Combinatorial optimization: Max-Cut, Min UnCut and Sparsest Cut Problems

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Max-Cut Problem

Given an undirected weighted graph $G = (V, E, W)$, where

$$V = \{1, \dots, n\} \text{ – set of vertices}$$
$$E \subseteq V \times V \text{ – set of edges}$$
$$W : E \to \mathbb{R} \text{ – weights}$$

One wants to find a partition $f : V \to \{0, 1\}$ in order to maximize the sum of edges in the cut:

$$\sum_{(i,j) \in E : f(i) \neq f(j)} w_{ij} \to \max,$$

where w_{ij} stands for weight of the edge (i, j).

Min UnCut Problem

Given an undirected weighted graph $G = (V, E, W)$, one wants to find a partition $f : \{0, 1\}$ in order to minimize the sum of edges out of the cut:

$$\sum_{(i,j)\in E: f(i)=f(j)} w_{ij} \to \min$$

REMARK. Let $Opt(MUC)$ and $Opt(MC)$ stand for optimal solutions of Min UnCut and Max-Cut problems respectively. Then it holds

$$Opt(MUC) + Opt(MC) = \sum_{(i,j)\in E} w_{ij}$$

Sparsest Cut Problem

Given an undirected weighted graph $G = (V, E$ and a capacity function $c : E \to \mathbb{R}_+$. Also given a set of demand pairs $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ and demand values $d_1, d_2, \ldots, d_k$. One wants to find a set $E' \subseteq E$ minimizing

$$\frac{c(E')}{D(E')} \to \min,$$

where

$$c(f) = \sum_{(i,j) \in E'} c_{ij}$$

$$D(f) = \sum_{i:(s_i, t_i) \text{ are separated by } E'} d_i$$

Outline

Max-Cut, Min UnCut and Sparsest Cut problems are **NP-hard**

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Approaches:

- Use naive algorithm of discrete optimization
- Use convex relaxations

Greedy algorithm

Idea:

- On the k-th iteration choose a point x_{k+1} from neighbourhood of the current position x_k, such that

$$\mathrm{Obj}(x_{k+1}) < \mathrm{Obj}(x_k)$$

- If there is no such point x_{k+1} stop and return x_k

Problems:

- How to choose the neighbourhood?
- How far will be the result from the solution?

Semi-Definite Programming

The following type of optimization problems is considered to be the SDP problems:

$$\min_{x \in \mathbb{R}^n} \; c^T x$$
$$\text{s.t. } A_0 - x_1 A_1 - \ldots x_n A_n \succeq 0$$

SDP relaxation

Suppose we have the initial problem:

$$\min_{x \in X} f(x),$$

where X is a feasible region. If we construct the SDP problem with the X' feasible region, s.t. $X \subseteq X'$, then this SDP problem is considered to be an SDP relaxation for the initial problem.

SDP relaxation for MaxCut

The following optimization problem represents the SDP relaxation for MaxCut

$$\begin{aligned} \min_X \ & \operatorname{tr}(WX), \\ \text{s.t. } & X \succeq 0, \\ & X_{ii} = 1 \ \forall i. \end{aligned}$$

Here W is a matrix of weights.

SDP relaxation with triangle constraints

In order to improve the SDP relaxation one can add triangle constraints like:

$$x_{ij} + x_{jk} + x_{ki} \leq 2,$$
$$x_{ij} + x_{jk} \geq x_{ki}.$$

Such constraints are appropriate for the cut problems on graphs.In these cases x_{ij} could be:

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ edge is in the cut,} \\ 0 & \text{otherwise.} \end{cases}$$

Interior Point Method

One can apply interior-point method to the problem

$$\begin{cases} \operatorname{tr}(WX) \to \min\limits_{X \in \mathbb{R}^{n \times n}} \\ X \succeq 0 \\ X_{ii} = 1, \quad i = \overline{1, n} \end{cases}$$

Use log-det barrier function and solve the problem

$$\begin{cases} \operatorname{tr}(WX) - \mu \log \det X \to \min\limits_{X \in \mathbb{R}^{n \times n}} \\ X_{ii} = 1, \quad i = \overline{1, n} \end{cases}$$

The duality gap is equal to μn

LP relaxations

We want to implement **LP relaxations** for Maximal Cut, Min UnCut and Sparsest cut problems.
For example:
MAXCUT can be phrased as the following integer program.

$$\max \sum_{(u,v)\in E} e_{uv}$$

$$x_u \in \{0,1\} \quad \forall u \in V$$

$$e_{uv} \in \{0,1\} \quad \forall (u,v) \in E$$

$$e_{uv} \leq \begin{cases} x_u + x_v \\ 2 - (x_u + x_v) \quad \forall (u,v) \in E \end{cases}$$

LP relaxation for MAXCUT

We relax $e_{uv} \in \{0, 1\}$ to $0 \leq e_{uv} \leq 1$ and $x_{u,v} \in \{0, 1\}$ to $0 \leq x_{u,v} \leq 1$ to obtain the following LP relation.

$$\max \sum_{(u,v) \in E} e_{uv}$$

$$x_u \in [0, 1] \quad \forall u \in V$$

$$e_{uv} \in [0, 1] \quad \forall (u, v) \in E$$

$$e_{uv} \leq \begin{cases} x_u + x_v \\ 2 - (x_u + x_v) & \forall (u, v) \in E \end{cases}$$

First order methods

We want to implement the following first order methods for SDP relaxations:

1. gradient descent;
2. ADMM.

Alternating direction method of multipliers

ADMM problem form (with f and g convex):

$$\min_{x,z} \; f(x) + g(z)$$
$$\text{s.t. } Ax + Bz = c$$

Augmented Lagrandian:

$$L_r(x, y, z) = f(z) + g(z) + y^\top (Ax + Bz - c) + \frac{r}{2}\|Ax + Bz - c\|_2^2$$

ADMM:

$$x^{k+1} = \operatorname*{argmin}_x L_r(x, z^k, y^k), \text{ x - update,}$$
$$z^{k+1} = \operatorname*{argmin}_z L_r(x^{k+1}, z, y^k), \text{ z - update,}$$
$$y^{k+1} = y^k + r(Ax^{k+1} + Bz^{k+1} - c), \text{ dual update}$$

Thank you for attention!

SDP relaxation for Sparsest cut Problem

Let $x_e = \mathbb{I}\{e \in E'\}$ and y_i represents whether or not the pair (s_i, t_i) should be separated, then we have the following SDP relaxation:

$$
\begin{aligned}
\min\ & t \\
\text{s.t.}\ & \\
& \begin{pmatrix} t & 1 \\ c^T x & d^T y \end{pmatrix} \succeq 0 \\
& \sum_{e \in p} x_e \geq y_i, && p \in \mathcal{P}_{s_i t_i} \\
& 1 \geq y_i \geq 0 && i \in \overline{1, k} \\
& 1 \geq x_e \geq 0 && e \in E
\end{aligned}
$$