DIGITAL LOGIC

Chapter 2:
Introduction to logic circuits

Ru Han

2.11 MINIMIZATION AND KARNAUGH MAPS

Karnaugh Maps (K-Map): Part I

- What are K-Maps?
- Minimization by Using K-Maps: SOP
- Minimization in POS form
- Don't Care Conditions
- Networks with Multiple Outputs
- Design Examples

KARNAUGH MAPS (K-MAP)

- Recall the problems you may confront in manipulation:
  - It is not obvious and tedious
  - Whether it is the simplest expression
- Distributive property and 7a/7b is mostly used
- Shown in Example 2.48

KARNAUGH MAPS (K-MAP)

Row number	x_1	x_2	x_3	f
0	0	0	0	1
1	0	0	1	0
2	0	1	0	1
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

Row number	x_1	x_2	x_3	f
0	0	0	0	1
2	0	1	0	1
4	1	0	0	1
6	1	1	0	1

Row number	x_1	x_2	x_3	f
4	1	0	0	1
5	1	0	1	1

Figure 2.48 The function $f(x_1, x_2, x_3) = \Sigma m(0, 2, 4, 5, 6)$.

KARNAUGH MAPS (K-MAP)

- A graphical way to represent Boolean functions
- Useful for minimizing functions
- Can generate SOP or POS forms
- Grid of squares:
  - Each square represents a different minterm (or maxterm).
  - Adjacent squares differ in only one variable

    e.g., $x\bar{y}\bar{z} \leftrightarrow xy\bar{z}$
  - Makes it easy to perform steps like

    e.g., $x\bar{y}\bar{z} + xy\bar{z} = x(\bar{y} + y)\bar{z} = x\bar{z}$

2-VARIABLE MAP: *f*(A, B)

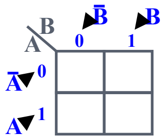
- 4 minterms:

B \ A	0	1
0	m_0	m_2
1	m_1	m_3

A	B	minterms	
0	0	$\bar{A}\bar{B}$	m_0
0	1	$\bar{A}B$	m_1
1	0	$A\bar{B}$	m_2
1	1	AB	m_3

2-VARIABLE MAP: f(A, B)

- 4 minterms:



- To represent f(A,B), put 1's in the squares for minterms that are included in f, and 0's in squares that are not included.

Ex: $f = \bar{A}B + AB$
$= (\bar{A} + A)B = B$

A \ B	0	1
0	0	1
1	0	1

Ex: $f = \bar{A}\bar{B} + \bar{A}B + A\bar{B}$
$= \bar{A} + \bar{B}$

A \ B	0	1
0	1	1
1	1	0

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EXAMPLE: HOW TO SIMPLIFY THE 2-VARIABLE MAP: f(X1, X2)

Ex: $f = \bar{x}_1\bar{x}_2 + \bar{x}_1 x_2 + x_1 x_2$

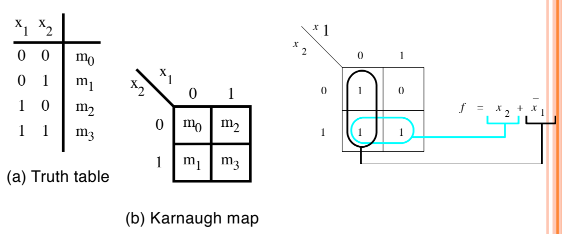

Figure 2.49. Location of two-variable minterms.

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3-VARIABLE MAP: f(A, B, C)

- 8 minterms:

C \ AB	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

A	B	C	minterms	
0	0	0	$\bar{A}\bar{B}\bar{C}$	m_0
0	0	1	$\bar{A}\bar{B}C$	m_1
0	1	0	$\bar{A}B\bar{C}$	m_2
0	1	1	$\bar{A}BC$	m_3
1	0	0	$A\bar{B}\bar{C}$	m_4
1	0	1	$A\bar{B}C$	m_5
1	1	0	$AB\bar{C}$	m_6
1	1	1	ABC	m_7

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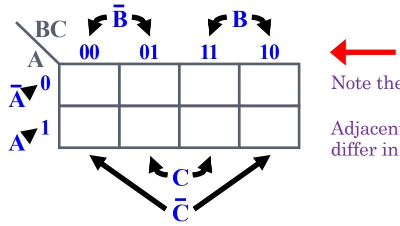
3-VARIABLE MAP: f(A, B, C)

- 8 minterms:



Note the unusual order!!

Adjacent squares must differ in only one variable

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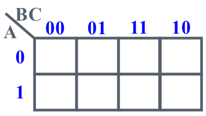
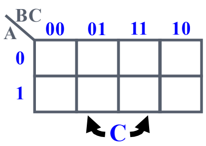
3-VARIABLE MAP: f(A, B, C)

Ex: $f = \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + \bar{A}BC + A\bar{B}C$



Ex: $f = \bar{A}\bar{B}C + \bar{A}BC + A\bar{B}C + ABC = \bar{A}C + AC = C$



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4-VARIABLE MAP: f(A, B, C, D)

- 16 minterms:

AB \ CD	00	01	11	10
00	$\bar{A}\bar{B}\bar{C}\bar{D}$	$\bar{A}\bar{B}\bar{C}D$	$\bar{A}\bar{B}CD$	$\bar{A}\bar{B}C\bar{D}$
01	$\bar{A}B\bar{C}\bar{D}$	$\bar{A}B\bar{C}D$	$\bar{A}BCD$	$\bar{A}BC\bar{D}$
11	$AB\bar{C}\bar{D}$	$AB\bar{C}D$	$ABCD$	$ABC\bar{D}$
10	$A\bar{B}\bar{C}\bar{D}$	$A\bar{B}\bar{C}D$	$A\bar{B}CD$	$A\bar{B}C\bar{D}$

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4-variable Map: $f(A, B, C, D)$

Ex:

$f = \overline{A}\,\overline{B}CD + \overline{A}B\overline{C}\,\overline{D} + \overline{A}B\overline{C}D + \overline{A}BCD + ABCD + A\overline{B}CD$

AB \ CD	00	01	11	10
00				
01				
11				
10				

Ex:

$f = AB\overline{C}\,\overline{D} + AB\overline{C}D + ABC\overline{D} + ABCD$

$= AB\overline{C} + ABC = AB$

AB \ CD	00	01	11	10
00				
01				
11				
10				

$F = \underset{(110\times)}{AB\overline{C}} + \underset{(\times\times11)}{CD} + \underset{(1\times\times\times)}{A}$

CD \ AB	00	01	11	10
00	0	0	1	1
01	0	0	1	1
11	1	1	1	1
10	0	0	1	1

$F = \underset{(01\times0)}{(A + \overline{B} + D)}\underset{(1\times0\times)}{(\overline{A} + C)}$

CD \ AB	00	01	11	10
00	1	0	0	0
01	1	1	0	0
11	1	1	1	1
10	1	0	1	1

2.12 Minimization Using K-maps: SOP

Principle

- Combine rectangles

  containing 2, 4, 8, or 16 adjacent 1's

- More bigger the circle is, more simpler the expression is
- Note: Squares at opposite edges are considered adjacent

Ex: $A\overline{B}\,\overline{C} \leftrightarrow AB\overline{C}$

A \ BC	00	01	11	10
0	$\overline{A}\,\overline{B}\,\overline{C}$	$\overline{A}\,\overline{B}C$	$\overline{A}BC$	$\overline{A}B\overline{C}$
1	$A\overline{B}\,\overline{C}$	$A\overline{B}C$	ABC	$AB\overline{C}$

Minimization Using K-maps: Jargon

- **Implicant**: Any rectangle containing 2^n adjacent ones that represents a product term.
- **Prime Implicant**: An implicant that is not part of a larger implicant containing fewer literals

Ex: $f(x, y, z) = \sum m(2,3,4,6,7)$

x \ yz	00	01	11	10
0	0	0	1	1
1	1	0	1	1

Implicants:

Prime Implicants:

Minimization Using K-maps: Jargon

- **Essential Prime Implicant:** A prime implicant that includes a square which is not part of any other prime implicant.
- **Cover**: Any collection of implicants that includes all squares where $f = 1$. Each cover corresponds to a SOP form for f.

x \ yz	00	01	11	10
0	0	0	1	1
1	1	0	1	1

$f = x\overline{z} + yz + \overline{x}y$

x \ yz	00	01	11	10
0	0	0	1	1
1	1	0	1	1

$f = y + x\overline{z}$

Minimization Using K-maps: Jargon

- **Cost**: Cost = # Gates + # Gate Inputs
- Except complements of input variables are available at zero cost (for simplicity, we don't count inverters.)

(i) minterms: $f = \overline{x}y\overline{z} + \overline{x}yz + x\overline{y}\,\overline{z} + xy\overline{z} + xyz$

$\text{Cost}_i = 6 + (3\times5 + 1\times5) = 6 + 20 = 26$

(ii) $f = x\overline{z} + yz + \overline{x}y$

x \ yz	00	01	11	10
0	0	0	1	1
1	1	0	1	1

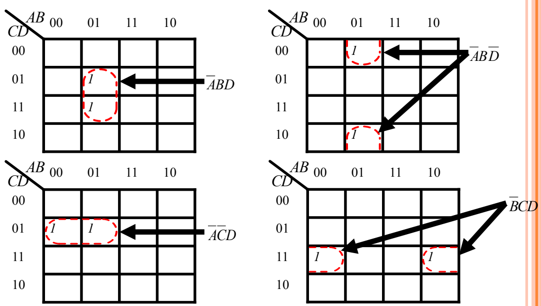
$\text{Cost}_{ii} = 4 + (2\times3 + 1\times3)$
$= 4 + 9 = 13$

(iii) $f = y + x\overline{z}$

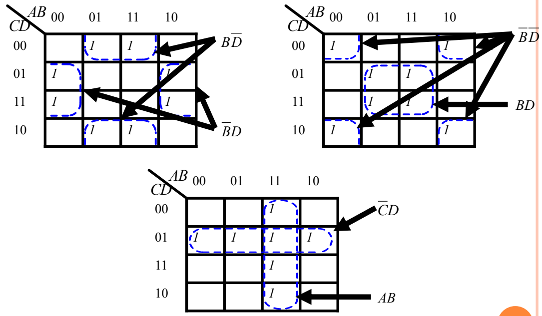
x \ yz	00	01	11	10
0	0	0	1	1
1	1	0	1	1

$\text{Cost}_{iii} = 2 + (1\times2 + 1\times2)$
$= 2 + 4 = 6$

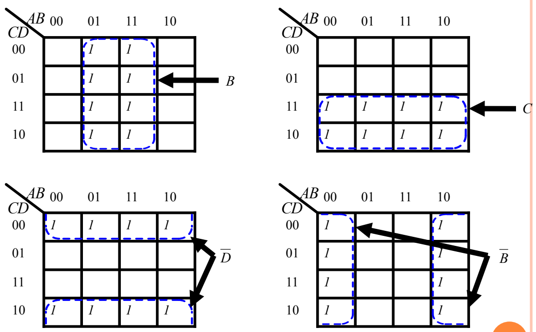
Principle of minimization Using K-maps



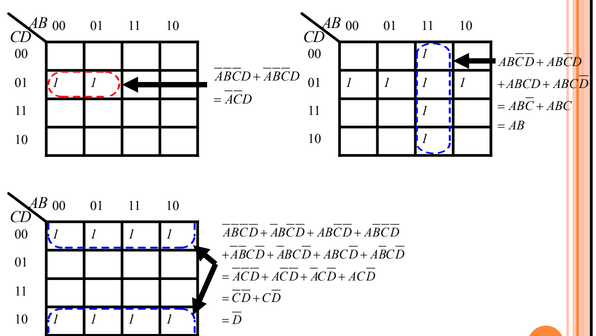
Principle of minimization Using K-maps



Principle of minimization Using K-maps

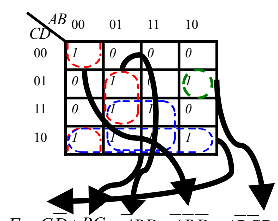


Principle of minimization Using K-maps



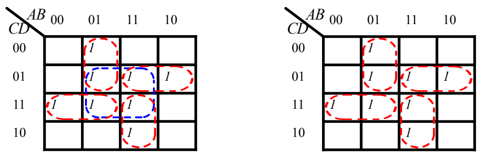
Principle of minimization Using K-maps

- Steps of the minimization using K-map



$$F=C\overline{D}+BC+\overline{A}BD+\overline{A}\overline{B}\overline{D}+A\overline{B}\overline{C}D$$

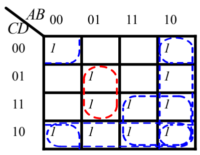
Minimization Using K-maps: example 2



$F=\overline{A}B\overline{C}+\overline{A}CD+A\overline{C}D+ABC+BD$ — ***Worse***

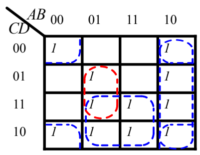
$F=\overline{A}B\overline{C}+\overline{A}CD+A\overline{C}D+ABC$ — ***Better***

Minimization Using K-maps: example 3



$F = \overline{B}\overline{D} + A\overline{B} + C\overline{D} + AC + \overline{A}BD$

Worse



$F = \overline{B}\overline{D} + A\overline{B} + BC + \overline{A}BD$

Better

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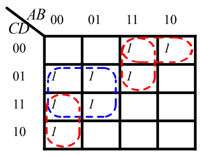
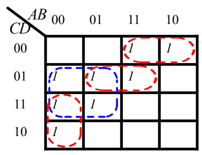
Minimization Using K-maps: example 4



$F = \overline{A}D + \overline{A}\overline{B}C + A\overline{C}\overline{D} + AB\overline{C}$

Better



$F = \overline{A}D + \overline{A}\overline{B}C + A\overline{C}\overline{D} + B\overline{C}D$

Better

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Minimization Using K-maps: SOP

Procedure:

- Build K-map for function *f*
- Combine adjacent squares containing 1's into rectangles of size 2, 4, 8, or 16
- Determine prime implicants, essential P.I.'s
- Find the largest circle with one different minterm

- Find all essential prime implicants
- Add the minimum number of non-essential prime implicants needed to cover *f*
- Form the sum of the selected implicants

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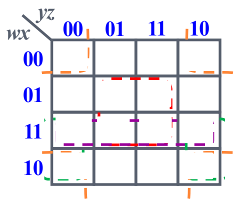
Minimization Using K-maps: SOP

Example: $f(w,x,y,z) = \sum m(0,2,5,7,8,10,12,13,14,15)$



Prime Implicants:

Essential Prime Implicants:

$f = xz + \overline{x}\overline{z} + wx$

$f = xz + \overline{x}\overline{z} + w\overline{z}$

Cost = $4 + 9 = 13$

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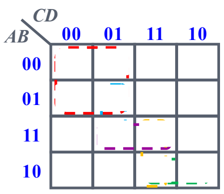
Minimization Using K-maps: SOP

Example:

$f(A,B,C,D) = \overline{A}\overline{B}\overline{C} + \overline{A}\overline{C}D + ABD + ACD + \overline{A}B\overline{C}\overline{D} + A\overline{B}C$



Prime Implicants:
$\overline{A}\overline{C}$, $B\overline{C}D$, ABD, ACD, $A\overline{B}C$

Essential: $\overline{A}\overline{C}$, $A\overline{B}C$

Cover 1:
$f = \overline{A}\overline{C} + A\overline{B}C + B\overline{C}D + ACD$
cost = 5 + 15 = 20

Cover 2:
$f = \overline{A}\overline{C} + A\overline{B}C + ABD$
cost = 4 + 11 = 15

***Rule of Thumb*: Minimize overlap among prime implicants!**

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2.13 Minimization Using K-maps: POS

Procedure:

- Build K-map for function *f*
- Combine adjacent squares **containing 0's** into rectangles of size 2, 4, 8, or 16
- Determine prime implicants, essential P.I.'s

- Find minimum cover of all squares where ***f* = 0**
- Express in terms of **sum** terms that correspond to selected prime implicants
- Compare cost to SOP realization

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Minimization Using K-maps: POS

Example: f(w,x,y,z) = $\sum$m(0,2,5,7,8,10,12,13,14,15)

wx \ yz	00	01	11	10
00	1	0	0	1
01	0	1	1	0
11	1	1	1	1
10	1	0	0	1

Prime Implicants: $\bar{x}z$, $\bar{w}x\bar{z}$

Essential Prime Implicants: $\bar{x}z$, $\bar{w}x\bar{z}$

$\bar{f} = \bar{x}z + \bar{w}x\bar{z}$

$f = \overline{\bar{x}z + \bar{w}x\bar{z}}$

$= (\overline{\bar{x}z})\,(\overline{\bar{w}x\bar{z}})$

$f = (x + \bar{z})\,(w + \bar{x} + z)$

Cost of POS = 3 + 7 = 10

Cost??

$f = xz + \bar{x}\bar{z} + wx$

$f = xz + \bar{x}\bar{z} + w\bar{z}$

Cost = 4 + 9 = 13

Less than SOP!

Minimization Using K-maps: POS

Example: f(w,x,y,z) = $\sum$m(0,2,5,7,8,10,12,13,14,15)

wx \ yz	00	01	11	10
00	1	0	0	1
01	0	1	1	0
11	1	1	1	1
10	1	0	0	1

Prime Implicants: $\bar{x}z$, $\bar{w}x\bar{z}$

Essential Prime Implicants: $\bar{x}z$, $\bar{w}x\bar{z}$

How to obtain the same result directly from 2 P.I.s ?

What sum term is 0 for a given P.I.?

$x = 0,\ z = 1$ — $(x + \bar{z})$

$w = 0,\ x = 1,\ z = 0$ — $(w + \bar{x} + z)$

$f = (x + \bar{z})\,(w + \bar{x} + z)$

More examples:

x_3 \ x_1x_2	00	01	11	10
0	1	1	0	0
1	1	1	1	0

$(\bar{x}_1 + x_3)$

$(\bar{x}_1 + x_2)$

Figure 2.60. POS minimization of $f(x_1, x_2, x_3) = \Pi M(4, 5, 6)$.

More examples:

x_3x_4 \ x_1x_2	00	01	11	10
00	0	0	0	0
01	0	1	1	0
11	1	1	0	1
10	1	1	1	1

$(x_3 + x_4)$

$(x_2 + x_3)$

$(\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \bar{x}_4)$

Figure 2.61. POS minimization of $f(x_1, \ldots, x_4) = \Pi M(0, 1, 4, 8, 9, 12, 15)$.

Conclusion

- K-map is an effective way to find the minimal-cost circuit
- Learn the way to simplify the 3/4 variables logic function
- Textbook Reading: Chapter 2.11 ~ 2.14
- Assignment: None

Karnaugh Maps (K-Map): Part II

- What are K-Maps?
- Minimization by Using K-Maps: SOP
- Minimization in POS form
- Don't Care Conditions
- Networks with Multiple Outputs
- Design Examples

2.14 Incompletely Specified Functions
Don't Care Conditions

- In some cases, particular input combinations can never occur. These are **Don't Care Conditions.**
- Example: Buffer with Set/Reset inputs

  Inputs: Set (S), Reset (R), Data (D)

  Output: f

If $S = R = 0$,	$f = D$
If $S = 0$, $R = 1$,	$f = 0$
If $S = 1$, $R = 0$,	$f = 1$
$S = R = 1$,	*Cannot occur*

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Don't Care Conditions

- When drawing K-maps, place a ***d*** in squares for don't care conditions.
- Use these ***d***'s as either 0's or 1's, in whatever way to create the largest prime implicants.
- Can count some as 1's and others as 0's.
- Example:

$S = R = 0$, $f = D$
$S = 0$, $R = 1$, $f = 0$
$S = 1$, $R = 0$, $f = 1$
$S = R = 1$, *Cannot occur*

D \ SR	00	01	11	10
0				
1				

SOP: $f = S + \bar{R}D$

D \ SR	00	01	11	10
0				
1				

POS: $f = \bar{R}\,(S + D)$

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Don't Care Conditions

Example: Let $wxyz$ represent a 4-bit number n

$$f = \begin{cases} 1 & \text{if } 2 \le n \le 5 \\ 0 & \text{otherwise} \end{cases}$$

wx \ yz	00	01	11	10
00				
01				
11				
10				

SOP: $f = \bar{w}\bar{x}y + \bar{w}x\bar{y}$

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Don't Care Conditions

Example: Let $wxyz$ represent a 4-bit number n

$$f = \begin{cases} 1 & \text{if } 2 \le n \le 5 \\ 0 & \text{otherwise} \end{cases}$$

wx \ yz	00	01	11	10
00	0	0	1	1
01	1	1	0	0
11	0	0	0	0
10	0	0	0	0

SOP: $f = \bar{w}\bar{x}y + \bar{w}x\bar{y}$

POS: $f = \bar{w}\,(\bar{x}+\bar{y})\,(x+y)$

$f = \bar{w}\,(x \oplus y)$

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Don't Care Conditions

Example: Now assume that n must have a decimal value 0-9. Other cases ⇒ don't care

It is a BCD (Binary Coded Decimal)

wx \ yz	00	01	11	10
00				
01				
11				
10				

SOP: $f = \bar{x}y + x\bar{y}$

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Don't Care Conditions

Example: Now assume that n must have a decimal value 0-9. Other cases ⇒ don't care

It is a BCD (Binary Coded Decimal)

wx \ yz	00	01	11	10
00	0	0	1	1
01	1	1	0	0
11	*d*	*d*	*d*	*d*
10	0	0	*d*	*d*

SOP: $f = \bar{x}y + x\bar{y}$

POS: $f = (\bar{x}+\bar{y})\,(x+y)$

$f = x \oplus y$

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Don't Care Conditions

SO:

- To Form the largest groups, you can assign proper value to the Don't s care Cell.
- The freedom in choosing the value of don't care leads to greatly simplified realizations
- Example in Figure 2.62
- Example 2.15

Don't Care Conditions

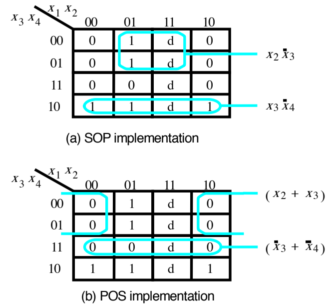


Figure 2.62. Two implementations of the function $f(x_1,\ldots,x_4) = \Sigma\, m(2, 4, 5, 6, 10) + D(12, 13, 14, 15)$.

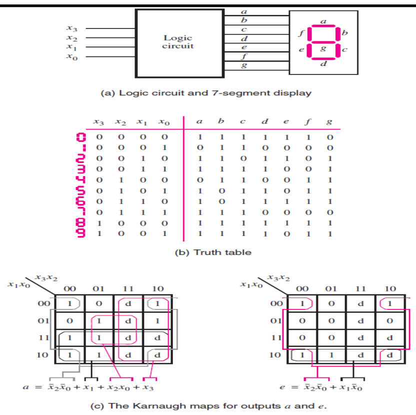


Figure 2.63. Using don't-care minterms when displaying BCD numbers.

2.15 Networks with Multiple Outputs

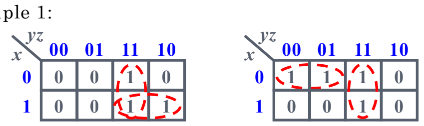
- Often, it is necessary to implement two or more functions of the same inputs

  Example: arithmetic

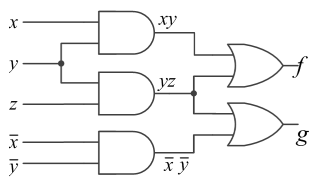
- In many cases, some gates can be shared between the two functions to keep the total cost down.
- Such cases can be recognized by finding prime implicants that are common to both K-maps.

Networks with Multiple Outputs

Example 1:



$f = xy + yz$ $\qquad$ $g = \overline{x}\overline{y} + yz$



Networks with Multiple Outputs

- Example 2: Minimal SOP – Separate circuits

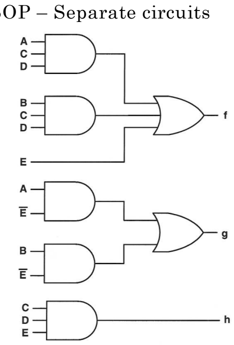
$f = ACD + BCD + E$

$g = A\overline{E} + B\overline{E}$

$h = CDE$

Inverter is 0 cost.

cost = 7 + 19 = 26



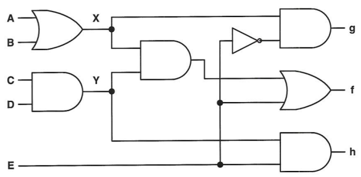
Networks with Multiple Outputs

$f = ACD + BCD + E = (A + B)CD + E$ ⇒ $f = XY + E$

$g = A\bar{E} + B\bar{E} = (A + B)\bar{E}$ ⇒ $g = X\bar{E}$

$h = CDE$ ⇒ $h = YE$

Let X = A + B, Y = CD



Inverter is 0 cost.
cost = 6 + 12 = 18

Example 2.16

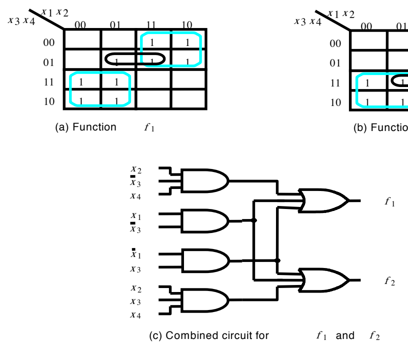


Figure 2.64. An example of multiple-output synthesis.

Example 2.17

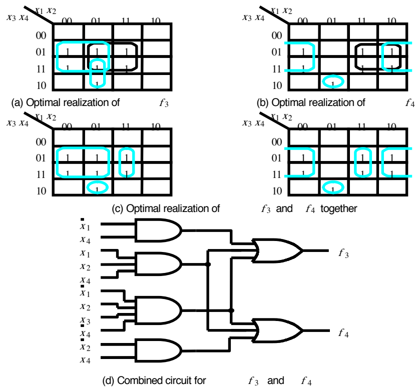


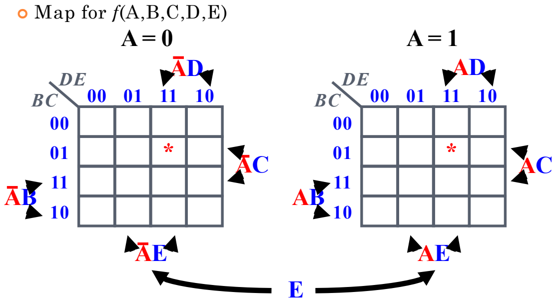
Figure 2.65. An example of multiple-output synthesis.

Five-Variable K-Map

- A K-map must have the property that two minterms which differ in only one variable lie in adjacent squares.
- For a two-dimensional map, this is not possible with more than 4 variables.
- However, we can create a 5-variable map as two 4-variable maps that lie on top of another.

Five-Variable K-Map

- Map for $f(A,B,C,D,E)$

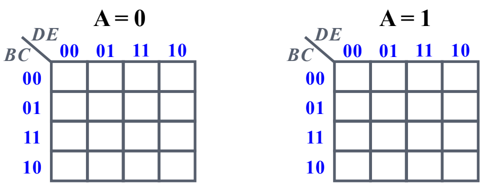


Squares that occupy the same relative position in each of the sub-maps are adjacent in the third dimension.

Five-Variable K-Map

- Example:

$f(A,B,C,D,E) = \sum(2,5,7,8,10,13,15,17,19,21,23,24,29,31)$



Five-variable K-map

- Example:

$f(A,B,C,D,E) = \sum(2,5,7,8,10,13,15,17,19,21,23,24,29,31)$

A = 0

BC \ DE	00	01	11	10
00	0	0	0	1
01	0	1	1	0
11	0	1*	1	0
10	1*	0	0	1

A = 1

BC \ DE	00	01	11	10
00	0	1	1	0
01	0	1	1	0
11	0	1*	1	0
10	1*	0	0	0

Prime Implicants: CE, $\bar{A}\bar{C}D\bar{E}$, $A\bar{B}E$, $B\bar{C}\bar{D}\bar{E}$, $\bar{A}B\bar{C}E$

$f = CE + \bar{A}\bar{C}D\bar{E} + A\bar{B}E + B\bar{C}\bar{D}\bar{E}$

2.17 Design Example

- BCD to Gray Code Converter

k	A	B	C	D	w	x	y	z
0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	1
2	0	0	1	0	0	0	1	1
3	0	0	1	1	0	0	1	0
4	0	1	0	0	0	1	1	0
5	0	1	0	1	1	1	1	0
6	0	1	1	0	1	0	1	0
7	0	1	1	1	1	0	1	1
8	1	0	0	0	1	0	0	1
9	1	0	0	1	1	0	0	0
10	1	0	1	0	X	X	X	X
11	1	0	1	1	X	X	X	X
12	1	1	0	0	X	X	X	X
13	1	1	0	1	X	X	X	X
14	1	1	1	0	X	X	X	X
15	1	1	1	1	X	X	X	X

Design Example

CD \ AB	00	01	11	10
00	0	0	X	1
01	0	1	X	1
11	0	1	X	X
10	0	1	X	X

$w = A + BD + BC$

CD \ AB	00	01	11	10
00	0	1	X	0
01	0	1	X	0
11	0	0	X	X
10	0	0	X	X

$x = B\bar{C}$

Design Example

CD \ AB	00	01	11	10
00	0	1	X	0
01	0	1	X	0
11	1	1	X	X
10	1	1	X	X

$y = B + C$

CD \ AB	00	01	11	10
00	0	0	X	1
01	1	0	X	0
11	0	1	X	X
10	1	0	X	X

$z = \bar{A}\bar{B}\bar{C}D + BCD + A\bar{D} + \bar{B}C\bar{D}$

Design Example

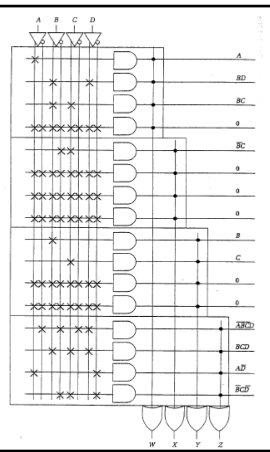
PAL Implementation

$w = A + BD + BC$

$x = B\bar{C}$

$y = B + C$

$z = \bar{A}\bar{B}\bar{C}D + BCD$
$+ A\bar{D} + \bar{B}C\bar{D}$



More example for Chapter 2

Pages: 102-111

Conclusions of this chapter

- Chapter 2 is the key of the book
- 2.16 Remarks:
  - The concept of logic circuit
  - Boolean algebra
  - Synthesis
  - CAD tool and Verilog
  - K-map

- Textbook Reading: Chapter 2.15 ~ 2.17
- Assignment : 2.37, 2.38, 2.40,
  2.45, 2.48, 2.69

  Note: using K-map method