Probability Notes: Cards, Coins, Dice, Random Variables, and Useful Techniques

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1 Card Problems

1.1 Drawing n Cards from a Deck

We begin with standard techniques used in drawing cards without replacement.

1.1.1 Expected Total Value

Let X_i be the value of the i-th drawn card. Because expectation is linear,

$$\mathbb{E}\big[X_1 + \cdots + X_n\big] = n\,\mathbb{E}[X_1].$$

In a standard 52-card deck with ranks $\{1, \ldots, 13\}$ equally represented,

$$\mathbb{E}[X_1] = \frac{1 + 2 + \cdots + 13}{13} = 7.$$

Intuition. Even though later draws depend on earlier ones, linearity of expectation does *not* require independence. That is why the formula remains simple.

1.1.2 Including Jokers

If two Jokers worth 100 are added (giving 54 cards),

$$\mathbb{E}[X_1] = \frac{4}{54}(1 + \cdots + 13) + \frac{2}{54} \cdot 100.$$

1.1.3 Replacing the Minimum of the First $n+1$ Cards

Suppose we draw $n + 1$ cards and replace the minimum card by a new card. The expected sum becomes

$$\mathbb{E}\Big[\sum_{i=1}^{n+1} X_i\Big] - \mathbb{E}[\min(X_1, \ldots, X_{n+1})].$$

To compute $\mathbb{E}[M]$ where $M = \min(X_1, \ldots, X_{n+1})$ and each X_i is uniform on $\{1, \ldots, 13\}$:

$$\mathbb{P}(M > k) = \left(1 - \frac{k}{13}\right)^{n+1}.$$

Then

$$\mathbb{E}[M] = \sum_{k=1}^{13} \mathbb{P}(M \geq k).$$

Intuition. The minimum is small precisely when at least one value is small. Because for discrete variables it is easier to compute $\mathbb{P}(M > k)$, we use a cumulative expectation trick.

1.1.4 Expected Position of the First Ace

Let X be the number of cards revealed before the first Ace appears. Let the 48 non-Ace cards be "ordinary" cards.

For each ordinary card i, define indicator:

$$I_i = \mathbf{1}\{\text{card } i \text{ appears before any Ace}\}.$$

Then

$$X = 1 + \sum_{i=1}^{48} I_i.$$

Because all orderings of the deck are equally likely, card i falls into one of the 5 intervals created by the 4 Aces with equal probability, so $\mathbb{P}(I_i = 1) = 1/5$. Hence

$$\mathbb{E}[X] = 1 + 48 \cdot \frac{1}{5} = \frac{53}{5}.$$

Generalization. With m ordinary items and n special items:

$$\mathbb{E}[\text{position of first special}] = 1 + \frac{m}{m+n}.$$

Intuition. Each ordinary card has probability $1/(m+n)$ of falling into any of the $m+n$ interval slots between special cards.

1.2 Dealing the Whole Deck

1.2.1 Counting Poker Hands

The total number of 5-card hands is $\binom{52}{5}$.

Below are standard counts, rewritten for clarity:

- **One Pair:**
$$\binom{13}{1}\binom{4}{2}\binom{12}{3}4^3.$$
- **Two Pairs:**
$$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}4.$$

- **Three of a Kind:**
$$\binom{13}{1}\binom{4}{3}\binom{12}{2}4^2.$$
- **Straight:**
$$10 \cdot (4^5 - 4).$$
- **Flush:**
$$4 \cdot \binom{13}{5} - 10 \cdot 4.$$
- **Full House:**
$$13\binom{4}{3} \cdot 12\binom{4}{2}.$$
- **Four of a Kind:**
$$13 \cdot 12 \cdot 4.$$
- **Straight Flush:**
$$10 \cdot 4 - 4.$$
- **Royal Flush:**
$$4.$$

Intuition. A straight flush is extremely rare: the probability is roughly 1 in 65,000.

1.2.2 Probability That Each Pile Receives Exactly One Ace

Shuffle a full deck of 52 cards uniformly and deal into four piles of 13. Track the locations of the four Aces among the 52 positions.

First Ace. The first Ace can appear in any position without restriction:

$$\Pr(A_1) = 1.$$

Second Ace. After the first Ace is placed, one pile contains an Ace and must not receive another. There are 51 remaining positions, of which 39 lie in the other three piles:

$$\Pr(A_2 \text{ in a new pile} \mid A_1) = \frac{39}{51}.$$

Third Ace. Two piles now contain one Ace each. Among the 50 remaining positions, 26 lie in the two piles that do not yet contain an Ace:

$$\Pr(A_3 \text{ in a new pile} \mid A_1, A_2) = \frac{26}{50}.$$

Fourth Ace. Three piles contain one Ace each. Among the 49 remaining positions, the final pile contains 13 available positions:

$$\Pr(A_4 \text{ in the last empty pile} \mid A_1, A_2, A_3) = \frac{13}{49}.$$

Final Probability. Multiplying the conditional probabilities gives

$$\Pr(\text{each pile gets exactly one Ace}) = 1 \times \frac{39}{51} \times \frac{26}{50} \times \frac{13}{49}.$$

2 Coin and Dice Models

2.1 Geometric Distribution

Let X be the number of tosses until the first success.

$$\mathbb{P}(X = k) = (1-p)^{k-1}p, \quad \mathbb{E}[X] = \frac{1}{p}.$$

Intuition. Each trial is a fresh chance to succeed; on average you need $1/p$ attempts.

2.2 Negative Binomial Distribution

Number of tosses to obtain n successes:

$$\mathbb{P}(X = k) = \binom{k-1}{n-1} p^n (1-p)^{k-n}, \qquad \mathbb{E}[X] = \frac{n}{p}.$$

2.3 Expected Time to n Consecutive Heads

We assume a fair coin, i.e. each toss is Heads (H) or Tails (T) with probability 1/2.

Let $g(k)$ denote the expected additional number of tosses needed to obtain n consecutive Heads, given that we currently have a run of k consecutive Heads. Thus

$$g(k) = \mathbb{E}[\text{time to reach } n \text{ Heads in a row} \mid \text{current run length } k],$$

for $k = 0, 1, \ldots, n$, and in particular we want $g(0)$. Once we already have n consecutive Heads, we are done, so

$$g(n) = 0.$$

Recurrence. For $0 \le k < n$, consider the next coin toss starting from state k:

- With probability $\frac{1}{2}$ we toss H, so the run of Heads increases from k to $k+1$, and the expected remaining time becomes $g(k+1)$.
- With probability $\frac{1}{2}$ we toss T, so the run is broken and we return to state 0, with expected remaining time $g(0)$.

In either case, we have used 1 toss. Hence, for $k = 0, 1, \ldots, n-1$,

$$g(k) = 1 + \frac{1}{2}g(k+1) + \frac{1}{2}g(0). \tag{$*$}$$

Difference equation. For $k = 0, \ldots, n-2$, write the recurrence (**??**) for k and $k+1$ and subtract:

$$\begin{aligned} g(k) &= 1 + \tfrac{1}{2}g(k+1) + \tfrac{1}{2}g(0), \\ g(k+1) &= 1 + \tfrac{1}{2}g(k+2) + \tfrac{1}{2}g(0). \end{aligned}$$

Subtracting the second from the first gives

$$g(k) - g(k+1) = \frac{1}{2}\big(g(k+1) - g(k+2)\big).$$

Define the differences

$$d_k := g(k) - g(k+1), \qquad k = 0, 1, \ldots, n-1.$$

Then the relation above becomes

$$d_k = \frac{1}{2}d_{k+1}, \qquad k = 0, 1, \ldots, n-2.$$

Thus the differences form a geometric progression:

$$d_{k+1} = 2d_k, \qquad d_k = 2^{k-(n-1)}d_{n-1}, \qquad k = 0, 1, \ldots, n-1.$$

Using $g(n) = 0$, we have

$$d_{n-1} = g(n-1) - g(n) = g(n-1),$$

so

$$d_k = g(k) - g(k+1) = 2^{k+1-n}\, g(n-1), \qquad k = 0, 1, \ldots, n-1.$$

Expressing $g(0)$ in terms of $g(n-1)$. Summing the differences from $k = 0$ to $n-1$ gives

$$g(0) - g(n) = \sum_{k=0}^{n-1} (g(k) - g(k+1)) = \sum_{k=0}^{n-1} d_k = g(n-1) \sum_{k=0}^{n-1} 2^{k+1-n}.$$

Since $g(n) = 0$, this becomes

$$g(0) = g(n-1)\, 2^{1-n} \sum_{k=0}^{n-1} 2^k = g(n-1)\, 2^{1-n} \big(2^n - 1\big) = g(n-1)\big(2 - 2^{1-n}\big). \tag{1}$$

On the other hand, for $k = n-1$ the recurrence (**??**) reads

$$g(n-1) = 1 + \frac{1}{2} g(n) + \frac{1}{2} g(0) = 1 + \frac{1}{2} g(0), \tag{2}$$

since $g(n) = 0$.

Solving for $g(0)$. Substitute (**??**) into (**??**):

$$g(0) = \big(1 + \tfrac{1}{2} g(0)\big)\big(2 - 2^{1-n}\big).$$

Expand:

$$g(0) = 2 - 2^{1-n} + g(0)\big(1 - 2^{-n}\big).$$

Move the $g(0)$ terms to one side:

$$g(0) - g(0)\big(1 - 2^{-n}\big) = 2 - 2^{1-n},$$

so

$$g(0)\, 2^{-n} = 2 - 2^{1-n}.$$

Multiplying both sides by 2^n gives

$$g(0) = 2^{n+1} - 2.$$

3 Coupon Collector

Expected time to collect all N coupon types:

$$\mathbb{E}[T] = N H_N.$$

Expected number of distinct types after n trials:

$$\mathbb{E}[\text{distinct}] = N \left(1 - \left(1 - \frac{1}{N}\right)^n\right).$$

4 Approximations and Sequence Counts

4.1 Occurrences of a Pattern

Expected occurrences of a specific sequence of length x in n tosses:

$$(n-x+1)\left(\frac{1}{2}\right)^x.$$

4.2 Binomial Approximation

$$X \sim \mathrm{Bin}(n,p) \approx \mathcal{N}(np, np(1-p)).$$

5 Normal Distribution and MGF

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right).$$

6 Sums and Products of Random Variables

6.1 Sum of Uniform Random Variables

For $X_i \sim U[0,1]$,

$$\mathbb{P}(X_1 + \cdots + X_n \le 1) = \frac{1}{n!}.$$

Intuition. The region $x_1 + \cdots + x_n \le 1$ in the n-cube is an n-simplex whose volume is $1/n!$.

7 Stick-Breaking Triangle Problem

Break at $X < Y$ with $X, Y \sim U[0,1]$. A triangle forms iff all pieces $< 1/2$. The probability is:

$$\frac{1}{4}.$$

8 Order Statistics

8.1 Max and Min

For i.i.d. variables with CDF F:

$$\mathbb{P}(M \ge x) = (1-F(x))^n, \qquad \mathbb{P}(Z \le x) = F(x)^n.$$

8.2 Correlation of Max and Min for Two Uniforms

Let $X_1, X_2 \sim U[0,1]$. Then:

$$\mathbb{E}[\min] = \frac{1}{3}, \qquad \mathbb{E}[\max] = \frac{2}{3}, \qquad \mathbb{E}[\min \cdot \max] = \frac{1}{4}.$$

9 Transformations and Jensen's Inequality

If $Y = g(X)$ and g is monotone, then

$$f_Y(y) = f_X(g^{-1}(y)) \frac{1}{|g'(g^{-1}(y))|}.$$

Jensen:

$$\mathbb{E}[g(X)] \geq g(\mathbb{E}[X]).$$

10 Moments and Sampling

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}.$$

11 Correlation, Covariance, and Portfolios

11.1 Constructing Correlated Gaussians

$$X = aZ_1, \qquad Y = b(\rho Z_1 + \sqrt{1-\rho^2}\, Z_2).$$

11.2 Minimum-Variance Hedge Ratio

Variance of $A - hB$:

$$\mathrm{Var}(A - hB) = \sigma_A^2 - 2h\rho\sigma_A\sigma_B + h^2\sigma_B^2.$$

Setting derivative 0 gives:

$$h^* = \rho \frac{\sigma_A}{\sigma_B}.$$