# Notes on Probability in Card Problems

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## Contents

# 1 Deck Problems

## 1.1 Draw n Cards from Deck

### 1.1.1 Expected Total Values

When drawing cards from a deck, the expected total value of $n$ cards is the sum of the expected values of individual cards:

$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + \cdots + E(X_n)$$

There is no reason for the distribution of $X_i$ to be different if drawing at the same time. Therefore, the formula simplifies to:

$$E(X_1 + X_2 + \cdots + X_n) = n \cdot E(X_i) = n \cdot \frac{(1 + 2 + \cdots + 13)}{13}$$

### 1.1.2 Including Joker Cards

If Joker cards are included in the deck, the expected value of the first card, $E(x_1)$, will be different. The calculation for $E(x_1)$ incorporating Jokers (assuming Jokers have a value of 100) is given as:

$$E(x_1) = \frac{4}{54} \times (1 + 2 + \cdots + 13) + \frac{2}{54} \times 100$$

### 1.1.3 Replacing the Minimum Card

The document also touches upon the concept of replacing the minimum card. A formula is presented for the expected value after such an operation:

$$E(x_1 + x_2 + \cdots + x_{n+1}) - E(\min(x_1, x_2, x_3))$$

The expected value of the minimum, $E(m)$, is calculated using the sum of probabilities:

$$\begin{aligned}
E(m) &= \sum_{k=1}^{13} P(m = k)k \\
&= P(m = 1) + \cdots + 13P(m = 13) \\
&= P(m = 1) + P(m = 2) + \cdots + P(m = 13) \\
&\quad + P(m = 2) + \cdots + P(m = 13) \\
&\quad + \ldots \\
&\quad + P(m = 13)
\end{aligned}$$

For a discrete uniform distribution, the probability that the minimum is greater than $k$ is given by:

$$\begin{aligned} P(m > k) &= (P(x_i \geq k))^n \\ &= (1 - P(x_i \leq k-1))^n \\ &= \left(1 - \frac{(k-1) - 1 + 1}{13}\right)^n \end{aligned}$$

### 1.1.4 Expected Number of Cards to See the First Ace

This section addresses the problem of finding the expected number of cards that need to be turned over to see the first Ace, given a deck with 4 Aces and 48 other cards.

Let $X_i$ be an indicator variable such that:

$$x_i = \begin{cases} 1 & \text{is card } i \text{ is turned over before 4 aces} \\ 0 & \text{otherwise} \end{cases}$$

The total number of cards needed to be turned over is $X = 1 + \sum_{i=1}^{48} x_i$. The expected value $E[X]$ is thus:

$$E[X] = 1 + 48E[X_i]$$

The card $i$ is equally likely to be in one of the five regions separated by the 4 Aces. Therefore, the probability $P(X_i)$ is:

$$E[X_i] = P(X_i) = \frac{1}{5}$$

**Generalization** The concept can be generalized to ordering $m$ ordinary cards and $n$ special cards. The expected position of the first special card is given by:

$$E[x] = 1 + m\frac{1}{m+1}$$

## 1.2 Distributed 52 Cards to Players

This section covers calculations related to distributing 52 cards to players, specifically focusing on poker hands and the probability of Aces in different piles.

### 1.2.1 Poker Hands

The total number of possible 5-card poker hands from a standard 52-card deck is given by $\binom{52}{5}$. The number of ways to form specific poker hands are as follows:

- **One Pair**: The number of ways to get one pair is calculated as:

$$\binom{13}{1} \times \binom{4}{2} \times \binom{12}{3} \times \binom{4}{1}^3$$

- **Two Pairs**: The number of ways to get two pairs is calculated as:

$$\binom{13}{2} \times \binom{4}{2} \times \binom{4}{2} \times \binom{11}{1} \times \binom{4}{1}$$

- **Three of a Kind**: The number of ways to get three of a kind is calculated as:

$$\binom{13}{1} \times \binom{4}{3} \times \binom{12}{2} \times \binom{4}{1}^2$$

- **Straight**: A straight consists of five cards of sequential rank, not all of the same suit. The calculation for straights involves choosing a low card from ranks 1 to 10. The number of such hands is given by:

$$\binom{10}{1}\left(\binom{4}{1}^5 - \binom{4}{1}\right)$$

  This formula accounts for choosing a low card out of 10 possibilities and then selecting suits for each of the five cards in the sequence ($4^5$), while deducting cases where all five cards are of the same suit (which would be a straight flush).

- **Flush**: A flush consists of five cards of the same suit, not all of sequential rank. The number of ways to get a flush is calculated as:

$$\binom{13}{5} \times \binom{4}{1} - \binom{10}{1} \times \binom{4}{1}$$

  This formula selects 5 ranks from 13 within one of the 4 suits, and then subtracts the number of straight flushes (including royal flushes) to ensure they are not counted as just flushes.

- **Full House**: A full house consists of three cards of one rank and two cards of another rank. The number of ways to get a full house is calculated as:

$$13 \times \binom{4}{3} \times 12 \times \binom{4}{2}$$

- **Four of a Kind**: The number of ways to get four of a kind is calculated as:

$$13 \times \binom{12}{1} \times \binom{4}{1}$$

- **Straight Flush**: A straight flush consists of five cards of sequential rank, all of the same suit. The formula given for straight flush (excluding royal flush) is:

$$\binom{10}{1} \times \binom{4}{1} - \binom{4}{1}$$

  This accounts for 10 possible low cards and 4 suits, then subtracts the 4 royal flushes.

- **Royal Flush**: A Royal Flush is a specific type of straight flush: 10, Jack, Queen, King, Ace of the same suit. The number of Royal Flushes is 4 (one for each suit).

### 1.2.2 Probability Each Pile Has an Ace

Consider distributing 52 cards into four piles, with each pile having $n = 13$ cards. We want to calculate the probability that each pile contains exactly one Ace.

1. **Probability the first pile has an Ace**: The probability that there is an Ace in the first pile is 1 (conceptually, we assume an Ace is placed in the first pile and calculate subsequent probabilities relative to this).

2. **Probability the second Ace belongs to a different pile**: After one Ace is placed, there are 51 cards remaining. Among these, $52 - 13 = 39$ cards are outside the first pile. The probability that the second Ace is in a different pile is:
$$\frac{52-13}{51} = \frac{39}{51}$$

3. **Probability the third Ace belongs to a different pile**: After two Aces are in two different piles, there are 50 cards left. Of these, $39 - 13 = 26$ cards are in the remaining piles that don't yet have an Ace. The probability the third Ace is in a new pile is:
$$\frac{39-13}{50} = \frac{26}{50}$$

4. **Probability the fourth Ace belongs to a different pile**: With three Aces in three different piles, there are 49 cards left. Of these, $26 - 13 = 13$ cards are in the last pile without an Ace. The probability the fourth Ace is in the last new pile is:
$$\frac{26-13}{49} = \frac{13}{49}$$

The total probability that each pile has an Ace is the product of these probabilities:
$$\text{Total Probability} = 1 \times \frac{39}{51} \times \frac{26}{50} \times \frac{13}{49}$$

# 2 Normal Distribution

## 2.1 Definition

A random variable $X$ is said to follow a normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$, denoted by
$$X \sim \mathcal{N}(\mu, \sigma^2),$$

if its probability density function (PDF) is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}.$$

### 2.2 Moment Generating Function

The moment generating function (MGF) of $X \sim \mathcal{N}(\mu, \sigma^2)$ is

$$M_X(t) = \mathbb{E}\big[e^{tX}\big] = \exp\big(\mu t + \tfrac{1}{2}\sigma^2 t^2\big), \quad t \in \mathbb{R}.$$

### 2.3 Special Case: Standard Normal

For the standard normal distribution $Z \sim \mathcal{N}(0,1)$:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad M_Z(t) = \exp\big(\tfrac{1}{2}t^2\big).$$