Cagniard de Hoop Method for Solving Green's Function

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1 Cagniard de Hoop Method for Line Source

Consider love wave initiated by a line source, the equation of motion could be written as :

$$\rho\frac{\partial^2 v}{\partial t^2} = A\delta(x)\delta(z)\delta(t) + \mu\nabla^2 v \tag{1.1}$$

We take Laplace transform for t and spatial Fourier transform for x, i.e.:

$$\tilde{v}(k,z,s) = \int_{-\infty}^{\infty} e^{-ikx}dx \int_0^{\infty} v(x,z,t)e^{-st}dt \tag{1.2}$$

Inset (1.2) into (1.1), utilize the derivative property, we get:

$$\frac{\partial^2\tilde{v}}{\partial z^2} - n^2\tilde{v} = -\frac{A}{\beta^2\rho}\delta(z) \tag{1.3}$$

where $n = \sqrt{k^2 + s^2/\beta^2}, Re(n) \geq 0$. Noting the continuous for $\tilde{v}$ and jumping condition $\frac{\partial\tilde{v}}{\partial z}$, as well as the regularity at infinity, we could get solution to (1.3):

$$\tilde{v} = \frac{A}{2\beta^2\rho n}e^{-n|z|} \tag{1.4}$$

Now do inverse Fourier transform for $\tilde{v}$:

$$\hat{v}(x,z,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\tilde{v}dk = \frac{A}{4\pi\beta^2\rho}\int_{-\infty}^{\infty} e^{ikx}\frac{e^{-n|z|}}{n}\tilde{v}dk \tag{1.5}$$

We change variable from k to p via $k = isp$, where $s \in \boldsymbol{R}^+$. So $n = s\sqrt{\beta^{-2} - p^2} = s\eta$, and:

$$\hat{v}(x,z,s) = \frac{A}{4\pi\beta^2\rho}\int_{-i\infty}^{i\infty} \frac{-ie^{-s(px+\eta|z|)}}{\eta}dp \tag{1.6}$$

Note that the integrand in (1.6) could be written as the form $-i[E(p) + iO(p)]$ where $E(p) = E(-p)$ and $O(p) = -O(-p)$. At the same time, note that :

$$\begin{aligned}\int_{-i\infty}^{i\infty} -i[E(p) + iO(p)]dp = -2i\int_{0}^{i\infty} E(p)dp = 2Im\int_{0}^{i\infty} Edp \\ = 2Im\int_{0}^{i\infty} (E + iO)dp\end{aligned} \tag{1.7}$$

(1.6) could be rewritten as :

$$\hat{v}(x,z,s) = \frac{A}{2\pi\beta^2\rho}Im\int_{0}^{i\infty} \frac{e^{-s(px+\eta|z|)}}{\eta}dp \tag{1.8}$$

Now we set $t = px+\eta|z|$ and change intergal path from imaginary axis to a path called cagniard path in which t is a real number, and then solve the relation between t and p:

$$p = \begin{cases} \frac{xt-|z|\sqrt{R^2/\beta^2-t^2}}{R^2} & t < R/\beta, \\ \frac{xt+i|z|\sqrt{t^2-R^2/\beta^2}}{R^2} & t > R/\beta. \end{cases} \tag{1.9}$$

where $R = \sqrt{x^2 + z^2}$. This path could be shown in Figure 1. In this figure we also show the branch point $1/\beta$. Because the turning point from real axis to p-plane is $p = p(R/\beta) = \frac{x}{R\beta} < 1/\beta$, the branch cut does not contribute to this integral. Also, the first part of this integral only get real number and could not contribute to the final results. So when we try to evaluate this integral, we only need to consider the second part ($t > R/\beta$)

Now we change variable from p to t, utilize (1.9), it is easily to show that:

$$\frac{dp}{dt} = \frac{i\eta}{\sqrt{t^2 - R^2/\beta^2}} \tag{1.10}$$

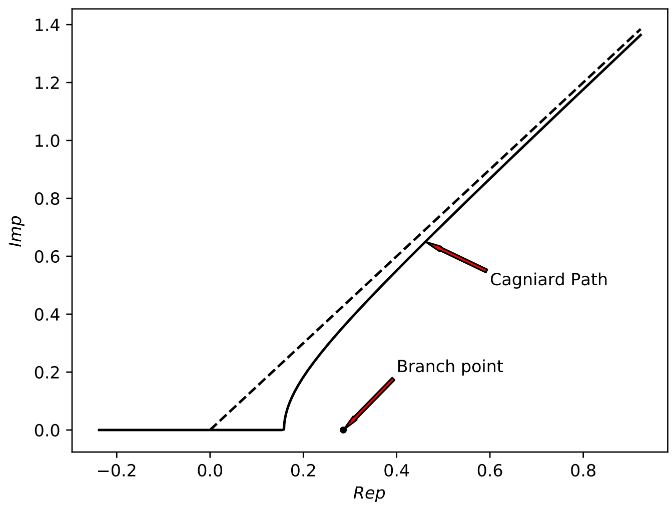

Figure 1: Cagniard Path

So (1.8) could be rewritten as:

$$\hat{v} = \frac{A}{2\pi\beta^2\rho} Im \int_{R/\beta}^{\infty} i \frac{e^{-st}}{\sqrt{t^2 - R^2/\beta^2}} dt = \frac{A}{2\pi\beta^2\rho} \int_0^{\infty} \frac{H(t - R/\beta)}{\sqrt{t^2 - R^2/\beta^2}} e^{-st} dt \quad (1.11)$$

this is the forward Laplace transform of $\frac{H(t-R/\beta)}{\sqrt{t^2-R^2/\beta^2}}$, so the solution to this problem is:

$$v(x, z, t) = \frac{H(t - R/\beta)}{\sqrt{t^2 - R^2/\beta^2}} \quad (1.12)$$

2 Half Space Green's Function

Note: the results here are quite complicated, so we here only show the critical steps. The all formulae could be found in Johnson(1974)

The assumptions and equations are all in (Johnson,1974), here we talk a little about some steps omitted in this paper.

2.1 Foumula (5) to Formula (9)

In this step, we want to inverse the matrix in (5) to find the solution of full space Green's function in frequency domain. Note the matrix could be written as :

$$A = \xi\xi^T + \frac{\mu}{\lambda+\mu}(\xi_3^2 - \nu_\beta^2)I \tag{2.1}$$

The inverse of A is:

$$A^{-1} = \frac{\lambda+\mu}{\mu(\xi_3^2-\nu_\beta^2)}\left(I - \frac{\xi\xi^T}{\mu/(\lambda+\mu)(\xi_3^2-\nu_\beta^2)+\xi^T\xi}\right) \tag{2.2}$$

This lead to solution to the full space Green's function:

$$G^F(\xi_1,\xi_2,\xi_3,s,0,x_3',0) = \begin{bmatrix} I_3^{(2)}(\nu_\alpha) - I_3^{(2)}(\nu_\beta) & I_1(\nu_\beta) - I_1(\nu_\alpha) & I_2^{(1)}(\nu_\beta) - I_2^{(1)}(\nu_\alpha) \\ sym & I_3^{(1)}(\nu_\alpha) - I_3^{(1)}(\nu_\beta) & I_2^{(2)}(\nu_\beta) - I_2^{(2)}(\nu_\alpha) \\ sym & sym & I_4(\nu_\alpha) - I_4(\nu_\beta) \end{bmatrix} \frac{F}{\rho s^2} \tag{2.3}$$

Where:

$$\begin{aligned} I_1(c) &= \frac{\xi_1\xi_2}{\xi_3^2-c^2}e^{-\xi_3 x_3'} \\ I_2^\eta(c) &= \frac{\xi_\eta\xi_3}{\xi_3^2-c^2}e^{-\xi_3 x_3'} \\ I_3^\eta(c) &= \left(\frac{\xi_\eta^2+\xi_3^2}{\xi_3^2-c^2} + \frac{\mu}{\lambda+\mu}\frac{-\nu_\beta^2+\xi_3^2}{\xi_3^2-c^2}\right)e^{-\xi_3 x_3'} \\ I_4(c) &= \left(\frac{\xi_3^2+\xi_2^2}{\xi_3^2-c^2} + \frac{\mu}{\lambda+\mu}\frac{-\nu_\beta^2+\xi_3^2}{\xi_3^2-c^2}\right)e^{-\xi_3 x_3'} \end{aligned} \tag{2.4}$$

Then by using mirror method, which suppose the virtual force is the combination of actual force, set equations to satisfy the boundary condition, we could get (9)

2.2 From (17) to (28)

Green's function for half space could be written by using double integral:

$$\begin{aligned} G(x_1,x_2,0,s;0,0,x_3',0) = \frac{-1}{4\pi^2\mu}\iint_{(-i\infty)^2}^{(i\infty)^2} &[e^{-\nu_\alpha x_3'}M(\xi_1,\xi_2,0,s,x_3') \\ &+ e^{-\nu_\beta x_3'}N(\xi_1,\xi_2,0,s,x_3')]\frac{F}{d}e^{\xi_1 x_1+\xi_2 x_2}d\xi_1 d\xi_2 \end{aligned} \tag{2.5}$$

Where $d = h^2 + 4\nu_\alpha\nu_\beta(\xi_1^2 + \xi_2^2)$, $h = \nu_\beta^2 - \xi_1^2 - \xi_2^2$,and $\nu_c = \sqrt{s^2/c^2 - \xi_1^2 - \xi_2^2}$ $(c = \alpha, \beta)$. We use the following steps to convert (2.5) to the form of Laplace forward transform:

1. Change coordinates from Cartesian coordinates to sphere ones, and variables from ξ_1, ξ_2 to p, q via :

$$\begin{aligned} &x_1 = r\sin\theta\cos\phi, x_2 = r\sin\theta\sin\phi, x_3' = r\cos\theta \\ &\xi_1 = sq\cos\phi - isp\sin\phi \\ &\xi_2 = sq\sin\phi + isp\cos\phi \end{aligned} \tag{2.6}$$

These lead to:

$$\begin{aligned} &\xi_1^2 + \xi_2^2 = s^2(q^2 - p^2), \nu_c = s\sqrt{1/c^2 - q^2 + p^2} = s\eta_c \\ &h = s^2(\eta_\beta^2 + p^2 - q^2) = s^2\gamma, d = s^4(\gamma^2 + 4\eta_\alpha\eta_\beta(q^2 - p^2)) = s^4\sigma \\ &-\nu_c x_3' + \xi_1 x_1 + \xi_2 x_2 = -s(-qr\sin\theta + \eta_c r\cos\theta) = -s\tau_c \\ &d\xi_1 d\xi_2 = -is^2 dpdq, M(\xi_1, \xi_2) = s^3 M(p, q), N(\xi_1, \xi_2) = s^3 N(p, q) \end{aligned}$$

It is easily to show that $p \in R, iq \in R$, use the symmetries in (2.5) similar to (1.6):

$$\begin{aligned} G(x_1, x_2, 0, s; 0, 0, x_3', 0) &= \frac{s}{\pi^2\mu} Im \int_0^\infty dp \int_0^{i\infty} \sigma^{-1}(e^{-s\tau_\alpha}M + e^{-s\tau_\beta}N)Fdq \\ &= G_\alpha + G_\beta \end{aligned} \tag{2.7}$$

The branch points of this problem are $q_{1,2} = \pm\sqrt{1/\beta^2 + p^2}$, $q_{3,4} = \pm\sqrt{1/\alpha^2 + p^2}$.

2. Deform the path from imaginary axis to Cagniard path:

First we tackle G_α, set $\tau_\alpha = -qr\sin\theta + \eta_\alpha r\cos\theta \in R$, solve the quadratic equation:

$$q = \begin{cases} -\tau_\alpha \sin\theta/r + \cos\theta\sqrt{1/\alpha^2 + p^2 - (\tau_\alpha/r)^2} & \tau_\alpha < r\sqrt{1/\alpha^2 + p^2}, \\ -\tau_\alpha \sin\theta/r + i\cos\theta\sqrt{(\tau_\alpha/r)^2 - 1/\alpha^2 - p^2} & \tau_\alpha > r\sqrt{1/\alpha^2 + p^2}. \end{cases} \tag{2.8}$$

Note that $\frac{dq}{d\tau_\alpha} = \frac{i\eta_\alpha}{r\sqrt{(\tau_\alpha/r)^2 - \alpha^{-2} - p^2}}$, $G_\alpha$could be written as:

$$G_\alpha = \frac{s}{\pi^2\mu r} Re \int_0^\infty dp \int_{r\sqrt{1/\alpha^2+p^2}}^\infty \eta_\alpha\sigma^{-1}\left[(\tau_\alpha/r)^2 - \alpha^{-2} - p^2\right]^{-1/2} MFd\tau_\alpha \tag{2.9}$$

Here we use the fact that the branch points are not on the cagniard path because when $\tau_\alpha = r\sqrt{1/\alpha^2+p^2}$, $q = -\sqrt{1/\alpha^2+p^2}\sin\theta$, which is at the left of the nearest branch point q_4. This is obvious when we use the physics of the initiation of inhomogenous waves. Because P wave cannot generate this kind of wave at free surface, we cannot hope for some kind of solution which is different from full space solution in section 1.

Then we tackle G_β. There is nothing new but at this time, SV wave could generate inhomogenous wave at free surface when it incidents at critical angle. To see how it can happen, we note that the turning point at Cagniard path is $q = -\sqrt{1/\beta^2+p^2}\sin\theta$, which means that this path could pass branch point q_4!

So the solution is dependent on different values of θ. First the path will not pass q_4, then all the steps are the same as G_α. But at the second time, because the imaginary parts of the integrand in G_β is not zero when it pass q_4. So we should divide this integral into 2 parts, the first one is the path after turning point, and the last is the path between branch point and the turning point. We mark the time corresponding to branch point as τ_0:

$$\tau_0 = \sqrt{1/\alpha^2+p^2}r\sin\theta + \sqrt{1/\beta^2-1/\alpha^2}r\cos\theta \tag{2.10}$$

3. Interchange the order of τ_c and p, note that:

$$\int_0^\infty dp\int_{\tau_0(p)}^\infty e^{-st}dt = \int_{\tau_0(p=0)}^\infty e^{-st}dt\int_0^{p(\tau_0)}dp = \int_0^\infty H[t-\tau_0(p=0)]e^{-st}dt\int_0^{p(\tau_0)}dp \tag{2.11}$$

So we can rewrite G_α as (2.11),and use the derivative property of Laplace transform:

$$\begin{aligned} G_\alpha(x_1,x_2,0,t,0,0,x_3',0) = \frac{1}{\pi^2\mu r}\frac{\partial}{\partial t}\int_0^{((t/r)^2-\alpha^{-2})^{-1/2}} & H(t-r/\alpha) \\ \times Re[\eta_\alpha\sigma^{-1}((t/r)^2-\alpha^{-2}-p^2)^{-1/2}M]F dp \end{aligned} \tag{2.12}$$

As for G_β, if $\sin\theta < \sqrt{\frac{1/\beta^2+p^2}{1/\alpha^2+p^2}}$, the expression is similar to (2.12):

$$
\begin{aligned}
G_\beta(x_1, x_2, 0, t, 0, 0, x_3', 0) = \frac{1}{\pi^2 \mu r}\frac{\partial}{\partial t}\int_0^{((t/r)^2-\beta^{-2})^{-1/2}} & H(t - r/\beta) \\
& \times Re[\eta_\beta \sigma^{-1}((t/r)^2 - \beta^{-2} - p^2)^{-1/2} N] F dp
\end{aligned}
\tag{2.13}
$$

Otherwise, we need to rewrite the second part in (2.7). Note that when $\tau_\beta < r\sqrt{1/\beta^2 + p^2}$, it is easily to show that :

$$
\frac{dq}{d\tau_\beta} = -\frac{\eta_\alpha}{r\sqrt{\alpha^{-2} + p^2 - (\tau_\alpha/r)^2}} \tag{2.14}
$$

the contribution part of integral path in x axis is from $q = -r\sqrt{\alpha^{-2} + p^2}$ to $q = -\sqrt{1/\alpha^2 + p^2}\sin\theta$, which corresponding to $\tau_\beta = \tau_0$ and $\tau_\beta = r\sqrt{1/\beta^2 + p^2}$. Then this part of G_β could be written as :

$$
\begin{aligned}
G_\beta &= -\frac{s}{\pi^2 \mu r} Im \int_0^\infty dp \int_{\tau_0}^{r\sqrt{1/\beta^2+p^2}} \eta_\beta \sigma^{-1}\left[\beta^{-2} - p^2 - (\tau_\beta/r)^2\right]^{-1/2} N F d\tau_\beta \\
&= -\frac{s}{\pi^2 \mu r}\int_0^\infty dp \int_{\tau_0}^{r\sqrt{1/\beta^2+p^2}} Im\left[\frac{N\eta_\beta}{\sigma\sqrt{\beta^{-2} - p^2 - (\tau_\beta/r)^2}}\right] F d\tau_\beta
\end{aligned}
\tag{2.15}
$$

Now we interchange the order of integral, but note that at present time the integrate area is between 2 curves instead of one curve with x axis(Figure 2). So this integral can be divide into 2 segments, according to the 2 vertical line in Figure 2.

$$
\begin{aligned}
G_\beta(x_1, x_2, 0, t, 0, 0, x_3', 0) = \frac{1}{\pi^2 \mu r}\frac{\partial}{\partial t}\int_0^{((t/r)^2-\alpha^{-2})^{-1/2}} & H(t - r/\alpha) \\
& \times Re[\eta_\alpha \sigma^{-1}((t/r)^2 - \alpha^{-2} - p^2)^{-1/2} M] F dp
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
G_\beta = \frac{1}{\pi^2 \mu r}\frac{\partial}{\partial t}\int_0^{\sqrt{(t/r)^2-1/\beta^2}} & H(t - r/\beta) \times Re[\eta_\beta \sigma^{-1}((t/r)^2 - \beta^{-2} - p^2)^{-1/2} N] F dp \\
- \frac{1}{\pi^2 \mu r}\frac{\partial}{\partial t}\int_0^{p_0} & H(\sin\theta - \frac{\beta}{\alpha})\left[H(t - t_2) - H(t - r/\beta)\right] Im\left[\frac{\eta_\beta N}{\sigma\sqrt{\beta^{-2} + p^2 - (t/r)^2}}\right] F dp \\
- \frac{1}{\pi^2 \mu r}\frac{\partial}{\partial t}\int_{\sqrt{(t/r)^2-1/\beta^2}}^{p_0} & H(\sin\theta - \frac{\beta}{\alpha}) H(t - r/\beta) Im\left[\frac{\eta_\beta N}{\sigma\sqrt{\beta^{-2} + p^2 - (t/r)^2}}\right] F dp
\end{aligned}
\tag{2.17}
$$

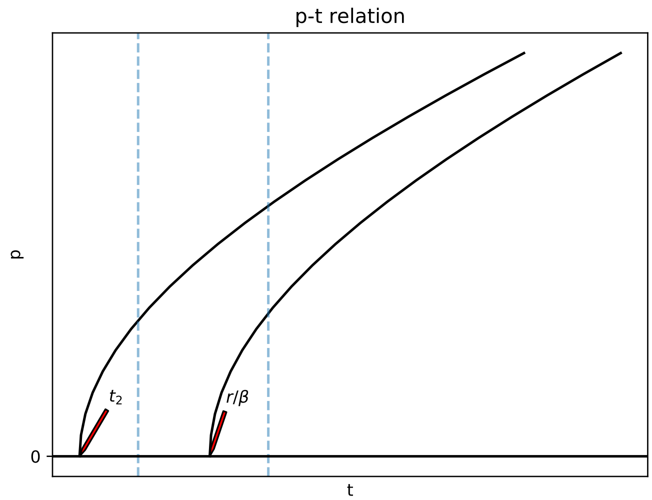

Figure 2: Change order of integral

where:

$$t_2 = \frac{r}{\alpha}\sin\theta + \sqrt{1/\beta^2 - 1/\alpha^2}\, r\cos\theta \tag{2.18}$$

$$p_0 = \left[\left(\frac{t/r - (\beta^{-2} - \alpha^{-2}\cos\theta)}{\sin\theta}\right)^2 - \alpha^{-2}\right]^{1/2} \tag{2.19}$$