CSS322: Scientific Computing
Midterm Mock Exam

curated by The Peanuts

Name. Peanuts12345 . .ID. . . . . .Section. . . . . .Seat No. . . . .

Conditions: Semi-closed Book

Directions:

1. This exam has 11 pages (including this page).
2. You may bring **two A4 cheat sheets**, written/printed/photocopied on both sides. They must be submitted with your exam paper. Treasure them well, for they will not return.
3. Calculators are allowed. Dictionaries are not.
4. Cheating is strictly prohibited.
5. For inspiration you may look at the ceiling, but not at your friend
6. Good luck!

For solution, ***click here***.
Will be available soon, probably the night before the real exam.

Question 1

(a) Let $v = \begin{bmatrix} -3 \\ 2 \\ 0 \\ -5 \end{bmatrix}$. Compute $\|v\|_1$, $\|v\|_2$, and $\|v\|_\infty$.

$$\|v\|_1 = |-3| + |2| + |0| + |-5| = 10$$

$$\|v\|_2 = \sqrt{(-3)^2 + 2^2 + (-5)^2} = \sqrt{9+4+25} = \sqrt{38}$$

$$\|v\|_\infty = \max\{|-3|, |2|, |0|, |-5|\} = 5$$

(b) Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 4 & -2 \\ -1 & 0 & 3 \end{bmatrix}$. Compute $\|A\|_F$, $\|A\|_1$, and $\|A\|_\infty$.

$$\|A\|_F = \sqrt{2^2 + (-3)^2 + 1^2 + 0^2 + 4^2 + (-2)^2 + (-1)^2 + 0^2 + 3^2} = \sqrt{44} = 2\sqrt{11}$$

$$\|A\|_1 = \max\{\overset{3}{|2| + |0| + |-1|}, \overset{7}{|-3| + |4| + |0|}, \overset{6}{|1| + |-2| + |3|}\} = 7$$

$$\|A\|_\infty = \max\{\overset{6}{|2| + |-3| + |1|}, \overset{6}{|0| + |4| + |-2|}, \overset{4}{|-1| + |0| + |3|}\} = 6$$

Question 2

Perform Gaussian Elimination with Partial Pivoting (GEPP) on the matrix

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 2 & 1 \\ 2 & -1 & 3 \end{bmatrix}$$

to find its $P^T LU$ factorization.

Step 1

$$A \leftarrow \begin{bmatrix} 1 & 4 & 2 \\ 3 & 2 & 1 \\ 2 & -1 & 3 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Column 1: Swap Row 1 and Row 2 of A and P (since 3 is max)

$$A \leftarrow \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 2 \\ 2 & -1 & 3 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eliminate Column 1 : Subtract $\frac{1}{3}$ of Row 1 from Row 2

Subtract $\frac{2}{3}$ of Row 1 from Row 3

$$A \leftarrow \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3.33 & 1.67 \\ 0 & -2.33 & 2.33 \end{bmatrix} \quad L \leftarrow \begin{bmatrix} 1 \\ 1/3 \\ 2/3 \end{bmatrix}$$

Eliminate column 2 : subtract $\frac{-2.33}{3.33}$ of Row 2 from Row 3

$$A \leftarrow \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3.33 & 1.67 \\ 0 & 0 & 3.5 \end{bmatrix} = U \quad \# \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & \frac{-2.33}{3.33} & 1 \end{bmatrix} \quad \#$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \#$$

$$[0 \;\; -2.33 \;\; 2.33] - \frac{-2.33}{3.33}[0 \;\; 3.33 \;\; 1.67]$$

$$\parallel$$

$$[0 \;\; 0 \;\; 3.5]$$

Question 3

We are solving the linear system

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} x = \begin{bmatrix} 6 \\ 5 \end{bmatrix} \qquad Ax = b$$

$\overset{b}{6} = b$

and have computed an approximate solution

$$x_0 = \begin{bmatrix} 1.1 \\ 1.9 \end{bmatrix}.$$

Perform one iteration of iterative refinement to obtain a better approximate solution x_1.

Step 1 : $r_0 = b - Ax_0 = \begin{bmatrix} 6 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1.1 \\ 1.9 \end{bmatrix} = \begin{bmatrix} 4.4+1.9 \\ 1.1+5.7 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix} - \begin{bmatrix} 6.3 \\ 6.8 \end{bmatrix} = \begin{bmatrix} -0.3 \\ -1.8 \end{bmatrix}$

$2 \times 2 \quad 2 \times 1$

Step 2 : $As_0 = r_0$

$$\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} s_0 = \begin{bmatrix} -0.3 \\ -1.8 \end{bmatrix}$$

$$s_0 = A^{-1} r_0$$

$$= \frac{1}{11}\begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} -0.3 \\ -1.8 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -0.9 + 1.8 \\ 0.3 + 7.2 \end{bmatrix} = \begin{bmatrix} 0.9/11 \\ 7.5/11 \end{bmatrix}$$

Step 3 : $x_1 = x_0 + s_0$

$$= \begin{bmatrix} 1.1 \\ 1.9 \end{bmatrix} + \begin{bmatrix} 0.9/11 \\ 7.5/11 \end{bmatrix} = \begin{bmatrix} 1.1\overline{8} \\ 2.58\overline{18} \end{bmatrix} \quad \#$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad ad - bc \neq 0$$

$$= \frac{1}{12-1}\begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}$$

Question 4

Given the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$:

(a) Find A^{-1}.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad ad-bc \neq 0$$

Check: $\det(A) = ad-bc = (3)(2)-(1)(1) = 5 \neq 0$ invertible

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \end{bmatrix}$$

(b) Compute the condition numbers $\text{cond}_1(A)$ and $\text{cond}_\infty(A)$.

$$\text{Cond}_1(A) = \underbrace{\|A\|_1}_{4} \cdot \|A^{-1}\|_1 = 4 \cdot 4.5 = 18$$

$\hookrightarrow \max\{|2/5| + |-1/5|, |-1/5| + |3/5|\}$

$= 4/5$

$$\text{Cond}_\infty(A) = \underbrace{\|A\|_\infty}_{4} \cdot \underbrace{\|A^{-1}\|_\infty}_{4/5} = 18$$

Question 5

Find the Cholesky factorization (LL^T factorization) of the symmetric positive definite matrix by the algorithm described in the lecture:

$$A = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -1 \\ 2 & -1 & 5 \end{bmatrix}$$

Show your calculations step by step.

$\ell_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$

$\ell_{21} = a_{21} / \ell_{11} = -2/2 = -1$

$\ell_{22} = \sqrt{2 - (-1)^2} = \sqrt{2-1} = 1$

$L = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

$\begin{matrix} k=3 \\ i=1 \end{matrix}$ $\ell_{31} = \dfrac{2-0}{2} = 1$

$\begin{matrix} k=3 \\ i=2 \end{matrix}$ $\ell_{32} = \dfrac{-1-(-1)(1)}{1} = 0$

$\begin{matrix} k=3 \\ i=3 \end{matrix}$ $\ell_{33} = \sqrt{5 - 1^2 - 0^2} = \sqrt{4} = 2$

$$A = LL^T = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & 2 \\ -2 & 2 & -1 \\ 2 & -1 & 5 \end{pmatrix} \quad \#$$

Question 6

Catastrophic Cancellation

เราจะคำนวณ

$f(x) = \sqrt{x+1} - \sqrt{x}$

ถ้า x **ใหญ่** ตัวเลขสองตัวที่เราลบกันคือ $\sqrt{x+1}$ กับ $\sqrt{x}$ ซึ่งมีค่า **ใกล้กันมาก**
การลบเลขที่เกือบเท่ากัน จะทำให้ "หลักที่เหมือนกันหายไป" เหลือผลต่างเล็ก ๆ
→ เกิด **catastrophic cancellation** (ตัวเลขสำคัญหายไปจากการปัดเศษ)

For what range of values of x is it difficult to compute

$$f(x) = \sqrt{x+1} - \sqrt{x}$$

accurately in floating-point arithmetic? Explain why.

algebra • multiply & devide by the conjunction

(a^2+b^2)

$x+1-x$

$$\sqrt{x+1} - \sqrt{x} = \frac{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}{\sqrt{x+1} + \sqrt{x}} = \frac{1}{\sqrt{x+1} + \sqrt{x}}$$

(มีค่าเท่ากับ 1)

⇓ test

Old formula: $f(9) = \sqrt{10} - \sqrt{9} = 3.1623 - 3 = 0.1623$

New formula: $f(9) = \frac{1}{\sqrt{10} + \sqrt{9}} = \frac{1}{3.1623 + 3} = 0.1623$

for large x

Old formula: $f(10^6) = \sqrt{1000001} - \sqrt{100000} = 0.000499999875$

New formula: $f(10^6) = \frac{1}{\sqrt{1000001} + \sqrt{100000}} = 0.000499999875$ ← ค่าเดียวกัน แต่ไม่ต้องลบเลขใหญ่

Cancellation_demo__direct_vs_rationalized

x	direct (float32)	stable (float32)	rel. error direct32	direct (float64)	stable (float64)	rel. error direct64	ref (high-precision)
1.0	0.41421353816986100	0.41421353816986100	5.84317762459163E-08	0.41421356237309500	0.4142135623730950	2.68031548330893E-16	0.41421356237309500
100.0	0.04987525939941410	0.04987562075257300	7.25247060748756E-06	0.04987562112088900	0.04987562112089030	6.53882610691021E-15	0.04987562112089030
10000.0	0.00499725341796875	0.004999875091016290	0.00052433076378558550	0.004999875006248540	0.004999875006249610	2.1302936831789E-13	0.004999875006249610
1000000.0	0.00048828125	0.0004999998491257430	0.02343725585943610	0.0004999998750463420	0.0004999998750000630	9.2558796285644E-11	0.0004999998750000630
100000000.0	0.0	4.99999987368938E-05	1.0	5.00000005558832E-05	4.9999999875E-05	1.36176632106052E-08	4.9999999875E-05
10000000000.0	0.0	4.99999987368938E-06	1.0	4.99999441672117E-06	4.999999999875E-06	1.11663076706565E-06	4.999999999875E-06
1000000000000.0	0.0	4.99999998737621E-07	1.0	5.00003807246685E-07	4.99999999999875E-07	7.61449361997803E-06	4.99999999999875E-07
100000000000000.0	0.0	5.00000005843049E-08	1.0	5.02914190292358E-08	4.99999999999999E-08	0.005828380584719370	4.99999999999999E-08
1E+16	0.0	4.99999996961265E-09	1.0	0.0	5E-09	1.0	5E-09

ทำไม่เป็น

Question 7

Find the Lagrange interpolant $p(t)$ for the following data points:

$$(0,2), \quad (1,3), \quad (3,1), \quad (4,5)$$

Also, evaluate $p(2)$.

T_n Y

4 data points : (0, 2) , (1, 3) , (3, 1) , (4, 5) $\rightarrow$ use polynomial of degree 3

$$P_3(t) = y_1 \ell_1(t) + y_2 \ell_2(t) + y_3 \ell_3(t) + y_4 \ell_4(t)$$

$$= 2\ell_1(t) + 3\ell_2(t) + 1\ell_3(t) + 5\ell_4(t)$$

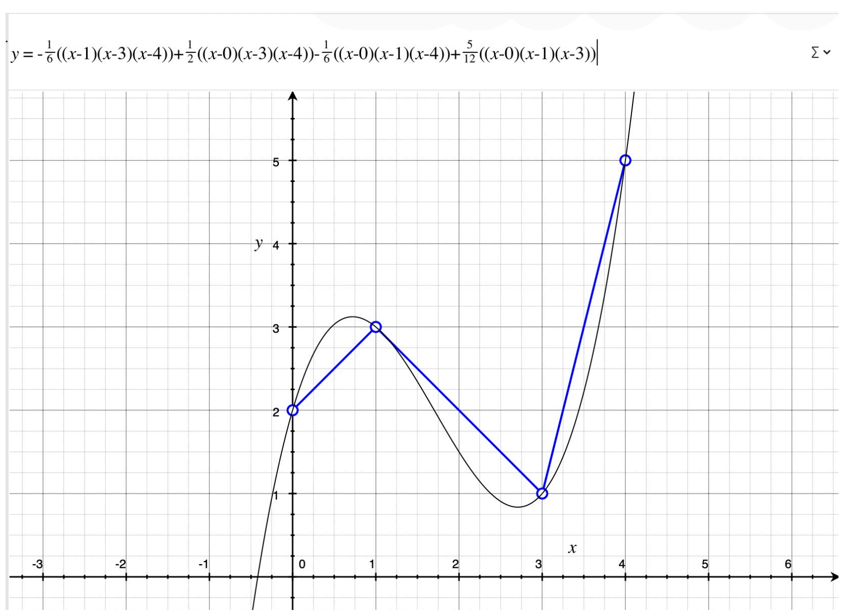
$$\ell_1(t) = \frac{(t-t_2)(t-t_3)(t-t_4)}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} = \frac{(t-1)(t-3)(t-4)}{(0-1)(0-3)(0-4)} = \frac{(t-1)(t-3)(t-4)}{-12}$$

$$\ell_2(t) = \frac{(t-t_1)(t-t_3)(t-t_4)}{(t_2-t_1)(t_2-t_3)(t_2-t_4)} = \frac{(t-0)(t-3)(t-4)}{(1-0)(1-3)(1-4)} = \frac{(t-0)(t-3)(t-4)}{6}$$

$$\ell_3(t) = \frac{(t-t_1)(t-t_2)(t-t_4)}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} = \frac{(t-0)(t-1)(t-4)}{(3-0)(3-1)(3-4)} = \frac{(t-0)(t-1)(t-4)}{-6}$$

$$\ell_4(t) = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_4-t_1)(t_4-t_2)(t_4-t_3)} = \frac{(t-0)(t-1)(t-3)}{(4-0)(4-1)(4-3)} = \frac{(t-0)(t-1)(t-3)}{12}$$

$$P_3(t) = 2\,\frac{(t-1)(t-3)(t-4)}{-12} + 3\,\frac{(t-0)(t-3)(t-4)}{6} + 1\,\frac{(t-0)(t-1)(t-4)}{-6} + 5\,\frac{(t-0)(t-1)(t-3)}{12} \quad \#$$



Question 8

Suppose we have the following six data points:

$$(-1, 3), \quad (0, -2), \quad (1, 1), \quad (2, 5), \quad (3, -4), \quad (5, 8)$$

(a) Set up the linear system that needs to be solved in order to compute the polynomial interpolant in Newton basis for the above data points. Write the full matrix equation $Ax = b$. No need to solve the system.

6 data points → Newton polynomial of degree 5

$$P_5(t) = x_1 + x_2(t-t_1) + x_3(t-t_1)(t-t_2) + x_4(t-t_1)(t-t_2)(t-t_3) + x_5(t-t_1)(t-t_2)(t-t_3)(t-t_4) + x_6(t-t_1)(t-t_2)(t-t_3)(t-t_4)(t-t_5)$$

$$P_5(t) = x_1 + x_2(t-(-1)) + x_3(t-(-1))(t-0) + x_4(t-(-1))(t-0)(t-1) + x_5(t-(-1))(t-0)(t-1)(t-2) + x_6(t-(-1))(t-0)(t-1)(t-2)(t-3)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 & 0 \\ 1 & 3 & 6 & 6 & 0 & 0 \\ 1 & 4 & 12 & 24 & 24 & 0 \\ 1 & 6 & 30 & 120 & 360 & 720 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 5 \\ -4 \\ 8 \end{bmatrix} \qquad x = \begin{bmatrix} 3 \\ -5 \\ 4 \\ -7/6 \\ -7/24 \\ 2/9 \end{bmatrix}$$

1 ตลอดไป

$$P_3(t) = 3 - 5(t+1) + 4(t+1)(t-0) - \frac{7}{6}(t+1)(t-0)(t-1) - \frac{7}{24}(t+1)(t-0)(t-1)(t-2) + \frac{2}{9}(t+1)(t-0)(t-1)(t-2)(t-3) \quad \#$$

(b) What algorithm should be used to solve this system and why? What is its computational complexity?

forward Substitution, $O(n^2)$

ลอก Cheatsheet.

Structure of the system

- The interpolation system in the **Newton basis** always produces a **lower triangular** matrix A.
- Each row corresponds to evaluating $N_k(x)$ at a data point x_i.
- Since all entries above the diagonal are zero, the matrix is **triangular and nonsingular** (because x_i are distinct).

(b) Algorithm

- The correct algorithm is **forward substitution** (sometimes called forward elimination).
- Reason: For lower-triangular systems $Lx = b$, forward substitution allows solving coefficient-by-coefficient sequentially without Gaussian elimination or matrix inversion.
- Each step requires only a few multiplications and additions.

Complexity

- For an $n \times n$ lower-triangular system, forward substitution requires about $\frac{1}{2}n^2$ arithmetic operations.
- So the computational complexity is $\mathcal{O}(n^2)$.

✅ **Answer (concise form):**

Use **forward substitution** because the matrix is lower-triangular. The algorithm works in $\mathcal{O}(n^2)$ time, much faster than Gaussian elimination ($\mathcal{O}(n^3)$).

Question 9

e^{-2t}

Consider interpolating $f(t) =$ [redacted] with a polynomial of degree 2, using sample points at $t = 0, 0.5, 1$. Give a reasonably tight upper bound on the error of the interpolant over the interval $[0, 1]$. **Hint:** *The derivative of* e^{-x} *is* $-e^{-x}$

$t = 0, 0.5, 1 \rightarrow n = 3$

$h = \max\{0.5 - 0,\ 1 - 0.5\} = 0.5$

Finding M (diff เท่า n)

$f'(t) = -2e^{-2t}$

$f''(t) = 4e^{-2t}$

$f'''(t) = -8e^{-2t}$

for all $t \in [0, 1]$

เลือกค่าจาก Domain ที่ทำให้ได้ค่ามากที่สุด

$|f'''(t)| = |-8e^{-2t}| \leq |-8e^{-2(0)}| = 8e^{0} = 8 = M$ $= 8e^{-2(1)} = 1.083$ X

$$\frac{Mh^n}{4n} = \frac{8e^{0}(0.5)^3}{4(3)} \approx \frac{1}{12} \approx 0.8\overline{33}$$

Question 10 ยากเกินไม่น่าออก / เหมือนจะมีวิธีใส่ Matrix

(a) Use the method of undetermined coefficients to derive a 3-point quadrature rule on the interval $[a, b]$ using

$n = 3$

$$x_1 = a, \quad x_2 = \frac{a+b}{2}, \quad x_3 = b$$

Set up the system of equations and solve for the weights.

ทำไมบอกว่า $w_1 = w_3$?

- เพราะตำแหน่งของจุด a และ b **สมมาตร** กันรอบจุดกึ่งกลาง $\frac{a+b}{2}$.
- ใน numerical integration ถ้าจุดอยู่ “สมมาตร” แบบนี้ ค่า weight (น้ำหนัก) ก็ต้องเท่ากันด้วย ไม่งั้นสูตรจะไม่ “บาลานซ์” รอบกึ่งกลาง
  👉 เช่น ถ้าเราให้ด้านซ้าย (ที่ a) หนักกว่าด้านขวา (ที่ b) ทั้งที่อยู่ห่างกึ่งกลางเท่ากัน มันก็จะไม่ถูกต้อง

ดังนั้น เราตั้งได้เลยว่า
$w_1 = w_3 =: w.$

$$\int_a^b f(x)\,dx \approx \omega_1 f(x_1) + \omega_2 f(x_2) + \omega_3 f(x_3)$$

Find $\omega_1, \omega_2, \omega_3 \rightarrow$ polynomials of degree ≤ 2 (since 3 points)

1) $f(x) = 1$

$\int_a^b 1\,dx = b - a = \omega_1 + \omega_2 + \omega_3 = 2\omega + \omega_2 \Rightarrow \omega_2 = (b-a) - 2\omega$ —(1)

2) $f(x) = x$

$\int_a^b x\,dx = \frac{b^2 - a^2}{2} = a\omega_1 + \frac{a+b}{2}\omega_2 + b\omega_3$

3) $f(x) = x^2$

$\int_a^b x^2 dx = \frac{b^3 - a^3}{3} = a^2\omega_1 + \left(\frac{a+b}{2}\right)^2 \omega_2 + b^2\omega_3$ from (1): $a^2\omega + \left(\frac{a+b}{2}\right)^2((b-a) - 2\omega) + b^2\omega$

collect ω using: $a^2 + b^2 - \frac{(a+b)^2}{2} = \frac{(a-b)^2}{2}$

$\Rightarrow \omega \cdot \frac{(a-b)^2}{2} + \frac{(a+b)^2}{4}(b-a) = \frac{b^3 - a^3}{3}$

Let $h = b - a$, $s = a + b$

Since $b^3 - a^3 = h(a^2 + ab + b^2)$

devide both size by h $(a \neq b)$

$$\omega \cdot \frac{h}{2} + \frac{s^2}{4} = \frac{a^2 + ab + b^2}{3}$$

$$\omega \cdot \frac{h}{2} + \frac{a^2 + 2ab + b^2}{4} = \frac{a^2 + ab + b^2}{3}$$

$$\omega \cdot \frac{h}{2} = \frac{a^2 + ab + b^2}{3} - \frac{a^2 + 2ab + b^2}{4}$$

$$\omega \cdot \frac{h}{2} = \frac{4a^2 + 4ab + 4b^2 - 3a^2 + 6ab + 3b^2}{12}$$

$$\omega \cdot \frac{h}{2} = \frac{a^2 - 2ab + b^2}{12} \quad (a-b)$$

$(a-b)^2 = (b-a)^2 = h^2$

$$\omega \cdot \frac{h}{2} = \frac{h^2}{12}$$

$$6\omega = h$$

$$\omega = \frac{h}{6}$$

$\omega_1 = \omega_3 =: \omega = \frac{b-a}{6}$ #

$\omega_2 = (b-a) - 2\left(\frac{b-a}{6}\right)$

$= (b-a)\left(1 - \frac{1}{3}\right)$

$\omega_2 = \frac{2}{3}(b-a)$ #

(b) Apply your quadrature rule from part (a) to approximate $\int_0^2 x^2 dx$.

$a = 0$, $b = 2$ $\quad \frac{a+b}{2}$

$x_1 = 0$, $x_2 = 1$, $x_3 = 2$

$$\begin{cases} \omega = \frac{b-a}{6} = \frac{1}{3} \\ \omega_2 = \frac{2}{3}(b-a) = \frac{4}{3} \end{cases}$$

Approximate :

$\int_0^2 x^2 dx \approx \omega_1 f(0) + \omega_2 f(1) + \omega_3 f(2)$

$f(0) = 0$, $f(1) = 1$, $f(2) = 4$

$= \frac{1}{3} \cdot 0 + \frac{4}{3} \cdot 1 + \frac{1}{3} \cdot 4 = \frac{8}{3}$ #

(c) Compare your result with the exact value of the integral.

- Exact Value

$\int_0^2 x^2 dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$

$\frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$

Error : $\left|\frac{8}{3} - \frac{8}{3}\right| = 0$ #

วิธีใช้เมทริกซ์หาค่าเวท (weights)

ให้จุดเก็บตัวอย่าง 3 จุดบนช่วง $[a,b]$:

$$x_1 = a, \quad x_2 = m = \frac{a+b}{2}, \quad x_3 = b$$

และต้องการ

$$\int_a^b f(x)\,dx \approx w_1 f(a) + w_2 f(m) + w_3 f(b).$$

บังคับให้ “อินทิเกรตได้ตรง” สำหรับพหุนาม $1, x, x^2$:

$$\int_a^b 1\,dx = b - a = w_1 + w_2 + w_3,$$
$$\int_a^b x\,dx = \frac{b^2-a^2}{2} = a\,w_1 + m\,w_2 + b\,w_3,$$
$$\int_a^b x^2\,dx = \frac{b^3-a^3}{3} = a^2 w_1 + m^2 w_2 + b^2 w_3.$$

เขียนเป็นเมทริกซ์ $A\mathbf{w} = \mathbf{b}$:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ a & m & b \\ a^2 & m^2 & b^2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}_{\mathbf{w}} \underbrace{\begin{bmatrix} b-a \\ \dfrac{b^2-a^2}{2} \\ \dfrac{b^3-a^3}{3} \end{bmatrix}}_{\mathbf{b}}.$$

แก้ระบบ (ใช้เกาส์/อินเวิร์สก็ได้) จะได้คำตอบทั่วไป:

$$w_1 = w_3 = \frac{b-a}{6}, \quad w_2 = \frac{2}{3}(b-a)$$

ซึ่งก็คือกฎซิมป์สันแบบ 1/3

ตัวอย่างช่วง $[0,2]$

- $a = 0,\ b = 2,\ m = 1$

เมทริกซ์และเวกเตอร์:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ \frac{8}{3} \end{bmatrix}.$$

แก้ได้

$$\mathbf{w} = \begin{bmatrix} \frac{1}{3} \\ \text{[2pt]}\,\frac{4}{3} \\ \text{[2pt]}\,\frac{1}{3} \end{bmatrix},$$

ดังนั้น

$$\int_0^2 f(x)\,dx \approx \frac{1}{3} f(0) + \frac{4}{3} f(1) + \frac{1}{3} f(2).$$