Coin/Dice problems

I Playing a game :

- stop when results achieved > expected value of that turn
- Use a coin to generate even odds : HT , TH .
- use a coin to generate 3 events :

$$\underbrace{HH, HT, TH}, TT \Rightarrow \text{Do again}$$

successing create events with $p = 0.75$

-> Expected no of tosses : p (first of success)

II No of tosses until:

- get a head / tail :
  - considered as first success => follow a geometrics distribution

$E[X^2] = E[X(X-1)] + E[X]$

- $p(x=k) = (1-p)^{k-1} p$ - $Var(X) = \frac{\cancel{1}-p}{p^2}$

- $E(X) = \sum_{k=1}^{\infty} (1-p)^{k-1} p . k = \frac{1}{p}$

$$= p \sum_{k=1}^{\infty} (1-p)^{k-1} k$$

$$= p \left(\sum_{k=1}^{\infty} (1-p)^{k-1} + \sum_{k=2}^{\infty} (1-p)^{k-2} + \dots \right)$$

$$= p \left(\frac{1}{1-(1-p)} + \frac{(1-p)}{p} + \frac{(1-p)}{p^2} + \dots \right)$$

⊕ get n heads / tails

- considered like a sum of independent geometric distributions

$\rightarrow E(X) = \frac{n}{p}$, $var(X) = \frac{n(1-p)}{p^2}$

- Followed a negative binomial distribution

$$P(X=k) = \binom{k-1}{n-1} p^n (1-p)^{k-n}$$

⊕ specific sequence: (e.g 2H / 2T)

$P(N=k) = P(k-1$ throws after 1st throw is determined$)$

$$= \left(\frac{1}{2}\right)^{k-1}$$

$$\rightarrow E(N) = \sum^{\infty} \left(\frac{1}{2}\right)^{k-1} k$$

k = min for the sequence to occur.

⊕ n heads in a row:

Approach 1: Markov chain

state (0) (1) (2) . . . (n)

0 head, 1 head, n heads

state x: obtained x heads in a row

$g(x)$: no of tosses to get n heads in a row

→ we need to find $g(0)$

$$g(0) = \frac{1}{2}(g(0)+1) + \frac{1}{2}(g(1)+1)$$

$$g(1) = \frac{1}{2}(g(0)+g(2)) + 1$$

$$g(x) = \frac{1}{2}(g(0)+g(x+1)) + 1$$

$$\vdots$$

$$g(n) = 0$$

$$g(0) = \frac{1}{2}g(0) + \frac{1}{2}g(1) + 1$$

$$= \frac{1}{2}g(0) + \frac{1}{2}\left(\frac{1}{2}g(0) + \frac{1}{2}g(2) + 1\right) + 1$$

$$= g(0)\left(\frac{1}{2}+\frac{1}{2^2}\right) + \left(1+\frac{1}{2}\right) + \frac{1}{2^2}g(2)$$

$$= g(0)\left(\frac{1}{2}+\frac{1}{2^2}\right) + \left(1+\frac{1}{2}\right) + \frac{1}{2^2}\left(\frac{1}{2}g(0) + \frac{1}{2}g(3)+1\right)$$

$$= \dots = g(0)\left(\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^n}\right) + \left(1+\frac{1}{2}+\dots+\frac{1}{2^{n-1}}\right)$$

$$= g(0)\left(\frac{1-\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}} - 1\right) + \left(\frac{1-\left(\frac{1}{2}\right)^{n}}{\frac{1}{2}}\right)$$

$$= g(0)\left(1-\left(\frac{1}{2}\right)^{n}\right) + 2 - \left(\frac{1}{2}\right)^{n-1}$$

$$\rightarrow \left(\frac{1}{2}\right)^{n} g(0) = 2 - \left(\frac{1}{2}\right)^{n-1}$$

$$\rightarrow g(0) = 2^{n+1} - 2$$

Approach 2 : Betting strategies :

construct a strategy so that :

⊕ Total payoff after each step is a martingale → use stopping time to calculate the stopping time when the sequence is reached.

⊕ payoff is bounded to use the above → at toss n^{th}, lose at most $-n$ (prove by induction)

i^{th} toss :

(1) $i = 1$ or $X_{i-1} = \text{Tail}$ → bet 1 on head

(2) $X_{i-1} = \text{head}$ → bet 3 on head

(3) $X_{i-2} = X_{i-1} = \text{head}$ → bet 7 on head

(4) stop when you get HHH

→ For every toss your expected payoff is 0

→ At stopping time n :

$$E(\text{Payoff}) = E(-(n-3) + 1 + 3 + 7) = 0$$

→ $n = 14$

④ <u>get N distinct types:</u>

- X_i, $i = 1, 2, \dots, N$ be the no of rolls/tosses needed to get the i^{th} distinct type after $(i-1)$ distinct types have been obtained

→ Total needed $= \sum_{i=1}^{N} X_i$

~~$p(X_i$~~ - X_i follows a geometric distribution with probability: $\frac{N-(i-1)}{N}$

→ $E(X_i) = \frac{N}{N-(i-1)}$

III <u>Toss/Roll n times</u>

⊕ <u>same outcome:</u>

- unlikely to be fair
- assume probability of being unfair is p

→ calculate probability of (unfair | all heads)

⊕ <u>At least k heads:</u>

$$\sum_{i=k}^{n} \binom{N}{i} p^i (1-p)^{N-i}$$

→ $Bi(n, p)$

for <u>large</u> n $\sim \underline{N(np, np(1-p))}$

95% of normal dist lies between 2 s.d + mean?

⊕ Expected no of strings of specific sequence

- x : length of sequence
- allowing overlap: $n-(x-1)$ slots this sequence can occur

$$\rightarrow p = (n-(x-1)) \times \underbrace{\left(\frac{1}{2}\right)^x}_{p(\text{occurs})}$$

⊕ probability for no specific sequence

- No of possible strings after n coin toss:

$$2^n$$

- G_n : no of string that do not have that sequence

→ find G_1, G_2

- consider first toss and deduce G_n from G_{n-1}, G_{n-2}

⊕ Dice with ↑ order

$\rightarrow 1 \times \frac{5}{6} \times \frac{4}{6}$

↑ → all different outcomes → P

⊗ for ↑ → follow a specific sequence

→ P(one specific sequence on all permutation) conditioned on different outcomes → $\frac{1}{3!}$

(+) no of distinct types after n tosses

Y : no of distict typects

$i = 1, \dots, N$: all types

$$I_i = \begin{cases} 1, & \text{appears} \\ 0, & \text{not appear} \end{cases}$$

$\Rightarrow Y = I_1 + \dots + I_N$

$\Rightarrow$ Need to find $E[I_i] = P(I_i = 1)$

$$P(I_i = 1) = 1 - P(I_i = 0)$$

$$= 1 - \left(\frac{N-1}{N}\right)^n$$

$\downarrow$

P(no I_i in all n tosses)