Combinatorial optimization: Max-Cut, Min UnCut and Sparsest Cut Problems

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Outline

Max-Cut Problem

Given an undirected weighted graph $G = (V, E, W)$, where

$$V = \{1, \dots, n\} \text{ – set of vertices}$$
$$E \subseteq V \times V \text{ – set of edges}$$
$$W : E \to \mathbb{R} \text{ – weights}$$

One wants to find a partition $f : V \to \{0, 1\}$ in order to maximize the sum of edges in the cut:

$$\sum_{(i,j) \in E : f(i) \neq f(j)} w_{ij} \to \max,$$

where w_{ij} stands for weight of the edge (i, j).

Min UnCut Problem

Given an undirected weighted graph $G = (V, E, W)$, one wants to find a partition $f : \{0, 1\}$ in order to minimize the sum of edges out of the cut:

$$\sum_{(i,j)\in E: f(i)=f(j)} w_{ij} \to \min$$

Remark. Let $Opt(MUC)$ and $Opt(MC)$ stand for optimal solutions of Min UnCut and Max-Cut problems respectively. Then it holds

$$Opt(MUC) + Opt(MC) = \sum_{(i,j)\in E} w_{ij}$$

Sparsest Cut Problem

Given an undirected weighted graph $G = (V, E$ and a capacity function $c : E \to \mathbb{R}_+$. Also given a set of demand pairs $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ and demand values $d_1, d_2, \ldots, d_k$. One wants to find a set $E' \subseteq E$ minimizing

$$\frac{c(E')}{D(E')} \to \min,$$

where

$$c(f) = \sum_{(i,j) \in E'} c_{ij}$$

$$D(f) = \sum_{i:(s_i,t_i) \text{ are separated by } E'} d_i$$

Outline

Max-Cut, Min UnCut and Sparsest Cut problems are NP-hard

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Approaches:

- Use naive algorithm of discrete optimization
- Use convex relaxations

Greedy algorithm

Idea:

- On the k-th iteration choose a point x_{k+1} from neighbourhood of the current position x_k, such that

$$\mathrm{Obj}(x_{k+1}) < \mathrm{Obj}(x_k)$$

- If there is no such point x_{k+1} stop and return x_k

Problems:

- How to choose the neighbourhood?
- How far will be the result from the solution?

Semi-Definite-Programming

The following type of optimisation problems is considered to be the SDP problems:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \; & c^T x \\ \text{s.t. } & A_0 - x_1 A_1 - \ldots x_n A_n \succeq 0 \end{aligned}$$

SDP relaxation

Suppose we have the initial problem:

$$\min_{x \in X} f(x),$$

where X is a feasible region. If we construct the SDP problem with the X' feasible region, s.t. $X \subseteq X'$, then this SDP problem is considered to be an SDP relaxation for the initial problem.

SDP relaxation with triangle constraints

In order to improve the SDP relaxation one can add triangle constraints like:

$$
\begin{aligned}
x_{ij} + x_{jk} + x_{ki} &\leq 2, \\
x_{ij} + x_{jk} &\geq x_{ki}.
\end{aligned}
$$

Such constraints are appropriate for the cut problems on graphs.In these cases x_{ij} could be:

$$
x_{ij} = \begin{cases} 1 & \text{if } (i, j) \text{ edge is in the cut,} \\ 0 & \text{otherwise.} \end{cases}
$$

Interior Point Method

- One can find solutions of SDP-relaxations via interior-point method
- Use log-det barrier function

LP relaxations

We want to implement **LP relaxations** for Maximal Cut, Min UnCut and Sparsest cut problems.

For example:

MAXCUT can be phrased as the following integer program.

$$\max \sum_{(u,v)\in E} e_{uv}$$

$$x_u \in \{0,1\} \quad \forall u \in V$$

$$e_{uv} \in \{0,1\} \quad \forall (u,v) \in E$$

$$e_{uv} \leq \begin{cases} x_u + x_v \\ 2 - (x_u + x_v) \quad \forall (u,v) \in E \end{cases}$$

LP relaxation for MAXCUT

We relax $e_{uv} \in \{0,1\}$ to $0 \leq e_{uv} \leq 1$ and $x_{u,v} \in \{0,1\}$ to $0 \leq x_{u,v} \leq 1$ to obtain the following LP relation.

$$\max \sum_{(u,v)\in E} e_{uv}$$

$$x_u \in [0,1] \quad \forall u \in V$$

$$e_{uv} \in [0,1] \quad \forall (u,v) \in E$$

$$e_{uv} \leq \begin{cases} x_u + x_v \\ 2 - (x_u + x_v) \quad \forall (u,v) \in E \end{cases}$$

First order methods

We want to implement the following first order methods for SDP relaxations:

1. gradient descent;
2. ADMM.

Alternating direction method of multipliers

ADMM problem form (with f and g convex):

$$\min_{x,z} \; f(x) + g(z)$$
$$\text{s.t. } Ax + Bz = c$$

Augmented Lagrandian:

$$L_r(x, y, z) = f(z) + g(z) + y^\top (Ax + Bz - c) + \frac{r}{2}\|Ax + Bz - c\|_2^2$$

ADMM:

$$\begin{aligned} x^{k+1} &= \operatorname*{argmin}_x L_r(x, z^k, y^k), \text{ x - update,} \\ z^{k+1} &= \operatorname*{argmin}_z L_r(x^{k+1}, z, y^k), \text{ z - update,} \\ y^{k+1} &= y^k + r(Ax^{k+1} + Bz^{k+1} - c), \text{ dual update} \end{aligned}$$

Thank you for attention!

SDP relaxation for MaxCut

The following optimisation problem represents the SDP relaxation for MaxCut

$$\begin{aligned} \min_X \;& \operatorname{tr}(WX), \\ \text{s.t. } & X \succeq 0, \\ & X_{ii} = 1 \;\forall i. \end{aligned}$$

Here W is a matrix of weights.

SDP relaxation for Sparsest cut Problem

Let $x_e = \mathbb{I}\{e \in E'\}$ and y_i represents whether or not the pair (s_i, t_i) should be separated, then we have the following SDP relaxation:

$$
\begin{aligned}
&\min\ t \\
&\text{s.t.} \\
&\qquad \begin{pmatrix} t & 1 \\ c^T x & d^T y \end{pmatrix} \succeq 0 \\
&\qquad \sum_{e \in p} x_e \geq y_i, && p \in \mathcal{P}_{s_i t_i} \\
&\qquad 1 \geq y_i \geq 0 && i \in \overline{1, k} \\
&\qquad 1 \geq x_e \geq 0 && e \in E
\end{aligned}
$$