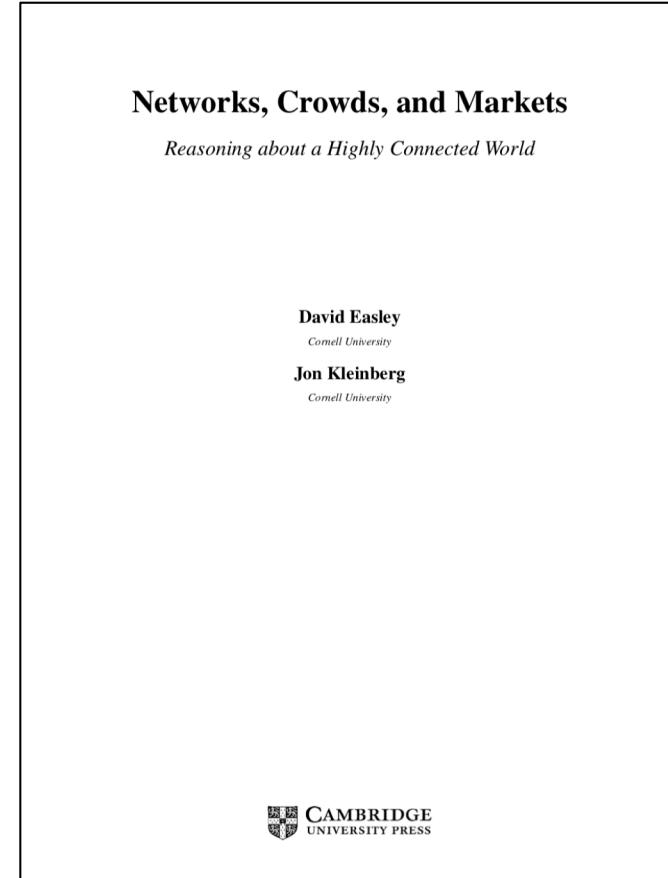
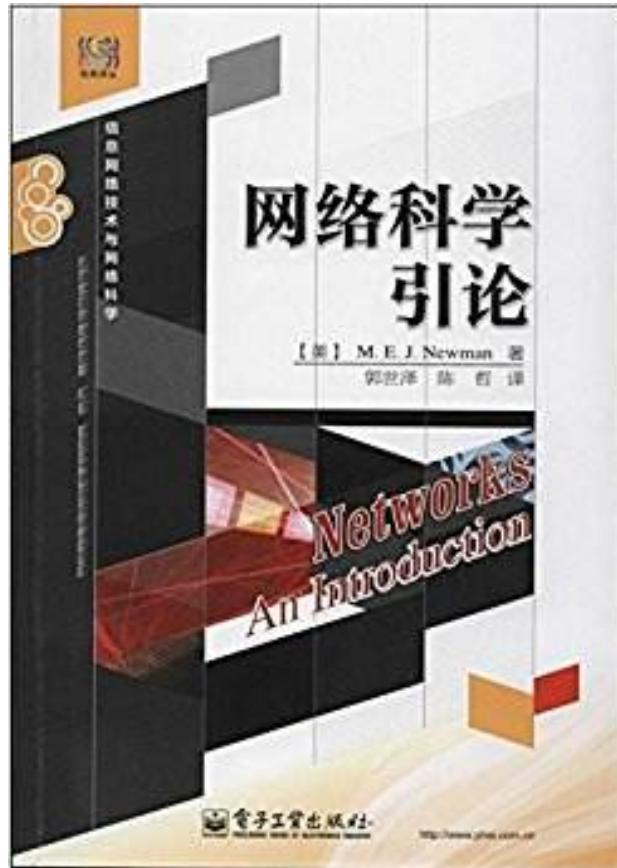


GRAPH | NETWORK ANALYSIS

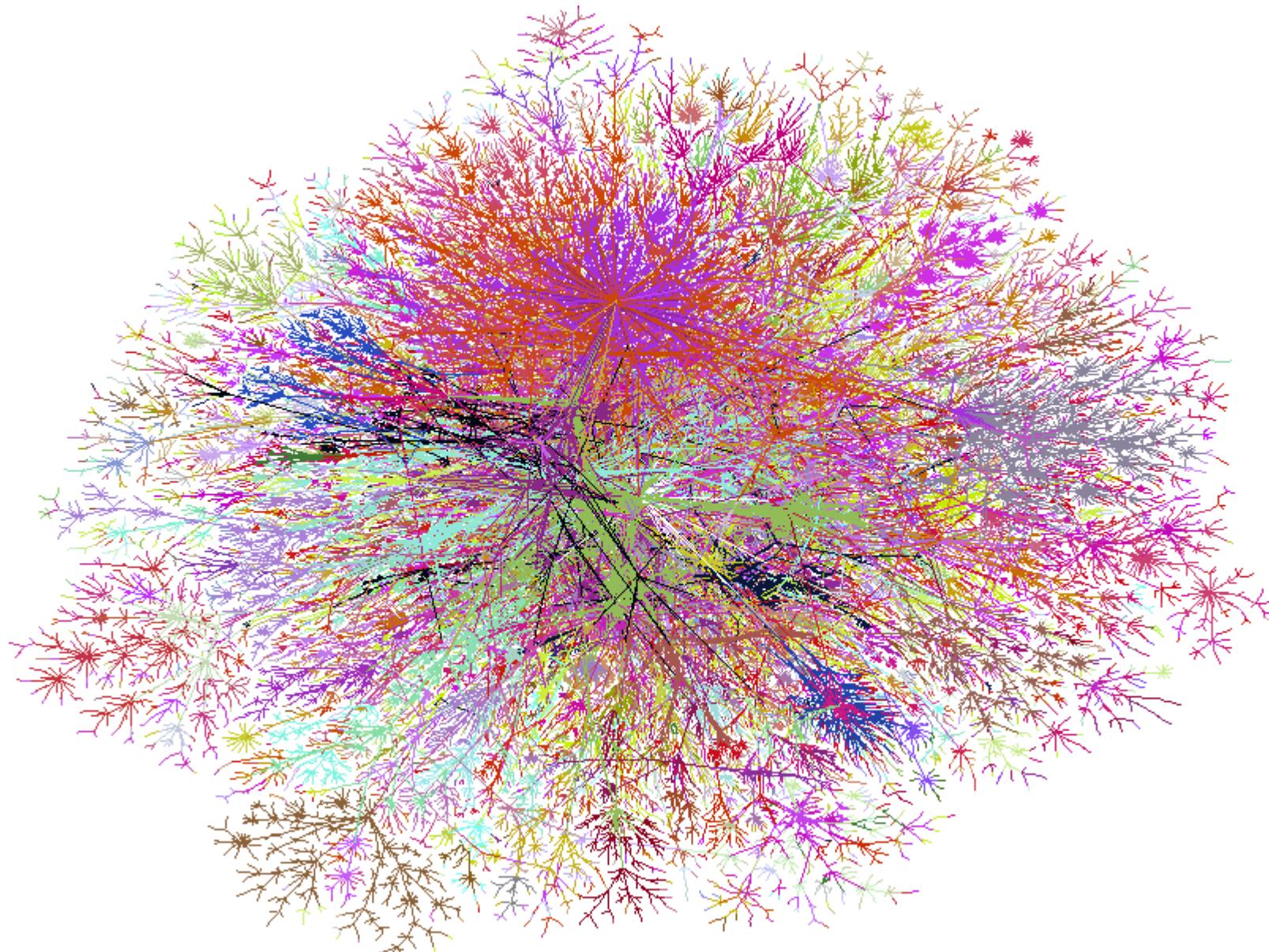
Peng Wang

Textbook



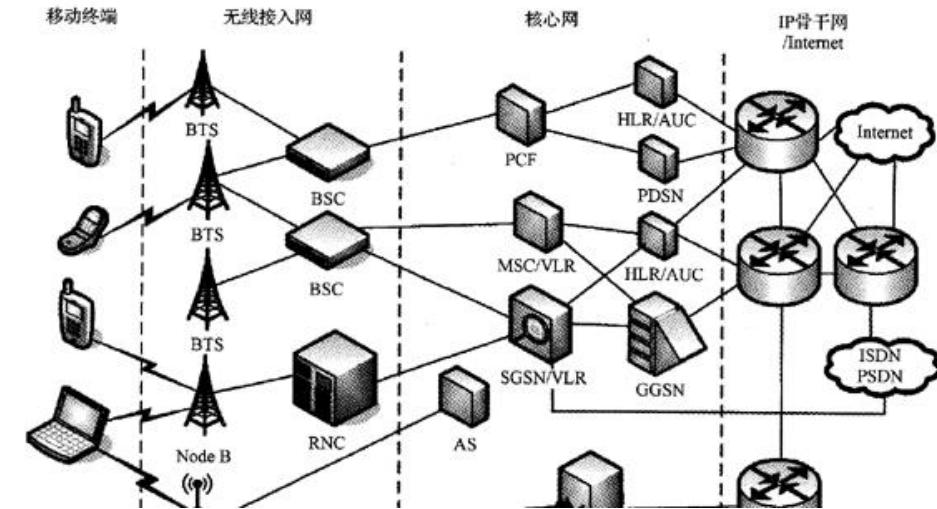
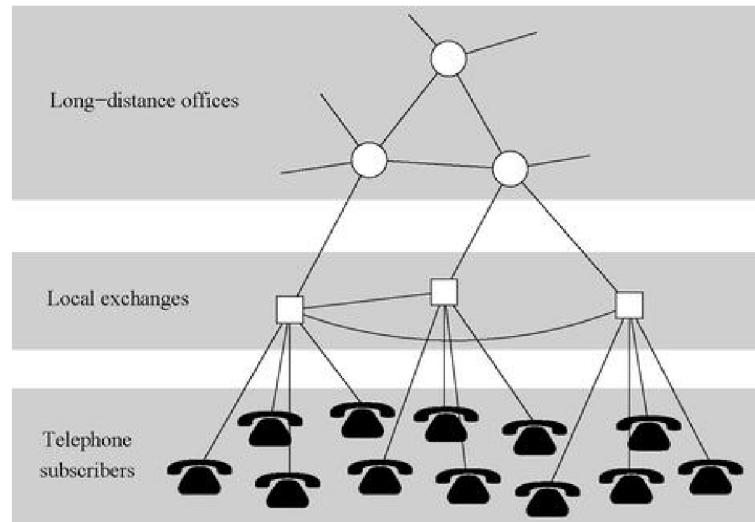
<https://www.cs.cornell.edu/home/kleinber/networks-book/>

TYPES OF NETWORKS



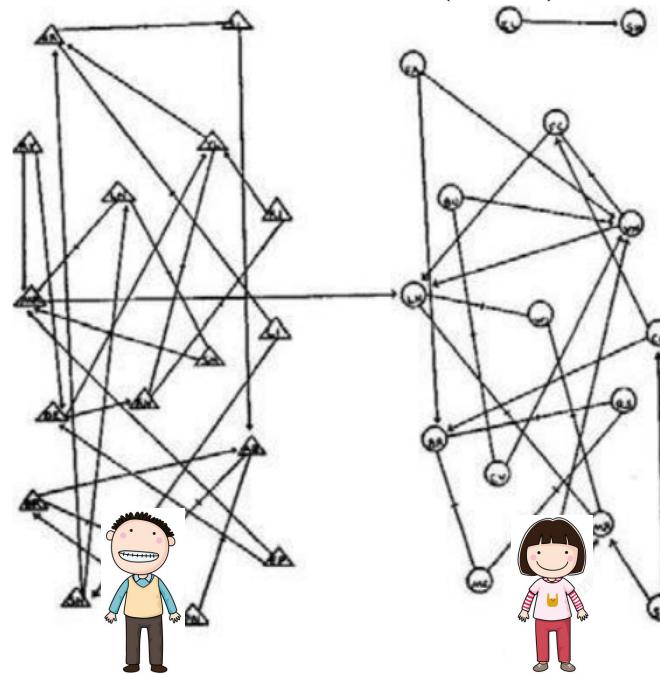
Technological Networks: The Internet

Technological Networks: The Telephone and Mobile Network

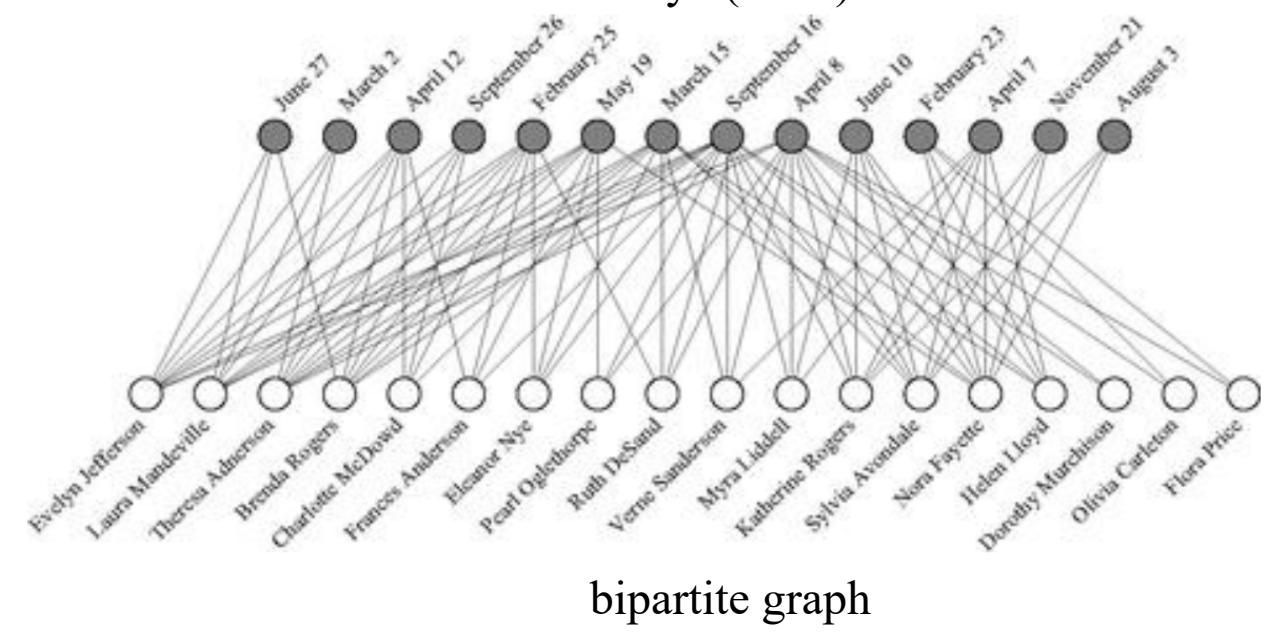


Social Networks:

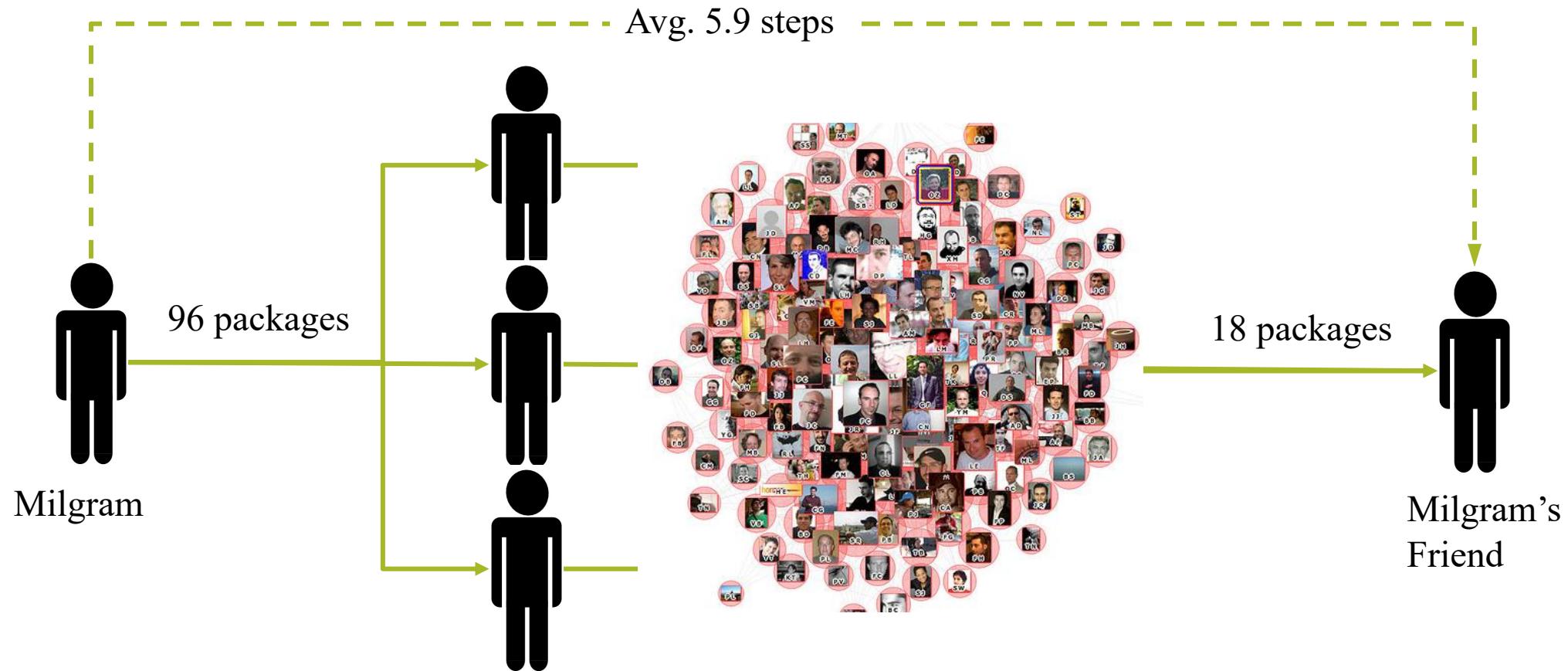
Moreno: friendships between schoolchildren (1930)



The affiliation network (隶属网络) of the “Southern Women Study” (1940)



A Small World Experiment



A Small World Experiment

An Experimental Study of the Small World Problem*

JEFFREY TRAVERS

Harvard University

AND

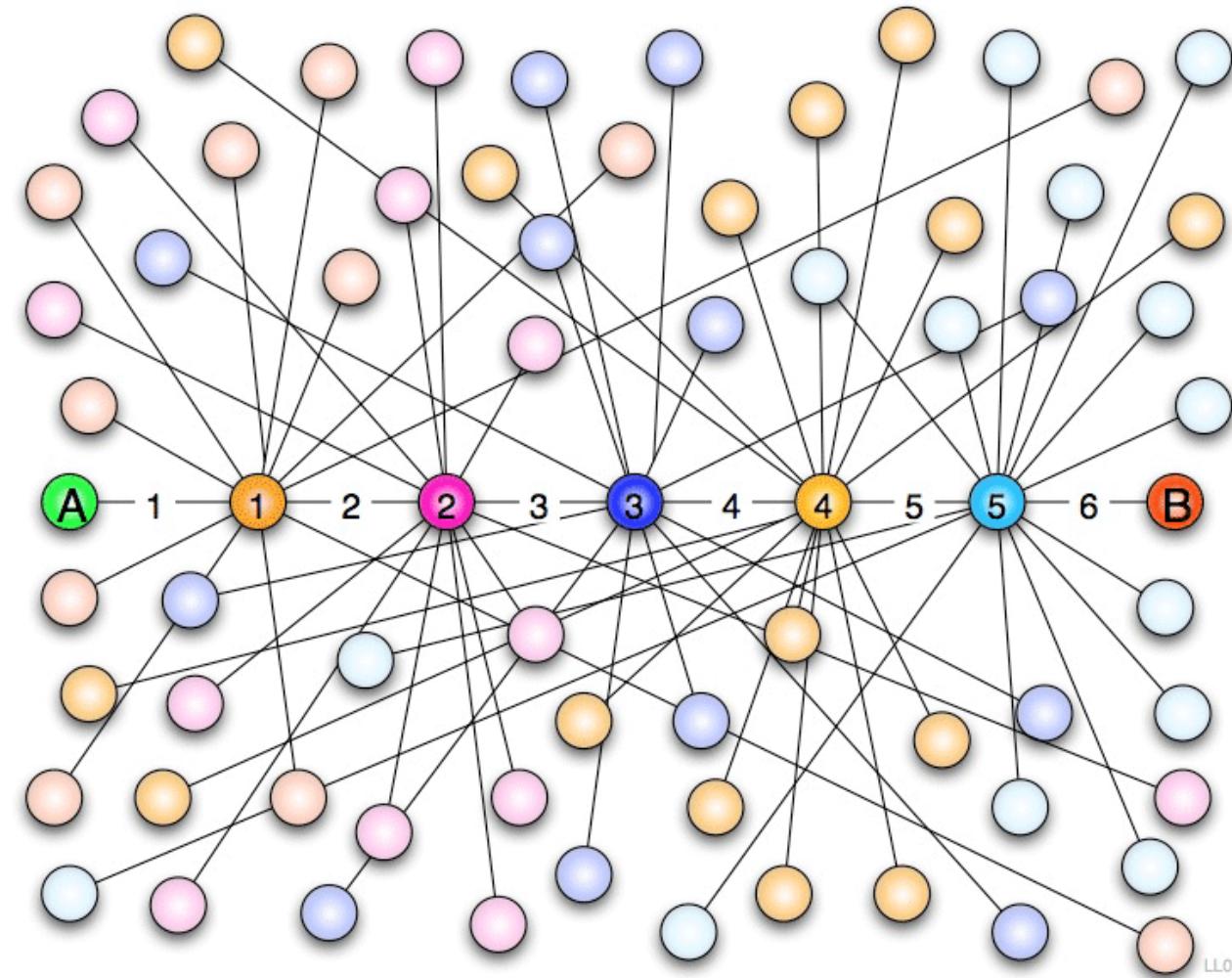
STANLEY MILGRAM

The City University of New York

Arbitrarily selected individuals ($N=296$) in Nebraska and Boston are asked to generate acquaintance chains to a target person in Massachusetts, employing “the small world method” (Milgram, 1967). Sixty-four chains reach the target person. Within this group the mean number of intermediaries between starters and targets is 5.2. Boston starting chains reach the target person with fewer intermediaries than those starting in Nebraska; subpopulations in the Nebraska group do not differ among themselves. The funneling of chains through sociometric “stars” is noted, with 48 per cent of the chains passing through three persons before reaching the target. Applications of the method to studies of large scale social structure are discussed.

A Small World Experiment

Six Degrees of Separation
六度分离理论



A Small World Experiment

- Microsoft MSN network

In 2006

Average 6.6 steps in 240,000,000 users

- Facebook network

In 2016

Average 3.74 steps in 721,000,000 users

Dunbar's number

- 150 Law

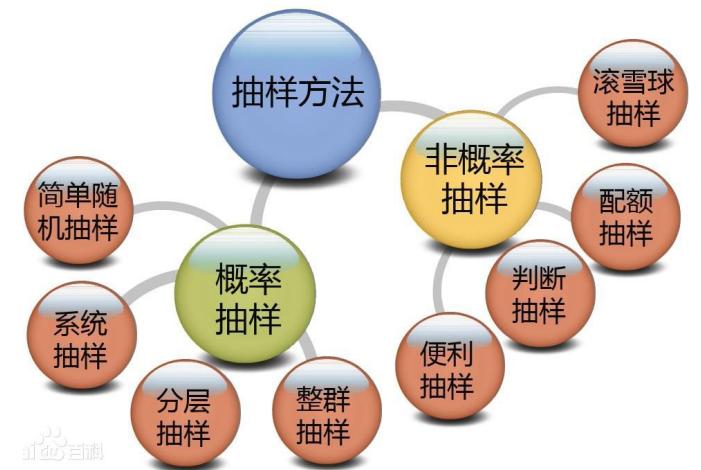
$$150^6 = 11,390,625,000,000$$

Bigger than population size on the earth

https://en.wikipedia.org/wiki/Dunbar%27s_number

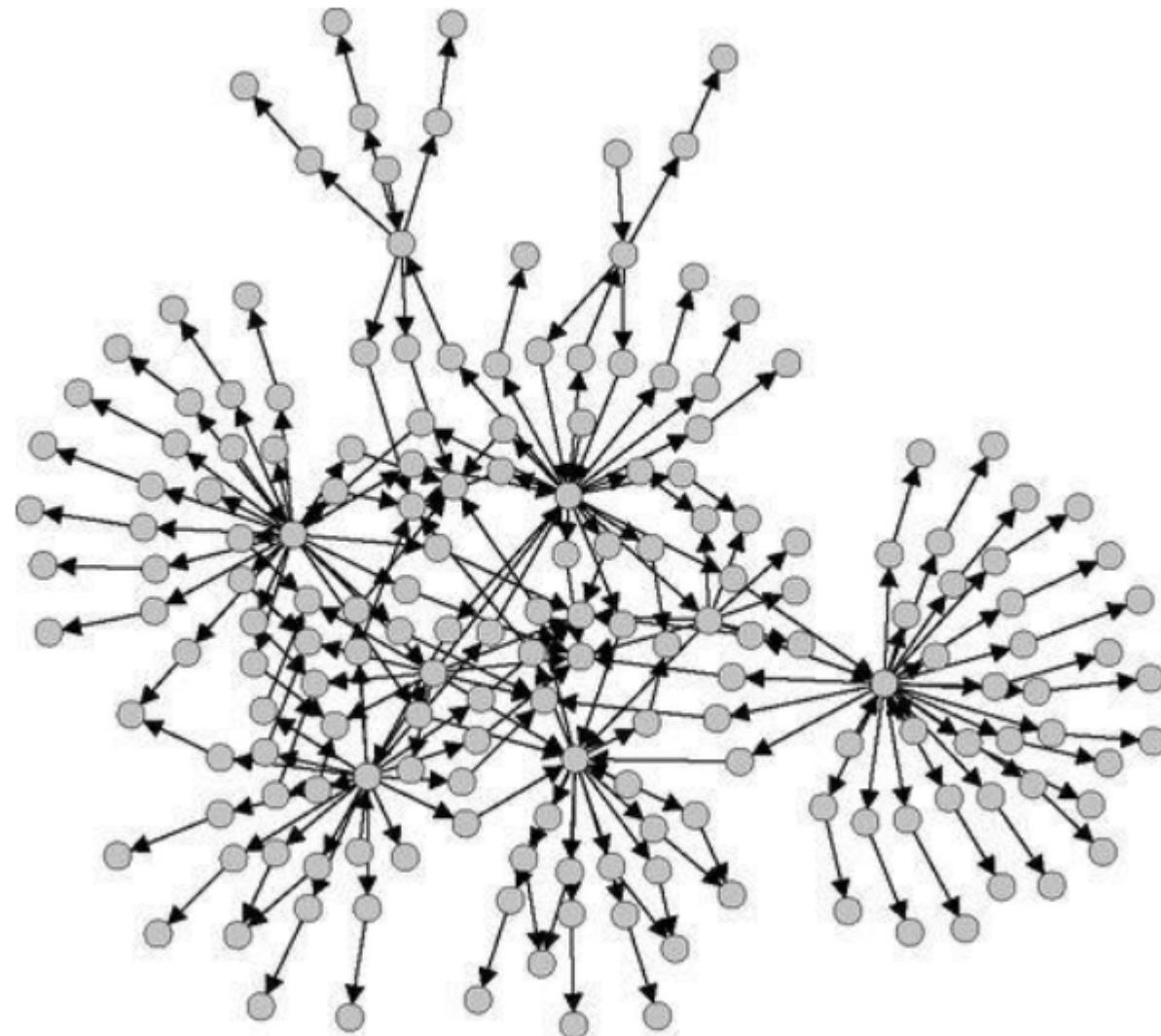
Sampling Hidden Populations

- Scenario: telephone survey - randomly select k men, and ask them if they use drugs.
- Problem:
 - random sampling may fail to find sparsely-distributed positive samples;
 - positive samples have no trust on random visitors;
- **Snowball Sampling** - non-probability sampling:
 - step1: find a real drug user;
 - step2: let the drug user help to find his neighboring drug users;
 - step3: iteration and propagation;
- **Contact Tracing** – closely related to Snowball Sampling:
 - scenarios: disease incidence (such as tracing HIV contacts)
 - step1: find a HIV sample
 - step2: find all of his contacts
 - step3: iteration and propagation



Sampling Hidden Populations

- shortage of Snowball Sampling and Contact Tracing
 - focus on local structures
 - sampling with strong bias
- **random-walk sampling**
 - step1: find a positive sample and its neighbors;
 - step2: randomly select *one and only one* neighbor;
 - step3: follow the selected neighbor and iteration;
 - shortage: slow
 - advantage: less bias. *Why?*
- *What if it is unethical to get a positive sample to name its neighbors?*
 - It is not allowed to publicly list HIV patients by their friends
 - **respondent-driven sampling**



Information Network:

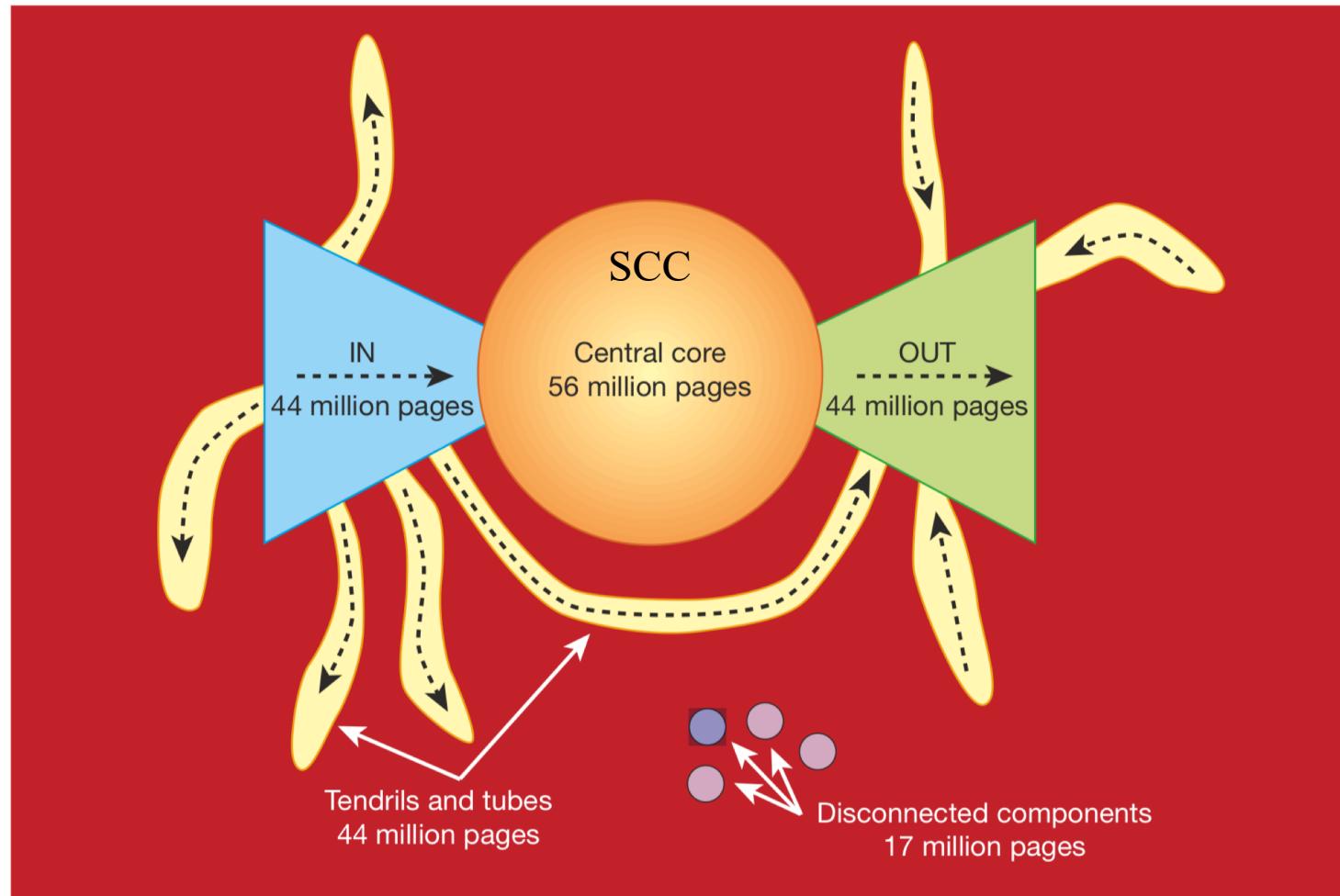
WWW: a network of pages
of a corporate website

Web Crawler

- A robot fetching pages on WWW by following links
- various crawling strategies:
 - breadth first search
 - depth first search
 - focused search
- robots.txt
- Problem: *how to get a proper set of seeds?*

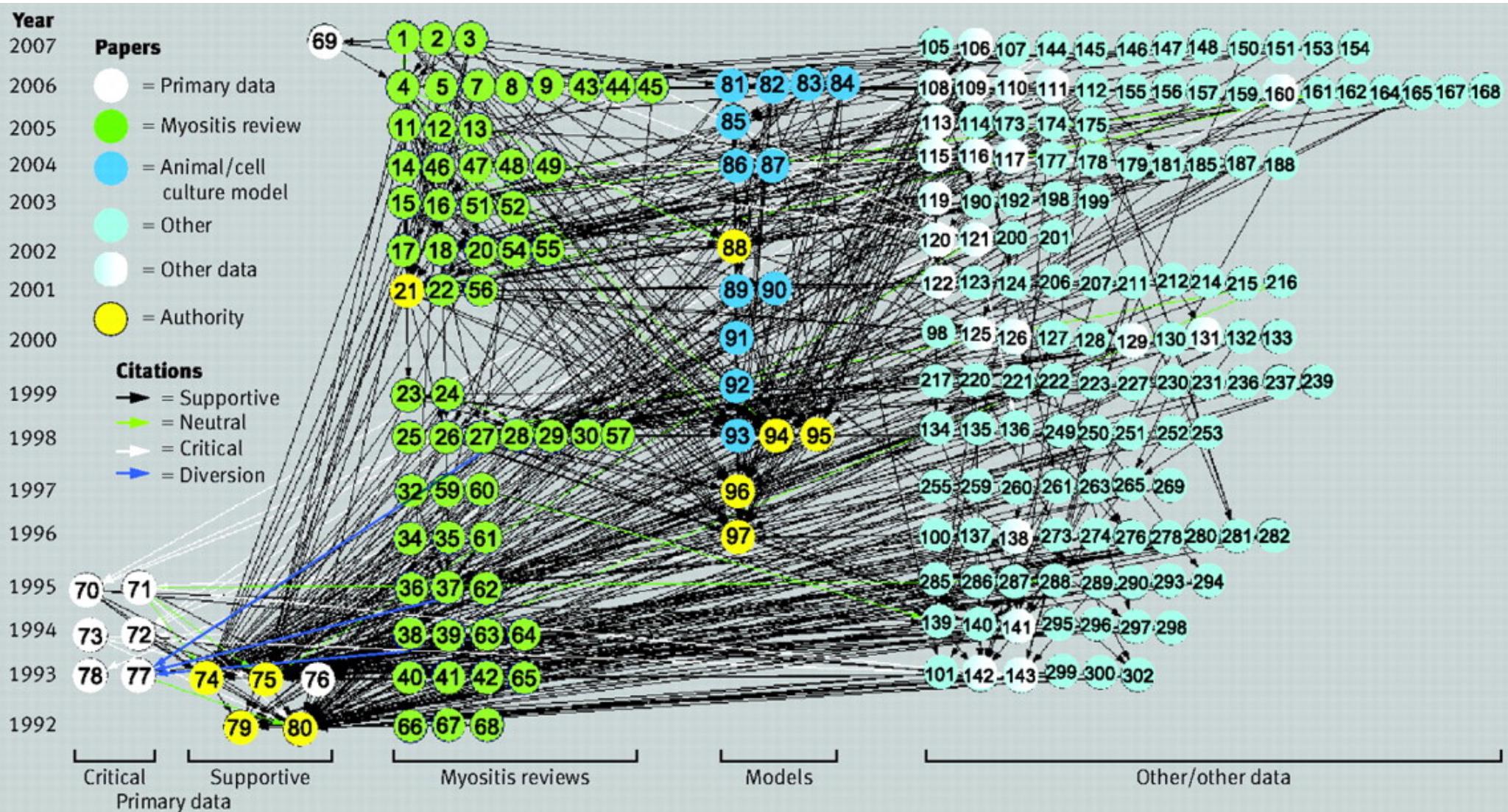
领结

Bow-tie structure of the WWW



<https://www.nature.com/articles/35012155>

Information Network: Citation Network





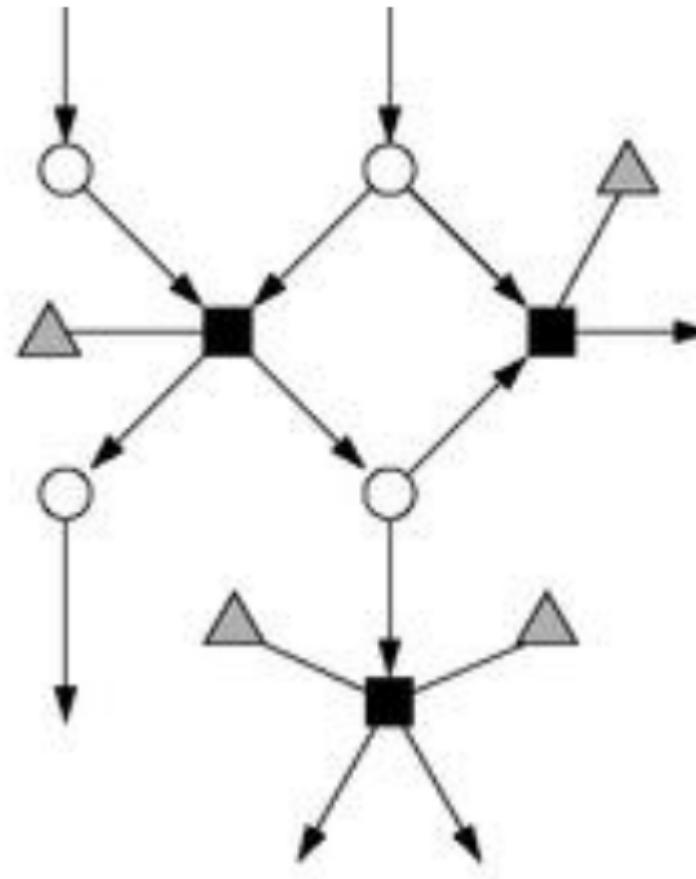
**What is the major difference
between Citation Network and
WWW network?**

Some fun data in SCI papers

- 47% of SCIs never get cited;
- 9% of remaining only cited once;
- 6% of remaining only cited twice;
- only 1% of remaining have >100 cites

Biological Networks

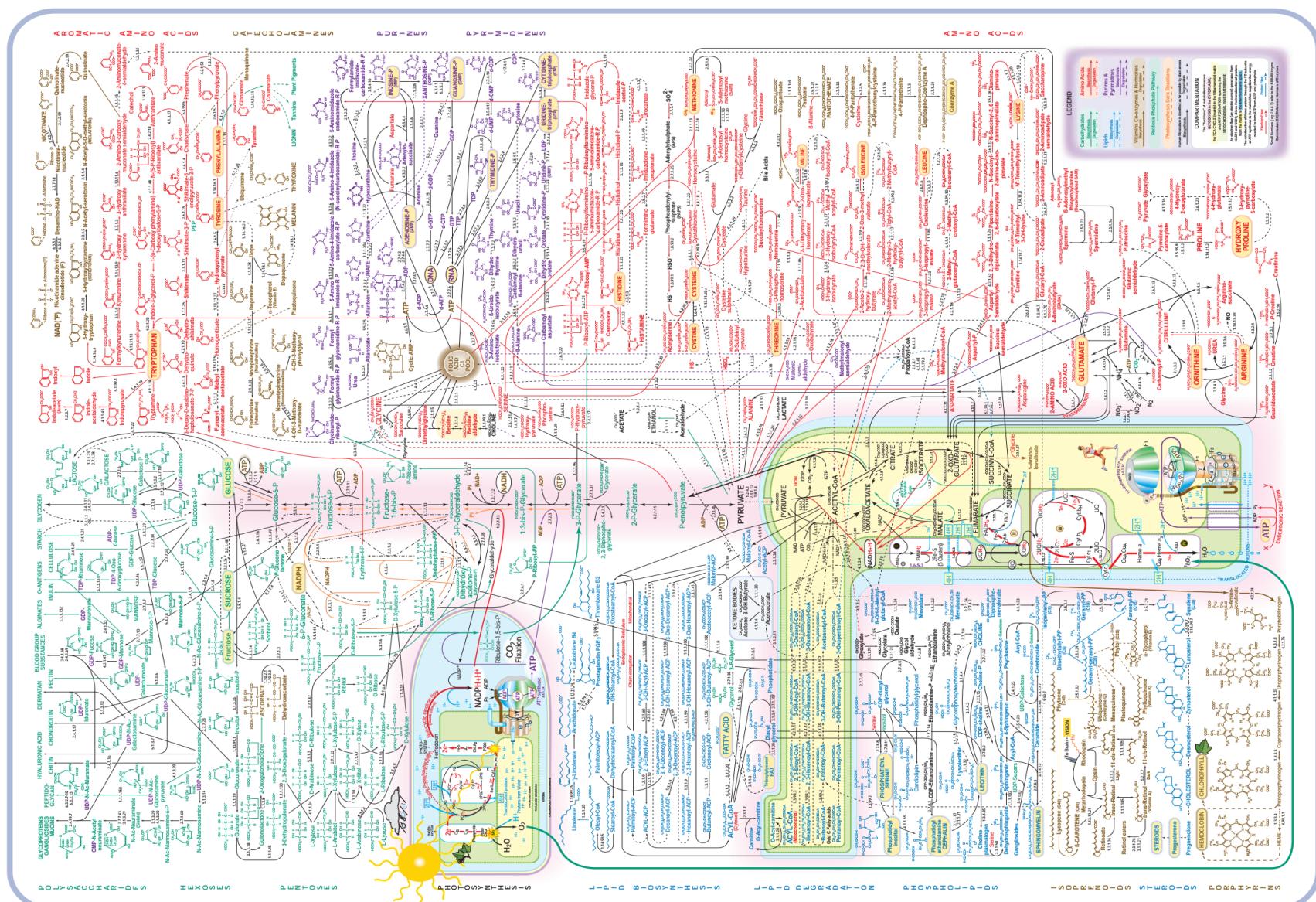
- Metabolite (代谢物)
- Reactions (化学反应)
- ▲ Enzyme (酶)



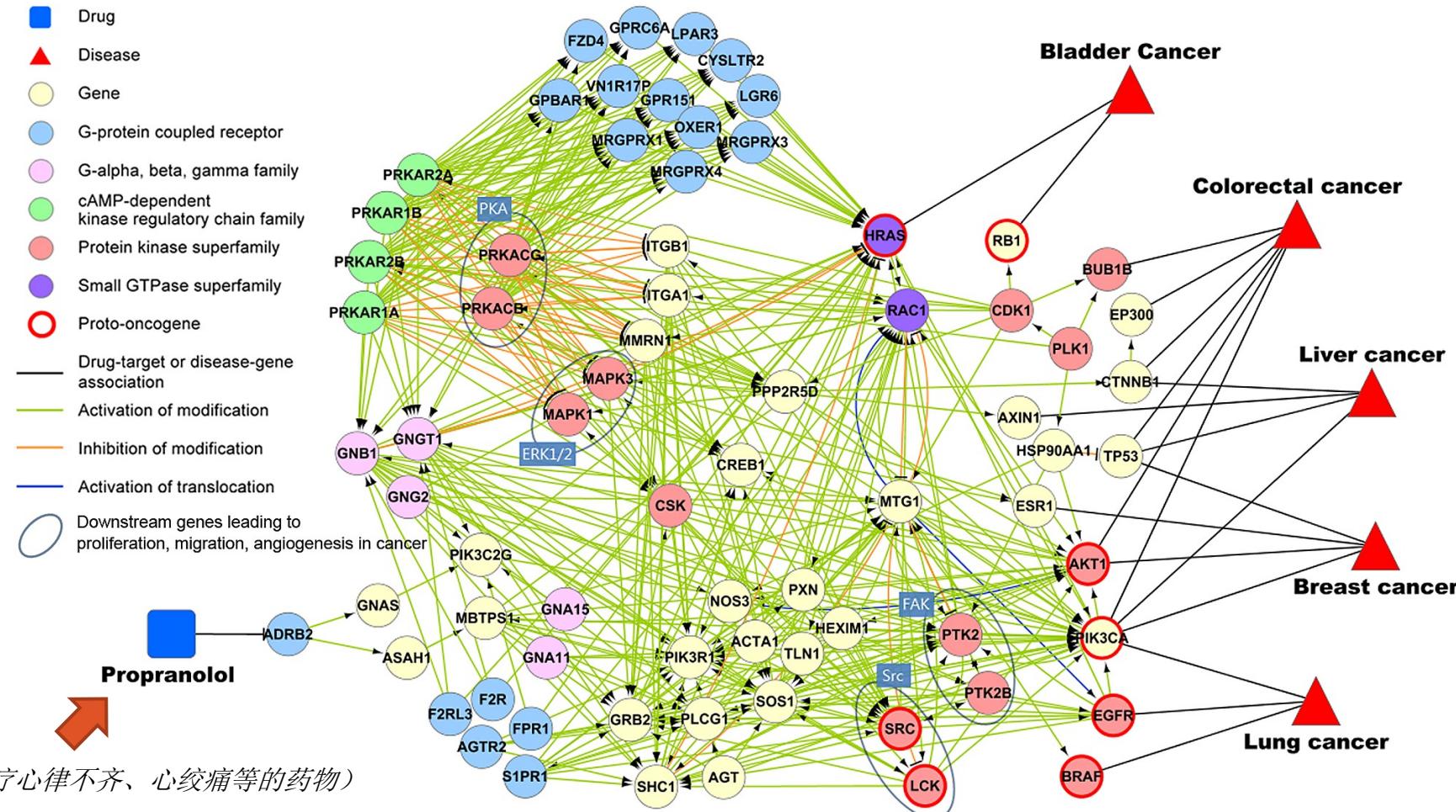
A tripartite representation of a portion of metabolic network

Metabolic Pathways

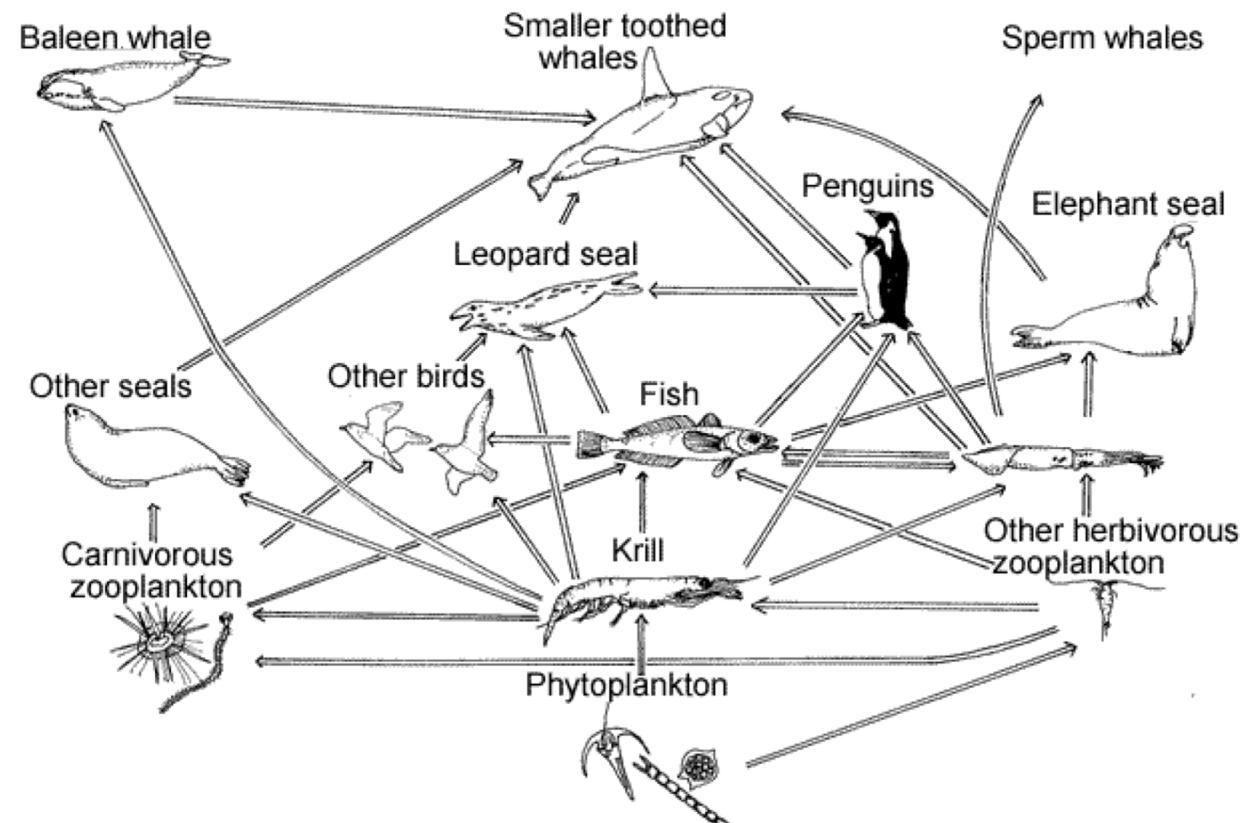
© 2003 International Union of Biochemistry and Molecular Biology www.iubmb.org



Network of Drugs, Diseases and Genes



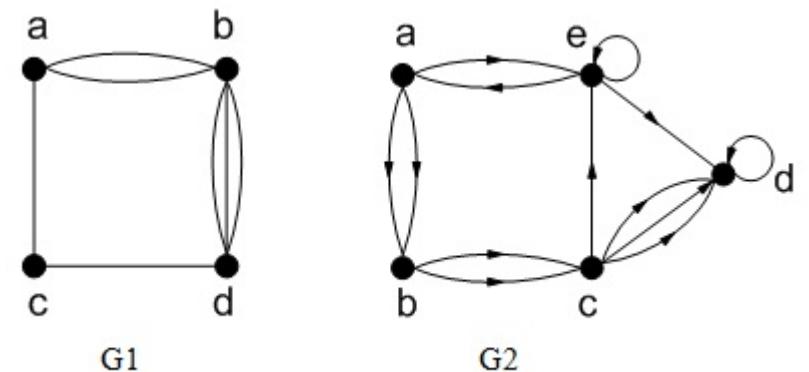
Antarctica Food Network



FUNDAMENTAL OF NETWORKS

Representation of Networks

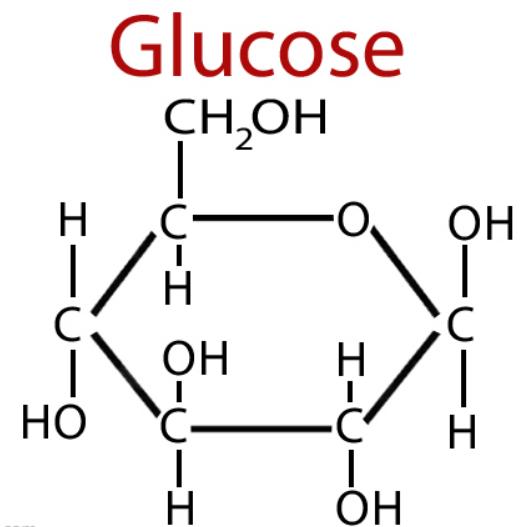
- *Network – Graph* (Mathematics)
- *Vertex, Edge – Node, Link* (CS) – *Site, Bond* (Physics) – *Actor, Tie* (Sociology)
- n for #node, m for #link
- *multi-edge*: multiple edges between two nodes
- *self-loop*: edges connecting nodes to themselves



Graph Terminology

Type	Direction	Multi-edge	Self-loop
simple graph	undirected	no	no
multigraph	undirected	yes	no
pseudograph	undirected	yes	yes
directed graph	directed	no	yes
directed multigraph	directed	yes	yes

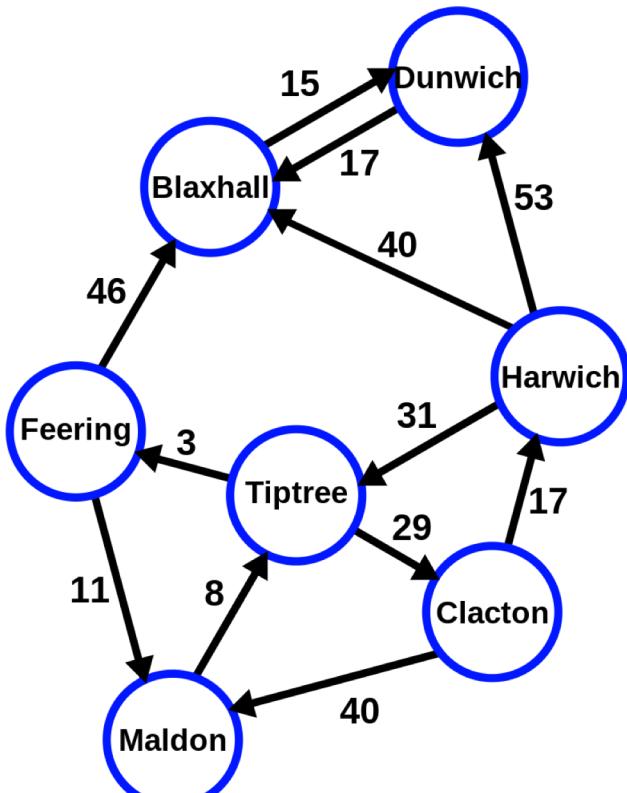
What kind of graph is **chemical formulas**?



Adjacency Matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Weighted | Directed | Labeled Networks



Adjacency Matrix of directed graph

$$A_{ij} = \begin{cases} 1, & \text{if there is an edge from } j \text{ to } i \\ 0, & \text{else} \end{cases}$$

Notice: $A_{ij} = 1$ if there is an edge from j to i , not i to j . It is for the mathematically convenience.

共引

文献耦合

Cocitation Network | Bibliographic Coupling

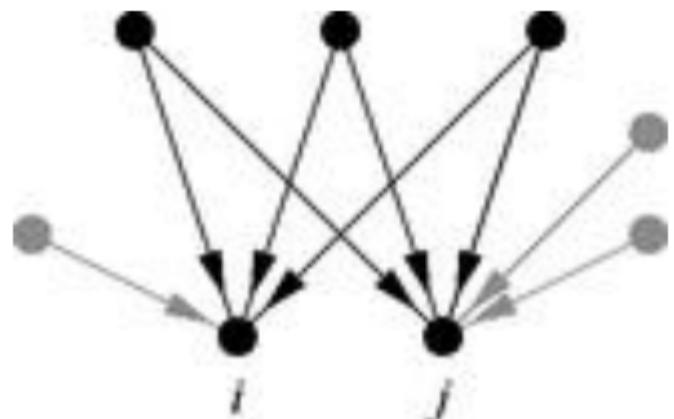
- Translating a directed graph into an undirected graph
- *cocitation* C_{ij} : #edge pointing to node i and node j

共引次数

$$C_{ij} = \sum_{k=1}^n A_{ik} A_{jk} = \sum_{k=1}^n A_{ik} A_{kj}^T \quad (A_{kj}^T \text{ is the element in } A^T)$$

$$\mathbf{C} = \mathbf{A}\mathbf{A}^T$$

- \mathbf{C} is called a *cocitation matrix*, and \mathbf{C} is symmetric (why?)
- A network constructed by \mathbf{C} is called a *cocitation network*



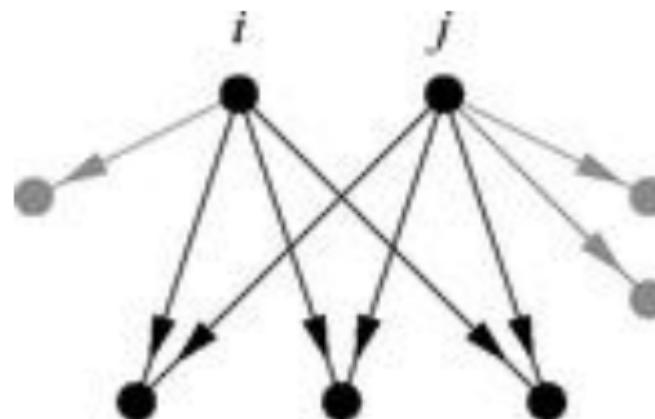
共引

文献耦合

Cocitation Network | Bibliographic Coupling

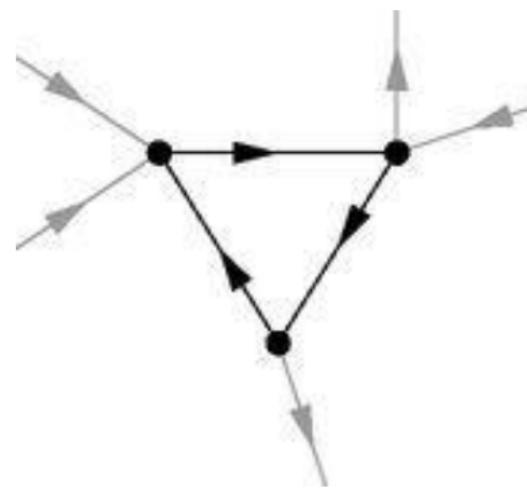
- *Bibliographic coupling* is a reverse translation of *cocitation*
- *Bibliographic coupling* B_{ij} : #edge from node i and node j

$$\mathbf{B} = \mathbf{A}^T \mathbf{A}$$

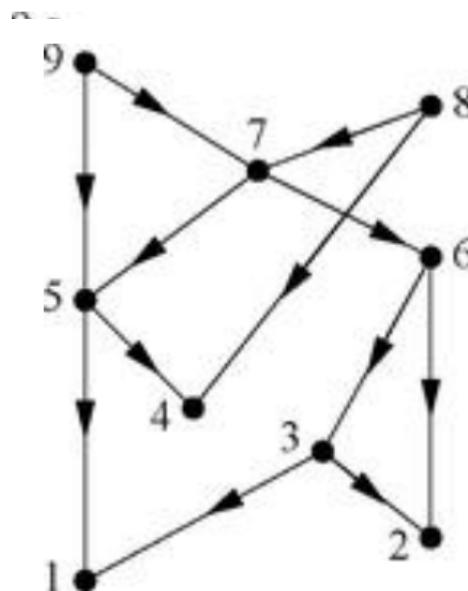


有向无环图

DAG: directed acyclic graph



A directed graph with cycle

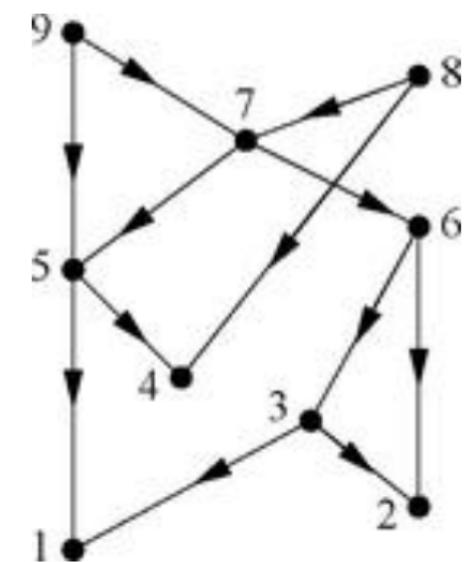


A directed acyclic graph

Judging and representing a DAG

- Each DAG can be drawn in a manner that all edges are pointing **downward**
- **Lemma:** At least one node in DAG has no **出边** outgoing edges.
- Using this lemma, we can draw the downward representation of each graph , and judge if it is DAG.

? HOW?



Adjacency Matrix of downward DAG

- adjacency matrix of a downward DAG is upper triangular
上三角矩阵 对角元
- considering that there is no self-loop, the diagonal elements are zero
严格三角矩阵
- Triangular matrices with zeros on the diagonal are called *strictly triangular*
特征值 充要条件
- all of its eigenvalues are zero (N&SC of DAG)
下图箭头指向此行
- the diagonal elements of a triangular matrix are its eigenvalues
幂零矩阵
- matrices with all eigenvalues zero are called *nilpotent matrices*.

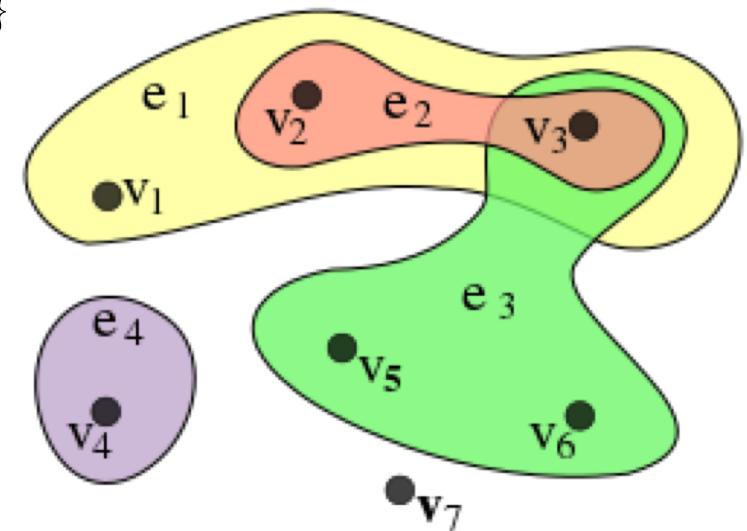
$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

超图

二部图

Hypergraph and Bipartite Graph

- In some networks, edges connect more than two nodes at a time.
- For example, in a social network, social actors are connected in a complex manner:
 - $X = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$
 - $E = \{e_1, e_2, e_3, e_4\} = \{\{v_1, v_2, v_3\}, \{v_2, v_3\}, \{v_3, v_5, v_6\}, \{v_4\}\}$
- edges can be generalized into *hyperedges*
k均匀子图
- *k-uniform hypergraph*: all its hyperedges have size k
- 2-uniform hypergraph: graph

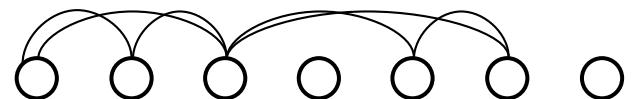
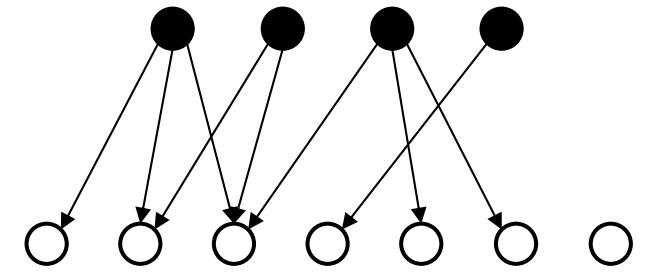


Hypergraph and Bipartite Graph

- hypergraphs can be represented by *bipartite graph*
双模网络
- a bipartite graph is also called a *two-mode network*
- a bipartite graph can be represented by a *incidence matrix*
-

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

- 单模投影
- *One mode projection*: deducing a single mode graph



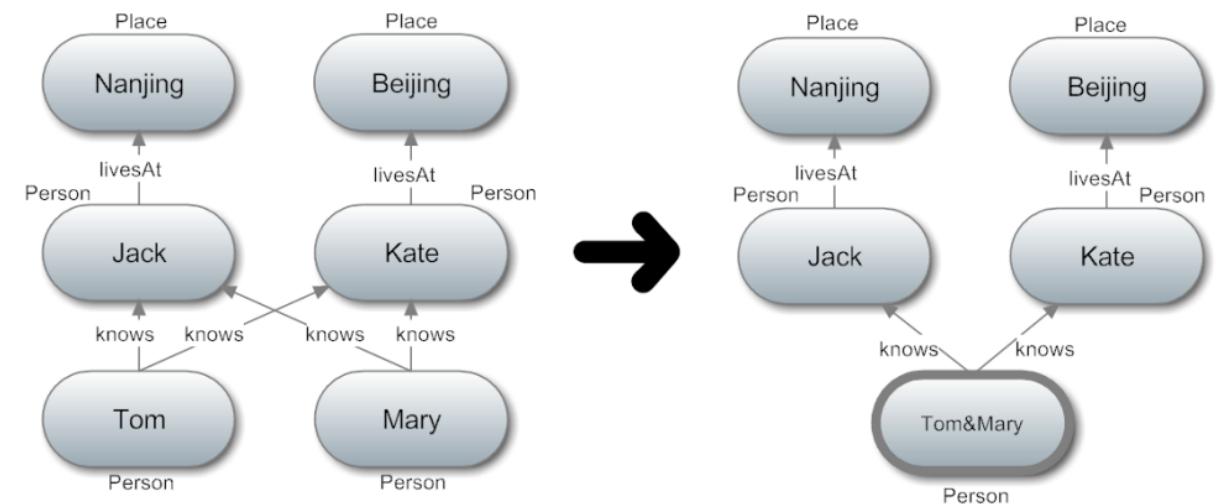
Using hypergraph in Semantic Web

■ scenario 1:

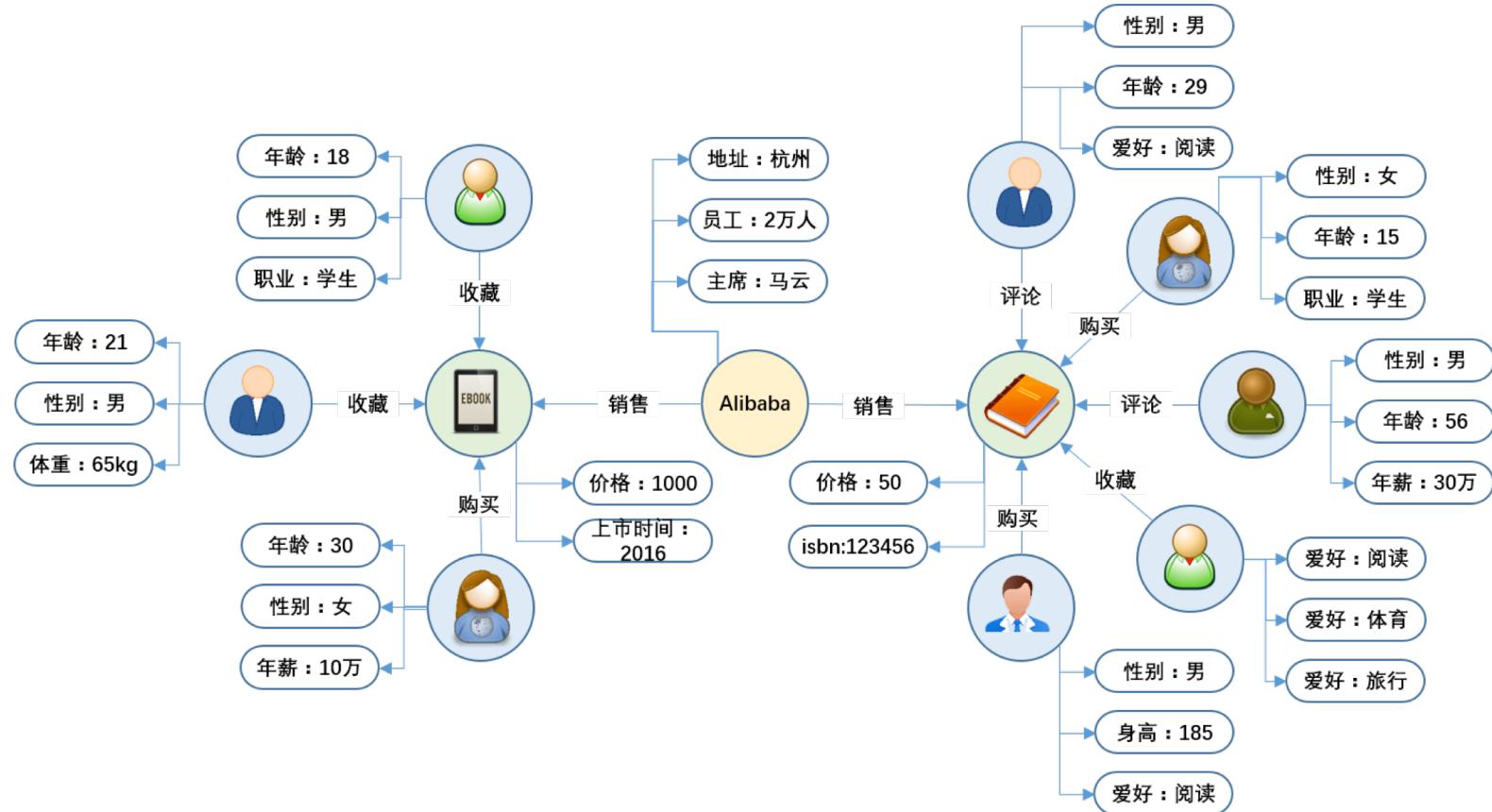
- given a set of statements: $\langle a, \text{isMarriedTo}, b \rangle, \langle a, \text{hasFather}, c \rangle \dots$
- how to describe WHO, WHEN and WHERE made these statements?
- Named Graph: https://en.wikipedia.org/wiki/Named_graph

■ scenario 2:

- given a large and redundant RDF graph
- some subgraphs are compressible
- how to represent compressed graph?



Multi-Mode and Multi-Dimensional Graph



Tree

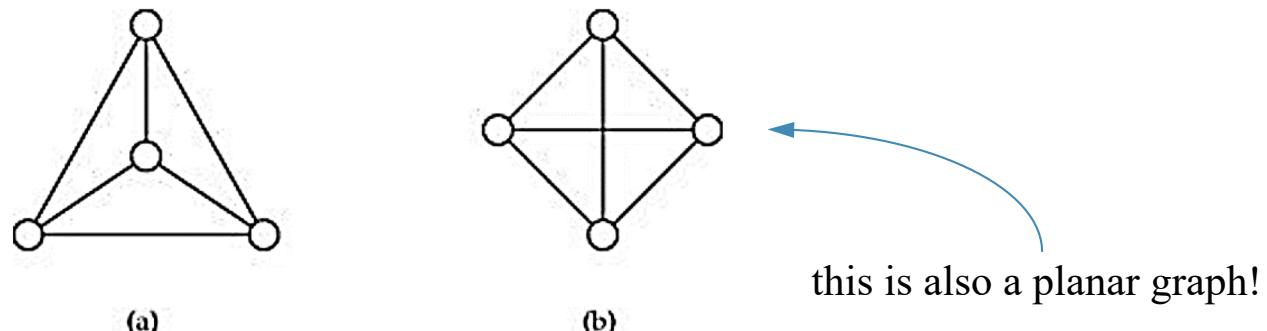
- property 1: there is **exactly one path** between any pair of vertices
- property 2: a tree of n vertices always has **exactly $n-1$ edges**

平面网络

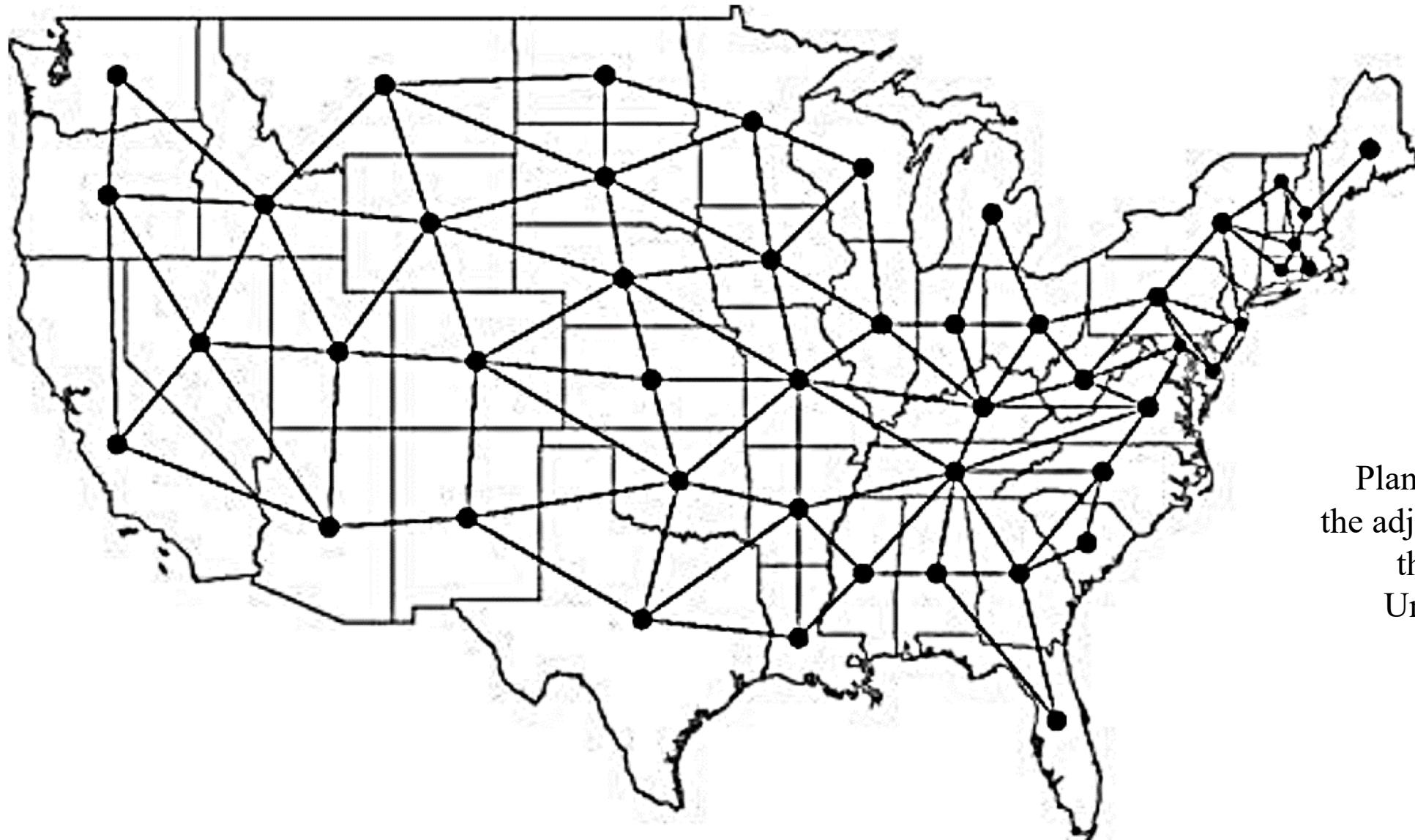
Planar Network

- A *planar network* is a network that can be drawn on a plane **without having any edges cross**

不存在交叉边



Two drawings of a planar graph



Planar graph of
the adjacencies of
the lower 48
United States

Planar Network

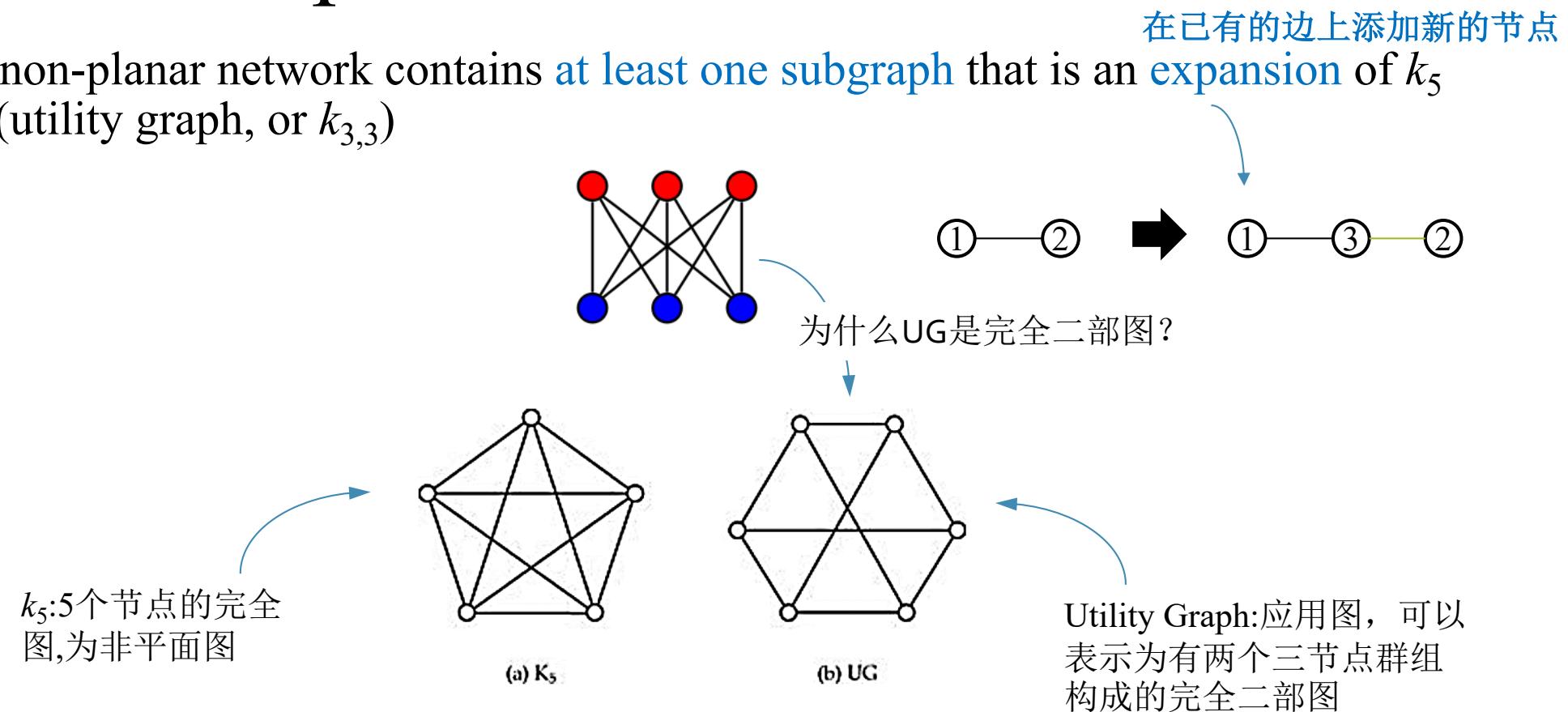
- Most real-world planar network is related to geography
四色定理
- Planer network is important to *four-color theorem*
-



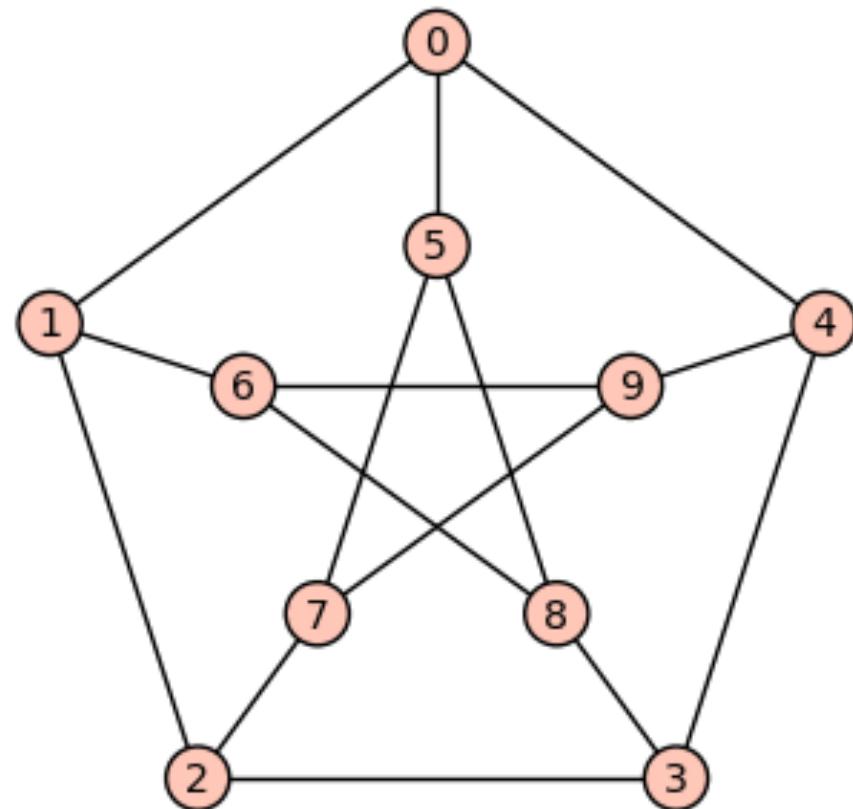
**How to determine a
network is planar or not**

Kuratowski's Theorem: Determine a planar network

- Every non-planar network contains **at least one subgraph** that is an **expansion of k_5** or UG (utility graph, or $k_{3,3}$)



Kuratowski's Theorem: Determine a planar network



<https://www.cs.sfu.ca/~ggbaker/zju/math/planar.html>

Degree

Undirected graph

$$k_i = \sum_{j=1}^n A_{ij}$$

$$2m = \sum_{i=1}^n k_i$$

$$c = \frac{1}{n} \sum_{i=1}^n k_i = \frac{2m}{n}$$

degree and
adjacency matrix

degree and
number of edges

average degree

Directed graph

$$k_i^{\text{in}} = \sum_{j=1}^n A_{ij}, k_j^{\text{out}} = \sum_{i=1}^n A_{ij}$$

$$m = \sum_{i=1}^n k_i^{\text{in}} = \sum_{i=1}^n k_i^{\text{out}} = \sum_{ij} A_{ij}$$

$$c_{\text{in}} = c_{\text{out}} = \frac{m}{n}$$

Density

- density ρ (for simple graph)

$$\rho = \frac{m}{\binom{n}{2}} = \frac{2m}{n(n-1)} = \frac{c}{n-1}$$

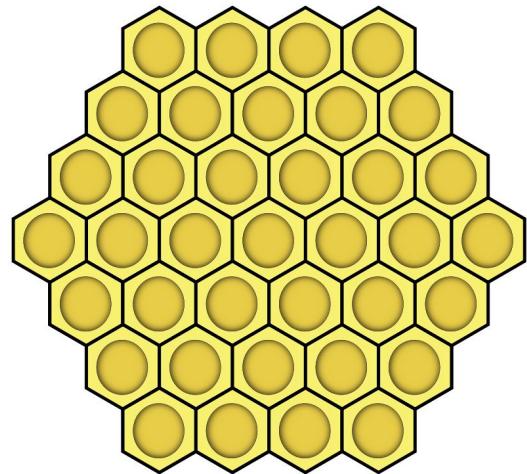
n个节点的图中最大可能边数  密集图

- if $n \rightarrow \infty$, $\rho \rightarrow$ constant, the graph is *dense*
- if $n \rightarrow \infty$, $\rho \rightarrow 0$, the graph is *sparse* 
- Internet, WWW, Social Networks (most networks) are sparse networks
- Food Network is dense network

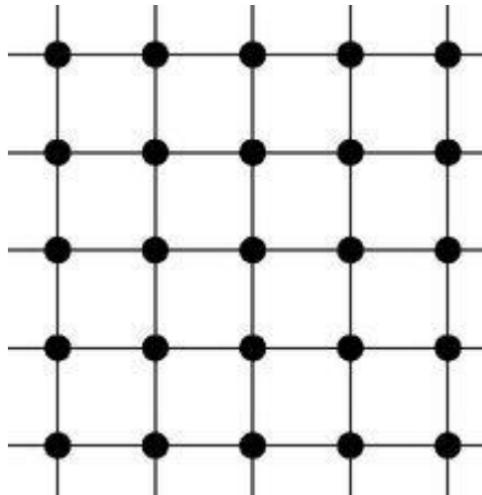
正则图

Regular Graph

- k -regular graph: all vertices have degree k



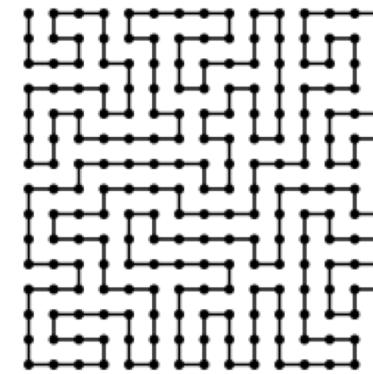
An infinite honeycomb is an example of a 3-regular graph



An infinite square lattice is an example of a 4-regular graph

Path

- self-avoid path: paths that do not intersect themselves
- path length: #edge, NOT #nodes
- calculating number of r -path between i and j :



A self-avoiding walk

i 和 j 之间长度为2的路径个数 $\rightarrow N_{ij}^{(2)} = \sum_{k=1}^n A_{ik} A_{kj} = [A^2]_{ij}$ 既链接*i*又链接*j*的节点*k*

$$N_{ij}^{(2)} = \sum_{k=1}^n A_{ik} A_{kj} = [A^2]_{ij}$$
$$N_{ij}^{(3)} = [A^3]_{ij}$$
$$N_{ij}^{(r)} = [A^r]_{ij}$$

无环图

Acyclic Graph

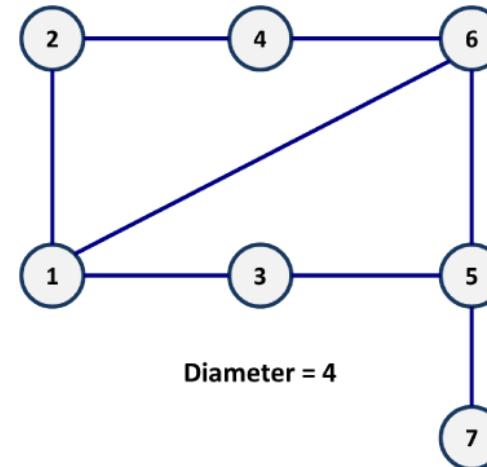
幂零矩阵

“the graph described by a nilpotent
adjacency matrix (i.e., a matrix
whose eigenvalues are all zero)
must be acyclic ”

测地路径

Geodesic Path (shortest path)

- simple graph: BFS
- directed and weighted graph
 - single-source shortest path: Dijkstra ($O(n^2)$)
 - all-pair shortest path: Floyd ($O(n^3)$)
- 图的直径
- *Graph Diameter*
 - length of longest shortest path

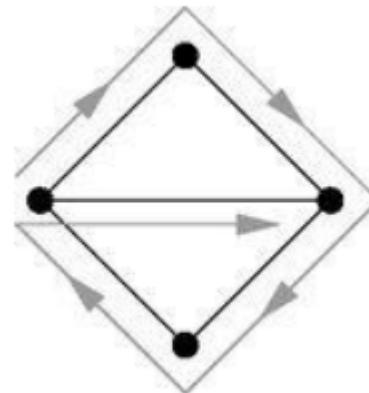


欧拉路径

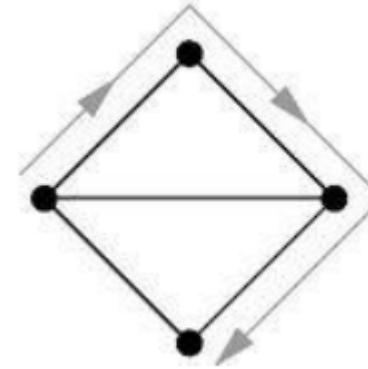
汉密尔顿路径

Eulerian Path and Hamiltonian Path

- Eulerian path: a path that traverses **each edge** in a network **exactly once**. Not necessary to be self-avoiding.
- Hamiltonian path: a path that visits **each vertex** exactly once. Must be a self-avoiding path.

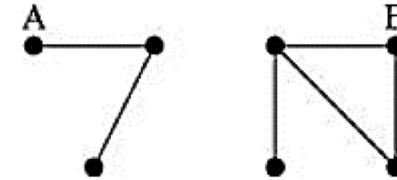


Eulerian path



Hamiltonian path

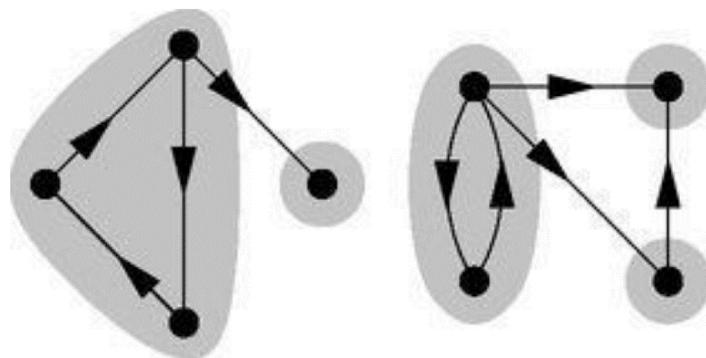
分支 Component



- The adjacency matrix of a network with more than one component can be written in block diagonal form 分块对角形式
 - non-zero elements of the matrix are confined to square blocks along the diagonal of the matrix 非零元素在沿对角线的正方形中
 - all other elements being zero 其他均为零元素

$$\mathbf{A} = \begin{pmatrix} & & \\ & \square & \\ & 0 & \dots \\ 0 & & \\ & \square & \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Components in Directed Graph



弱连通分支

A graph with WCC and SCC

强连通分支(任意两个顶点存在有向路径)

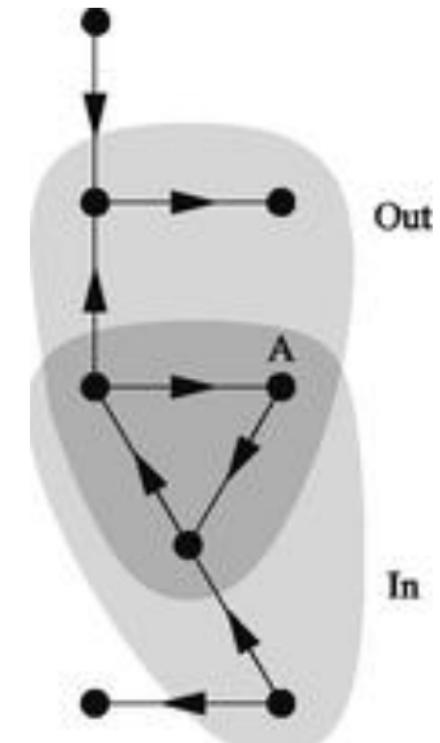
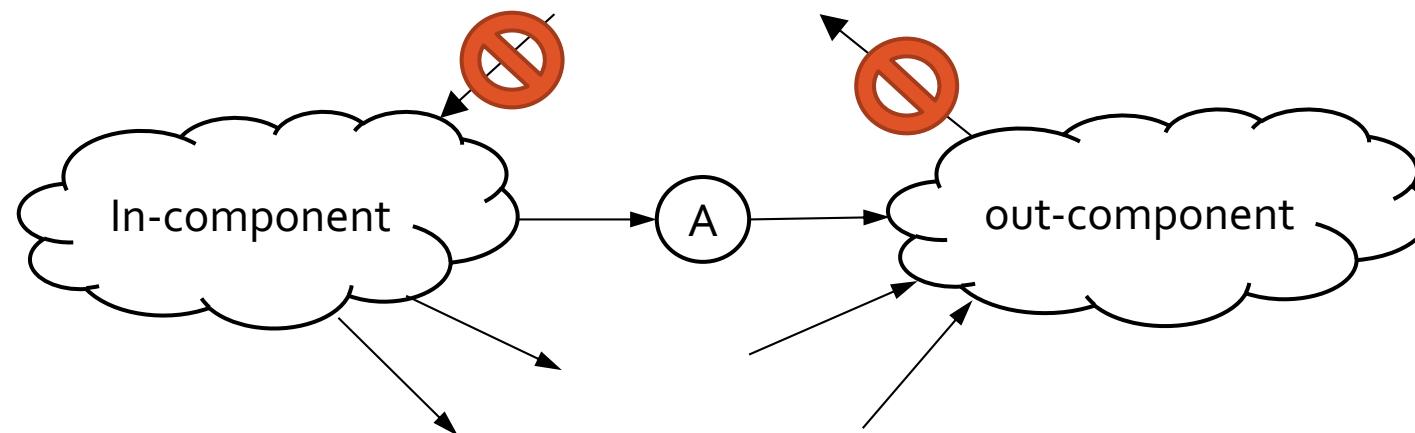
1. there must be circles in a SCC with $\#node > 1$;
2. every node in these SCC must belong to a circle;
3. there is no SCC in DAG

外向分支

内向分支

Out-component and In-component

- Out-component: reachable nodes from a specific node
 - edges connecting OC to other vertices only point **inward**, never **outward**
- In-component: nodes that can reach to a specific node
 - edges connecting IC to other only point **outward**, never **inward**



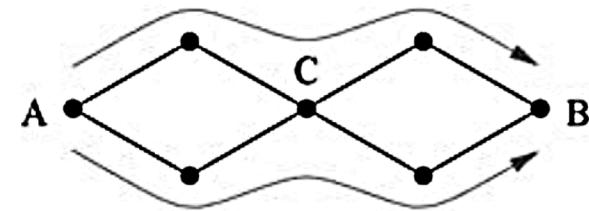
The in- and out-components of a vertex A in a small directed network

独立路径

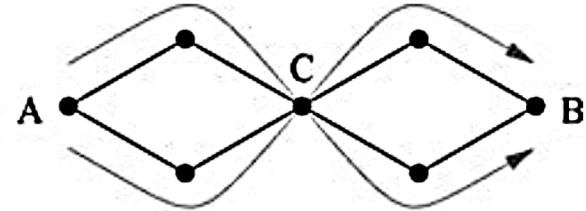
Independent Path

用来评估两个点之间的连接强度

- there are more than one path between node A and B
- some paths share no vertex (vertex-independent) or edge (edge-independent)



(a)



(b)

From A to B , there are **two** edge-independent paths, but only **one** vertex-independent path

连通度

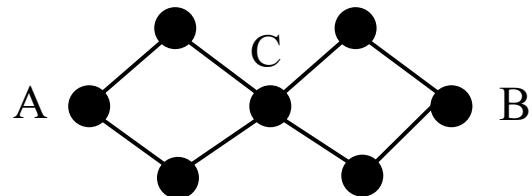
Connectivity

- edge connectivity = # edge-independent path
- vertex connectivity = # vertex-independent path
- e-connectivity(A,B)=2; v-connectivity(A,B)=1

割集

Cut Set

- Given A and B are connected,
 - the cut set is a set of vertex $\{v_1, v_2 \dots v_k\}$
 - removing the cut set will make A and B disconnected
 - 最小割集 minimum cut set (MCS) is the smallest cut set
 - $\{C\}$ is the MCS of A and B



门格尔定理

Menger's Theorem implies:

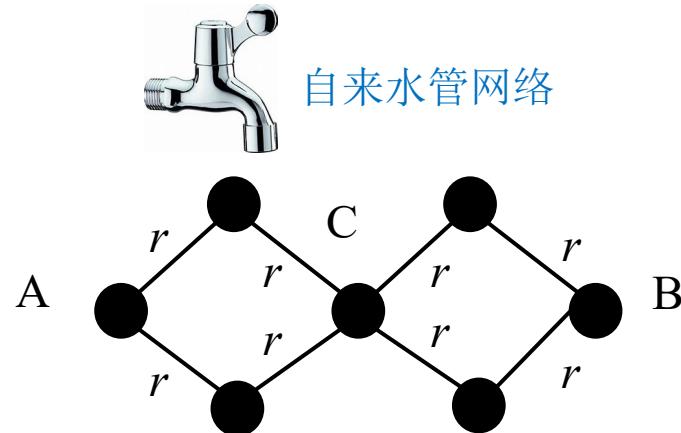
$$|\text{MCS}(A,B)| = \text{v-connectivity}(A,B)$$

最大流

Maximum Flow

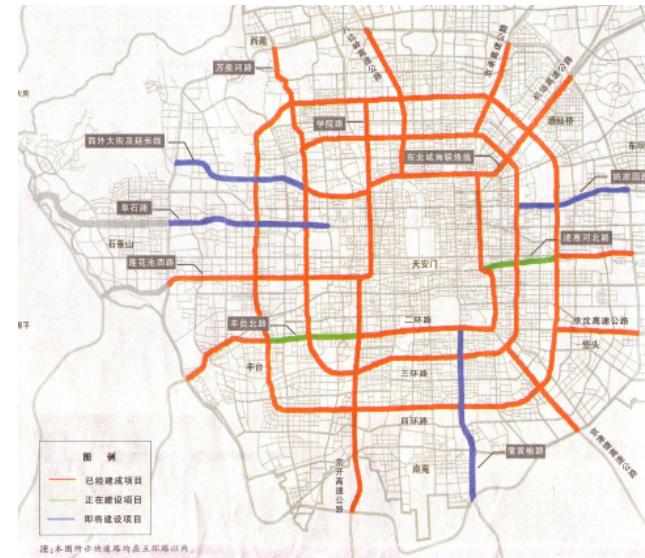
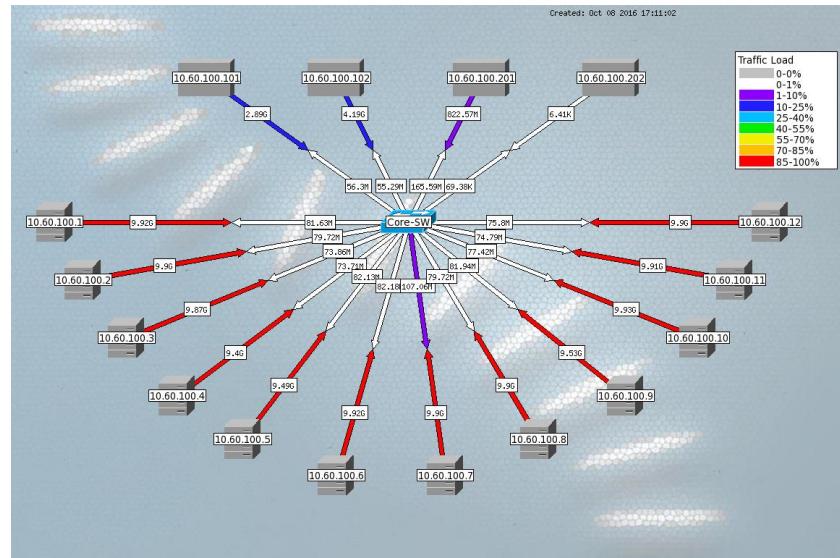
- a water pipeline from A to B
- max rate on each edge: r
- max rate from A to B:
$$\text{max-rate}(A,B) = \text{e-connectivity}(A,B) * r$$

增广路径算法
- Augment Path Algorithm



Application of min Cut | max Flow

- Finding bottleneck in communication network 通信网络中的带宽瓶颈问题
- Predicting maximum traffic in transportation planning 交通规划中的最大流量问题
- border detection in image segmentation 图像分割中的边缘检测问题



图拉普拉斯矩阵

Graph Laplacian Matrix

扩散过程

- diffusion process (gas, viewpoint, disease)

类似于气体扩散方程中的拉普拉斯算子 ∇^2

- Graph Laplacian Matrix: describe the diffusion process in a graph

$$\mathbf{L} = \mathbf{D} - \mathbf{A}$$

对角矩阵 邻接矩阵

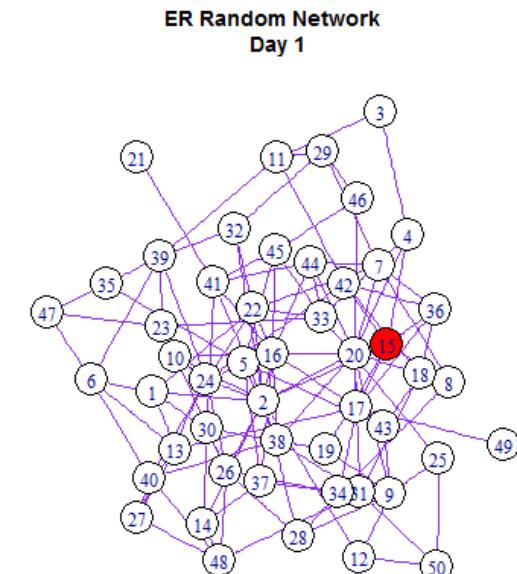
第*i*个顶点的度

$$\mathbf{D} = \begin{pmatrix} k_1 & 0 & 0 & \dots \\ 0 & k_2 & 0 & \dots \\ 0 & 0 & k_3 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

*t*时间点第*i*个节点的状态

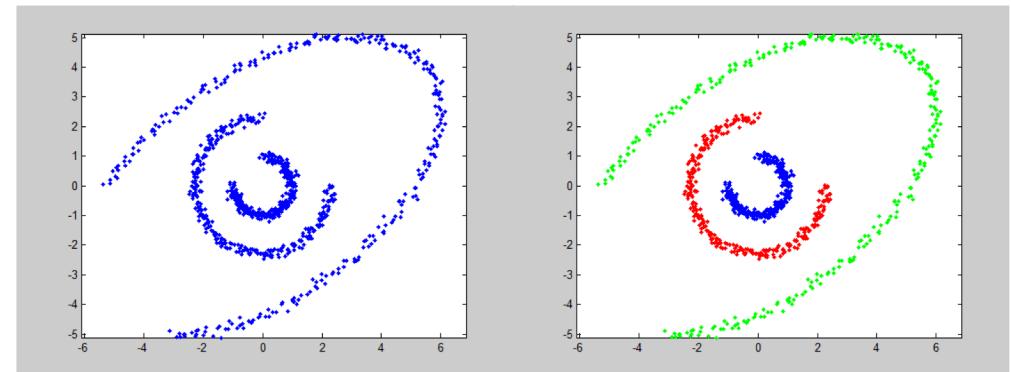
$$a_i(t) = a_i(0)e^{-c\lambda_i t}$$

第*i*个拉普拉斯特征值
扩散速率
初始状态



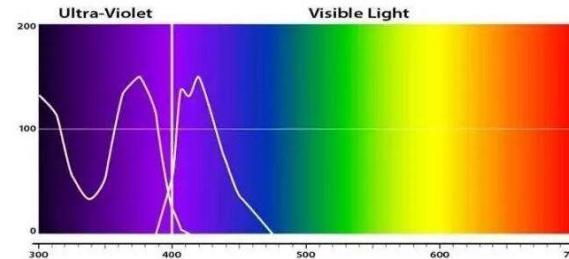
Application of Graph Laplacian Matrix

- calculating the number of components
 - the number of components equals to the number of non-zero eigenvalues of laplacian matrix
随机游走
- analysis of random walk
 - *given*: a random walk that starts at a specified vertex i and takes t random steps
 - *goal*: $p_i(t)$, the probability that the walk is at vertex i at time t
 - link analysis
- graph clustering
 - spectrum clustering
谱聚类



图谱理论

Graph Spectrum Theory



代数图论

- Algebraic Graph Theory: using algebra to analyze graph structures
分治法
- divide-and-conquer: divide a complex structure into separated components
 - in optics: optical spectrum 光谱
 - in graph: graph spectrum 图谱
- understanding graph structure in temporal dimension and spatial dimension
时间维度 空间维度



ASSIGNMENT

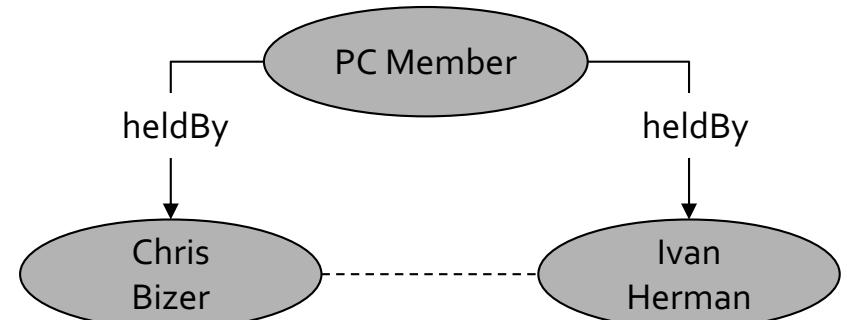


Text Similarity

- Given two text document (such as news reports) D_1 and D_2
- How to calculate the similarity between D_1 and D_2
- Solutions:
 - Jaccard Similarity
 - TF/IDF + Cosine Similarity
 - Word Embedding + Cosine Similarity
 - Knowledge-based Similarity
- Paper writing: a team paper can be collaborated written including:
 - abstract / problem definition / approach overview
 - details of algorithms / experiments / related works / reference
 - LNCS template with latex
 - PPT

Community Detection

- Title: Finding Communities in a (Social) Network
- Dataset: Stanford Large Network Dataset Collection (<https://snap.stanford.edu/data/>)
- Goal: to detect communities in given networks
- Requirements:
 - a paper in LNCS
 - implements more than two clustering algorithms
 - quantitative evaluation
 - compare the results of different clustering algorithms
 - PPT



ARRANGEMENT OF DISCUSSION AND PRESENTATION

- Time: Next Wednesday afternoon (2 pm ~ 5 pm)
- Style: Online (Tencent online conference)