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Low eigenvalues of Laplacian matrices of large random graphs

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Abstract For each $n \geq 2$, let $\mathbf{A}_n = (\xi_{ij})$ be an $n \times n$ symmetric matrix with diagonal entries equal to zero and the entries in the upper triangular part being independent with mean μ_n and standard deviation σ_n. The Laplacian matrix is defined by $\mathbf{\Delta}_n = \mathrm{diag}(\sum_{j=1}^{n} \xi_{ij})_{1 \leq i \leq n} - \mathbf{A}_n$. In this paper, we obtain the laws of large numbers for $\lambda_{n-k}(\mathbf{\Delta}_n)$, the $(k+1)$-th smallest eigenvalue of $\mathbf{\Delta}_n$, through the study of the order statistics of weakly dependent random variables. Under certain moment conditions on ξ_{ij}'s, we prove that, as $n \to \infty$,

(i) $\frac{\lambda_{n-k}(\mathbf{\Delta}_n) - n\mu_n}{\sigma_n \sqrt{n \log n}} \to -\sqrt{2} \quad a.s.$

for any $k \geq 1$. Further, if $\{\mathbf{\Delta}_n; n \geq 2\}$ are independent with $\mu_n = 0$ and $\sigma_n = 1$, then,

(ii) the sequence $\left\{\frac{\lambda_{n-k}(\mathbf{\Delta}_n)}{\sqrt{n \log n}}; n \geq 2\right\}$ is dense in $\left[-\sqrt{2 + 2(k+1)^{-1}}, -\sqrt{2}\right]$ $a.s.$

for any $k \geq 0$. In particular, (i) holds for the Erdös–Rényi random graphs. Similar results are also obtained for the largest eigenvalues of $\mathbf{\Delta}_n$.

Keywords Random graph · Random matrix · Laplacian matrix · Extreme eigenvalues · Order statistics

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1 Introduction

Let G be a non-oriented graph with n different vertices $\{v_1, \ldots, v_n\}$. We assume that G does not have loops or multiple edges, its Laplacian matrix $\mathbf{\Delta} = (l_{ij})_{n\times n}$ is defined by

$$l_{ij} = \begin{cases} \deg(v_i), & \text{if } i = j; \\ -1, & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j; \\ 0, & \text{otherwise,} \end{cases}$$

where $\deg(v_i)$ is the degree of v_i, that is, the number of vertices v_j, $j \neq i$, which have edges with v_i. Evidently, $\mathbf{\Delta} = \operatorname{diag}(\deg(v_i))_{1\le i\le n} - \mathbf{A}$, where $\mathbf{A} = (\epsilon_{ij})_{n\times n}$ is the adjacency matrix of the graph with $\epsilon_{ij} = 1$ if vertices v_i and v_j are connected, and $\epsilon_{ij} = 0$ if not. In other words, the Laplacian matrix of graph G is the difference of the degree matrix and the adjacency matrix. The matrix $\mathbf{\Delta}$ is sometimes also called the admittance matrix or Kirchhoff matrix in literature.

For a random graph G, the corresponding $\mathbf{\Delta}$ is a symmetric random matrix. For instance, suppose G is an *Erdös–Rényi random graph* $G(n, p_n)$, that is, a (non-oriented) graph with n vertices, and for each pair of vertices v_i and v_j with $i \neq j$, an edge between them is formed randomly with chance p_n and independently of other edges, see [13,14]. Then

$$\mathbf{\Delta}_n = \begin{pmatrix} \sum_{j\neq 1} \xi_{1j}^{(n)} & -\xi_{12}^{(n)} & \cdots & -\xi_{1n}^{(n)} \\ -\xi_{21}^{(n)} & \sum_{j\neq 2} \xi_{2j}^{(n)} & \cdots & -\xi_{2n}^{(n)} \\ \vdots & \vdots & \vdots & \vdots \\ -\xi_{n1}^{(n)} & -\xi_{n2}^{(n)} & \cdots & \sum_{j\neq n} \xi_{nj}^{(n)} \end{pmatrix}, \tag{1.1}$$

where $\{\xi_{ij}^{(n)};\ 1 \le i < j \le n\}$ are independent random variables with $P(\xi_{ij}^{(n)} = 1) = 1 - P(\xi_{ij}^{(n)} = 0) = p_n$ for all $1 \le i < j \le n$ and $n \ge 2$.

Remark 1.1 By construction zero is an eigenvalue of $\mathbf{\Delta}_n$ (the corresponding eigenvector is $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^n$), and if ξ_{ij}'s are all non-negative then $-\mathbf{\Delta}_n$ is the generator of a Markov process on $\{1, 2, \ldots, n\}$, hence all the eigenvalues of $\mathbf{\Delta}_n$ are non-negative, i.e., $\mathbf{\Delta}_n$ is non-negative definite.

The Kirchhoff theorem from [20] establishes the relationship between the number of spanning trees of G and the eigenvalues of $\mathbf{\Delta}_n$; the second smallest eigenvalue relates to the algebraic connectivity of the graph, see, e.g., [16].

Bryc et al. [6] and Ding and Jiang [11] show that the empirical distribution of suitably normalized eigenvalues of $\mathbf{\Delta}_n$ converges to a deterministic probability distribution, which is the free convolution of the semi-circle law and the standard normal distribution. Note that the second smallest eigenvalue of $\mathbf{\Delta}_n$ stands for the algebraic connectivity of a graph G as mentioned earlier, it is our purpose here to study the properties of the second smallest eigenvalue as well as other low eigenvalues of $\mathbf{\Delta}_n$.

For weighted random graphs, $\{\xi_{ij}^{(n)};\ 1 \le i < j \le n\}$ in (1.1) are independent random variables and each of them is the product of a Bernoulli random variable $Ber(p_n)$ and a nice random variable, for instance, a Gaussian random variable or a random variable with all finite moments (see, e.g., [18,19]). For the sign model studied in [3,19,25,26], $\xi_{ij}^{(n)}$ are independent random variables taking three values: 0, 1, -1. From this perspective, to make our results more applicable, we investigate the spectral properties of $\mathbf{\Delta}_n$ under more general conditions on the entries $\{\xi_{ij}^{(n)};\ 1 \le i < j \le n\}$ given below.

Assumption A_p Let $\{\xi_{ij}^{(n)};\ 1 \le i \ne j \le n,\ n \ge 2\}$ be random variables defined on the same probability space and $\{\xi_{ij}^{(n)};\ 1 \le i < j \le n\}$ be independent for each $n \ge 2$ (not necessarily identically distributed) with $\xi_{ij}^{(n)} = \xi_{ji}^{(n)}$, $E(\xi_{ij}^{(n)}) = \mu_n$, $Var(\xi_{ij}^{(n)}) = \sigma_n^2 > 0$ for all $1 \le i < j \le n$ and $n \ge 2$, and $\sup_{1\le i<j\le n,\, n\ge 2} E|(\xi_{ij}^{(n)} - \mu_n)/\sigma_n|^t < \infty$ for some $t > p > 0$.

We will state our main results next. Before that some notation is needed. Given an $n \times n$ symmetric matrix $\mathbf{M}$, let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ be the eigenvalues of $\mathbf{M}$. Sometimes this is also written as $\lambda_1(\mathbf{M}) \ge \lambda_2(\mathbf{M}) \ge \cdots \ge \lambda_n(\mathbf{M})$ for clarity. For an $n \times n$ matrix $\mathbf{M}$, we use $\|\mathbf{M}\| = \sup_{\mathbf{x}\in\mathbb{R}^n:\, \|\mathbf{x}\|_2=1} \|\mathbf{M}\mathbf{x}\|_2$ to denote its spectral norm, where $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$ for $\mathbf{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n$.

Throughout this paper, $\log x = \log_e x$ for $x > 0$. Keep in mind our application for the Erdös–Rényi random graph $G(n, p_n)$, which will be given after Theorem 2, the mean μ_n of $\xi_{ij}^{(n)}$ is not equal to zero in general. The following step reduces the general case to that with $\mu_n = 0$. Fix $n \ge 2$. Set

$$\begin{aligned}
&\eta_{ij}^{(n)} = \frac{\xi_{ij}^{(n)} - \mu_n}{\sigma_n},\ \ \eta_{ii}^{(n)} = 0 \text{ for } 1 \le i \ne j \le n,\ \ \mathbf{B}_n = \left(\eta_{ij}^{(n)}\right)_{1\le i,j\le n},\\
&X_{n,i} = \sum_{j=1}^{n} \eta_{ij}^{(n)},\ \ X_{n,(1)} \ge \cdots \ge X_{n,(n)} \text{ is the order statistics of } \{X_{n,i};\ 1 \le i \le n\}.
\end{aligned} \tag{1.2}$$

Easily, $\mathbf{\Delta}_n = \sigma_n \cdot \mathrm{diag}(X_{n,i})_{1\le i\le n} - \sigma_n \mathbf{B}_n + \mu_n \mathbf{H}_n$ where all of the off-diagonal entries of $\mathbf{H}_n$ are equal to -1 and all of the diagonal entries are identical to $n-1$. Based on this, the following proposition establishes a connection between the eigenvalues of $\mathbf{\Delta}_n$ and the order statistics $\{X_{n,(i)}\}$.

Proposition 1.1 *Suppose Assumption A_0 holds. Then the following are true for all $n \ge 3$.*

(i) *For all $2 \le i \le n-1$, we have that*

$$X_{n,(i+1)} - \|\mathbf{B}_n\| \le \frac{\lambda_i(\mathbf{\Delta}_n) - n\mu_n}{\sigma_n} \le X_{n,(i-1)} + \|\mathbf{B}_n\|.$$

(ii) *Further, if* $\mu_n \geq 0$, *then, for all* $n \geq 2$,

$$X_{n,(2)} - \|\mathbf{B}_n\| \leq \frac{\lambda_1(\mathbf{\Delta}_n) - n\mu_n}{\sigma_n} \leq X_{n,(1)} + \|\mathbf{B}_n\|.$$

(iii) *Under Assumption* A_6, $\lim_{n\to\infty} \frac{1}{\sqrt{n\log n}}\|\mathbf{B}_n\| = 0$ *a.s.*

The moment assumption in (iii) seems optimal from its proof via a general theorem on the spectrum of a random matrix (Lemma 2.1). It would be interesting to see a proof to confirm this. On the other hand, we see from this proposition that the limiting behavior of $(\lambda_i(\mathbf{\Delta}_n) - n\mu_n)/(\sigma_n\sqrt{n\log n})$ can be obtained from $X_{n,(i)}/\sqrt{n\log n}$ if the latter is understood. In fact we have the following results about $X_{n,(i)}$.

Proposition 1.2 *Let* $\{\eta_{ij}^{(n)}\}$ *and* $X_{n,(i)}$ *be as in* (1.2).

(i) *If Assumption* A_6 *holds, then, for any* $k \geq 1$, $X_{n,(k)}/\sqrt{n\log n} \to \sqrt{2}$ *in probability as* $n \to \infty$.
(ii) *Suppose Assumption* A_p *holds for all* $p > 0$. *Let* $\{k_n;\ n \geq 1\}$ *be a sequence of integers such that* $k_n \to +\infty$ *and* $\log k_n = o(\log n)$ *as* $n \to \infty$. *Then* $X_{n,(k_n)}/\sqrt{n\log n} \to \sqrt{2}$ *a.s. as* $n \to \infty$.
(iii) *Given integer* $k \geq 1$, *let Assumption* A_{4k+4} *hold. If* $\{\mathbf{\Delta}_2, \mathbf{\Delta}_3 \dots\}$ *are independent, then*

$$\liminf_{n\to\infty} \frac{X_{n,(k)}}{\sqrt{n\log n}} = \sqrt{2}\ \ a.s.\ \ \textit{and}\ \ \limsup_{n\to\infty} \frac{X_{n,(k)}}{\sqrt{n\log n}} = \sqrt{2+2k^{-1}}\ \ a.s.\ \textit{and}$$
$$\textit{the sequence}\ \left\{\frac{X_{n,(k)}}{\sqrt{n\log n}};\ n \geq 2\right\}\ \textit{is dense in}\ [\sqrt{2},\ \sqrt{2+2k^{-1}}]\ \ a.s.$$

The $(4k+4)$-th moment assumption in (iii) above comes from a condition to guarantee a large deviation result (Lemma 3.6). It is not known if it is the best moment condition. The order statistics of independent random variables are understood quite well, see, e.g., [10]. However, as the situation in Proposition 1.2, when random variables are not independent and their joint density is not known, there seems not much investigation in the literature. Observe

$$\lambda_{k+1}(-\mathbf{M}) = -\lambda_{n-k}(\mathbf{M}) \tag{1.3}$$

for any $n \times n$ symmetric matrix $\mathbf{M}$ and $0 \leq k \leq n-1$, particularly the corresponding equality for order statistics holds when $\mathbf{M}$ is diagonal. Also, if $\mu_n \equiv 0$ and $\sigma_n \equiv 1$, then $\mathbf{\Delta}_n = \mathrm{diag}(X_{n,i})_{1\leq i\leq n} - \mathbf{B}_n$. We know from (iii) of Proposition 1.1 and Weyl's perturbation theorem (see (iii) of Lemma 2.2) that

$$\lim_{n\to\infty}\left(\frac{\lambda_{k_n}(\mathbf{\Delta}_n)}{\sqrt{n\log n}} - \frac{X_{n,(k_n)}}{\sqrt{n\log n}}\right) = 0\ \ a.s. \tag{1.4}$$

under Assumption A_6 for any $\{k_n;\ n \geq 1\}$ with $1 \leq k_n \leq n-1$ and $n \geq 2$. By using (1.3) and (1.4) we immediately have the following result from Propositions 1.1 and 1.2.

Theorem 1 (i) *Suppose Assumption A_6 holds. Then*

$$\frac{n\mu_n - \lambda_{n-k}(\mathbf{\Delta}_n)}{\sigma_n\sqrt{n\log n}} \to \sqrt{2} \tag{1.5}$$

in probability as $n \to \infty$ for any $k \geq 1$. Further, the conclusion still holds if "$n\mu_n - \lambda_{n-k}(\mathbf{\Delta}_n)$" is replaced by "$\lambda_{k+1}(\mathbf{\Delta}_n) - n\mu_n$" for all $k \geq 1$.

(ii) *Assuming all the conditions in* (i) *hold and also $\mu_n \geq 0$ for all $n \geq 2$, then* (1.5) *holds if "$n\mu_n - \lambda_{n-k}(\mathbf{\Delta}_n)$" is replaced by "$\lambda_1(\mathbf{\Delta}_n) - n\mu_n$."*

(iii) *Assuming the conditions in* (ii) *of Proposition* 1.2 *hold. Then* (1.5) *holds almost surely with k replaced by k_n, and it also holds if "$n\mu_n - \lambda_{n-k}(\mathbf{\Delta}_n)$" in* (1.5) *is replaced by "$\lambda_{k_n}(\mathbf{\Delta}_n) - n\mu_n$."*

Note that (1.5) holds only for $k \geq 1$. When $k = 0$, the statement may not hold in general. In fact, when the entries of $\mathbf{\Delta}_n$ are all non-negative, $\lambda_n(\mathbf{\Delta}_n) = 0$, see Remark 1.1.

Relating Proposition 1.1 and (iii) of Proposition 1.2 to (1.3) and (1.4), we easily get the following result with the almost sure convergence.

Theorem 2 *Let $k \geq 1$. Suppose $\{\mathbf{\Delta}_2, \mathbf{\Delta}_3, \ldots\}$ are independent. Under Assumption A_{4k+12}, the following holds.*

(i) *If $\mu_n = 0$ and $\sigma_n = 1$ for all $n \geq 2$ then*

$$\liminf_{n\to\infty} \frac{\lambda_k(\mathbf{\Delta}_n)}{\sqrt{n\log n}} = \sqrt{2} \ \ a.s. \ \ and \ \ \limsup_{n\to\infty} \frac{\lambda_k(\mathbf{\Delta}_n)}{\sqrt{n\log n}} = \sqrt{2+2k^{-1}} \ \ a.s. \ and$$

$$\text{the sequence } \left\{\frac{\lambda_k(\mathbf{\Delta}_n)}{\sqrt{n\log n}};\ n \geq 2\right\} \text{ is dense in } [\sqrt{2},\ \sqrt{2+2k^{-1}}] \ \ a.s.$$

(ii) $\liminf_{n\to\infty} \frac{n\mu_n - \lambda_{n-k}(\mathbf{\Delta}_n)}{\sigma_n\sqrt{n\log n}} = \sqrt{2}$ *a.s. and*

$$\sqrt{2+2(k+2)^{-1}} \leq \limsup_{n\to\infty} \frac{n\mu_n - \lambda_{n-k}(\mathbf{\Delta}_n)}{\sigma_n\sqrt{n\log n}} \leq \sqrt{2+2k^{-1}} \ \ a.s.$$

(iii) *If $\mu_n \geq 0$ for all $n \geq 2$, then* $\liminf_{n\to\infty} \frac{\lambda_1(\mathbf{\Delta}_n) - n\mu_n}{\sigma_n\sqrt{n\log n}} = \sqrt{2}$ *a.s. and*

$$\sqrt{3} \leq \limsup_{n\to\infty} \frac{\lambda_1(\mathbf{\Delta}_n) - n\mu_n}{\sigma_n\sqrt{n\log n}} \leq 2 \ \ a.s.$$

(iv) *The conclusion in* (i) *also holds if "$\lambda_k(\mathbf{\Delta}_n)$" is replaced by "$-\lambda_{n-k+1}(\mathbf{\Delta}_n)$"; the conclusion in* (ii) *also holds if "$n\mu_n - \lambda_{n-k}(\mathbf{\Delta}_n)$" is replaced by "$\lambda_{k+1}(\mathbf{\Delta}_n) - n\mu_n$."*

The $(4k+12)$-th moment condition in the above theorem comes from replacing A_{4k+4} in Proposition 1.2 with $A_{4(k+2)+4}$ when applying (i) of Proposition 1.1.

It is interesting to observe that the "limsup" for $\lambda_k(\mathbf{\Delta}_n)$ depends on k, however, the "liminf" remains the same for any k. This is very different from the corresponding results for the classical random matrices such as the Gaussian Orthogonal Ensembles or the Gaussian Unitary Ensembles, see the comments between Theorem 1 and Corollary 1.1 in [11].

Notice $\xi_{ij}^{(n)} \geq 0$ for all $1 \leq i < j \leq n$ for the Erdös–Rényi random graph $G(n, p_n)$. Also, if $\inf_{n\geq 2}\{p_n, 1-p_n\} > 0$, Assumption A_p obviously holds with $\sup_{1\leq i<j\leq n, n\geq 2} E|\frac{\xi_{ij}^{(n)}-p_n}{\sqrt{p_n(1-p_n)}}|^t < \infty$ for all $t \geq 1$. Then, Theorems 1 and 2 hold with $\mu_n = p_n$ and $\sigma_n = \sqrt{p_n(1-p_n)}$ for all $n \geq 2$. Now we give some comments.

Remark 1.2 The result for $k = 1$ in (i) of Theorem 2 is obtained in [11], the conclusions for $k \geq 2$ in Theorem 2 are new. The proofs for the two cases are quite different. In fact, the proof of the result in [11] relies on the study of the maximum of weakly dependent random variables. For the proof of the case "$k \geq 2$" in Theorem 2, we need to analyze the behavior of k-th order statistics of weakly dependent random variables. The computation becomes more involved, for example, a large deviation result is developed in Lemma 3.6.

Remark 1.3 The above results apply to the extreme eigenvalues of the generators of the Markov processes on large finite state space and random symmetric jump rates. For the spectra and the gaps of the eigenvalues of $\mathbf{\Delta}_n$ generated by the reversible Markov chains, see [5].

In this paper we focus on the eigenvalues of the Laplacian matrices. There are a lot of other interests for the random graphs. For reference, one can see [4,7–9,12,17,21,24] for book-length studies.

The organization of the rest of paper is as follows. We prove Proposition 1.1 in Sect. 2; we prove Proposition 1.2 in Sect. 3.

2 Proof of Proposition 1.1

As mentioned in Introduction, for an $n \times n$ symmetric matrix $\mathbf{M}$, we write $\lambda_1(\mathbf{M}) \geq \lambda_2(\mathbf{M}) \geq \cdots \geq \lambda_n(\mathbf{M})$ for the eigenvalues of $\mathbf{M}$.

Lemma 2.1 (Remark 2.7 in [2]) *Suppose, for each $n \geq 1$, $\{\omega_{ij}^n;\ 1 \leq i \leq j \leq n\}$ are independent random variables (not necessarily identically distributed) with mean $\mu = 0$ and variance no larger than σ^2. Assume there exist constants $b > 0$ and $\delta_n \downarrow 0$ such that $\sup_{1\leq i,j\leq n} E|\omega_{ij}^n|^l \leq b(\delta_n\sqrt{n})^{l-3}$ for all $n \geq 1$ and $l \geq 3$. Set $\mathbf{W}_n = (\omega_{ij}^n)_{n\times n}$. Then* $\limsup_{n\to\infty} \frac{\lambda_1(\mathbf{W}_n)}{n^{1/2}} \leq 2\sigma$ *a.s.*

The inequalities in the following lemma are standard in matrix theory, see, e.g., p. 62 in [1] or p. 184 in [15].

Lemma 2.2 *Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be two Hermitian matrices. Then*

(i) $\lambda_i(\mathbf{M}_1 + \mathbf{M}_2) \leq \lambda_j(\mathbf{M}_1) + \lambda_{i-j+1}(\mathbf{M}_2)$ *for* $1 \leq j \leq i \leq n$*;*
(ii) $\lambda_i(\mathbf{M}_1 + \mathbf{M}_2) \geq \lambda_j(\mathbf{M}_1) + \lambda_{i-j+n}(\mathbf{M}_2)$ *for* $1 \leq i \leq j \leq n$*;*
(iii) (*Weyl's perturbation theorem*) $\max_{1\leq j\leq n} |\lambda_j(\mathbf{M}_1) - \lambda_j(\mathbf{M}_2)| \leq \|\mathbf{M}_1 - \mathbf{M}_2\|$.

Recall $\mathbf{\Delta}_n$ defined in (1.1). Suppose Assumption A_0 holds. For $n \geq 2$, recalling (1.2), set

$$\tilde{\mathbf{\Delta}}_n = \mathrm{diag}(X_{n,i})_{1\leq i\leq n} - \mathbf{B}_n. \tag{2.1}$$

In other words, $\tilde{\mathbf{\Delta}}_n$ is defined by replacing $\xi_{ij}^{(n)}$ from $\mathbf{\Delta}_n$ in (1.1) with $\eta_{ij}^{(n)}$ for all $1 \leq i < j \leq n$ and $n \geq 2$.

Proof of Proposition 1.1 (i) Reviewing (1.1) and (1.2), we have

$$\mathbf{\Delta}_n = \sigma_n \tilde{\mathbf{\Delta}}_n + \mu_n \mathbf{H}_n \quad \text{and} \quad \tilde{\mathbf{\Delta}}_n = \mathrm{diag}(X_{n,i})_{1\leq i\leq n} - \mathbf{B}_n \tag{2.2}$$

where all of the off-diagonal entries of $\mathbf{H}_n$ are equal to -1 and all of the diagonal entries are identical to $n-1$. It is easy to check that the eigenvalues of $\mu_n\mathbf{H}_n$ are equal to $n\mu_n$ with $n-1$ folds, and 0 with one fold. Observe that $\lambda_i(\mathrm{diag}(X_{n,i})_{1\leq i\leq n}) = X_{n,(i)}$ for $1 \leq i \leq n$. Apply (i) in Lemma 2.2 to the first identity in (2.2) to get

$$\begin{aligned}\lambda_i(\mathbf{\Delta}_n) &\leq \lambda_{i-1}(\sigma_n \tilde{\mathbf{\Delta}}_n) + \lambda_2(\mu_n \mathbf{H}_n)\\ &= \sigma_n \cdot \lambda_{i-1}(\tilde{\mathbf{\Delta}}_n) + n\mu_n\\ &\leq \sigma_n \cdot X_{n,(i-1)} + \sigma_n\|\mathbf{B}_n\| + n\mu_n\end{aligned}$$

for any $2 \leq i \leq n-1$, where, in the last step, (iii) of Lemma 2.2 is applied to the second equality in (2.2). On the other hand, applying (ii) in Lemma 2.2 to the first identity in (2.2), we have that

$$\begin{aligned}\lambda_i(\mathbf{\Delta}_n) &\geq \lambda_{i+1}(\sigma_n \tilde{\mathbf{\Delta}}_n) + \lambda_{n-1}(\mu_n \mathbf{H}_n)\\ &= \sigma_n \cdot \lambda_{i+1}(\tilde{\mathbf{\Delta}}_n) + n\mu_n\\ &\geq \sigma_n \cdot X_{n,(i+1)} - \sigma_n\|\mathbf{B}_n\| + n\mu_n\end{aligned} \tag{2.3}$$

for any $1 \leq i \leq n-1$. The two inequalities conclude (i).

(ii) Notice $\lambda_j(\mu_n\mathbf{H}_n) = n\mu_n$ for $1 \leq j \leq n-1$ as $\mu_n \geq 0$. From (2.2), we know $\mathbf{\Delta}_n = \sigma_n \cdot \mathrm{diag}(X_{n,i})_{1\leq i\leq n} - \sigma_n\mathbf{B}_n + \mu_n\mathbf{H}_n$. For any symmetric matrices $\mathbf{M}_1$ and $\mathbf{M}_2$, we know $\lambda_1(\mathbf{M}_1 + \mathbf{M}_2) \leq \lambda_1(\mathbf{M}_1) + \lambda_1(\mathbf{M}_2)$, it follows that

$$\begin{aligned}\lambda_1(\mathbf{\Delta}_n) &\leq \lambda_1(\sigma_n \cdot \mathrm{diag}(X_{n,i})_{1\leq i\leq n}) + \lambda_1(-\sigma_n\mathbf{B}_n) + \lambda_1(\mu_n\mathbf{H}_n)\\ &\leq \sigma_n \cdot X_{n,(1)} + \sigma_n \cdot \|\mathbf{B}_n\| + n\mu_n.\end{aligned}$$

Take $i = 1$ in (2.3), we have that $\lambda_1(\mathbf{\Delta}_n) \geq \sigma_n \cdot X_{n,(2)} - \sigma_n\|\mathbf{B}_n\| + n\mu_n$. So (ii) follows.

(iii) The proof of this part is relatively long, we put it into Lemma 2.3 below which is slightly stronger than what we need. □

Lemma 2.3 *For $n \geq 2$, let $\mathbf{U}_n = (u_{ij}^{(n)})$ be an $n \times n$ symmetric random matrix with $u_{ii}^{(n)} = 0$ for all $1 \leq i \leq n$. Suppose $\{u_{ij}^{(n)};\ 1 \leq i < j \leq n,\ n \geq 2\}$ are defined on the* *same probability space, and for each* $n \geq 2$, $\{u_{ij}^{(n)}; 1 \leq i < j \leq n\}$ *are independent random variables with mean zero and variance one, and* $\sup_{1\leq i<j\leq n,\, n\geq 2} E|u_{ij}^{(n)}|^{6+\delta} < \infty$ *for some* $\delta > 0$. *Then* $\limsup_{n\to\infty} \|\mathbf{U}_n\|/\sqrt{n} \leq 2$ *a.s.*

Proof We claim that it suffices to show

$$\limsup_{n\to\infty} \frac{\lambda_1(\mathbf{U}_n)}{\sqrt{n}} \leq 2 \;\; a.s. \tag{2.4}$$

In fact, once this is true, by applying it to $-\mathbf{U}_n$, we see that

$$\limsup_{n\to\infty} \frac{-\lambda_n(\mathbf{U}_n)}{\sqrt{n}} = \limsup_{n\to\infty} \frac{\lambda_1(-\mathbf{U}_n)}{\sqrt{n}} \leq 2 \;\; a.s.$$

Noticing $\|\mathbf{U}_n\| = \max\{\lambda_1(\mathbf{U}_n), -\lambda_n(\mathbf{U}_n)\}$, it follows that $\limsup_{n\to\infty} \|\mathbf{U}_n\|/\sqrt{n} \leq 2\;\; a.s.$

Now we prove (2.4). Define

$$\delta_n = \frac{1}{\log(n+1)}, \quad \tilde{u}_{ij}^{(n)} = u_{ij}^{(n)} I(|u_{ij}^{(n)}| \leq \delta_n\sqrt{n}) \text{ and } \tilde{\mathbf{U}}_n = (\tilde{u}_{ij}^{(n)})_{1\leq i,j\leq n}$$

for $1 \leq i \leq j \leq n$ and $n \geq 2$. By the Markov inequality,

$$\begin{aligned} P(\mathbf{U}_n \neq \tilde{\mathbf{U}}_n) &\leq P(|u_{ij}^{(n)}| > \delta_n\sqrt{n} \text{ for some } 1 \leq i < j \leq n) \\ &\leq n^2 \max_{1\leq i<j\leq n} P(|u_{ij}^{(n)}| > \delta_n\sqrt{n}) \leq \frac{K(\log(n+1))^{6+\delta}}{n^{1+(\delta/2)}} \end{aligned}$$

where $K := \sup_{1\leq i<j\leq n,\, n\geq 2} E|u_{ij}^{(n)}|^{6+\delta} < \infty$. Therefore, by the Borel–Cantelli lemma,

$$P(\mathbf{U}_n = \tilde{\mathbf{U}}_n \text{ for sufficiently large } n) = 1. \tag{2.5}$$

From $Eu_{ij}^{(n)} = 0$, we have that

$$|Eu_{ij}^{(n)} I(|u_{ij}^{(n)}| \leq \delta_n\sqrt{n})| = |Eu_{ij}^{(n)} I(|u_{ij}^{(n)}| > \delta_n\sqrt{n})| \leq \frac{K}{(\delta_n\sqrt{n})^{5+\delta}} \tag{2.6}$$

for any $1 \leq i \leq j \leq n$ and $n \geq 2$. Note that $\lambda_1(\mathbf{A}) \leq n \cdot \max_{1\leq i,j\leq n} |a_{ij}|$ for any $n \times n$ symmetric matrices $\mathbf{A} = (a_{ij})$. We have from (2.6) that

$$\begin{aligned} \lambda_1(\tilde{\mathbf{U}}_n) - \lambda_1(\tilde{\mathbf{U}}_n - E(\tilde{\mathbf{U}}_n)) &\leq \lambda_1(E\tilde{\mathbf{U}}_n) \\ &\leq n \max_{1\leq i<j\leq n} |Eu_{ij}^{(n)} I(|u_{ij}^{(n)}| \leq \delta_n\sqrt{n})| \leq \frac{K}{\delta_n^{5+\delta}(\sqrt{n})^{3+\delta}} \end{aligned}$$

for any $n \geq 2$. This and (2.5) indicate that

$$\limsup_{n\to\infty} \frac{\lambda_1(\mathbf{U}_n)}{\sqrt{n}} = \limsup_{n\to\infty} \frac{\lambda_1(\tilde{\mathbf{U}}_n)}{\sqrt{n}} \leq \limsup_{n\to\infty} \frac{\lambda_1(\tilde{\mathbf{U}}_n - E\tilde{\mathbf{U}}_n)}{\sqrt{n}}$$

almost surely. Note that $|\tilde{u}_{ij}^{(n)}| \leq |u_{ij}^{(n)}|$ and $Var(\tilde{u}_{ij}^{(n)}) \leq E(u_{ij}^{(n)})^2 = 1$, to save notation, without loss of generality, we will prove (2.4) by assuming that

$$E(u_{ij}^{(n)}) = 0,\ E(u_{ij}^{(n)})^2 \leq 1,\ |u_{ij}^{(n)}| \leq \frac{2\sqrt{n}}{\log(n+1)} \text{ and } \max_{1\leq i,j\leq n, n\geq 2} E|u_{ij}^{(n)}|^{6+\delta} < \infty$$

for all $1 \leq i, j \leq n$ and $n \geq 2$. Now, $\max_{i,j,n} E|u_{ij}^{(n)}|^3 \leq \max_{i,j,n}(E|u_{ij}^{(n)}|^{6+\delta})^{3/(6+\delta)} = K^{3/(6+\delta)}$ by the Hölder inequality. Write $E|u_{ij}^{(n)}|^l = E\left(|u_{ij}^{(n)}|^{l-3} \cdot |u_{ij}^{(n)}|^3\right)$. Then,

$$\max_{1\leq i,j\leq n} E|u_{ij}^{(n)}|^l \leq K^{3/(6+\delta)} \cdot \left(\frac{2\sqrt{n}}{\log(n+1)}\right)^{l-3} \tag{2.7}$$

for all $n \geq 2$ and $l \geq 3$, where K is a constant. By Lemma 2.1, we get (2.4). □

3 Order statistics of weakly dependent random variables

In this section, we will prove Proposition 1.2. First, we collect some facts for a preparation. The following is a Bonferroni-type inequality, see, e.g., pp. 4–5 from [22].

Lemma 3.1 *Let k and n be two integers with $n \geq k+1$. Then, for any events $A_1, \ldots, A_n$,*

$$\begin{aligned} &\left| P\left(\bigcup_{1\leq i_1<\cdots<i_k\leq n} A_{i_1}\cdots A_{i_k}\right) - \sum_{1\leq i_1<\cdots<i_k\leq n} P(A_{i_1}\cdots A_{i_k})\right| \\ &\quad \leq k \sum_{1\leq m_1<\cdots<m_{k+1}\leq n} P(A_{m_1}\cdots A_{m_{k+1}}). \end{aligned} \tag{3.1}$$

The next result, due to Sakhanenko, is called a strong approximation theorem, see Theorem 5 in Section 6 from [28] or Corollary 5 in Section 5 from [27]. It establishes a connection between the sum of independent random variables and the sum of independent Gaussian random variables.

Lemma 3.2 *Let $\{\xi_i; i = 1, 2, \ldots\}$ be a sequence of independent random variables with mean zero and variance σ_i^2. If $E|\xi_i|^p < \infty$ for some $p > 2$, then there exists a constant $C > 0$ and $\{\eta_i; i = 1, 2, \ldots\}$, a sequence of independent normally distributed random variables with $\eta_i \sim N(0, \sigma_i^2)$ such that*

$$P\left(\max_{1\leq k\leq n} |S_k - T_k| > x\right) \leq \frac{C}{1+|x|^p} \sum_{i=1}^n E|\xi_i|^p$$

for any n and $x > 0$, *where* $S_k = \sum_{i=1}^k \xi_i$ *and* $T_k = \sum_{i=1}^k \eta_i$.

Recall that the density function and the cumulative distribution function of $N(0,1)$ are $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$ and $\Phi(x) = \int_{-\infty}^x \phi(t)\,dt,\ x \in \mathbb{R}$, respectively.

Lemma 3.3 *Let* $U_1, U_2, \ldots, U_n$ *be i.i.d.* $N(0,1)$*-distributed r.v.'s with* $U_{(1)} \ge U_{(2)} \ge \cdots \ge U_{(n)}$ *as the order statistics. Then* $P(U_{(k)} \le t) \le n^k \Phi(t)^{n-k}$ *for any* $1 \le k \le n$ *and* $t \in \mathbb{R}$. *In particular, if* $\{k_n;\ n \ge 1\}$ *is a sequence of integers such that* $1 \le k_n \le n/2$ *for all* $n \ge 1$, *then* $P(U_{(k_n)} \le b\sqrt{\log n}) \le n^{k_n} \cdot \exp\{-n^{1-(b^2/2)}/\log n\}$ *for any* $b > 0$ *as n is sufficiently large.*

Proof It is known (see, e.g., p. 10 from [10]) that the density function of $U_{(k)}$ is

$$f_k(x) = \frac{n!}{(k-1)!(n-k)!}\Phi(x)^{n-k}(1-\Phi(x))^{k-1}\phi(x),\ x \in \mathbb{R},$$

for any $x \in \mathbb{R}$ and $1 \le k \le n$. Note that $n!/((k-1)!(n-k)!) \le n^k$ and $(1-\Phi(x))^{k-1} \le 1$ for any $1 \le k \le n$ and $x \in \mathbb{R}$. We have

$$\begin{aligned} P(U_{(k)} \le t) &\le n^k \int_{-\infty}^{t} \Phi(x)^{n-k}\phi(x)\,dx \\ &= n^k \int_0^{\Phi(t)} y^{n-k}\,dy \le n^k \Phi(t)^{n-k} \end{aligned}$$

for any $t \in \mathbb{R}$. The first part of the lemma is proved. Now, write $\Phi(t)^{n-k} = (1-(1-\Phi(t)))^{n-k}$. By the fact that $1 - x \le e^{-x}$ for $x \in \mathbb{R}$, we get from the above that

$$P(U_{(k)} \le t) \le n^k \cdot e^{-(n-k)(1-\Phi(t))}. \tag{3.2}$$

By using the fact that $1 - \Phi(t) \sim \frac{1}{\sqrt{2\pi}t}e^{-t^2/2}$ as $t \to +\infty$, the second part of the lemma follows. □

Lemma 3.4 *For each* $n \ge 1$, *let* $U_{n,1}, \ldots, U_{n,n}$ *be independent random variables with mean zero, variance one and* $S_n = U_{n,1} + \cdots + U_{n,n}$. *Suppose* $K := \sup_{1\le i\le n<\infty} E(|U_{n,i}|^p) < \infty$ *for some* $p > 2$. *Given* $t > 0$, *set* $\alpha = \min\{t^2/2, (p/2)-1\}$. *Then, for any* $\beta < \alpha$, *there exists a constant* $C_{\beta,t}$ *depending only on* β *and t such that*

$$\sup_{n\ge 2} n^\beta P\left(\frac{|S_n|}{\sqrt{n\log n}} \ge t\right) \le C_{\beta,t}K.$$

Proof By Lemma 3.2, for each $n \geq 1$, there exist i.i.d. $N(0,1)$-distributed random variables $\{\epsilon_{n,i};\ 1 \leq i \leq n\}$ such that for any $\delta > 0$,

$$P\left(\left|S_n - \sum_{i=1}^{n}\epsilon_{n,i}\right| \geq \delta\sqrt{n\log n}\right) \leq \frac{C}{1+(\delta\sqrt{n\log n})^p}\sum_{i=1}^{n}E|U_{n,i}|^p$$
$$\leq \frac{CK}{\delta^p(\log n)^{p/2}}\cdot\frac{1}{n^{p/2-1}}$$

as $n \geq 2$, where C is a universal constant. Since $|S_n| \leq \left|\sum_{i=1}^{n}\epsilon_{n,i}\right| + \left|S_n - \sum_{i=1}^{n}\epsilon_{n,i}\right|$, noticing $\sum_{i=1}^{n}\epsilon_{n,i} \sim \sqrt{n}\cdot N(0,1)$, we get

$$\begin{aligned} &P(|S_n| \geq t\sqrt{n\log n}) \\ &\leq P\left(\left|S_n - \sum_{i=1}^{n}\epsilon_{n,i}\right| \geq \delta\sqrt{n\log n}\right) + P\left(\left|\sum_{i=1}^{n}\epsilon_{n,i}\right| \geq (t-\delta)\sqrt{n\log n}\right) \\ &\leq \frac{CK}{\delta^p(\log n)^{p/2}}\cdot\frac{1}{n^{p/2-1}} + P(|N(0,1)| \geq (t-\delta)\sqrt{\log n}) \end{aligned} \tag{3.3}$$

as $n \geq 2$. It is well known that

$$\frac{x}{\sqrt{2\pi}(1+x^2)}e^{-x^2/2} \leq P(N(0,1) \geq x) \leq \frac{1}{\sqrt{2\pi}x}e^{-x^2/2} \tag{3.4}$$

for any $x > 0$. Therefore, for any $0 < \delta < t$,

$$P(|N(0,1)| \geq (t-\delta)\sqrt{\log n}) \leq \frac{1}{(t-\delta)\sqrt{\log n}}\cdot\frac{1}{n^{(t-\delta)^2/2}}$$

as $n \geq 2$. This together with (3.3) yields that

$$P(|S_n| \geq t\sqrt{n\log n}) \leq \frac{CK}{\delta^p(\log n)^{p/2}}\cdot\frac{1}{n^{p/2-1}} + \frac{1}{(t-\delta)\sqrt{\log n}}\cdot\frac{1}{n^{(t-\delta)^2/2}}$$

for any $\delta \in (0,t)$ and $n \geq 2$. Since $(t-\delta)^2/2 \to t^2/2$ as $\delta \to 0^+$, for $\beta < \alpha \leq t^2/2$, choose $\delta_0 > 0$ such that $(t-\delta_0)^2/2 > \beta$. Then

$$n^\beta P(|S_n| \geq t\sqrt{n\log n}) \leq \frac{CK}{\delta_0^p(\log n)^{p/2}} + \frac{1}{(t-\delta_0)\sqrt{\log n}}$$

for all $n \geq 2$. The conclusion then follows. □

Now we introduce a method that will be used in later proofs. Let $\{\eta_{ij}^{(n)}\}$ and $X_{n,(i)}$ be as in (1.2). Set $m_n = [n/\log n]$ for $n \geq 3$,

$$V_n = \max_{1\leq i\leq n}\left|\sum_{j=1}^{m_n}\eta_{ij}^{(n)}\right| \quad \text{and} \quad Y_{n,i} = \sum_{j=m_n+1}^{n}\eta_{ij}^{(n)}, \quad 1 \leq i \leq n. \tag{3.5}$$

Obviously, $\max_{1\le i\le n}|X_{n,i}-Y_{n,i}|\le V_n$ for $n\ge 2$. Applying (iii) of Lemma 2.2 to the two diagonal matrices: $\mathrm{diag}(X_{n,i})_{1\le i\le n}$ and $\mathrm{diag}(Y_{n,i})_{1\le i\le n}$, we obtain that

$$\max_{1\le i\le n}|X_{n,(i)}-Y_{n,(i)}|\le V_n \tag{3.6}$$

for $n\ge 2$. Let $Z_{n,(1)}\ge Z_{n,(2)}\ge\cdots\ge Z_{n,(m_n)}$ be the order statistics of $Y_{n,i},\ 1\le i\le m_n$, which is a small portion of $Y_{n,i},\ 1\le i\le n$. Thus, $Y_{n,(i)}\ge Z_{n,(i)}$ for $i=1,2,\ldots,m_n$. This together with (3.6) yields a useful inequality:

$$Z_{n,(k)}-V_n\le X_{n,(k)}\le \max_{1\le j\le n}\{X_{n,j}\} \tag{3.7}$$

for any $1\le k\le m_n$. The idea of connecting $X_{n,(k)}$ to $Z_{n,(k)}$ is that $\{Z_{n,(i)}\}_{i=1}^{m_n}$ are order statistics of independent random variables $\{Y_{n,i},\ 1\le i\le m_n\}$, and V_n is negligible. We next give some tail probabilities of V_n and $Z_{n,(k)}$.

Lemma 3.5 *Suppose Assumption A_6 holds. Let $\{\eta_{ij}^{(n)}\}$ and $X_{n,(i)}$ be as in (1.2), and $\{k_n;\,n\ge 1\}$ be positive integers such that $\log k_n=o(\log n)$ as $n\to\infty$. Then, for any $\epsilon\in(0,1/2)$, there exists $\gamma=\gamma(\epsilon)>1$ not depending on n such that*

$$P\left(\frac{V_n}{\sqrt{n\log n}}\ge\epsilon\right)\le\frac{1}{n^{\gamma}}\quad\text{and}\quad P\left(\frac{Z_{n,(k_n)}}{\sqrt{n\log n}}\le\sqrt{2}-2\epsilon\right)=O\left(\frac{1}{n(\log n)^{3.5}}\right) \tag{3.8}$$

as n is sufficiently large.

Proof Note that $\alpha=\min\{p/2-1,10^2/2\}>2$ since $p>6$. By Lemma 3.4, for any $\beta\in(2,\alpha)$, notifying $\eta_{i,i}^{(n)}=0$ for each $1\le i\le n$,

$$\begin{aligned}P\left(\frac{V_n}{\sqrt{n\log n}}\ge\epsilon\right)&\le n\cdot\max_{1\le i\le n}P\left(\left|\sum_{j=1}^{m_n}\eta_{ij}^{(n)}\right|\ge 10\sqrt{m_n\log m_n}\right)\\&\le\frac{n}{(m_n-1)^{\beta}}\le\frac{1}{n^{\gamma}}\end{aligned} \tag{3.9}$$

as n is large enough because $\beta-1>\beta/2:=\gamma>1$.

Now we prove the second assertion in (3.8). By Lemma 3.2, for each $1\le i\le m_n$ and $n\ge 2$, there exist i.i.d. $N(0,1)$-distributed random variables $\{\zeta_{ij}^{(n)};\ 1\le i\le n,\ m_n<j\le n\}$ such that, for any $\epsilon>0$,

$$\begin{aligned}P\left(\left|\sum_{j=m_n+1}^{n}\eta_{ij}^{(n)}-\sum_{j=m_n+1}^{n}\zeta_{ij}^{(n)}\right|\ge\epsilon\sqrt{n\log n}\right)&\le\frac{C}{1+(\epsilon\sqrt{n\log n})^6}\sum_{j=m_n+1}^{n}E|\eta_{ij}^{(n)}|^6\\&\le\frac{1}{n^2(\log n)^{2.5}}\end{aligned} \tag{3.10}$$

uniformly for any $1 \le i \le m_n$ as n is sufficiently large. Define $W_{n,i} = \sum_{j=m_n+1}^{n} \zeta_{ij}^{(n)}$ for $1 \le i \le m_n$. Then $\{W_{n,i};\ 1 \le i \le m_n\}$ are i.i.d. $N(0, n-m_n)$-distributed random variables. Recalling that $Z_{n,(1)} \ge Z_{n,(2)} \ge \cdots \ge Z_{n,(m_n)}$ are the order statistics of $Y_{n,i},\ 1 \le i \le m_n$, similar to (3.6), we know that $\max_{1\le i\le m_n} |Z_{n,(i)} - W_{n,(i)}| \le \max_{1\le i\le m_n} |\sum_{j=m_n+1}^{n} \eta_{ij}^{(n)} - \sum_{j=m_n+1}^{n} \zeta_{ij}^{(n)}|$. Thus, by (3.10), for any $\epsilon > 0$,

$$\begin{aligned} &P\left(\max_{1\le i\le m_n} |Z_{n,(i)} - W_{n,(i)}| \ge \epsilon\sqrt{n\log n}\right) \\ &\quad\le P\left(\max_{1\le i\le m_n}\left|\sum_{j=m_n+1}^{n} \eta_{ij}^{(n)} - \sum_{j=m_n+1}^{n} \zeta_{ij}^{(n)}\right| \ge \epsilon\sqrt{n\log n}\right) \\ &\quad\le \frac{m_n}{n^2(\log n)^{2.5}} \le \frac{1}{n(\log n)^{3.5}} \end{aligned} \tag{3.11}$$

as n is sufficiently large. Fix $\beta > 0$ and $\epsilon \in (0, 2\beta)$. Use the fact $\sqrt{n}/\sqrt{n-m_n} \to 1$ as $n \to \infty$, we see that

$$\begin{aligned} &P(Z_{n,(k)} \le (\beta - 2\epsilon)\sqrt{n\log n}) \\ &\quad\le P(W_{n,(k)} \le (\beta - \epsilon)\sqrt{n\log n}) + P(|Z_{n,(k)} - W_{n,(k)}| \ge \epsilon\sqrt{n\log n}) \\ &\quad\le P(U_{(k)} \le (\beta - (\epsilon/2))\sqrt{\log n}) + \frac{1}{n(\log n)^{3.5}} \end{aligned} \tag{3.12}$$

uniformly for all $1 \le k \le m_n$ as n is sufficiently large, where $U_{(k)}$ is as in Lemma 3.3. By the same lemma, we have $P(U_{(k_n)} \le (\beta - (\epsilon/2))\sqrt{\log n}) \le n^{k_n} \exp\{-n^{1-(\beta-2^{-1}\epsilon)^2/2}/\log n\}$ as n is sufficiently large. Therefore,

$$P(Z_{n,(k_n)} \le (\beta - 2\epsilon)\sqrt{n\log n}) \le n^{k_n} \exp\left\{-n^{1-(\beta-2^{-1}\epsilon)^2/2}/\log n\right\} + \frac{1}{n(\log n)^{3.5}} \tag{3.13}$$

as n is sufficiently large. Now, take $\beta = \sqrt{2}$, then $2\rho := 1 - (\beta - 2^{-1}\epsilon)^2/2 > 0$ for $\epsilon \in (0, \sqrt{2})$. By the condition on k_n, the first term on the right hand side of (3.13) is bounded by $e^{-n^{\rho}}$ as n is sufficiently large. Therefore,

$$P\left(\frac{Z_{n,(k_n)}}{\sqrt{n\log n}} \le \sqrt{2} - 2\epsilon\right) = O\left(\frac{1}{n(\log n)^{3.5}}\right) \tag{3.14}$$

as $n \to \infty$ for all $0 < \epsilon < \sqrt{2}/2$. □

Lemma 3.6 *Let $\{\eta_{ij}^{(n)}\}$ and $\{X_{n,(i)}\}$ be as in (1.2). Fix integer $k \ge 1$ and $a > \sqrt{2}$. Suppose Assumption A_{2ka^2} holds. Then,*

$$\lim_{n\to\infty} \frac{1}{\log n} \cdot \log P\left(\frac{X_{n,(k)}}{\sqrt{n\log n}} \ge a\right) = -k\left(\frac{a^2}{2} - 1\right).$$

Proof Let $a_n = a\sqrt{n \log n}$, $n \geq 2$. Observe that

$$P(X_{n,(k)} \geq a_n) = P\left(\bigcup_{1\leq l_1<\cdots<l_k\leq n} \{X_{n,l_1} \geq a_n, \ldots, X_{n,l_k} \geq a_n\}\right)$$

for $1 \leq k \leq n$. Define

$$\alpha_{n,j} = \sum_{1\leq l_1<\cdots<l_j\leq n} P(X_{n,l_1} \geq a_n, \ldots, X_{n,l_j} \geq a_n) \tag{3.15}$$

for $j = k, k+1, \ldots, n$ and $n \geq 2$. By Lemma 3.1,

$$|P(X_{n,(k)} \geq a_n) - \alpha_{n,k}| \leq k\alpha_{n,k+1}. \tag{3.16}$$

Step 1. Now we study $\{\alpha_{n,j}\}$. Remember $X_{n,i} = \sum_{j=1}^{n} \eta_{ij}^{(n)}$ with $\eta_{ii}^{(n)} = 0$ for all $1 \leq i \leq n$. Similar to (3.5), set

$$V_n = \max_{1\leq i\leq 2k} \left|\sum_{j=1}^{2k} \eta_{ij}^{(n)}\right| \quad \text{and} \quad Y_{n,i} = \sum_{j=2k+1}^{n} \eta_{ij}^{(n)}$$

for $1 \leq i \leq n$. A key observation is that $Y_{n,1} \ldots, Y_{n,2k}$ are independent. Moreover,

$$\max_{1\leq i\leq 2k} |X_{n,i} - Y_{n,i}| \leq V_n \tag{3.17}$$

for all $n \geq 2$. Evidently, by a convex inequality,

$$\max_{1\leq i\leq 2k} E\left|\sum_{j=1}^{2k} \eta_{ij}^{(n)}\right|^p \leq (2k)^p \cdot \sup_{1\leq i<j\leq n} E|\eta_{ij}^{(n)}|^p := C_k < \infty. \tag{3.18}$$

So by the Markov inequality, for any $\epsilon > 0$, there exists a constant $K_1 = K_1(\epsilon) > 0$ satisfying

$$P(V_n \geq \epsilon\sqrt{n \log n}) \leq 2k \cdot \max_{1\leq i\leq 2k} P\left(\left|\sum_{j=1}^{2k} \eta_{ij}^{(n)}\right| \geq \epsilon\sqrt{n \log n}\right) \leq \frac{2kC_k\epsilon^{-p}}{(n \log n)^{p/2}} \leq \frac{1}{n^{p/2}} \tag{3.19}$$

for all $n \geq K_1$. Now, by (3.17) and then independence, we see that, uniformly for all $k \leq j \leq 2k$,

$$\begin{aligned} P(X_{n,1} \geq a_n, \ldots, X_{n,j} \geq a_n) &\leq P(V_n \geq \epsilon\sqrt{n \log n}) + P(Y_{n,1} \geq b_n, \ldots, Y_{n,j} \geq b_n) \\ &\leq \max_{1\leq i\leq j} P(Y_{n,i} \geq b_n)^j + \frac{1}{n^{p/2}} \end{aligned} \tag{3.20}$$

as $n \geq K_1$, where $b_n = (a-\epsilon)\sqrt{n\log n}$ for $\epsilon \in (0, a)$. Similarly,

$$\begin{aligned}&P(Y_{n,1} \geq \beta\sqrt{n\log n}, \dots, Y_{n,j} \geq \beta\sqrt{n\log n})\\&\quad\leq P(V_n \geq \epsilon\sqrt{n\log n}) + P(X_{n,1} \geq (\beta-\epsilon)\sqrt{n\log n}, \dots, X_{n,j} \geq (\beta-\epsilon)\sqrt{n\log n})\end{aligned}$$

for any $0 < \epsilon < \beta$. This leads to that, uniformly for all $k \leq j \leq 2k$,

$$P(X_{n,1} \geq a_n, \dots, X_{n,j} \geq a_n) \geq \min_{1\leq i\leq j} P(Y_{n,i} \geq c_n)^j - \frac{1}{n^{p/2}} \tag{3.21}$$

as $n \geq K_1$, where $c_n = (a+\epsilon)\sqrt{n\log n}$. From (3.20) and (3.21), to estimate $P(X_{n,1} \geq a_n, \dots, X_{n,j} \geq a_n)$, it is enough to study $P(Y_{n,i} \geq t\sqrt{n\log n})$ for any $t > 0$.

Step 2. Now we estimate $P(Y_{n,i} \geq t\sqrt{n\log n})$. By Lemma 3.2, for each $i = 1, 2, \dots, 2k$ and $n \geq 2$, there exist i.i.d. $N(0,1)$-distributed random variables $\{\epsilon_{ij}^{(n)};\ 2k+1 \leq j \leq n\}$ and an universal constant $C > 0$ such that

$$\begin{aligned}P\left(\left|Y_{n,i} - \sum_{j=2k+1}^{n} \epsilon_{ij}^{(n)}\right| \geq \epsilon\sqrt{n\log n}\right) &\leq \frac{C}{1+(\epsilon\sqrt{n\log n})^p} \sum_{j=2k+1}^{n} E|\eta_{ij}^{(n)}|^p\\&\leq \frac{C'}{n^{(p/2)-1}(\log n)^{p/2}}\end{aligned} \tag{3.22}$$

uniformly for $i = 1, 2, \dots, 2k$, where $C' = CC_k\epsilon^{-p}$ and C_k is as in (3.18). It is easy to see that

$$P(\eta \geq x+2y) - P(|\xi-\eta| \geq y) \leq P(\xi \geq x+y) \leq P(\eta \geq x) + P(|\xi-\eta| \geq y) \tag{3.23}$$

for any random variables ξ and η, and constants $x > 0$ and $y > 0$. Thus, for any $0 < \epsilon < t$, by (3.22),

$$\begin{aligned}&P\left(\sum_{j=2k+1}^{n} \epsilon_{ij}^{(n)} \geq (t+\epsilon)\sqrt{n\log n}\right) - \frac{C'}{n^{(p/2)-1}(\log n)^{p/2}}\\&\quad\leq P(Y_{n,i} \geq t\sqrt{n\log n}) \leq P\left(\sum_{j=2k+1}^{n} \epsilon_{ij}^{(n)} \geq (t-\epsilon)\sqrt{n\log n}\right)\\&\qquad+\frac{C'}{n^{(p/2)-1}(\log n)^{p/2}}\end{aligned}$$

for all $1 \leq i \leq 2k$. Clearly, $\sum_{j=2k+1}^{n} \epsilon_{ij}^{(n)} \sim \sqrt{n-2k}\cdot N(0,1)$ for each $1 \leq i \leq 2k$. By using (3.4), we see that

$$P\left(\sum_{j=2k+1}^{n} \epsilon_{ij}^{(n)} \geq (t-\epsilon)\sqrt{n\log n}\right) \leq P(N(0,1) \geq (t-\epsilon)\sqrt{\log n}) \leq \frac{1}{n^{(t-\epsilon)^2/2}}$$

and

$$P\left(\sum_{j=2k+1}^{n}\epsilon_{ij}^{(n)}\geq(t+\epsilon)\sqrt{n\log n}\right)\geq P(N(0,1)\geq(t+1.5\epsilon)\sqrt{\log n})\geq\frac{1}{n^{(t+2\epsilon)^2/2}}$$

for all $1\leq i\leq 2k$ as $n\geq K_2$ for some constant $K_2:=K_2(k,\epsilon,t)\geq K_1$. The last three inequalities give

$$\begin{aligned}\frac{1}{n^{(t+2\epsilon)^2/2}}-\frac{C'}{n^{(p/2)-1}(\log n)^{p/2}}&\leq P(Y_{n,i}\geq t\sqrt{n\log n})\\&\leq\frac{1}{n^{(t-\epsilon)^2/2}}+\frac{C'}{n^{(p/2)-1}(\log n)^{p/2}}\end{aligned}$$

uniformly for all $1\leq i\leq 2k$ and $t>\epsilon>0$ as $n\geq K_2$. Taking $t=a-\epsilon$ and $a+\epsilon$ respectively, and noticing $(p/2)-1>a^2/2$ from the condition $p>2ka^2$, we have that, uniformly for all $1\leq i\leq 2k$,

$$\frac{1}{n^{(a+\delta)^2/2}}\leq P(Y_{n,i}\geq c_n)\leq P(Y_{n,i}\geq b_n)\leq\frac{1}{n^{(a-\delta)^2/2}}\tag{3.24}$$

as $n\geq K_3:=K_3(k,\epsilon,a)$ and as $\delta=\delta(\epsilon)>0$ is small enough.

Step 3. Now we make a summary. Based on the assumption $p>2ka^2$, we know that $j(a-\delta)^2/2<j(a+\delta)^2/2<p/2$ for all $k\leq j\leq 2k$ as δ is small enough. Recalling (3.20) and (3.21), we have from (3.24) that, for $l_1=1,l_2=2,\ldots,l_j=j$, uniformly for all $k\leq j\leq 2k$,

$$\frac{1}{n^{j(a+\delta)^2/2}}\leq P(X_{n,l_1}\geq a_n,\ldots,X_{n,l_j}\geq a_n)\leq\frac{1}{n^{j(a-\delta)^2/2}}\tag{3.25}$$

as $n\geq K_3$ and as $\delta=\delta(a,p)$ small enough.

Fix $1\leq l_1<l_2<\cdots<l_{2k}\leq n$. Recall $X_{n,i}=\sum_{j=1}^{n}\eta_{ij}^{(n)}$, $\eta_{ii}^{(n)}=0$ and $\eta_{ij}^{(n)}=\eta_{ji}^{(n)}$ for all $1\leq i,j\leq n$. It is not difficult to see that all possible common $\eta_{ij}^{(n)}$'s in the expression of $X_{n,i}$ for $i=l_1,l_2,\ldots,l_{2k}$ are $\{\eta_{l_i l_j}^{(n)};\ 1\leq i<j\leq 2k\}$. Write $X_{n,l_i}=\sum_{j=1}^{2k}\eta_{l_i l_j}^{(n)}+\tilde{Y}_{n,i}$ for $i=1,2,\ldots,2k$, then $\tilde{Y}_{n,1},\ldots,\tilde{Y}_{n,2k}$ are independent, and each of which is a sum of $n-2k$ independent random variables with mean zero and variance one. With this and reviewing the whole process of getting (3.25), we find that (3.25) holds uniformly for all indices $1\leq l_1<\cdots<l_j\leq n$ and $k\leq j\leq 2k$. Since $\binom{n}{j}\sim n^j/j!$ as $n\to\infty$ for any j, relating to (3.15), we obtain that

$$\frac{1}{n^{j((a+\delta)^2/2-1)}}\leq j!\cdot\alpha_{n,j}\leq\frac{1}{n^{j((a-\delta)^2/2-1)}},\quad j=k,k+1,\ldots,2k$$

as n is sufficiently large and as $\delta>0$ is small enough. Taking $\delta>0$ small enough, since $a>\sqrt{2}$, one can see that, for n sufficiently large, $\alpha_{n,j}=o(\alpha_{n,k})$ for all $j\geq k+1$

as $n \to \infty$. Noting that $1/n^{ka_1} < 1/n^{k(a^2/2-1)} < 1/n^{ka_2}$ as $a_1 > a^2/2 - 1 > a_2$, then by combining (3.16) and the above assertion, we obtain

$$\frac{1}{n^{ka_1}} \leq j! \cdot P\left(\frac{X_{n,(k)}}{\sqrt{n \log n}} \geq a\right) \leq \frac{1}{n^{ka_2}}$$

as n is sufficiently large. The desired conclusion follows by taking the logarithm for the three terms above and then letting $a_1 \downarrow a^2/2 - 1$ and $a_2 \uparrow a^2/2 - 1$. □

Proof of Proposition 1.2 (i) First, observe that $X_{n,j}$ is a sum of $n-1$ random variables with mean zero and variance one. Second, $\alpha := \min\{(\sqrt{2}+\epsilon)^2/2,\ p/2-1\} > 1+\sqrt{2}\epsilon := \beta$ for any $\epsilon \in (0, 1)$ since $p > 6$. Then by Lemma 3.4, we obtain

$$\begin{aligned} P\left(\frac{X_{n,(k)}}{\sqrt{n \log n}} \geq \sqrt{2}+\epsilon\right) &\leq P\left(\frac{\max_{1\leq j\leq n}\{X_{n,j}\}}{\sqrt{n \log n}} \geq \sqrt{2}+\epsilon\right) \\ &\leq n \cdot \max_{1\leq j\leq n} P\left(\frac{|X_{n,j}|}{\sqrt{n \log n}} \geq \sqrt{2}+\epsilon\right) \\ &\leq \frac{n}{(n-1)^{1+\sqrt{2}\epsilon}} \leq \frac{1}{n^{\epsilon}} \end{aligned}$$

as n is large enough. On the other hand, taking $k_n \equiv k$ in Lemma 3.5, we have from (3.7) and (3.8) that

$$\begin{aligned} P\left(\frac{X_{n,(k)}}{\sqrt{n \log n}} \leq \sqrt{2}-3\epsilon\right) &\leq P\left(\frac{V_n}{\sqrt{n \log n}} \geq \epsilon\right) + P\left(\frac{Z_{n,(k)}}{\sqrt{n \log n}} \leq \sqrt{2}-2\epsilon\right) \\ &= O\left(\frac{1}{n(\log n)^2}\right) \end{aligned}$$

as $n \to \infty$ for sufficiently small ϵ. The above two inequalities imply (i).

(ii) Fix integer $k \geq 1$. For any $\epsilon > 0$, let $t = \sqrt{2+2k^{-1}}+\epsilon$, then $k(t^2/2-1) > 1+\sqrt{2}\epsilon$. By Lemma 3.6, $P(X_{n,(k_n)} \geq t\sqrt{n \log n}) \leq P(X_{n,(k)} \geq t\sqrt{n \log n}) \leq 1/n^{1+\epsilon}$ as n is sufficiently large. It follows that $\sum_{n\geq 2} P(X_{n,(k_n)} \geq t\sqrt{n \log n}) < \infty$ for any $\epsilon > 0$. This implies that $\limsup_{n\to\infty} X_{n,(k_n)}/\sqrt{n \log n} \leq \sqrt{2+2k^{-1}}$ a.s. for any $k \geq 1$. Let $k \to +\infty$, we get

$$\limsup_{n\to\infty} \frac{X_{n,(k_n)}}{\sqrt{n \log n}} \leq \sqrt{2}\ \ a.s. \tag{3.26}$$

To complete the proof, it suffices to show

$$\liminf_{n\to\infty} \frac{X_{n,(k_n)}}{\sqrt{n \log n}} \geq \sqrt{2}\ \ a.s. \tag{3.27}$$

Recalling (3.7) and Lemma 3.5, for any $\epsilon > 0$, there exits $\gamma > 1$ such that $P\left(V_n/\sqrt{n \log n} \geq \epsilon\right) \leq n^{-\gamma}$ as n is sufficiently large, then by the Borel–Cantelli lemma that $\lim_{n\to\infty} V_n/\sqrt{n\log n} = 0$ a.s. Looking at (3.7) again, to finish the proof, it is enough to prove

$$\liminf_{n\to\infty} \frac{Z_{n,(k_n)}}{\sqrt{n\log n}} \geq \sqrt{2}\ \ a.s. \tag{3.28}$$

Clearly, the condition $\log k_n = o(\log n)$ implies that $k_n = o(m_n)$ as $n \to \infty$. By Lemma 3.5,

$$P(Z_{n,(k_n)} \leq (\sqrt{2}-2\epsilon)\sqrt{n\log n}) = O\left(\frac{1}{n(\log n)^{3.5}}\right) \tag{3.29}$$

as n is large enough and $\epsilon > 0$ is small enough, it follows that $\sum_{n\geq 2} P(Z_{n,(k_n)} \leq (\sqrt{2}-2\epsilon)\sqrt{n\log n}) < \infty$ for $\epsilon > 0$ small enough. Therefore (3.28) is yielded by the Borel–Cantelli lemma.

(iii) The given assumptions say that

$$\begin{aligned} &\{\eta_{ij}^{(n)};\ 1 \leq i < j \leq n,\ n \geq 2\} \text{ are independent random variables,}\\ &\eta_{ii}^{(n)} = 0,\quad E\eta_{ij}^{(n)} = 0,\quad E(\eta_{ij}^{(n)})^2 = 1 \text{ and } \sup_{1\leq i<j\leq n\, n\geq 2} E|\eta_{ij}^{(n)}|^p < \infty \end{aligned} \tag{3.30}$$

for all $1 \leq i < j \leq n$ and $p > 4k+4$. By the condition $p > 4k+4 \geq 6$ and Lemma 3.5, for any $\epsilon > 0$, there exists $\gamma > 1$ such that $P(V_n \geq \epsilon\sqrt{n\log n}) \leq n^{-\gamma}$ as n is large enough. Consequently, $\sum_{n\geq 2} P(V_n \geq \epsilon\sqrt{n\log n}) < \infty$ for any $\epsilon > 0$. By the Borel–Cantelli lemma, $V_n/\sqrt{n\log n} \to 0$ almost surely as $n \to \infty$. Thus, recalling (3.7), to prove the lemma, it suffices to show that

$$\limsup_{n\to\infty} \frac{X_{n,(k)}}{\sqrt{n\log n}} \leq \sqrt{2+2k^{-1}}\ a.s. \text{ and } \liminf_{n\to\infty} \frac{Z_{n,(k)}}{\sqrt{n\log n}} \geq \sqrt{2}\, a.s. \text{ and} \tag{3.31}$$

$$\sum_{n=2}^{\infty} P\left(\frac{X_{n,(k)}}{\sqrt{n\log n}} \in [a,b)\right) = \infty \tag{3.32}$$

for any $[a,b) \subset (\sqrt{2}, \sqrt{2+2k^{-1}})$. In fact, since $\{X_{n,(k)};\ n \geq 2\}$ are independent, by the *second* Borel–Cantelli lemma, (3.32) implies that, with probability one, $\frac{X_{n,(k)}}{\sqrt{n\log n}} \in [a,b)$ for infinitely many n's.

Since $p > 4k+4$, $\sqrt{p/2k} > \sqrt{2+2k^{-1}}$. For any $0 < \epsilon < \sqrt{p/2k} - \sqrt{2+2k^{-1}}$, let $a = \sqrt{2+2k^{-1}} + \epsilon$. Then $p > 2ka^2$ and $k(a^2/2-1) > 1+\sqrt{2}\epsilon$. By Lemma 3.6, $P(X_{n,(k)} \geq a\sqrt{n\log n}) \leq n^{-1-\epsilon}$ as n is sufficiently large. It follows that $\sum_{n\geq 2} P(X_{n,(k)} \geq a\sqrt{n\log n}) < \infty$ for any $\epsilon > 0$. Thus, the first inequality in (3.31) follows from the Borel–Cantelli lemma.

From (3.14) we see that $\sum_{n\geq 2} P(Z_{n,(k)} \leq (\sqrt{2}-2\epsilon)\sqrt{n\log n}) < \infty$ for any $\epsilon > 0$ small enough. By the Borel–Cantelli again, the second inequality in (3.31) is obtained. Now we prove (3.32).

Given $[a, b) \subset (\sqrt{2}, \sqrt{2+2k^{-1}})$. Trivially, $k(2^{-1}a^2 - 1) \in (0, 1)$. Choose $\epsilon > 0$ such that $\epsilon < \min\{1 - k(2^{-1}a^2 - 1),\ k(b^2 - a^2)/4\}$. Then

$$0 < k\left(\frac{a^2}{2} - 1\right) + \epsilon < 1 \quad \text{and} \quad k\left(\frac{a^2}{2} - 1\right) + \epsilon < k\left(\frac{b^2}{2} - 1\right) - \epsilon. \tag{3.33}$$

By Lemma 3.6,

$$P\left(\frac{X_{n,(k)}}{\sqrt{n \log n}} \geq a\right) \geq \frac{1}{n^{k(a^2/2-1)+\epsilon}} \quad \text{and} \quad P\left(\frac{X_{n,(k)}}{\sqrt{n \log n}} \geq b\right) \leq \frac{1}{n^{k(b^2/2-1)-\epsilon}}$$

as n is sufficiently large. Therefore, from the second inequality in (3.33),

$$\begin{aligned} P\left(\frac{X_{n,(k)}}{\sqrt{n \log n}} \in [a, b)\right) &= P\left(\frac{X_{n,(k)}}{\sqrt{n \log n}} \geq a\right) - P\left(\frac{X_{n,(k)}}{\sqrt{n \log n}} \geq b\right) \\ &\geq \frac{1}{2n^{k(a^2/2-1)+\epsilon}} \end{aligned}$$

as n is sufficiently large. Then (3.32) is proved by using the first inequality in (3.33). □

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