{R}_+}$ is solution to (17.1). Figure 17.9 presents a random simulation of the zero-coupon bond price (17.33) in the Vasicek model with $\sigma = 10\%$, $r_0 = 2.96\%$, $b = 0.5$, and $a = 0.025$. The graph of the corresponding deterministic zero-coupon bond price with $r = r_0 = 2.96\%$ is also shown in Figure 17.9.

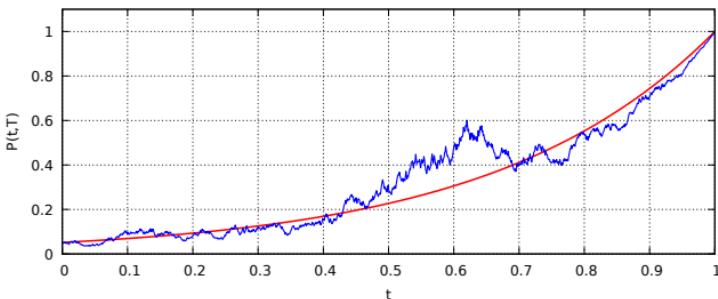
Fig. 17.9: Graphs of $t \mapsto F(t, r_t) = P(t, T)$ vs. $t \mapsto e^{-r_0(T-t)}$.

Figure 17.10 presents a random simulation of the coupon bond price (17.17) in the Vasicek model with $\sigma = 2\%$, $r_0 = 3.5\%$, $b = 0.5$, $a = 0.025$, and coupon rate $c = 5\%$. The graph of the corresponding deterministic coupon bond price (17.18) with $r = r_0 = 3.5\%$ is also shown in Figure 17.10.

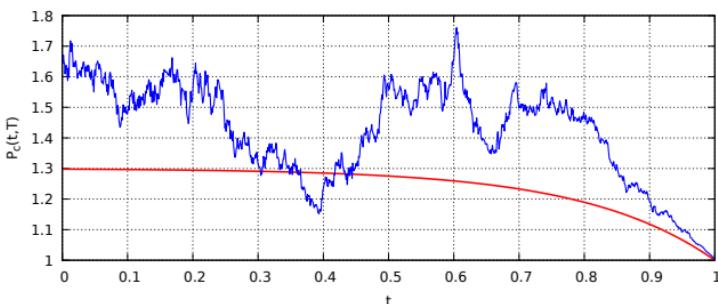
Fig. 17.10: Graph of $t \mapsto P_c(t, T)$ for a bond with a 5% coupon rate.

Figure 17.11 presents market price data for a coupon bond with coupon rate $c = 6.25\%$.

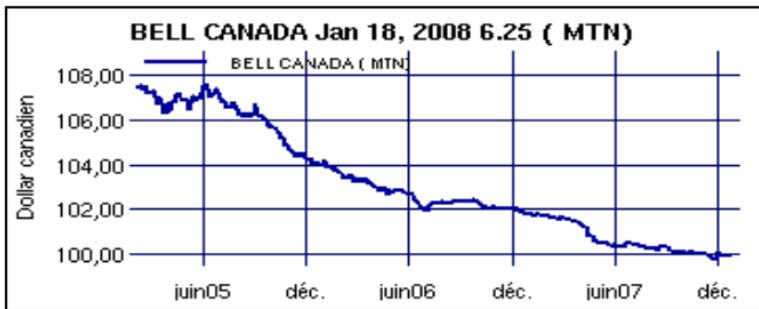


Fig. 17.11: Bond price graph with maturity 01/18/08 and coupon rate 6.25%.

Zero-coupon bond price and yield data

The following zero-coupon public bond price [data](#) was downloaded from [EMMA](#) at the Municipal Securities Rulemaking Board.

ORANGE CNTY CALIF PENSION OBLIG CAP APPREC-TAXABLE-REF-SER A (CA)
CUSIP: 68428LBB9

Dated Date: 06/12/1996 (June 12, 1996)

Maturity Date: 09/01/2016 (September 1st, 2016)

Interest Rate: 0.0 %

Principal Amount at Issuance: \$26,056,000

Initial Offering Price: 19.465

```

1 library(quantmod);getwd()
2 bondprice <- read.table("bond_data_R.txt",col.names =
3   c("Date","HighPrice","LowPrice","HighYield","LowYield","Count","Amount"))
4 head(bondprice)
5 time <- as.POSIXct(bondprice$Date, format = "%Y-%m-%d")
6 price <- xts(x = bondprice$HighPrice, order.by = time)
7 yield <- xts(x = bondprice$HighYield, order.by = time)
8 dev.new(width=10,height=7);
9 chartSeries(price,up.col="blue",theme=="white")
10 chartSeries(yield,up.col="blue",theme=="white")
```

	Date	HighPrice	LowPrice	HighYield	LowYield	Count	Amount
1	2016-01-13	99.082	98.982	1.666	1.501	2	20000
2	2015-12-29	99.183	99.183	1.250	1.250	1	10000
3	2015-12-21	97.952	97.952	3.014	3.014	1	10000
4	2015-12-17	99.141	98.550	2.123	1.251	5	610000
5	2015-12-07	98.770	98.770	1.714	1.714	2	10000
6	2015-12-04	98.363	98.118	2.628	2.280	2	10000



Fig. 17.12: Orange Cnty Calif bond prices.

The next Figure 17.13 plots the *bond yield* $y(t, T)$, defined as

$$y(t, T) := -\frac{\log P(t, T)}{T - t}, \quad \text{or} \quad P(t, T) = e^{-(T-t)y(t, T)}, \quad 0 \leq t \leq T,$$

see also [here](#) for another source of bond price market data.

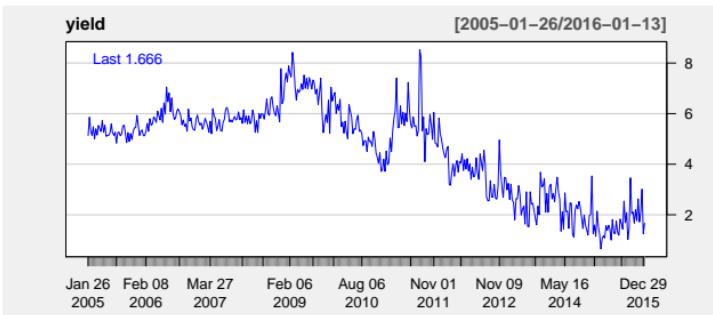


Fig. 17.13: Orange Cnty Calif bond yields.

Bond pricing in the Dothan model

In the [Dothan \(1978\)](#) model, the short-term interest rate process $(r_t)_{t \in \mathbb{R}_+}$ is modeled according to a geometric Brownian motion

$$dr_t = \mu r_t dt + \sigma r_t dB_t, \tag{17.38}$$

where the volatility $\sigma > 0$ and the drift $\mu \in \mathbb{R}$ are constant parameters and $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion. In this model the short-term interest rate r_t remains always positive, while the proportional volatility term

σr_t accounts for the sensitivity of the volatility of interest rate changes to the level of the rate r_t .

On the other hand, the Dothan model is the only lognormal short rate model that allows for an analytical formula for the zero-coupon bond price

$$P(t, T) = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T.$$

For convenience of notation we let $p = 1 - 2\mu/\sigma^2$ and rewrite (17.38) as

$$dr_t = (1-p)\frac{\sigma^2}{2}r_t dt + \sigma r_t dB_t,$$

with solution

$$r_t = r_0 e^{\sigma B_t - p\sigma^2 t/2}, \quad t \geq 0. \quad (17.39)$$

By the Markov property of $(r_t)_{t \in \mathbb{R}_+}$, the bond price $P(t, T)$ is a function $F(t, r_t)$ of r_t and time $t \in [0, T]$:

$$P(t, T) = F(t, r_t) = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \mid r_t \right], \quad 0 \leq t \leq T. \quad (17.40)$$

By computation of the conditional expectation (17.40) using (13.10) we easily obtain the following result, cf. Proposition 1.2 of Pintoux and Privault (2011), where the function $\theta(v, t)$ is defined in (13.8).

Proposition 17.4. *The zero-coupon bond price $P(t, T) = F(t, r_t)$ is given for all $p \in \mathbb{R}$ by*

$$\begin{aligned} & F(t, x) \\ &= e^{-\sigma^2 p^2 (T-t)/8} \int_0^\infty \int_0^\infty e^{-ux} \exp \left(-2 \frac{(1+z^2)}{\sigma^2 u} \right) \theta \left(\frac{4z}{\sigma^2 u}, \frac{(T-t)\sigma^2}{4} \right) \frac{du}{u} \frac{dz}{z^{p+1}}, \end{aligned} \quad (17.41)$$

$x > 0$.

Proof. By Proposition 13.5, cf. Proposition 2 in Yor (1992), the probability distribution of the time integral $\int_0^{T-t} e^{\sigma B_s - p\sigma^2 s/2} ds$ is given by

$$\begin{aligned} & \mathbb{P} \left(\int_0^{T-t} e^{\sigma B_s - p\sigma^2 s/2} ds \in dy \right) \\ &= \int_{-\infty}^\infty \mathbb{P} \left(\int_0^t e^{\sigma B_s - p\sigma^2 s/2} ds \in dy, B_t - p\sigma t/2 \in dz \right) \\ &= \frac{\sigma}{2} \int_{-\infty}^\infty e^{-p\sigma z/2 - p^2 \sigma^2 t/8} \exp \left(-2 \frac{1 + e^{\sigma z}}{\sigma^2 y} \right) \theta \left(\frac{4e^{\sigma z/2}}{\sigma^2 y}, \frac{\sigma^2 t}{4} \right) \frac{dy}{y} dz \end{aligned}$$



$$= e^{-(T-t)p^2\sigma^2/8} \int_0^\infty \exp\left(-2\frac{1+z^2}{\sigma^2 y}\right) \theta\left(\frac{4z}{\sigma^2 y}, \frac{(T-t)\sigma^2}{4}\right) \frac{dz}{z^{p+1}} \frac{dy}{y}, \quad y > 0,$$

where the exchange of integrals is justified by the Fubini theorem and the nonnegativity of integrands. Hence, by (13.10) and (17.39) we find

$$\begin{aligned} F(t, r_t) &= P(t, T) \\ &= \mathbb{E}^* \left[\exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}^* \left[\exp \left(-r_t \int_t^T e^{\sigma(B_s - B_t) - \sigma^2 p(s-t)/2} ds \right) \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}^* \left[\exp \left(-x \int_t^T e^{\sigma(B_s - B_t) - \sigma^2 p(s-t)/2} ds \right) \right]_{x=r_t} \\ &= \mathbb{E}^* \left[\exp \left(-x \int_0^{T-t} e^{\sigma B_s - \sigma^2 p s / 2} ds \right) \right]_{x=r_t} \\ &= \int_0^\infty e^{-r_t y} \mathbb{P} \left(\int_0^{T-t} e^{\sigma B_s - p \sigma^2 s / 2} ds \in dy \right) \\ &= e^{-(T-t)p^2\sigma^2/8} \int_0^\infty e^{-r_t y} \int_0^\infty \exp\left(-2\frac{1+z^2}{\sigma^2 y}\right) \theta\left(\frac{4z}{\sigma^2 y}, \frac{(T-t)\sigma^2}{4}\right) \frac{dz}{z^{p+1}} \frac{dy}{y}. \end{aligned}$$

□

The zero-coupon bond price $P(t, T) = F(t, r_t)$ in the Dothan model can also be written for all $p \in \mathbb{R}$ as

$$\begin{aligned} F(t, x) &= \frac{(2x)^{p/2}}{2\pi^2\sigma^p} \int_0^\infty u e^{-(p^2+u^2)\sigma^2 t/8} \sinh(\pi u) \left| \Gamma \left(-\frac{p}{2} + i\frac{u}{2} \right) \right|^2 K_{iu} \left(\frac{\sqrt{8x}}{\sigma} \right) du \\ &\quad + \frac{(2x)^{p/2}}{\sigma^p} \sum_{k \geq 0} \frac{2(p-2k)^+}{k!(p-k)!} e^{\sigma^2 k(k-p)t/2} K_{p-2k} \left(\frac{\sqrt{8x}}{\sigma} \right), \quad x > 0, \quad t > 0, \end{aligned}$$

cf. Corollary 2.2 of [Pintoux and Privault \(2010\)](#), see also [Privault and Uy \(2013\)](#) for numerical computations. Zero-coupon bond prices in the Dothan model can also be computed by the conditional expression

$$\mathbb{E} \left[\exp \left(- \int_0^T r_t dt \right) \right] = \int_0^\infty \mathbb{E} \left[\exp \left(- \int_0^T r_t dt \right) \middle| r_T = z \right] d\mathbb{P}(r_T \leq z), \tag{17.42}$$

where r_T has the lognormal distribution

$$d\mathbb{P}(r_T \leq z) = d\mathbb{P}(r_0 e^{\sigma B_T - p\sigma^2 T / 2} \leq z) = \frac{1}{z\sqrt{2\pi\sigma^2 T}} e^{-(p\sigma^2 T / 2 + \log(z/r_0))^2 / (2\sigma^2 T)}.$$

In Proposition 17.5 we note that the conditional Laplace transform

$$\mathbb{E} \left[\exp \left(- \int_0^T r_t dt \right) \middle| r_T = z \right]$$

cf. (17.46) above, can be computed by a closed-form integral expression based on the modified Bessel function of the second kind

$$K_\zeta(z) := \frac{z^\zeta}{2^{\zeta+1}} \int_0^\infty \exp \left(-u - \frac{z^2}{4u} \right) \frac{du}{u^{\zeta+1}}, \quad \zeta \in \mathbb{R}, \quad z \in \mathbb{C}, \quad (17.43)$$

cf. e.g. [Watson \(1995\)](#) page 183, provided that the real part $\mathcal{R}(z^2)$ of $z^2 \in \mathbb{C}$ is positive.

Proposition 17.5. ([Privault and Yu \(2016\)](#), Proposition 4.1). *Taking $r_0 = 1$, for all $\lambda, z > 0$ we have*

$$\begin{aligned} \mathbb{E} \left[\exp \left(-\lambda \int_0^T r_s ds \right) \middle| r_T = z \right] &= \frac{4e^{-\sigma^2 T/8}}{\pi^{3/2} \sigma^2 p(z)} \sqrt{\frac{\lambda}{T}} \\ &\times \int_0^\infty e^{2(\pi^2 - \xi^2)/(\sigma^2 T)} \sin \left(\frac{4\pi\xi}{\sigma^2 T} \right) \sinh(\xi) \frac{K_1(\sqrt{8\lambda} \sqrt{1 + 2\sqrt{z} \cosh \xi + z}/\sigma)}{\sqrt{1 + 2\sqrt{z} \cosh \xi + z}} d\xi. \end{aligned} \quad (17.44)$$

Note however that the numerical evaluation of (17.44) can fail for small values of $T > 0$, and for this reason the integral can be estimated by a gamma approximation as in (17.45) below. Under the gamma approximation we can approximate the conditional bond price on the Dothan short rate r_t as

$$\mathbb{E} \left[\exp \left(-\lambda \int_0^T r_t dt \right) \middle| r_T = z \right] \simeq (1 + \lambda \theta(z))^{-\nu(z)},$$

where, letting

$$\Lambda_T := \int_0^T r_t dt,$$

the parameters $\nu(z)$ and $\theta(z)$ are determined by conditional moment fitting to a gamma distribution, as

$$\theta(z) := \frac{\text{Var}[\Lambda_T \mid S_T = z]}{\mathbb{E}[\Lambda_T \mid S_T = z]}, \quad \nu(z) := \frac{(\mathbb{E}[\Lambda_T \mid S_T = z])^2}{\text{Var}[\Lambda_T \mid S_T = z]} = \frac{\mathbb{E}[\Lambda_T \mid S_T = z]}{\theta},$$

cf. [Privault and Yu \(2016\)](#), which yields

$$\mathbb{E} \left[\exp \left(-\lambda \int_0^T r_s ds \right) \right] \simeq \int_0^\infty (1 + \lambda \theta(z))^{-\nu(z)} d\mathbb{P}(r_T \leq z). \quad (17.45)$$



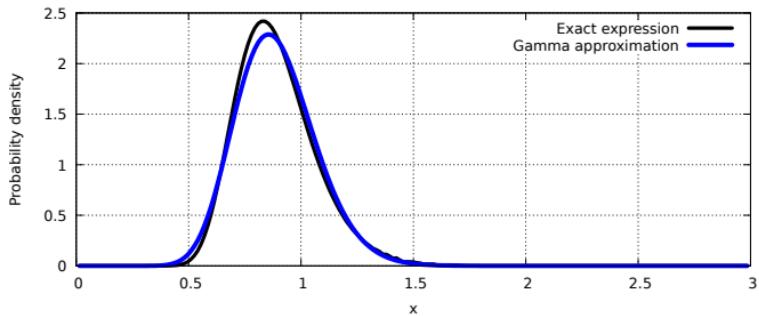
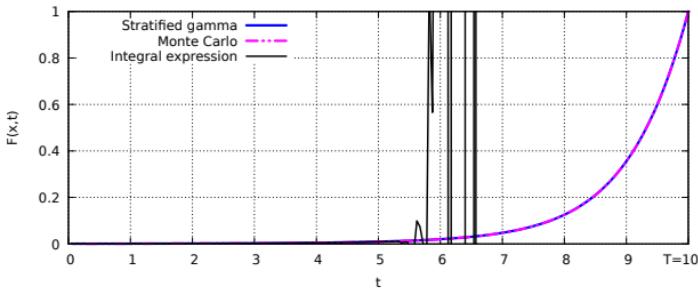


Fig. 17.14: Fitting of a gamma probability density function.

The quantity $\theta(z)$ is also known in physics as the *Fano factor* or *dispersion index* that measures the dispersion of the probability distribution of Λ_T given that $S_T = z$. Figures 17.15 shows that the stratified gamma approximation (17.45) matches the Monte Carlo estimate, while the use of the integral expressions (17.42) and (17.44) leads to numerical instabilities.

Fig. 17.15: Approximation of Dothan bond prices $t \mapsto F(t, x)$ with $\sigma = 0.3$ and $T = 10$.

Related computations for yield options in the Cox et al. (1985) (CIR) model can also be found in Prayoga and Privault (2017).

Path integrals in option pricing

Let \hbar denote the Planck constant, and let $S(x(\cdot))$ denote the action functional given as

$$S(x(\cdot)) = \int_t^T L(x(s), \dot{x}(s), s) ds = \int_t^T \left(\frac{1}{2} m(\dot{x}(s))^2 - V(x(s)) \right) ds,$$

where $L(x(s), \dot{x}(s), s)$ is the Lagrangian

$$L(x(s), \dot{x}(s), s) := \frac{1}{2}m(\dot{x}(s))^2 - V(x(s)).$$

In physics, the Feynman path integral

$$\psi(y, t) := \int_{x(t)=x, x(T)=y} \mathcal{D}x(\cdot) \exp\left(\frac{i}{\hbar} S(x(\cdot))\right)$$

solves the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t}(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}(x, t) + V(x(t))\psi(x, t).$$

After the Wick rotation $t \mapsto -it$, the function

$$\phi(y, t) := \int_{x(t)=x, x(T)=y} \mathcal{D}x(\cdot) \exp\left(-\frac{1}{\hbar} S(x(\cdot))\right)$$

solves the heat equation

$$\hbar \frac{\partial \phi}{\partial t}(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial x^2}(x, t) + V(x(t))\phi(x, t).$$

By reformulating the action functional $S(x(\cdot))$ as

$$\begin{aligned} S(x(\cdot)) &= \int_t^T \left(\frac{1}{2}m(\dot{x}(s))^2 + V(x(s)) \right) ds \\ &\simeq \sum_{i=1}^N \left(\frac{(x(t_i) - x(t_{i-1}))^2}{2(t_i - t_{i-1})^2} + V(x(t_{i-1})) \right) \Delta t_i, \end{aligned}$$

we can rewrite the Euclidean path integral as

$$\begin{aligned} \phi(y, t) &= \int_{x(t)=x, x(T)=y} \mathcal{D}x(\cdot) \exp\left(-\frac{1}{\hbar} S(x(\cdot))\right) \\ &= \int_{x(t)=x, x(T)=y} \mathcal{D}x(\cdot) \exp\left(-\frac{1}{2\hbar} \sum_{i=1}^N \frac{(x(t_i) - x(t_{i-1}))^2}{2\Delta t_i} - \frac{1}{\hbar} \sum_{i=1}^N V(x(t_{i-1}))\right) \\ &= \mathbb{E}^* \left[\exp\left(-\frac{1}{\hbar} \int_t^T V(B_s) ds\right) \middle| B_t = x, B_T = y \right]. \end{aligned}$$

This type of path integral computation

$$\phi(y, t) = \mathbb{E}^* \left[\exp\left(-\int_t^T V(B_s) ds\right) \middle| B_t = x, B_T = y \right]. \quad (17.46)$$

is particularly useful for bond pricing, as (17.46) can be interpreted as the price of a bond with short-term interest rate process $(r_s)_{s \in \mathbb{R}_+} :=$



$(V(B_s))_{s \in \mathbb{R}_+}$ conditionally to the value of the endpoint $B_T = y$, cf. (17.44) below. It can also be useful for exotic option pricing, cf. Chapter 13, and for risk management, see *e.g.* Kakushadze (2015). The path integral (17.46) can be estimated either by closed-form expressions using Partial Differential Equations (PDEs) or probability densities, by approximations such as (conditional) Moment matching, or by Monte Carlo estimation, from the paths of a Brownian bridge as shown in Figure 17.16.

Fig. 17.16: Brownian bridge.

Exercises

Exercise 17.1 We consider the stochastic differential equation

$$dr_t = (a - br_t)dt + \sigma dB_t, \quad (17.47)$$

where $a, \sigma \in \mathbb{R}$, $b > 0$.

a) Show that the solution of (17.47) is

$$r_t = r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt}) + \sigma \int_0^t e^{-(t-s)b} dB_s, \quad t \geq 0. \quad (17.48)$$

b) Show that the Gaussian $\mathcal{N}(a/b, \sigma^2/(2b))$ distribution is the *invariant* (or stationary) distribution of $(r_t)_{t \in \mathbb{R}_+}$.

Exercise 17.2 (Huang (2017)) Consider an asset whose price S_t follows the stochastic differential equation

$$dS_t = r_t S_t dt + \sigma S_t dB_t$$

with solution

$$S_T = S_0 e^{\sigma B_T + \int_0^T r_s ds - \sigma^2 T / 2},$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under a risk-neutral probability measure \mathbb{P}^* , and $(r_t)_{t \in \mathbb{R}_+}$ is a stochastic interest rate process.

a) Express the price

$$C(t, T) = \mathbb{E}^* \left[(S_T - S_t) e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right] \quad (17.49)$$

of the **contract for difference (CFD)** in terms of S_t and of the zero-coupon bond price

$$P(t, T) := \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T.$$

b) Express the price $F(t, T)$ of the **futures contract** on S_T in terms of S_t and of the zero-coupon bond price $P(t, T)$, from the no-arbitrage relation

$$\mathbb{E}^* \left[F(t, T) e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right] = \mathbb{E}^* \left[S_T e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T.$$

c) Evaluate the **CFD** price $C(t, T)$ in terms of $F(t, T)$ and $P(t, T)$ only.

Exercise 17.3 Consider the Chan-Karolyi-Longstaff-Sanders (CKLS) interest rate model ([Chan et al. \(1992\)](#)) parametrized as

$$dr_t = (a - br_t)dt + \sigma r_t^\gamma dB_t,$$

and time-discretized as

$$\begin{aligned} r_{t_{k+1}} &= r_{t_k} + (a - br_{t_k})\Delta t + \sigma r_{t_k}^\gamma Z_k \\ &= a\Delta t + (1 - b\Delta t)r_{t_k} + \sigma r_{t_k}^\gamma Z_k, \quad k \geq 0, \end{aligned}$$

where $\Delta t := t_{k+1} - t_k$ and $(Z_k)_{k \geq 0}$ is an *i.i.d.* sequence of $\mathcal{N}(0, \Delta t)$ random variables. Assuming that $a, b, \gamma > 0$ are known, find an unbiased estimator $\tilde{\sigma}^2$ for the variance coefficient σ^2 , based on a market data set $(\tilde{r}_{t_k})_{k=0,1,\dots,n}$.

Exercise 17.4 ([Brody et al. \(2018\)](#)) In the mean-reverting Vasicek model (17.48) with $b > 0$, compute:

i) The asymptotic bond yield, or exponential long rate of interest

$$r_\infty := - \lim_{T \rightarrow \infty} \frac{\log P(t, T)}{T - t}.$$

ii) The long-bond return

$$L_t := \lim_{T \rightarrow \infty} \frac{P(t, T)}{P(0, T)}.$$

Hint: Start from $\log(P(t, T)/P(0, T))$.



Exercise 17.5 Let $(B_t)_{t \in \mathbb{R}_+}$ denote a standard Brownian motion started at 0 under the risk-neutral probability measure \mathbb{P}^* . We consider a short-term interest rate process $(r_t)_{t \in \mathbb{R}_+}$ in a Ho-Lee model with constant deterministic volatility, defined by

$$dr_t = adt + \sigma dB_t, \quad (17.50)$$

where $a \in \mathbb{R}$ and $\sigma > 0$. Let $P(t, T)$ denote the arbitrage-free price of a zero-coupon bond in this model:

$$P(t, T) = \mathbb{E}^* \left[\exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T. \quad (17.51)$$

a) State the bond pricing PDE satisfied by the function $F(t, x)$ defined via

$$F(t, x) := \mathbb{E}^* \left[\exp \left(- \int_t^T r_s ds \right) \middle| r_t = x \right], \quad 0 \leq t \leq T.$$

b) Compute the arbitrage-free price $F(t, r_t) = P(t, T)$ from its expression (17.51) as a conditional expectation.

Hint: One may use the *integration by parts* relation

$$\begin{aligned} \int_t^T B_s ds &= TB_T - tB_t - \int_t^T s dB_s \\ &= (T-t)B_t + (B_T - B_t)T - \int_t^T s dB_s \\ &= (T-t)B_t + \int_t^T (T-s) dB_s, \end{aligned}$$

and the Gaussian moment generating function $\mathbb{E}[e^{\lambda X}] = e^{\lambda^2 \eta^2 / 2}$ for $X \sim \mathcal{N}(0, \eta^2)$.

c) Check that the function $F(t, x)$ computed in Question (b) does satisfy the PDE derived in Question (a).

Exercise 17.6 Consider the [Courtadon \(1982\)](#) model

$$dr_t = \beta(\alpha - r_t)dt + \sigma r_t dB_t, \quad (17.52)$$

where α, β, σ are nonnegative, which is a particular case of the Chan-Karolyi-Longstaff-Sanders (CKLS) model ([Chan et al. \(1992\)](#)) with $\gamma = 1$.

a) Show that the solution of (17.52) is given by

$$r_t = \alpha\beta \int_0^t \frac{S_u}{S_u} du + r_0 S_t, \quad t \geq 0, \quad (17.53)$$

where $(S_t)_{t \in \mathbb{R}_+}$ is the geometric Brownian motion solution of $dS_t = -\beta S_t dt + \sigma S_t dB_t$ with $S_0 = 1$.

- b) Given that the discounted bond price process is a martingale, derive the bond pricing PDE satisfied by the function $F(t, x)$ such that

$$F(t, r_t) = P(t, T) = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right] = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \mid r_t \right].$$

Exercise 17.7 Consider the [Marsh and Rosenfeld \(1983\)](#) short-term interest rate model

$$dr_t = (\beta r_t^{\gamma-1} + \alpha r_t) dt + \sigma r_t^{\gamma/2} dB_t$$

where $\alpha \in \mathbb{R}$ and $\beta, \sigma, \gamma > 0$.

- a) Letting $R_t := r_t^{2-\gamma}$, $t \geq 0$, find the stochastic differential equation satisfied by $(R_t)_{t \in \mathbb{R}_+}$.
 b) Given that the discounted bond price process is a martingale, derive the bond pricing PDE satisfied by the function $F(t, x)$ such that

$$F(t, r_t) = P(t, T) = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right] = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \mid r_t \right].$$

Exercise 17.8 Consider the [Cox et al. \(1985\)](#) (CIR) process $(r_t)_{t \in \mathbb{R}_+}$ solution of

$$dr_t = -ar_t dt + \sigma \sqrt{r_t} dB_t,$$

where $a, \sigma > 0$ are constants $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion started at 0.

- a) Write down the bond pricing PDE for the function $F(t, x)$ given by

$$F(t, x) := \mathbb{E}^* \left[\exp \left(- \int_t^T r_s ds \right) \mid r_t = x \right], \quad 0 \leq t \leq T.$$

Hint: Use Itô calculus and the fact that the discounted bond price is a martingale.

- b) Show that the PDE found in Question (a) admits a solution of the form $F(t, x) = e^{A(T-t)+xC(T-t)}$ where the functions $A(s)$ and $C(s)$ satisfy ordinary differential equations to be also written down together with the values of $A(0)$ and $C(0)$.

Exercise 17.9 Convertible bonds. Consider an underlying asset price process $(S_t)_{t \in \mathbb{R}_+}$ given by

$$dS_t = r S_t dt + \sigma S_t dB_t^{(1)},$$

and a short-term interest rate process $(r_t)_{t \in \mathbb{R}_+}$ given by

$$dr_t = \gamma(t, r_t) dt + \eta(t, r_t) dB_t^{(2)},$$



where $(B_t^{(1)})_{t \in \mathbb{R}_+}$ and $(B_t^{(2)})_{t \in \mathbb{R}_+}$ are two correlated Brownian motions under the risk-neutral probability measure \mathbb{P}^* , with $dB_t^{(1)} \cdot dB_t^{(2)} = \rho dt$. A convertible bond is made of a corporate bond priced $P(t, T)$ at time $t \in [0, T]$, that can be exchanged into a quantity $\alpha > 0$ of the underlying company's stock priced S_τ at a future time τ , whichever has a higher value, where α is a *conversion rate*.

- Find the payoff of the convertible bond at time τ .
- Rewrite the convertible bond payoff at time τ as the linear combination of $P(\tau, T)$ and a call option payoff on S_τ , whose strike price is to be determined.
- Write down the convertible bond price at time $t \in [0, \tau]$ as a function $C(t, S_t, r_t)$ of the underlying asset price and interest rate, using a discounted conditional expectation, and show that the discounted corporate bond price

$$e^{-\int_0^t r_s ds} C(t, S_t, r_t), \quad t \in [0, \tau],$$

is a martingale.

- Write down $d(e^{-\int_0^t r_s ds} C(t, S_t, r_t))$ using the Itô formula and derive the pricing PDE satisfied by the function $C(t, x, y)$ together with its terminal condition.
- Taking the bond price $P(t, T)$ as a numéraire, price the convertible bond as a European option with strike price $K = 1$ on an underlying asset priced $Z_t := S_t/P(t, T)$, $t \in [0, \tau]$ under the forward measure $\hat{\mathbb{P}}$ with maturity T .
- Assuming the bond price dynamics

$$dP(t, T) = r_t P(t, T) dt + \sigma_B(t) P(t, T) dB_t,$$

determine the dynamics of the process $(Z_t)_{t \in \mathbb{R}_+}$ under the forward measure $\hat{\mathbb{P}}$.

- Assuming that $(Z_t)_{t \in \mathbb{R}_+}$ can be modeled as a geometric Brownian motion, price the convertible bond using the Black-Scholes formula.

Exercise 17.10 Bond duration. Compute the duration

$$D_c(0, n) := -\frac{1+r}{P_c(0, n)} \frac{\partial}{\partial r} P_c(0, n)$$

of a discrete-time coupon bond priced as

$$\begin{aligned} P_c(0, n) &= \frac{1}{(1+r)^n} + c \sum_{k=1}^n \frac{1}{(1+r)^k} \\ &= \frac{c}{r} + \left(1 - \frac{c}{r}\right) \frac{1}{(1+r)^n}, \end{aligned}$$

where $r > 0$, and $c \geq 0$ denotes the coupon rate. What happens when n becomes large?

Exercise 17.11 Let $(r_t)_{t \in \mathbb{R}_+}$ denote a short-term interest rate process. For any $T > 0$, let $P(t, T)$ denote the price at time $t \in [0, T]$ of a zero-coupon bond defined by the stochastic differential equation

$$\frac{dP(t, T)}{P(t, T)} = r_t dt + \sigma_t^T dB_t, \quad 0 \leq t \leq T, \quad (17.54)$$

under the terminal condition $P(T, T) = 1$, where $(\sigma_t^T)_{t \in [0, T]}$ is an adapted process. We define the forward measure $\hat{\mathbb{P}}$ by

$$\mathbb{E}^* \left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} \mid \mathcal{F}_t \right] = \frac{P(t, T)}{P(0, T)} e^{-\int_0^t r_s ds}, \quad 0 \leq t \leq T.$$

Recall that

$$B_t^T := B_t - \int_0^t \sigma_s^T ds, \quad 0 \leq t \leq T,$$

is a standard Brownian motion under $\hat{\mathbb{P}}$.

- a) Solve the stochastic differential equation (17.54).
- b) Derive the stochastic differential equation satisfied by the discounted bond price process

$$t \mapsto e^{-\int_0^t r_s ds} P(t, T), \quad 0 \leq t \leq T,$$

and show that it is a martingale.

- c) Show that

$$\mathbb{E}^* \left[e^{-\int_0^T r_s ds} \mid \mathcal{F}_t \right] = e^{-\int_0^t r_s ds} P(t, T), \quad 0 \leq t \leq T.$$

- d) Show that

$$P(t, T) = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T.$$

- e) Compute $P(t, S)/P(t, T)$, $0 \leq t \leq T$, show that it is a martingale under the forward measure $\hat{\mathbb{P}}$ with maturity T , and that

$$P(T, S) = \frac{P(t, S)}{P(t, T)} \exp \left(\int_t^T (\sigma_s^S - \sigma_s^T) dB_s^T - \frac{1}{2} \int_t^T (\sigma_s^S - \sigma_s^T)^2 ds \right).$$

- f) Assuming that $(\sigma_t^T)_{t \in [0, T]}$ and $(\sigma_t^S)_{t \in [0, S]}$ are deterministic functions of time, compute the price

$$\mathbb{E}^* \left[e^{-\int_t^T r_s ds} (P(T, S) - \kappa)^+ \mid \mathcal{F}_t \right] = P(t, T) \hat{\mathbb{E}} \left[(P(T, S) - \kappa)^+ \mid \mathcal{F}_t \right]$$



of a bond option with strike price K .

Recall that if X is a Gaussian random variable with mean $m(t)$ and variance $v^2(t)$ given \mathcal{F}_t , we have

$$\begin{aligned} & \mathbb{E}[(e^X - K)^+ | \mathcal{F}_t] \\ &= e^{m(t) + v^2(t)/2} \Phi\left(\frac{v(t)}{2} + \frac{1}{v(t)}(m(t) + v^2(t)/2 - \log K)\right) \\ &\quad - K \Phi\left(-\frac{v(t)}{2} + \frac{1}{v(t)}(m(t) + v^2(t)/2 - \log K)\right) \end{aligned}$$

where $\Phi(x)$, $x \in \mathbb{R}$, denotes the Gaussian cumulative distribution function.

Exercise 17.12 (Exercise 4.20 continued). Write down the bond pricing PDE for the function

$$F(t, x) = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \mid r_t = x \right]$$

and show that in case $\alpha = 0$ the corresponding bond price $P(t, T)$ equals

$$P(t, T) = e^{-r_t B(T-t)}, \quad 0 \leq t \leq T,$$

where

$$B(x) := \frac{2(e^{\gamma x} - 1)}{2\gamma + (\beta + \gamma)(e^{\gamma x} - 1)}, \quad x \in \mathbb{R},$$

with $\gamma = \sqrt{\beta^2 + 2\sigma^2}$.

Exercise 17.13 Consider a zero-coupon bond with prices $P(1, 2) = 91.74\%$ and $P(0, 2) = 83.40\%$ at times $t = 0$ and $t = 1$.

- Compute the corresponding yields $y_{0,1}$, $y_{0,2}$ and $y_{1,2}$ at times $t = 0$ and $t = 1$.
- Assume that \$0.1 coupons are paid at times $t = 1$ and $t = 2$. Price the corresponding coupon bond at times $t = 0$ and $t = 1$ using the yields y_0 and y_1 .

Exercise 17.14 Let $(r_t)_{t \in \mathbb{R}_+}$ denote the Vasicek process solution of the equation

$$dr_t = (a - br_t)dt + \sigma dB_t. \tag{17.55}$$

- Consider the discretization

$$r_{t_{k+1}} := r_{t_k} + (a - br_{t_k})\Delta t \pm \sigma\sqrt{\Delta t}, \quad k = 0, 1, 2, \dots$$

of the equation (17.55) with $p(r_{t_0}) = p(r_{t_1}) = 1/2$ and

$$\mathbb{E}[\Delta r_{t_1}] = (a - br_{t_0})\Delta t + \sigma p(r_{t_0})\sqrt{\Delta t} - \sigma q(r_{t_0})\sqrt{\Delta t} = (a - br_{t_0})\Delta t$$

and

$$\mathbb{E}[\Delta r_{t_2}] = (a - br_{t_1})\Delta t + \sigma p(r_{t_0})\sqrt{\Delta t} - \sigma q(r_{t_0})\sqrt{\Delta t} = (a - br_{t_1})\Delta t.$$

Does this discretization lead to a (recombining) binomial tree?

- b) Using the Girsanov Theorem, find a probability measure \mathbf{Q} under which the process $(r_t/\sigma)_{t \in [0, T]}$ with

$$\frac{dr_t}{\sigma} = \frac{a - br_t}{\sigma} dt + dB_t$$

is a standard Brownian motion.

Hint: By the Girsanov Theorem, the process $X_t = X_0 + \int_0^t u_s ds + B_t$ is a martingale under the probability measure \mathbf{Q} with Radon-Nikodym density

$$\frac{d\mathbf{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T u_t dB_t - \frac{1}{2} \int_0^T (u_t)^2 dt\right)$$

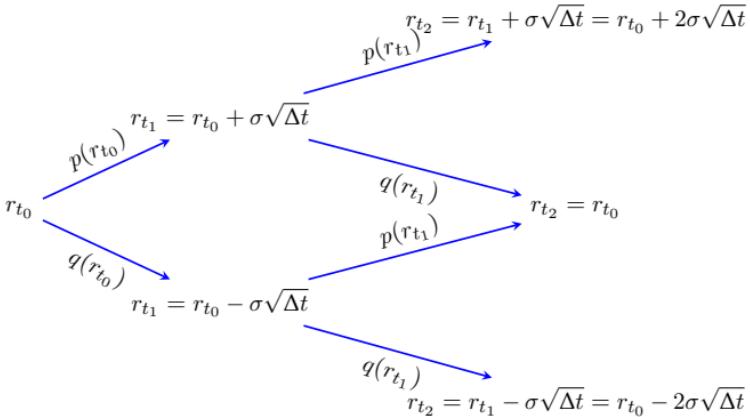
with respect to \mathbb{P} .

- c) Prove the Radon-Nikodym derivative approximation

$$\begin{aligned} & 2^{T/\Delta T} \prod_{0 < t < T} \left(\frac{1}{2} \pm \frac{a - br_t}{2\sigma} \sqrt{\Delta t} \right) \\ & \simeq \exp\left(\frac{1}{\sigma^2} \int_0^T (a - br_t) dr_t - \frac{1}{2\sigma^2} \int_0^T (a - br_t)^2 dt\right). \end{aligned} \quad (17.56)$$

- d) Using (17.56), show that the Vasicek process can be discretized along the *binomial tree*





with

$$\mathbb{E}[\Delta r_{t_1} | r_{t_0}] = (a - br_{t_0})\Delta t \quad \text{and} \quad \mathbb{E}[\Delta r_{t_2} | r_{t_1}] = (a - br_{t_1})\Delta t,$$

where $\Delta r_{t_1} := r_{t_1} - r_{t_0}$, $\Delta r_{t_2} := r_{t_2} - r_{t_1}$, and the probabilities $p(r_{t_0})$, $q(r_{t_0})$, $p(r_{t_1})$, $q(r_{t_1})$ will be computed explicitly.

The use of binomial (or recombining) trees can make the implementation of the Monte Carlo method easier as their size grows linearly instead of exponentially.

Exercise 17.15 Black-Derman-Toy model (Black et al. (1990)). Consider a one-step interest rate model in which the short-term interest rate r_0 on $[0, 1]$ can turn into two possible values $r_1^u = r_0 e^{\mu\Delta t + \sigma\sqrt{\Delta t}}$ and $r_1^d = r_0 e^{\mu\Delta t - \sigma\sqrt{\Delta t}}$ on the time interval $[1, 2]$ with equal probabilities $1/2$ at time $\Delta t = 1$ year and volatility $\sigma = 22\%$ per year, and a zero-coupon bonds with prices $P(0, 1)$ and $P(0, 2)$ at time $t = 0$.

- Write down the value of $P(1, 2)$ using r_1^u and r_1^d .
- Write down the value of $P(0, 2)$ using r_1^u , r_1^d and r_0 .
- Estimate the value of r_0 from the market price $P(0, 1) = 91.74$.
- Estimate the values of r_1^u and r_1^d from the market price $P(0, 2) = 83.40$.

Exercise 17.16 Consider a yield curve $(f(t, t, T))_{0 \leq t \leq T}$ and a bond paying coupons c_1, c_2, \dots, c_n at times T_1, T_2, \dots, T_n until maturity T_n , and priced as

$$P(t, T_n) = \sum_{k=1}^n c_k e^{-(T_k - t)f(t, t, T_k)}, \quad 0 \leq t \leq T_1,$$

where c_n is inclusive of the last coupon payment and the nominal \$1 value of the bond. Let $\tilde{f}(t, t, T_n)$ denote the *compounded yield to maturity* defined by equating

$$P(t, T_n) = \sum_{k=1}^n c_k e^{-(T_k - t)\tilde{f}(t, t, T_n)}, \quad 0 \leq t \leq T_1, \quad (17.57)$$

i.e. $\tilde{f}(t, t, T_n)$ solves the equation

$$F(t, \tilde{f}(t, t, T_n)) = P(t, T_n), \quad 0 \leq t \leq T_1,$$

with

$$F(t, x) := \sum_{k=1}^n c_k e^{-(T_k - t)x}, \quad 0 \leq t \leq T_1.$$

The *bond duration* $D(t, T_n)$ is the relative sensitivity of $P(t, T_n)$ with respect to $\tilde{f}(t, t, T_n)$, defined as

$$D(t, T_n) := -\frac{1}{P(t, T_n)} \frac{\partial F}{\partial x}(t, \tilde{f}(t, t, T_n)), \quad 0 \leq t \leq T_1.$$

The *bond convexity* $C(t, T_n)$ is defined as

$$C(t, T_n) := \frac{1}{P(t, T_n)} \frac{\partial^2 F}{\partial x^2}(t, \tilde{f}(t, t, T_n)), \quad 0 \leq t \leq T_1.$$

- a) Compute the bond duration in case $n = 1$.
- b) Show that the *bond duration* $D(t, T_n)$ can be interpreted as an average of times to maturity weighted by the respective discounted bond payoffs.
- c) Show that the *bond convexity* $C(t, T_n)$ satisfies

$$C(t, T_n) = (D(t, T_n))^2 + (S(t, T_n))^2,$$

where

$$(S(t, T_n))^2 := \sum_{k=1}^n w_k (T_k - t - D(t, T_n))^2$$

measures the dispersion of the duration of the bond payoffs around the portfolio duration $D(t, T_n)$.

- d) Consider now the zero-coupon yield defined as

$$P(t, t + \alpha(T_n - t)) = \exp(-\alpha(T_n - t)f_\alpha(t, t, T_n)),$$

where $\alpha \in (0, 1)$, i.e.

$$f_\alpha(t, t, T_n) := -\frac{1}{\alpha(T_n - t)} \log P(t, t + \alpha(T_n - t)), \quad 0 \leq t \leq T_n.$$



Compute the bond duration associated to the yield $f_\alpha(t, t, T_n)$ in affine bond pricing models of the form

$$P(t, T) = e^{A(T-t)+r_t B(T-t)}, \quad 0 \leq t \leq T.$$

- e) ([Wu \(2000\)](#)) Compute the bond duration associated to the yield $f_\alpha(t, t, T_n)$ in the Vasicek model, in which

$$B(T-t) := \frac{1 - e^{-(T-t)b}}{b}, \quad 0 \leq t \leq T.$$