

Simplex method for Linear Programming problem.

- ① \oplus $\max z = c^T x$
 s.t. $Ax \leq b$ (in inequalities)
 $A: mxn, x: nx1, b: mx1$
 $x_i \geq 0$
- \oplus Introduce slack variables $x_{n+1}, x_{n+2}, \dots, x_{n+m} \geq 0$
- $\rightarrow [A; I] x = b$
 $x = [x_1, \dots, x_{n+m}]^T$
- \oplus Assume $b_i \geq 0 \ \forall i \rightarrow$ initial feasible solution is
- $x_{n+1} = b_1$
 $x_{n+2} = b_2$
 \vdots
 $x_{n+m} = b_m$
 $x_1 = x_2 = \dots = x_n = 0$

\oplus

$$\left\{ \begin{array}{l} x_{n+1} = b_1 - a_{11}x_1 - \dots - a_{1n}x_n \geq 0 \\ x_{n+m} = b_m - a_{m1}x_1 - \dots - a_{mn}x_n \geq 0 \\ z = c_1x_1 + c_2x_2 + \dots + c_nx_n \text{ (objective fn)} \end{array} \right.$$

- \oplus If $c_i > 0 \rightarrow z \uparrow$ if $x_i \uparrow$ with $x_j = 0, j \neq i$
 \oplus If $c_i > c_j, j=1, 2, \dots, n \rightarrow \uparrow x_i$ but cannot go infinitely
 \oplus If $c_i \geq c_j, j=1, 2, \dots, n$ since $x_{n+1}, \dots, x_{n+m} \geq 0$

$$0 \leq x_i \leq \frac{b_i}{c_i} = d_i, i = 1, \dots, m$$

④ Let $j = k$ yields the strictest constraint for x_i

$$\rightarrow x_{n+k} = 0 \rightarrow x_i = \frac{b_k}{a_{ki}} = d_k$$

Transform the k^{th} entry

$$\rightarrow x_i = \frac{b_k}{a_{ki}} - \sum_j \frac{a_{kj}}{a_{ki}} x_j - \frac{1}{a_{ki}} x_{n+k}$$

→ New set of tableau and objective function

④ Repeat till $c_i \leq 0 \ \forall i \rightarrow$ cannot choose an incoming variable

② Auxiliary Problem for infeasible problem

④ If $b_j < 0 \quad j = 1 \dots m$

First solve the auxiliary problem:

$$\max W = -x_0$$

$$\text{s.t. } \sum_{i=1}^n a_{ji} x_i - x_0 \leq b_j, \quad j = 1, 2, \dots, m$$
$$x_i \geq 0, \quad i = 0, 1, 2, \dots, n$$

choose x_0 large enough \rightarrow can find a feasible solution.

③ Degeneracy:

⊕ More than 1 outgoing variable
 \Leftrightarrow more than 1 stictest constraints

⊕ Two basic (original variables) = 0

\rightarrow No improvement in obj \rightarrow degenerate iteration

\rightarrow Infinite loop.

④ Revised simplex method:

$$\max z = c^T x$$

$$Ax \leq b, x \geq 0$$

⊕ Introduce slack variable.

$$\tilde{A}x = [A_B \ A_N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b$$

A_B : ~~max~~ $m \times n$

A_N : $m \times n$

$$\begin{bmatrix} a_{11} & \dots & a_{1n+m} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn+m} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$m \times (m+n)$ $(m+n) \times 1$

$$\oplus \quad z = c_B^T x_B + c_N^T x_N$$

$$A_B x_B + A_N x_N = b$$

$$\Rightarrow x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$

$$\begin{aligned}
 Z &= c_B^T x_B + c_N^T x_N \\
 &= c_B^T (A_B^{-1} b - A_B^{-1} A_N x_N) + c_N^T x_N \\
 &= c_B^T A_B^{-1} b + (c_N^T - c_B^T A_B^{-1} A_N) x_N \\
 &\quad (A_B^{-1} = B) \\
 \rightarrow \text{Tabelau: } &x_B = \overbrace{B^{-1} b - B^{-1} A_N x_N}^{x_B^*} \geq 0 \\
 Z &= \underbrace{c_B^T B^{-1} b}_{x^*} + (c_N^T - c_B^T B^{-1} A_N) x_N
 \end{aligned}$$