BIOS13 - Question 3

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a Workers population

The equation $\frac{dW}{dt} = rW$ can also be expressed as:

$$\frac{1}{W}dW = rdt$$

Integrate to get the equation of individuals:

$$\int \frac{1}{W}dW = \int rdt$$

$$\Leftrightarrow ln(W) = rt + C \tag{1}$$

Initially, we have W=1 (the colony start with 1) at time t=0, we get:

$$ln(1) = r * 0 + C$$

$$\Leftrightarrow C = 0 \tag{2}$$

Substitute (2) into (1), we have:

$$lnW = rt$$

$$\Leftrightarrow W = e^{rt}$$

We have an equation of population of worker W corresponding to time t, thus at time t_s, we have:

$$W(t_\mathrm{s}) = e^{rt_\mathrm{s}} \tag{3}$$

b Number of queens

The number of queens function $\frac{dQ}{dt} = cW$ can also be expressed as:

$$dQ = cWdt$$

The queens will be produced after switching time t_s, thus the number of queens at time T is the result of:

$$Q_\mathrm{T} = \int_{t_\mathrm{s}}^{T} cW\,dt$$

$$\Leftrightarrow Q_{\mathrm{T}} = cW(T - t_{\mathrm{s}}) \tag{4}$$

From (3), (4) can be written as:

$$\Leftrightarrow Q_{\mathrm{T}} = ce^{rt_{\mathrm{s}}}(T - t_{\mathrm{s}}) \tag{5}$$

c Optimal switching time

Find the first derivative of (5) with respect to t_s :

$$Q_{\mathrm{T}}' = cre^{rt_{\mathrm{s}}}(T - t_{\mathrm{s}}) - ce^{rt_{\mathrm{s}}}$$

Find the value of t_s when Q_T'=0

$$ce^{rt_{\mathrm{s}}}(r(T - t_{\mathrm{s}}) - 1) = 0$$

$$\Leftrightarrow ce^{rt_{\mathrm{s}}} = 0 \vee r(T - t_{\mathrm{s}}) - 1 = 0$$

$$\Leftrightarrow t_{\mathrm{s}} = T - \frac{1}{r} \tag{6}$$

Find the second derivative of Q_T with respect to t_s :

$$Q_{\mathrm{T}}'' = cr^2e^{rt_{\mathrm{s}}}(T - t_{\mathrm{s}}) - cre^{rt_{\mathrm{s}}} - cre^{rt_{\mathrm{s}}}$$

$$\Leftrightarrow Q_{\mathrm{T}}'' = r(cre^{rt_{\mathrm{s}}}((T - t_{\mathrm{s}}) - ce^{rt_{\mathrm{s}}} - ce^{rt_{\mathrm{s}}})$$

$$\Leftrightarrow Q_{\mathrm{T}}'' = r(Q_{\mathrm{T}}' - ce^{rt_{\mathrm{s}}}) \tag{7}$$

We know that when substitute (6) to (7), Q_T' will be equal to 0, thus we have:

$$Q_{\mathrm{T}}'' = -rce^{r(T - \frac{1}{r})}$$

Since r,c is a positive number, and $e^{r(T-\frac{1}{r})}$ is never smaller than 0, $Q_T" < 0$. Thus there is a local maxima of Q_T when $t_{\mathrm{s}} = T - \frac{1}{r}$

d Extend model

Let's call a is the dying rate of workers bee, and b is the dying rate of queen bees.
Before t_s, the growth of worker bees is:

$$\frac{dW}{dt} = rW - aW$$

After t_s, worker bees stop reproducing and start to die:

$$\frac{dW}{dt} = -aW$$

Before t_s, there is no reproduction of queen bees:

$$\frac{dQ}{dt} = 0$$

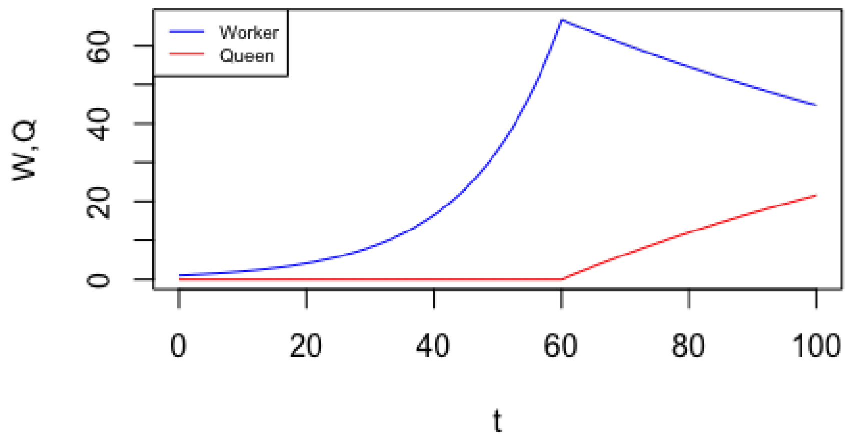
After t_s, queen bees start reproducing with a rate depends on the number of worker and dying at the same time:

$$\frac{dQ}{dt} = cW - bQ$$

e R script

```
rm(list=ls())
library(deSolve)
Q3e = function(ts,T,r,c,a,b){
  WQ = function(t,wp,P){
    W = wp[1]
    Q = wp[2]
    if (t<=ts) {
      dWdt = P$r*W - P$a*W
      dQdt = 0
    } else if (t>ts) {
      dWdt = -P$a*W
      dQdt = P$c*W - P$b*Q
    }
    return(list(c(dWdt,dQdt)))
  }
  timevec = seq(0,T,by=1)
  P = list(r=r,c=c,a=a,b=b)
  wp0 = c(W=1,Q=0)
  out = ode(y=wp0,func=WQ,parms=P,times=timevec)

  time <- out[,'time']
  w <- out[,'W']
  q <- out[,'Q']
  plot( time, w, type='l', col='blue',xlim=c(0,100),
        ylim=c(0,max(w)),ylab="W,Q",xlab="t")
  lines( time, q, col='red')
  legend("topleft", legend = c("Worker", "Queen"),
         lwd = 1, col = c("blue", "red"), cex=0.6)
}
Q3e(60,100,0.08,0.01,0.01,0.001)
```



The plot is illustrated by the following code *Q3e(60,100,0.08,0.01,0.01,0.001)*