P vs Q Measures and Yield Curve Modeling: Practical Summary

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1 Big Picture: P vs Q

In pricing (risk–neutral, $\mathbb{Q}$), models enforce no–arbitrage and treat primitive rates (short rate $r(t)$ or instantaneous forwards $f(t,T)$) as states so that discounted tradables are martingales. In real–world forecasting (physical, $\mathbb{P}$), desks typically model observed yields at fixed tenors or a low–dimensional factorization (level/slope/curvature), focusing on prediction, risk premia, and communication.

- **Primitive vs overlapping:** Forwards are non–overlapping primitives; yields and par swaps are overlapping averages of forwards.
- **Implication:** Directly splining par rates/yields can create shape artifacts; modeling primitives keeps static relations clean under $\mathbb{Q}$.

2 Static Curve Construction for Linear Desks

For linear products, the practical notion of "arbitrage–free" is an internally consistent discount curve $D(T)$ that reprices inputs and is well–behaved.

2.1 Recommended workflow

1. **Bootstrap** discount factors $D(T_i)$ on all cashflow dates from liquid quotes (deposits/FRAs/futures/swaps).
2. **Interpolate the right object**: use log–linear on discounts (linear in $\ln D(T)$) or a shape–preserving method (e.g., monotone–convex) between nodes.
3. **Reprice inputs** exactly (or within tight tolerances); keep $D(0) = 1$ and $D(T) > 0$.
4. **Sanity checks**: implied simple/inst. forwards are stable (no spurious oscillations from interpolation).

2.2 Notes

- Log–linear on $\ln D(T)$ implies piecewise–constant instantaneous forwards $f(T) = -\frac{d}{dT}\ln D(T)$ on each interval between nodes; no spline overshoot.
- Non–increasing $D(T)$ (non–negative forwards) is a *policy* choice, not a mathematical necessity in regimes that admit negative rates.

3 Dynamic Consistency and Why It Matters

Fitting today's curve is not enough when hedging options or path–dependent payoffs. An arbitrage–free term–structure *evolution* provides:

- **Time consistency**: prices evolve as discounted $\mathbb{Q}$ expectations; no systematic model P&L from "time passing".
- **Cross–tenor coherence**: forwards across maturities move jointly so that FRA/swap replications track.
- **Hedge validity**: Greeks/hedges depend on volatilities and correlations across tenors.

In Heath–Jarrow–Morton (HJM) form, if the instantaneous forward has volatility $\sigma(t,T)$, the drift is pinned by no–arbitrage:

$$\mu_f(t,T) \;=\; \sigma(t,T)\int_t^T \sigma(t,u)\,du. \tag{1}$$

Violating these ties by shocking tenors ad hoc can create roll/calendar arbitrage and unstable hedges.

4 Arbitrage–Free Factor Models

Research and practice favor low–dimensional, arbitrage–free factor models that:

- **Preserve day–1 fit**: include a deterministic shift so $P(0,T) = D(T)$ from the bootstrapped curve.
- **Impose no–arb dynamics**: HJM/LMM/affine families ensure drift restrictions like (1).
- **Provide joint dynamics**: a small state (level/slope/curvature) drives vol/correlation across tenors.
- **Support P–Q mapping**: link $\mathbb{Q}$ to $\mathbb{P}$ via market price of risk for forecasting and risk.

Examples: arbitrage–free Nelson–Siegel (AFNS), affine term structure (Dai–Singleton), HJM/LMM with the initial forward curve taken from the static bootstrap.

5 "Curve + Vols" vs Factor Dynamics

Calibrating vols per tenor on top of a static curve can fit cap/swaption surfaces, but without a factorization you lack a unique, coherent joint law. An arbitrage–free factor model is superior when you need:

- Stable cross–tenor hedges and risk attribution.
- Pricing/hedging of multi–period or path–dependent products (Bermudans, callable bonds, CMS).
- Smooth interpolation/extrapolation across sparse option quotes.

You still bootstrap the curve from linear instruments exactly; the factor model adds the dynamically consistent layer that matches options while preserving the curve.

6 Key Identities (for intuition)

$$y(t,T) = \frac{1}{T-t}\int_t^T f(t,u)\,du \quad \text{(yield is an average of forwards)} \tag{2}$$

$$r(t) = f(t,t) \quad \text{(short rate)} \tag{3}$$

$$\mathrm{FRA}(T_i,T_{i+1}) = \frac{D(T_i)}{D(T_{i+1})} - 1 \bigg/ \Delta_{i,i+1} \quad \text{(simple forward)} \tag{4}$$

Mapping $\mathbb{Q}$ to $\mathbb{P}$ is often summarized heuristically as $\mu_{\mathbb{P}} = \mu_{\mathbb{Q}} + \Sigma\,\lambda$, where λ is the (state–dependent) market price of risk and Σ are vol loadings.

7 Practical Workflow Summary

1. Build a clean static curve: bootstrap $D(T)$; interpolate in $\ln D(T)$ or with monotone–convex; reprice inputs; sanity check forwards.
2. Choose an arbitrage–free factor family (AFNS/HJM/LMM/affine) that preserves $D(T)$ at $t = 0$.
3. Calibrate vol parameters to options; monitor fit errors and surface no–arb (calendar/strike convexity).
4. If needed, estimate a $\mathbb{P}$ overlay (market price of risk) for forecasting and stress.