

Simplex method for Linear programming problem.

①. $\oplus \quad \max \quad z = c^T X$

s.t. $Ax \leq b \quad (m \text{ inequalities})$

$A: m \times n, \quad x: n \times 1, \quad b: m \times 1$

$x_i \geq 0$

\oplus Introduce slack variables $x_{n+1}, x_{n+2}, \dots, x_{n+m} \geq 0$

$\rightarrow [A; I] x = b$

$x = [x_1, \dots, x_{n+m}]^T$

\oplus Assume $b_i \geq 0 \quad \forall i \rightarrow$ initial feasible solution is

$x_{n+1} = b_1$

$x_{n+2} = b_2$

\vdots

$x_{n+m} = b_m$

$x_1 = x_2 = \dots = x_n = 0$

$\oplus \quad \left\{ \begin{array}{l} x_{n+1} = b_1 - a_{11}x_1 - \dots - a_{1n}x_n \geq 0 \\ x_{n+m} = b_m - a_{m1}x_1 - \dots - a_{mn}x_n \geq 0 \\ z = c_1x_1 + c_2x_2 + \dots + c_nx_n \quad (\text{objective func}) \end{array} \right.$

\oplus If $c_i > 0 \rightarrow z \uparrow$ if $x_i \uparrow$ with $x_j = 0, j \neq i$

\oplus If $c_i \geq c_j, j = 1, 2, \dots, n \rightarrow \uparrow x_i$ but cannot go infinitely since $x_{n+1}, \dots, x_{n+m} \geq 0$

$0 \leq x_i \leq \frac{b_j}{a_{ji}} = d_j, j = 1, \dots, m$

④ Let $j = k$ yields the strictest constraint for x_i

$$\rightarrow x_{n+k} = 0 \rightarrow x_i = \frac{b_k}{a_{ki}} = d_k$$

Transform the k^{th} entry

$$\rightarrow x_i = \frac{b_k}{a_{ki}} - \sum_{j \neq k} \frac{a_{kj}}{a_{ki}} x_j - \frac{1}{a_{ki}} x_{n+k}$$

\rightarrow New set of tableau and objective function

④ Repeat till $c_i \leq 0 \forall i \rightarrow$ cannot choose an incoming variable

②. Auxiliary Problem for infeasible problem

④ If $\exists b_j < 0 \quad j = 1, 2, \dots, m$

④ First solve the auxiliary problem:

$$\max W = -x_0$$

$$\text{s.t.} \quad \sum_{i=1}^n a_{ji} x_i - x_0 \leq b_j, \quad j = 1, 2, \dots, m$$

$$x_i \geq 0, \quad i = 1, 2, \dots, n$$

choose x_0 large enough \rightarrow can find a feasible solution.

③ Degeneracy:

⊕ more than 1 outgoing variable

⇒ more than 1 stickest constraints

⊕ two basic (original variables) = 0

→ no improvement in obj → degenerate iteration

→ Infinite loop.

④ Revised simplex method:

$$\max z = c^T x$$

$$Ax \leq b, x \geq 0$$

⊕ Introduce slack variable.

$$\tilde{A}x = \begin{bmatrix} A_B & A_N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b$$

$$A_n: \text{m} \times \text{m} \times \text{m}$$

$$A_B: \text{m} \times \text{n}$$

$$\begin{bmatrix} a_{11} & \dots & a_{1n} & 0 & \dots & 0 \\ \vdots & & \vdots & & & \\ a_{m1} & \dots & a_{mn} & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\text{m} \times (\text{m} + \text{n})$$

$$(\text{m} + \text{n}) \times 1$$

$$\oplus \quad z = c_B^T x_B + c_N^T x_N$$

$$A_B x_B + A_N x_N = b$$

$$\Rightarrow x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$

$$z = c_B^T x_B + c_N^T x_N$$

$$= c_B^T (A_B^{-1} b - A_B^{-1} A_N x_N) + c_N^T x_N$$

$$= c_B^T A_B^{-1} b + (c_N^T - c_B^T A_B^{-1} A_N) x_N$$

$$(A_B^{-1} = B)$$

→ Tableau:

$$x_B = \overbrace{B^{-1} b}^{x_B^*} - B^{-1} A_N x_N \geq 0$$

$$z = \underbrace{c_B^T B^{-1} b}_{x^*} + (c_N^T - c_B^T B^{-1} A_N) x_N$$