Chaos and the Double Pendulum

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Abstract

In order to explore concepts of Chaos Theory we analyzed the motion of the Double Pendulum. We first derive the governing equations of motion using Lagrangian Mechanics. We then, using numerical methods, find the approximate solution of the equations of motion. Finally we analyze the motion of a physical double pendulum using tracking software and compare our results to our simulated double pendulum.

Introduction

Chaos

Science is no stranger to complexity, in fact it seems that in the study of the natural world complex systems are inescapable. Through science and mathematics we have begun to uncover the fundamental properties of the universe and we always seem to have clever tricks to circumnavigate the inherent complexity of what we study. Yet, throughout the natural world there are a set of systems that have a hidden layer to them that makes analyzing them especially difficult, even when they don't seem complex at first glance. We describe these systems as chaotic.

Chaos Theory is a branch of mathematics that deals with dynamic systems that are deemed chaotic. This includes a wide variety of systems within many fields of science such as weather systems, population models, and celestial mechanics. Even systems typically outside the purview of science such as Financial Markets and Economies can exhibit chaos [4], but what exactly does it mean to be chaotic?

There are three conditions that must be met for a system to be considered chaotic.

- Sensitivity to initial conditions
- Topological Mixing
- Density of Periodic Orbits

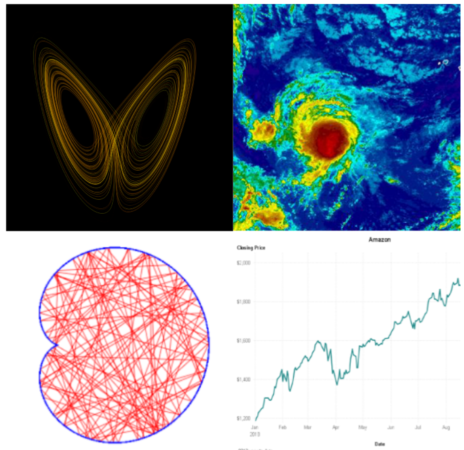

Figure 1: Systems that exhibit chaos. Top-Left: Lorentz Attractor, Top-Right: Weather System, Bottom-Left: Chaotic Billiards, Bottom-Right: Market Cycles

Sensitivity to Initial Conditions

In simplest terms a sensitivity to initial conditions means that if you adjust the initial conditions of the system by even a small amount, you will see a large change in the evolution of the system over time.

Suppose we have some dynamical system and we want to evaluate the system at initial condition y_0 and again at y_0'. We will define the separation between these initial states and subsequent states as it evolves in time as $\Delta y(t)$ and we can show how this changes in time with the following relationship

$$\Delta y(t) = e^{\lambda t} \Delta y(0)$$

[3]

A system's sensitivity to initial conditions is dependent on λ which is known as the Lyapunov exponent. There are three possible values for a Lyapunov exponent, which indicate different separation behaviours.

- If $\lambda < 0$ our system is dissipative, so our trajectories will eventually converge.
- If $\lambda = 0$ the separation will remain unchanged as our system evolves, which we might see in the case of simple harmonic motion.
- If $\lambda > 0$ our trajectories will begin to diverge exponentially as it evolves, which is a good indicator that our system is chaotic.

It may be tempting to say that this is the only condition we need for a system to be chaotic as surely minute changes growing over time seems chaotic, but this is not the case.

Consider the the following system.

$$y_{n+1} = 3y_n + 1$$

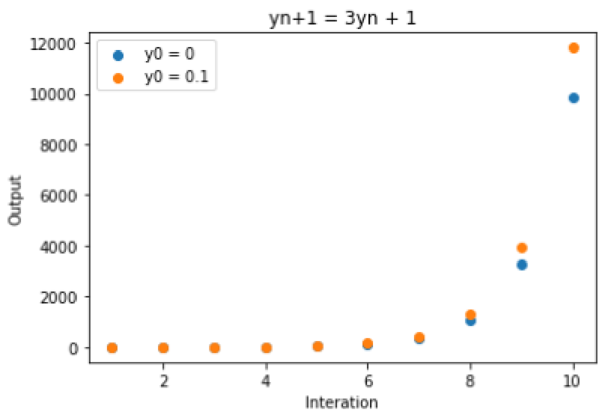

Figure 2: This system is sensitive to the change in initial conditions but it is not chaotic

We can see in Figure 1 how the system evolves over each iteration for two separate initial conditions. After only eight iterations the output diverges significantly even though they were initially separated by 0.1. This certainly qualifies as a sensitivity to initial conditions but the separation is monotonically increasing. No matter what our initial conditions are the system approaches positive or negative infinity, making the system not chaotic.

Topological Mixing

Topological Mixing helps to exclude cases, such as the system in Figure 1, from the definition of chaos. If a system exhibits Topological Mixing then any open set in the phase space of our system will at some point intersect with any other open set within the phase space. This means that as the set of possible states for one set of initial conditions evolves over time it will always (eventually) ”smear out” and intersect with the set of states for any other set of initial conditions.

Density of Periodic Orbits

To determine the Density of Periodic Orbits for a given dynamical system we must again look at the phase space. A system with dense periodic orbits means that if we look at any state in the phase space, it will be arbitrarily close to another set of initial conditions that lead to a different periodic orbit. This makes the system irreversibly as if you look at a certain state you have no way of determining what initial conditions it evolved from.

One could argue that there only needs to be two conditions for chaos as Topological Mixing and Dense Periodic Orbits actually implies a sensitivity to initial conditions, but we leave that for mathematicians to argue.

The Double Pendulum

The chaotic system we explored is the Double Pendulum, which is simply a pendulum with a second pendulum attached to the end of the first pendulum. The system seems simple but the effect of the two pendulum masses on each other cause complex behaviours. It was first formally studied by Euler and Daniel Bernoulli in 1738 and even generalised the system to n-pendulums. However it wasn't until the advent of electric computers that systems such as the double pendulum could be studied in more detail since the solutions to its governing equations in most cases can only be solved numerically.

There are some applications to studying the double pendulum specifically beyond simply exploring Chaos Theory. Test masses are suspended at LIGO comprised of a double pendulum system; its purpose is to help filter seismic noise from the gravitational waves awaiting detection [1]. The double pendulum can also be used to roughly approximate human arms and model the swinging motion in games such as golf [2].

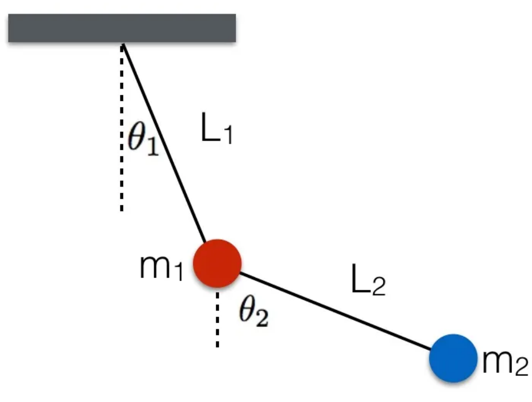

Figure 3: A graphic showing the parameters and variables of the Double Pendulum

The Lyapunov exponent can actually vary in the double pendulum. If we were to dampen the system we would expect $\lambda < 0$ and the separation in trajectories to tighten. When we use a small angle release as our initial conditions we will see $\lambda \approx 0$ which gives us seemingly regular behavior. Our large angle release or forced system will give us $\lambda > 0$ and we see the characteristic chaotic behavior of the double pendulum.

One should note that while the Lyapunov exponent is a good indicator of chaotic behavior, in general it is not a sufficient condition for chaos. For this case though, since we know the double pendulum satisfies the other conditions we use for chaos, we can use the value of the Lyapunov exponent to tell us when chaotic behavior will arise.

Lagrangian Formulation of the Equations of Motion

The standard derivation of the equations of motion for our system is using Lagrangian Formalism to analyze the system using energy rather than forces. We first need the kinetic and potential energies of the system. We can get these by adding the respective energies of the two pendulums together.

$$T = \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)]$$

$$V = -(m_1 + m_2)gl_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$

From there we can form our lagrangian.

$$\mathcal{L} = \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2 l_2^2\dot{\theta}_2^2+m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)+(m_1+m_2)gl_1\cos\theta_1+m_2 g l_2\cos\theta_1$$

We then use generalized coordinates with the Euler-Lagrange equation to find the equations of motion. In the double pendulum we have two degrees of freedom, where $q_1 = \theta_1$ and $q_2 = \theta_2$.

For θ_1 the Euler-Lagrange equation becomes,

$$(m_1+m_2)l_1\ddot{\theta}_1+m_2 l_2\ddot{\theta}_2\cos(\theta_1-\theta_2)+m_2 l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)+(m_1+m_2)g\sin\theta_1 = 0.$$

For θ_2 the Euler-Lagrange equation becomes,

$$l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - l_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) + g\sin\theta_2 = 0.$$

These highly non-linear differential equations should be our first clue that the behaviour of the double pendulum is going to be quite complicated.

Methodology

Simulation

In order to solve our system numerically we first convert the two, second order ODE's to a system of four, first order ODE's.

$$\theta_1' = \omega_1$$
$$\theta_2' = \omega_2$$
$$\omega_1' = \frac{g(2m_1+m_2)\sin\theta_1 - m_2 g\sin(\theta 1-2\theta_2) - 2\sin(\theta_1-\theta_2)m_2(\omega_2^2 l_2+\omega_1^2 l_1\cos(\theta_1-\theta_2))}{l_1(2m_2+m_2-m_2\cos(2\theta_2-2\theta_2))}$$
$$\omega_2' = \frac{2\sin(\theta_1-\theta_2)(\omega_1^2 l_1(m_1+m_2)+g(m_1+m_2)\cos\theta_1+\omega_2^2 l_2 m_2\cos(\theta_1-\theta_2))}{l_2(2m_1+m_2-m_2\cos(\theta_1-2\theta_2))}$$

The initial conditions we used are:

$$y_0 = \begin{bmatrix} (3\pi/2) - 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Which puts the first arm of the pendulum horizontal in the negative $\hat{x}$ direction and lets the second arm hang vertically. The large initial angle ensures we see chaotic behavior. We also start both pendulums entirely at rest to simulate it being suddenly released. From there we use Scipy's odeint solver method to obtain our numerical results.

Experimental Set-up

For the physical set-up we obtained an STL file for a double pendulum online and 3d-printed it. It comes in three parts: a longer rod for the first pendulum, a shorter rod for the second pendulum, and a base at the top for them to attach to. To finish our physical apparatus, we attached the top base of the double pendulum to a stand to let the pendulums fall freely below.

In order to collect data from the set-up, we place the double pendulum in front of a large sheet of paper and took several high-speed videos of the pendulum swinging from the same angle. Then using Tracker software we were able to specify the pendulums, select our origin, and track their position over time, giving us 1500 data point for each run.

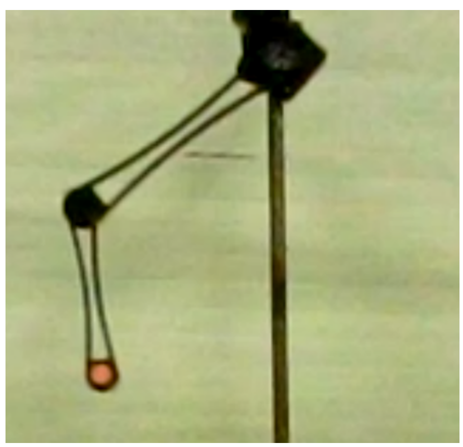

Figure 4: Our 3d-printed pendulum in action.

Data Analysis

After the positional data was collected it was processed largely with Python's Numpy package and plotted using Matplotlib. The Tracker software gave the position of the pendulums in Cartesian coordinates so the trajectories of the double pendulum in position space needed no processing and was simply plotted as they were in Figure 5.

For the phase space we decided to graph angular position versus angular velocity, so we first took the positional data in Cartesian coordinates and converted them to Polar coordinates. With our angular positions we use Numpy's Gradient function in order to numerically find the angular velocity from our data points.

In some of the trials a number of data points needed to be removed from the phase space as the camera we used had some problems with skipping frames. This did not affect the position plots but when calculating the angular velocity our data points were not technically evenly spaced, giving us much faster velocities than what was likely happening.

Results

With all the data processed we plot both the position and phase space of our simulated pendulum as well as for three runs of the real pendulum.

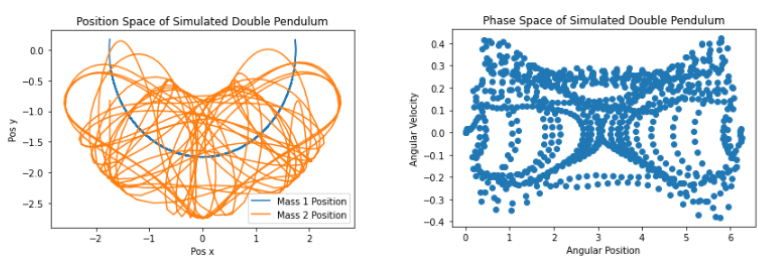

Figure 5: Phase and Position Space for Simulated Pendulum

If we first compare the position space we can see that there are clearly differences, which we expect to see. Firstly the simulation does not take friction into account so it is completely undamped while the real pendulum has slight damping. Secondly we cannot set the initial angle of the real pendulum with the infinite precision necessary to have identical results. The nature of the chaotic system means that if we deviate from the starting conditions even slightly our separation of trajectories will grow exponentially. It should be noted that even in the simulation, the round-off error and machine precision can lead to different results from what seems like the same initial conditions.

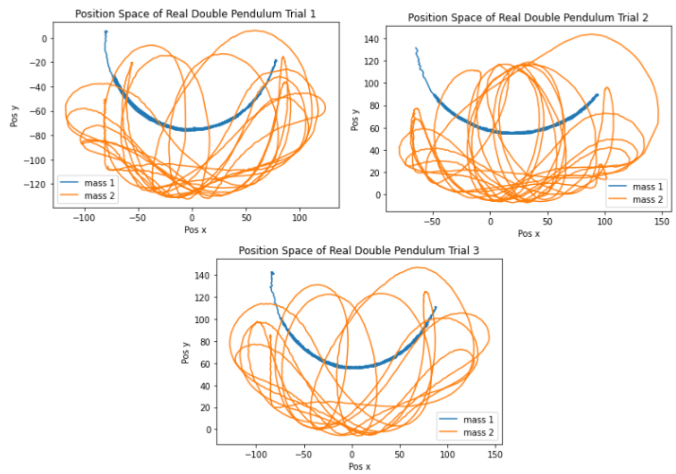

Figure 6: Position Space for a real Pendulum

When we look at Figure 7 and compare the phase spaces we can see further evidence of the chaos of the double pendulum. The phase space for Trial one actually has geometry reminiscent of our simulated phase space. With only 1500 data points the set of possible states is already ”smearing out” across phase space which we would expect to see in a system with topological mixing. We can even get a sense of the density of periodic orbits in the system. If you pick out any state in the phase space in any of the trials you can find see that it is nearly in one of a number of nearby trajectories.

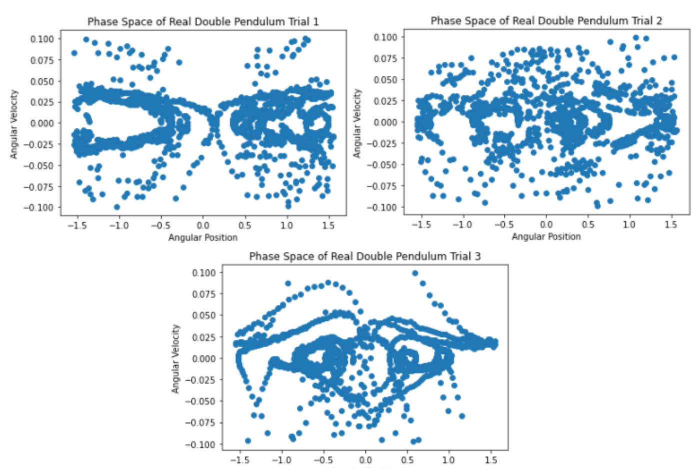

Figure 7: Phase Space for a real Pendulum

Conclusion

It is astonishing that such a simple mechanical device can exhibit these rich and complicated behaviours. The double pendulum equations of motion derived produced a pair of highly nonlinear coupled differential equations of the second order which hinted at its complexity. Using these equations we have successfully carried out investigations demonstrating the chaotic nature of the double pendulum. These results showed that even though there are a set of definite equations governing this system, it is still intrinsically unpredictable in the long run due to inevitable errors introduced either physically or numerically. Further explorations with the double pendulum could be undertaken, such as calculating the Lyapunov exponent for different initial conditions as well as studying a damped or driven system.

References

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