Chapter 5 Integrals

OUTLINE

In this chapter, we study about:

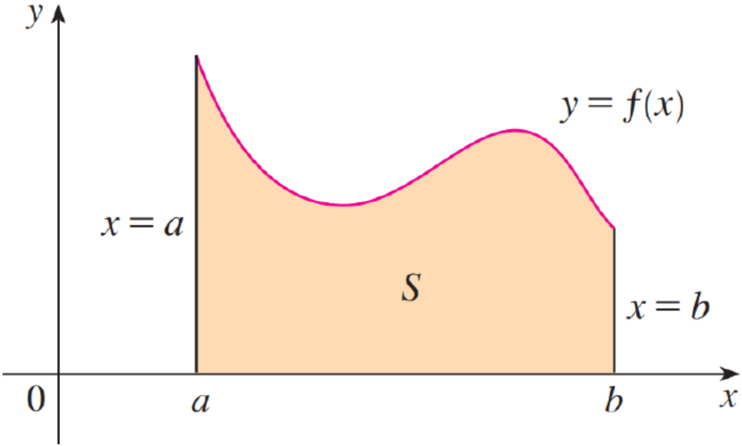
1. Definite Integral
2. Techniques of integration
3. Approximate integration
4. Improper integral of type 1
5. Improper integral of type 2

Chapter 5. Integrals

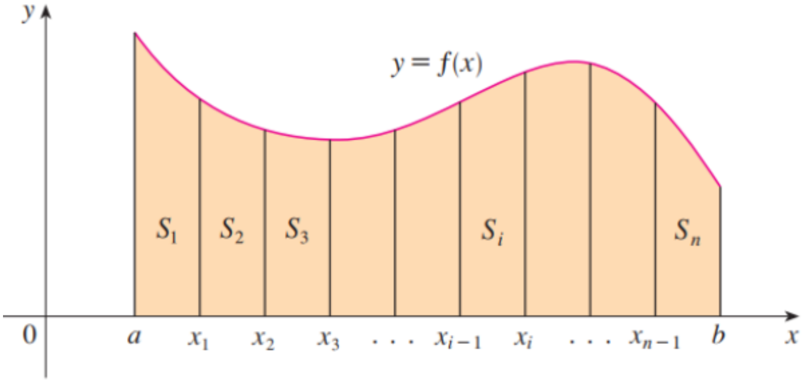
Definite Integral

THE AREA PROBLEM

Find the area of the region S that lies under the curve $y = f(x)$ from $x = a$ to $x = b$.

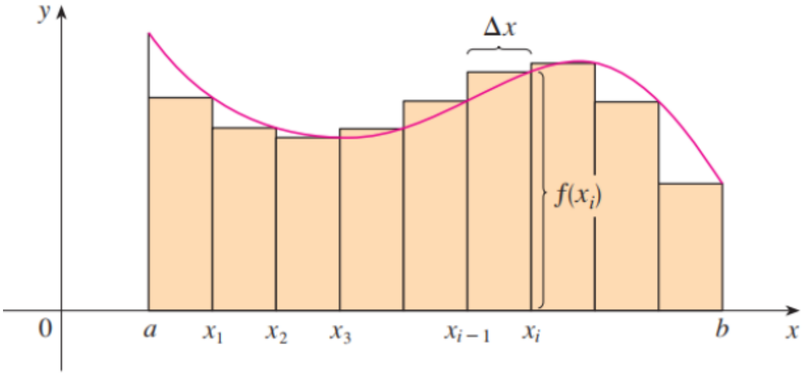


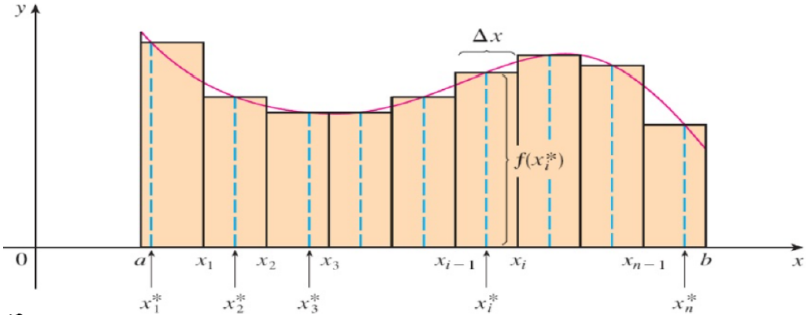
We start by subdividing S into strips $S_1; S_2; \ldots; S_n$ of equal width as in Figure below



- ⋆ Let's approximate the th strip S_i by a rectangle with width Δx and height $f(x_i)$. Then the area of the rectangle is $f(x_i)\Delta x$.
- ⋆ So, the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$





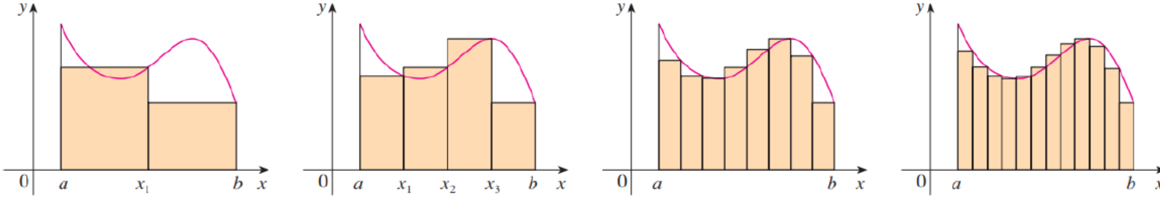
On each rectangle S_i takes any number x_i^* in the subinterval $[x_{i-1}, x_i]$. We have

$$S = S_1 + S_2 + \cdots + S_n$$

$$S \simeq f\left(x_1^*\right)\Delta x + f\left(x_2^*\right)\Delta x + \cdots + f\left(x_n^*\right)\Delta x$$

$$S \simeq \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$$

Figure below shows this approximation for n =2; 4; 8 and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to +\infty$



(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

DEFINITION

Definition. Given a function $f(x)$ that is continuous on the interval $[a,b]$ we divide the interval into n subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then the definite integral of $f(x)$ from a to b is

$$I = \lim_{n\to+\infty}\left(\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i\right) = \int_a^b f(x)dx.$$

Remark: the sum $\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$ is call **Riemann sum.**

The function is the integrand.

Upper limit of integration

Integral sign

x is the variable of integration.

$$\int_a^b \underbrace{f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have evaluated the integral.

If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx = \lim_{n_i \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x$$

PROPERTIES OF DEFINITE INTEGRALS

1. Order of Integration: $\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. Zero Width Interval: $\int_a^a f(x)dx = 0$
3. Constant Multiple: $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
4. Sum and Difference: $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. Additivity: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

PROPERTIES OF DEFINITE INTEGRALS

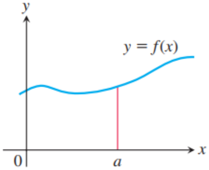
6. Max-Min Inequality: If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m \leq \frac{1}{b-a} \int_a^b f(x)dx \leq M.$$

7. Domination:

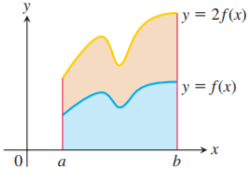
   ⋆ $f(x) \geq g(x)$ on $[a,b] \Rightarrow \displaystyle\int_a^b f(x)dx \geq \int_a^b g(x)dx.$

   ⋆ $f(x) \geq 0$ on $[a,b] \Rightarrow \displaystyle\int_a^b f(x)dx \geq 0.$



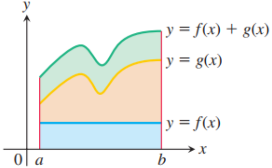
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



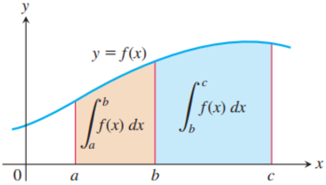
(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



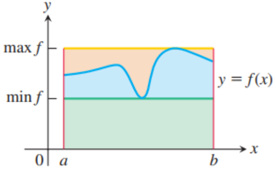
(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



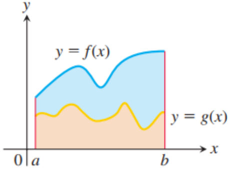
(d) *Additivity for Definite Integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a)$$



(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

Chapter 5. Integrals

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

Suppose f is continuous on $[a, b]$.

- ⋆ If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
- ⋆ $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

FTC 1 – DERIVATIVES OF INTEGRALS

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then the function g defined by

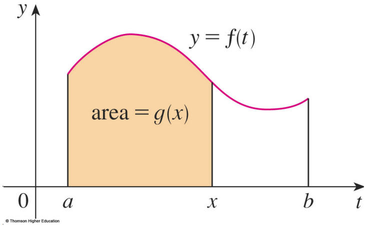
$$g(x) = \int_a^x f(t)dt \quad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $\boxed{g'(x) = f(x)}$.

Using Leibniz notation for derivatives, we can write the FTC1 as

$$\boxed{\frac{d}{dx}\int_a^x f(t)dt = f(x)}$$

when f is continuous.



Example. Use the Fundamental Theorem to find dy/dx if

a) $y = \displaystyle\int_a^x \left(t^3 + 1\right) dt$ b) $y = \displaystyle\int_x^5 3t \sin t dt$

Solution.

(a) $\dfrac{dy}{dx} = \dfrac{d}{dx} \displaystyle\int_a^x \left(t^3 + 1\right) dt = x^3 + 1$ Eq. (2) with $f(t) = t^3 + 1$

(b)
$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx} \int_x^5 3t \sin t dt &= \frac{d}{dx} \left(- \int_5^x 3t \sin t dt \right) \\ &= -\frac{d}{dx} \int_5^x 3t \sin t dt \\ &= -3x \sin x \end{aligned}$$

Eq. (2) with $f(t) = 3t \sin t$

Leibniz's Rule

If f is continuous on $[a, b]$ and if $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$, then

- $\star$ $\dfrac{d}{dx}\displaystyle\int_a^{u(x)} f(t)dt = f(u(x))\dfrac{du}{dx}$
- $\star$ $\dfrac{d}{dx}\displaystyle\int_{v(x)}^{u(x)} f(t)dt = f(u(x))\dfrac{du}{dx} - f(v(x))\dfrac{dv}{dx}$

Example.

1. Let $F(x) = \int_1^{\sqrt{x}} \sin t dt$. Find $F'(x)$.

   Solution. We have
$$F'(x) = \frac{\sin\sqrt{x}}{2\sqrt{x}}$$

2. Let $F(x) = \int_{\sqrt{x}}^{3x} t^2 \sin\left(1+t^2\right) dt$. Find $F'(x)$

   Solution. This will use the final formula that we derived above.

$$\left(\int_{\sqrt{x}}^{3x} t^2 \sin\left(1+t^2\right) dt\right)' = (3)(3x)^2 \sin\left(1+(3x)^2\right) - \frac{1}{2\sqrt{x}}(\sqrt{x})^2 \sin\left(1+(\sqrt{x})^2\right)$$
$$= 27x^2 \sin\left(1+9x^2\right) - \frac{1}{2}\sqrt{x}\sin(1+x).$$

The Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

NET CHANGE THEOREM

So, we can reformulate $\int_a^b f(x)dx = F(b) - F(a)$ as follows.

The integral of a rate of change is the net change:

$$\boxed{\int_a^b F'(x)dx = F(b) - F(a)}$$

NET CHANGE THEOREM

If $c(x)$ is the cost of producing x units of a certain commodity, then $c'(x)$ is the marginal cost. From Net change theorem,

$$\int_{x_1}^{x_2} c'(x)dx = c(x_2) - c(x_1),$$

which is the cost of increasing production from x_1 units to x_2 units.

NET CHANGE THEOREMS

If the rate of growth of a population is dn/dt, then

$$\int_{t_1}^{t_2} \frac{dn}{dt} dt = n(t_2) - n(t_1)$$

is the net change in population during the time period from t_1 to t_2.

- ⋆ The population increases when births happen and decreases when deaths occur.
- ⋆ The net change takes into account both births and deaths.

NET CHANGE THEOREM

If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$$

is the net **change of position**, or **displacement**, of the particle during the time period from t_1 to t_2.

If we want to calculate the distance the object travels during that time interval, then the distance is

$$\int_{t_1}^{t_2} |v(t)|dt = \textbf{total distance traveled}$$

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Example. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6\ m/s$.

1. Find the displacement of the particle during the time period $1 \le t \le 4$.
2. Find the distance traveled during this time period.

Solution. 1. The displacement is

$$s(4) - s(1) = \int_1^4 v(t)dt = \int_1^4 \left(t^2 - t - 6\right) dt$$
$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = -\frac{9}{2}$$

This means that the particle moved 4.5 m toward the left

2. The distance traveled is

$$\begin{aligned}
\int_1^4 |v(t)|dt &= \int_1^3 [-v(t)]dt + \int_3^4 v(t)dt \\
&= \int_1^3 \left(-t^2 + t + 6\right) dt + \int_3^4 \left(t^2 - t - 6\right) dt \\
&= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 \\
&= \frac{61}{6} \approx 10.17m
\end{aligned}$$

The Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)dx$$

that is,

$$\int_a^b f(x)dx = f(c)(b-a)$$

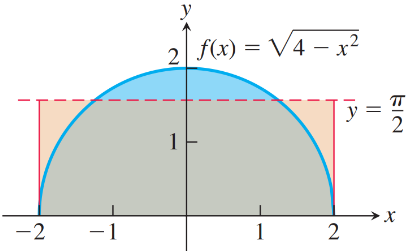
Example.

a) Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

b) Find the average value of the function $f(x) = x^2 + 3$ on the interval $[2, 5]$

Solution. a)

$$f_{\text{ave}} = \frac{1}{2 - (-2)} \int_{-2}^{2} \sqrt{4 - x^2} dx = \frac{1}{4}(2\pi) = \frac{\pi}{2}$$



The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$

Chapter 5. Integrals

Techniques of integration

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INTEGRATION FOMULARS

1. $$\int a dx = ax + C$$

2. $$\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + C$$

3. $$\int \frac{1}{x}dx = \ln|x| + C$$

4. $$\int e^{ax}dx = \frac{1}{a}e^{ax} + C$$

5. $$\int a^x dx = \frac{a^x}{\ln a} + C$$

6. $$\int \sin ax dx = -\frac{1}{a}\cos ax + C$$
$$\int \cos ax dx = \frac{1}{a}\sin ax + C$$

7. $$\int \frac{dx}{\cos^2 x} = \tan x + C$$
$$\int \frac{dx}{\sin^2 x} = -\cot x + C$$

8. $$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + C$$

9. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C = -\arccos\frac{x}{a} + C$$

10. $$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left(x + \sqrt{x^2 \pm a^2}\right) + C \quad a \neq 0$$

Example. Find

a) $\displaystyle\int \left(x + \sin 2x + e^x + \frac{1}{x}\right) dx$

b) $\displaystyle\int x(x+1)^2 dx$

c) $\displaystyle\int \frac{x^2 + xe^{2x} - 1}{x} dx$

THE SUBSTITUTION RULE

If $u = u(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(u(x))u'(x)dx = \int f(u)du$$

Example. Find $I = \int 2x\sqrt{1+x^2}dx$

Solution. Let $u = \sqrt{1+x^2}$

Then $u^2 = 1 + x^2$, so $2udu = 2xdx$

There for,

$$I = \int 2u^2 du = \frac{2}{3}u^3 + C = \frac{2}{3}\left(\sqrt{1+x^2}\right)^3 + C$$

Example. Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$

- Let $u = 3 - 5x$.

- Then, $du = -5dx$, so $dx = -du/5$.

- When $x = 1, u = -2$, and when $x = 2, u = -7$.

Exercise.

a) $\displaystyle\int x^3 \cos\left(x^4+2\right) dx$ b) $\displaystyle\int \sqrt{2x+1}dx$ c) $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}dx$

d) $\displaystyle\int \sqrt{1+x^2}x^5 dx$ e) $\displaystyle\int \tan x dx$

Exercise.

a) $\displaystyle\int_0^1 t^3\left(1+t^4\right)^3 dt$ b) $\displaystyle\int_0^{\sqrt{7}} t\left(t^2+1\right)^{1/3} dt$

c) $\displaystyle\int_{-1}^1 \frac{5r}{\left(4+r^2\right)^2}dr$ d) $\displaystyle\int_0^1 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}dv$

INTEGRATION BY PARTS

$$\boxed{\int u dv = uv - \int v du}$$

Example.

a) $\displaystyle\int xe^x dx$ b) $\displaystyle\int \ln x dx$ c) $\displaystyle\int x \sin 2x dx$

REVIEW

Exercise.

a) $I = \int_0^1 \left(x^2 - e^{2x} + 1\right) dx$

b) $I = \int_0^{2\sqrt{2}} \sqrt{1 + x^2} \cdot x dx$

c) $I = \int_1^2 x \ln x dx$

d) $I = \int_0^{\frac{\pi}{4}} x(1 + \sin 2x) dx$

e) $I = \int_1^3 \frac{xe^{2x} + 1}{x} dx$

Chapter 5. Integrals

Approximate Integration

APPROXIMATING DEFINITE INTEGRALS USING RIEMANN SUMS

- The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x := \Delta x_i = \frac{b-a}{n}$$

- These strips divide the interval $[a, b]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. The right endpoints of the subintervals are

$$x_1 = a + \Delta x,$$
$$x_2 = a + 2\Delta x,$$
$$x_3 = a + 3\Delta x,$$

So

$$\boxed{\int_a^b f(x)dx = S \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x}$$

- If $x_i^* = x_i$ for any $i \in \overline{1, n}$, then

$$R_n := S \approx \sum_{i=1}^{n} f(x_i) \Delta x \Longrightarrow \text{Right Endpoint Rule}$$

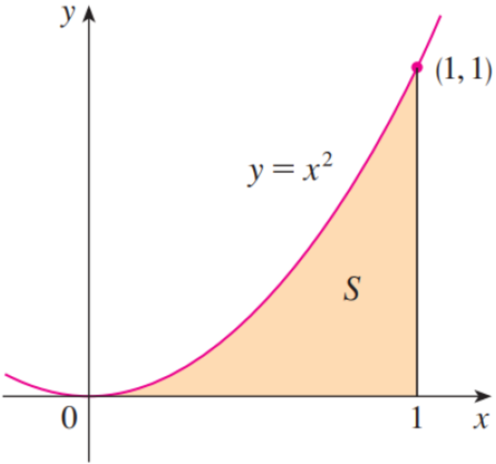
- If $x_i^* = x_{i-1}$ for any $i \in \overline{1, n}$, then

$$L_n := S \approx \sum_{i=1}^{n} f(x_{i-1}) \Delta x. \quad \Longrightarrow \text{Left Endpoint Rule}$$

- If $x_i^* = \dfrac{x_{i-1} + x_i}{2}$, for any $i \in \overline{1, n}$, then

$$M_n := S \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right) \Delta x \quad \Longrightarrow \text{Midpoint Rule}$$

Example. Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1 . Use right and Left Endpoint rules.



Dividing the interval $[0,1]$ into n subintervals. So, $\Delta x = \dfrac{1}{n}$ and $x_i = \dfrac{i}{n}$ with $i = \overline{1,n}$.

$$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$L_n = \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^{n} (i-1)^2 = \frac{2n^2-3n+1}{6n^2}.$$

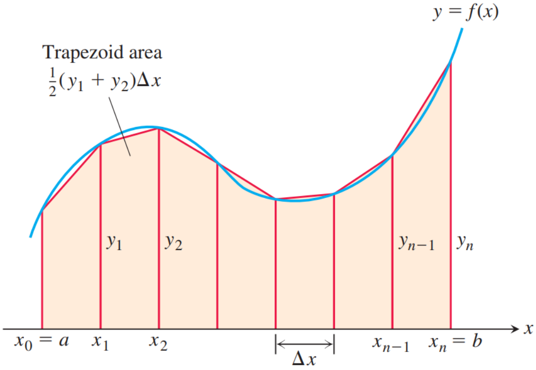
The table below shows the results of some calculations.

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

TRAPEZOIDAL RULE

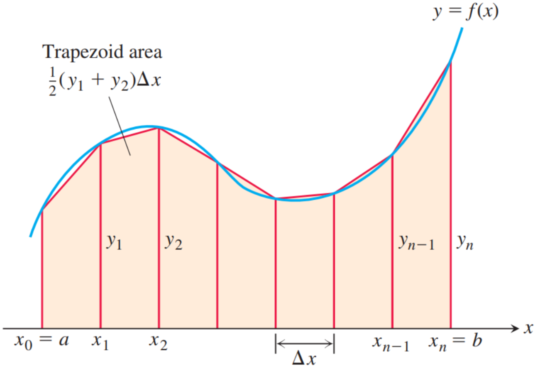
The area of the trapezoid that lies above the i th subinterval is:

$$\Delta x\left(\frac{f\left(x_{i-1}\right)+f\left(x_i\right)}{2}\right)=\frac{\Delta x}{2}\left[f\left(x_{i-1}\right)+f\left(x_i\right)\right]$$



If we add the areas of all these trapezoids, we get the right side of the Trapezoidal Rule

TRAPEZOIDAL RULE



$$\boxed{\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]}$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$

Example. Use the Trapezoidal Rule with $n = 4$ to estimate $\displaystyle\int_1^2 x^2 dx$.

Partition [1, 2] into four subintervals of equal length (Figure below). Then evaluate $y = x^2$ at each partition point (Table below).

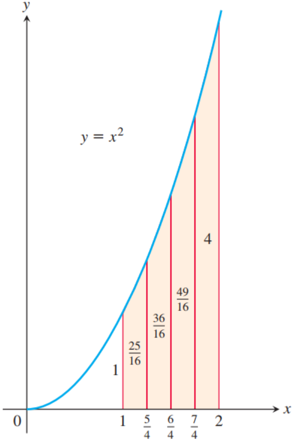


TABLE 1

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

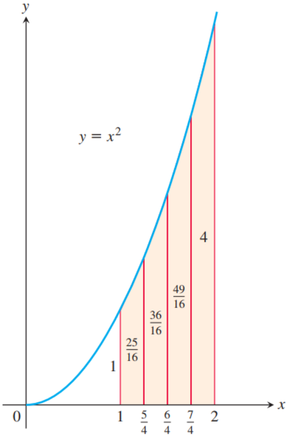


TABLE 1

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

$n = 4,\ a = 1,\ b = 2$

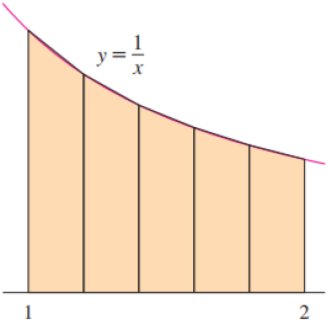
$\Delta x = (2-1)/4 = 1/4$

We have

$$\int_1^2 x^2 dx \approx T_4 = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right)$$

$$= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right) = \frac{75}{32} = 2.34375.$$

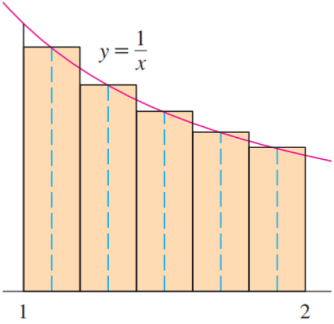
Example. Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with $n = 5$ to approximate the integral $\displaystyle\int_1^2 \frac{1}{x}dx$.



(a) With $n = 5, a = 1$, and $b = 2$, we have $\Delta x = \dfrac{2-1}{5} = 0.2$, and so the Trapezoidal Rule gives

$$\int_1^2 \frac{1}{x}dx \approx T_5 = \frac{0.2}{2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]$$

$$= 0.1\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right)$$



(b) The midpoints of the five subintervals are $1.1, 1.3, 1.5, 1.7$, and 1.9, so the Midpoint Rule give

$$\int_1^2 \frac{1}{x}dx \approx M_5 = \Delta x[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$$

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$\approx 0.691908$$

ERROR BOUNDS

Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$.

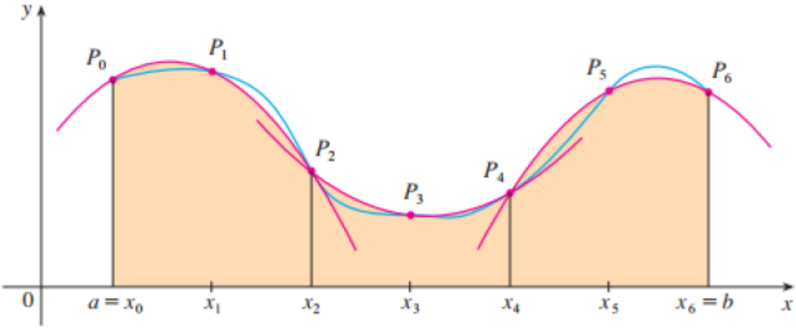
If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \text{ and } |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

SIMPSON'S RULE

This is called Simpson's Rule – after the English the English mathematician Thomas Simpson (1710–1761).

SIMPSON'S RULE



$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots \right.$$
$$\left. + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

where n **is even** and $\Delta x = \dfrac{b-a}{n}$.

ERROR BOUND (SIMPSON'S RULE)

Suppose that $\left|f^{(4)}(x)\right| \leq K$ for $a \leq x \leq b$.

If E_s is the error involved in using Simpson's Rule, then

$$|E_s| \leq \frac{K(b-a)^5}{180n^4}$$

Example. Use Simpson's Rule with $n = 10$ to approximate $\displaystyle\int_1^2 \frac{1}{x}dx$.

Solution. Putting $f(x) = \dfrac{1}{x}$, $n = 10$, and $\Delta x = 0.1$ in Simpson's Rule, we obtain

$$
\begin{aligned}
\int_1^2 \frac{1}{x}dx &\approx S_{10} \\
&= \frac{\Delta x}{3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + \cdots + 2f(1.8) + 4f(1.9) + f(2)] \\
&= \frac{0.1}{3}\left(\frac{1}{1} + \frac{4}{1.1} + \frac{2}{1.2} + \frac{4}{1.3} + \frac{2}{1.4} + \frac{4}{1.5} + \frac{2}{1.6} + \frac{4}{1.7} + \frac{2}{1.8} + \frac{4}{1.9} + \frac{1}{2}\right) \\
&\approx 0.693150.
\end{aligned}
$$

TRAPEZOIDAL RULE VS SIMPSON'S RULE

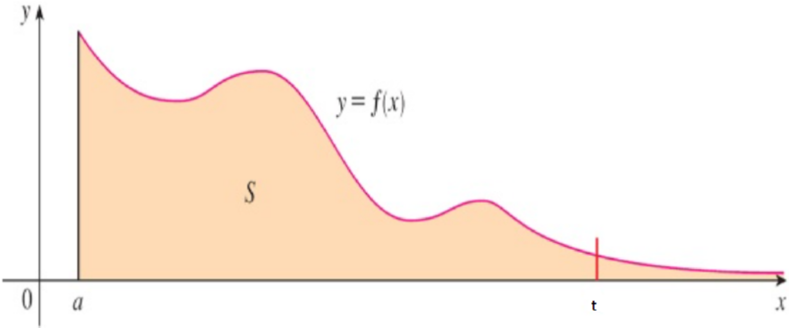
TABLE 8.4 Trapezoidal Rule approximations (T_n) and Simpson's Rule approximations (S_n) of $\ln 2 = \int_1^2 (1/x)\,dx$

n	T_n	$\lvert$Error$\rvert$ less than . . .	S_n	$\lvert$Error$\rvert$ less than . . .
10	0.6937714032	0.0006242227	0.6931502307	0.0000030502
20	0.6933033818	0.0001562013	0.6931473747	0.0000001942
30	0.6932166154	0.0000694349	0.6931472190	0.0000000385
40	0.6931862400	0.0000390595	0.6931471927	0.0000000122
50	0.6931721793	0.0000249988	0.6931471856	0.0000000050
100	0.6931534305	0.0000062500	0.6931471809	0.0000000004

Chapter 5. Integrals

Improper integral

THE AREA PROBLEM



Find the area of the region S that lies under the curve $y = f(x)$, above the -axis, and to the right of the line $x = a$.

$$S = \int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx$$

IMPROPER INTEGRALS OF TYPE 1

Definition. Let $y = f(x)$ be a function defined on $[a, +\infty)$, and there exists $\int_a^t f(x)dx$ for every number $t \geq a$, then the integral

$$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx$$

is called an **improper integrals of type 1**.

Similarly, we have the following improper integral of type 1

$$\int_{-\infty}^{a} f(x)dx = \lim_{t \to -\infty} \int_t^a f(x)dx$$

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{+\infty} f(x)dx$$

CONVERGENCE AND DIVERGENCE OF THE IMPROPER INTEGRAL OF TYPE 1

The improper integrals $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^{a} f(x)dx$ are called:

- ⋆ **Convergent** if the corresponding limit exists.
- ⋆ **Divergent** if the limit does not exist.

Two problems with improper integral:

- ⋆ Calculate the improper integrals;
- ⋆ Check convergence of the improper integrals.

Example. Calculate and determine whether the improper integral $\int_1^{+\infty} \frac{1}{x} dx$ is convergent or divergent.

Solution. By definition, we have

$$\int_1^{+\infty} \frac{1}{x} dx = \lim_{t \to +\infty} \int_1^t \frac{1}{x} dx = \lim_{t \to +\infty} \ln|x| \Big|_1^t$$

$$= \lim_{t \to +\infty} (\ln t - \ln 1) = \lim_{t \to +\infty} \ln t = +\infty$$

The limit does not exist as a real number, so the improper integral $\int_1^{+\infty} \frac{1}{x} dx$ is divergent.

Example. Calculate and determine whether the improper integral $\int_1^{+\infty} \frac{1}{x^p} dx, \forall p$ is convergent or divergent.

Solution. if $p = 1$ then the improper integral $\int_1^{+\infty} \frac{1}{x} dx$ is divergent. So, we assume $p \neq 1$, then

$$\int_1^{+\infty} \frac{1}{x^p} dx = \lim_{t \to +\infty} \left(\int_1^t x^{-p} dx \right) = \lim_{t \to +\infty} \left(\frac{x^{-p+1}}{-p+1} \right) \Bigg|_{x=1}^{x=t}$$

$$= \lim_{t \to +\infty} \frac{1}{1-p} \left(\frac{1}{t^{p-1}} - 1 \right).$$

- If $p > 1$ then $p - 1 > 0 \Longrightarrow t^{p-1} \to +\infty$ and $\frac{1}{t^{p-1}} \to 0$ as $t \to +\infty$, hence

$$\int_1^{+\infty} \frac{1}{x^p} dx = \frac{1}{p-1}, \text{ if } p > 1$$

so, the improper integral is convergent.

- If $p < 1$, then $p - 1 < 0$ and so

$$\frac{1}{t^{p-1}} = t^{1-p} \to +\infty \text{ khi } t \to +\infty$$

thus the improper integral is divergent.

Example. Calculate and determine whether the improper integral $\int_{-\infty}^{0} xe^x dx$ is convergent or divergent.

Solution. By definition, we have $\int_{-\infty}^{0} xe^x dx = \lim_{t\to-\infty} \int_{t}^{0} xe^x dx$.

- We use integration by parts with $u = x, dv = e^x dx \Rightarrow du = dx, v = e^x$:

$$\int_{t}^{0} xe^x dx = xe^x\Big|_t^0 - \int_t^0 e^x dx = -te^t - 1 + e^t$$

$e^t \to 0$ as $t \to -\infty$. Follow L'Hopital rule, then

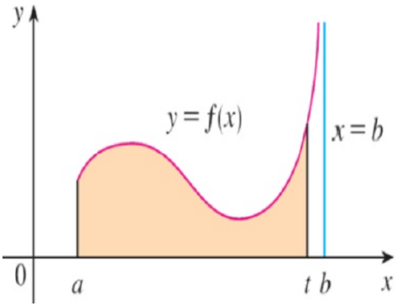
$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to-\infty} \frac{t'}{(e^{-t})'} = \lim_{t\to-\infty} \frac{1}{-e^{-t}} = \lim_{t\to-\infty} \left(-e^t\right) = 0$$

Thus

$$\int_{-\infty}^{0} xe^x dx = \lim_{t\to-\infty} \left(-te^t - 1 + e^t\right) = -0 - 1 + 0 = -1$$

so, the improper integral is convergent.

IMPROPER INTEGRAL OF TYPE 2

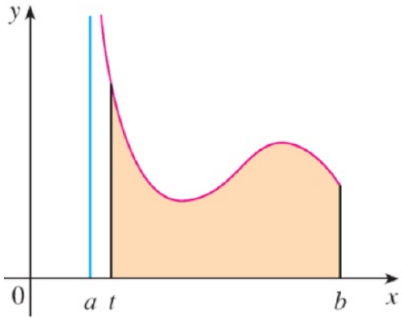


If f is continuous on $[a; b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

if this limit exists as a finite number.

IMPROPER INTEGRAL OF TYPE 2

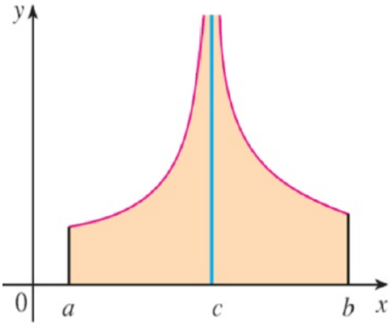


If f is continuous on $(a; b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

if this limit exists as a finite number.

IMPROPER INTEGRAL OF TYPE 2



If f has a discontinuity at c, where $a < c < b$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$
$$= \lim_{t \to c^-} \int_a^t f(x)dx + \lim_{t \to c^+} \int_t^c f(x)dx$$

CONVERGENCE AND DIVERGENCE OF THE IMPROPER INTEGRAL OF TYPE 2

The improper integral $\displaystyle\int_a^b f(x)dx$ is called:

- $\star$ **Convergent** if the corresponding limit exists.
- $\star$ **Divergent** if the limit does not exist.

Example. Calculate and determine whether the improper integral $\int_2^5 \frac{1}{\sqrt{x-2}}dx$ is convergent or divergent.

Solution. The function $f(x) = \frac{1}{\sqrt{x-2}}$ is discontinuous at $x = 2$. So

$$\begin{aligned}\int_2^5 \frac{1}{\sqrt{x-2}}dx &= \lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} \\ &= \lim_{t\to 2^+} 2\sqrt{x-2}\Big|_t^5 \\ &= \lim_{t\to 2^+} 2(\sqrt{3} - \sqrt{t-2}) \\ &= 2\sqrt{3}\end{aligned}$$

Hence, the improper integral is convergent.

Example. Calculate and determine whether the improper integral $I = \displaystyle\int_0^3 \frac{dx}{x-1}$ is convergent or divergent.

Solution. The function $f(x) = \dfrac{dx}{x-1}$ is discontinuous at $x = 1$. Thus,

$$I = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1} = I_1 + I_2$$

Moreover,

$$I_1 = \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-} \ln|x-1|\Big|_0^t = \lim_{t\to 1^-} \ln|t-1| = -\infty$$

So the improper integral I is divergent.

COMPARISON THEOREM

Theorem. Direct Comparison Test

Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

a) If $\int_a^{\infty} f(x)dx$ is convergent, then $\int_a^{\infty} g(x)dx$ is convergent.

b) If $\int_a^{\infty} g(x)dx$ is divergent, then $\int_a^{\infty} f(x)dx$ is divergent.

Example.

(a) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}dx$ converges because

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \text{ on } [1,\infty) \text{ and } \int_1^\infty \frac{1}{x^2}dx \quad \text{converges.}$$

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2-0.1}}dx$ diverges because

$$\frac{1}{\sqrt{x^2-0.1}} \ge \frac{1}{x} \text{ on } [1,\infty) \text{ and } \int_1^\infty \frac{1}{x}dx \quad \text{diverges.}$$

Example.

(c) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}} dx$ converges because

$$0 \leq \frac{\cos x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} \quad \text{on} \quad \left[0, \frac{\pi}{2}\right]$$

and

$$\begin{aligned}
\int_0^{\pi/2} \frac{dx}{\sqrt{x}} &= \lim_{a \to 0^+} \int_a^{\pi/2} \frac{dx}{\sqrt{x}} \\
&= \lim_{a \to 0^+} \sqrt{4x}\Big]_a^{\pi/2} \\
&= \lim_{a \to 0^+} (\sqrt{2\pi} - \sqrt{4a}) = \sqrt{2\pi} \quad \text{converges.}
\end{aligned}$$

COMPARISON THEOREM

Theorem. **Limit Comparison Test**

If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)dx \quad \text{and} \quad \int_a^\infty g(x)dx$$

both converge or both diverge.

Example. Show that

$$\int_1^\infty \frac{dx}{1+x^2}$$

converges by comparison with $\displaystyle\int_1^\infty \frac{1}{x^2}dx$.

The functions $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{1+x^2}$ are positive and continuous on $[1, \infty)$.

Also,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1/x^2}{1/\left(1+x^2\right)} = \lim_{x\to\infty} \frac{1+x^2}{x^2}$$

$$= \lim_{x\to\infty} \left(\frac{1}{x^2}+1\right) = 0+1 = 1,$$

a positive finite limit.

Therefore, $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$ converges because $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges.

Thank you for your attention.

Prob 1. Estimate the area under the graph of $y = f(x)$ using 6 rectangles and left endpoints:

a) $f(x) = \dfrac{1}{x} + x, x \in [1, 4]$

b) A table of values for f is given

x	1	2	3	4	5	6	7
$f(x)$	5	6	3	2	7	1	2

Prob 2. Repeat Prob 1 using right endpoints.

Prob 3. Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule to approximate the given integral with the specified value of n.

a) $\displaystyle\int_0^3 \sqrt{x}dx, n = 4$

b) $\displaystyle\int_1^3 \frac{\sin x}{x}dx, n = 6$

Prob 4. Find the derivative of the function $g(x) = \int_0^x \sqrt{t^2+1}dt$

Prob 5. Find g'

a) $g(x) = \int_1^{x^4} \frac{1}{\cos t} dt$

b) $g(x) = \int_1^{\sqrt{x}} \frac{\sin u}{u} du$

Prob 6. Find the average value of the function on the given interval

a) $f(x) = x^2, [-1, 1]$

b) $f(x) = \frac{1}{x}, [1, 5]$

Prob 7. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (m/s)

a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

b) Find the distance traveled during this time period.

Prob 8. Suppose the acceleration function and intial velocity are $a(t) = t + 3$ (m/s^2), $v(0) = 5$ (m/s). Find the velocity at time t and the distance traveled when $0 \leq t \leq 5$
Hint $a(t) = v'(t)$)

Prob 9. A particle moves along a line with velocity function $v(t) = t^2 - t$ (m/s). Find the displacement and the distance traveled by the particle during the time interval $t \in [0, 2]$.

Prob 10. Evaluate the integral $I = \int_0^2 x^2\sqrt{x^3+1}dx$.

Prob 11. Suppose $f(x)$ is differentiable, $f(1) = 4$ and $\int_0^1 f(x)dx = 5$. Find $\int_0^1 xf'(x)dx$

Prob 12. Suppose $f(x)$ is differentiable, $f(1) = 3, f(3) = 1$ and $\int_1^3 xf'(x)dx = 13$. What is the average value of f on the interval $[1,3]$?

Prob 13. Let $f(x) = \begin{cases} -x-1 & \text{if } -3 \leq x \leq 0 \\ -\sqrt{1-x^2} & \text{if } 0 < x \leq 1 \end{cases}$. Evaluate $\int_{-3}^1 f(x)dx$

Prob 14. Calculate the improper integral $\displaystyle\int_2^{+\infty} \frac{2}{x+1}dx$

a) $+\infty$ b) $-\infty$ c) $\ln 3$ d) $\ln(2)/3$

Prob 15. Calculate the improper integral $\displaystyle\int_1^{+\infty} \frac{1}{x^4}dx$

Prob 16. Calculate the improper integral $\displaystyle\int_1^{+\infty} \frac{1}{x^5}dx$

a) $-1/4$ b) $1/2$ c) 1 d) $1/4$

Prob 17. Calculate the improper integral $\displaystyle\int_0^{+\infty} xe^x dx$

Prob 18. Calculate the improper integral $\displaystyle\int_{-2}^{+\infty} \frac{dx}{x^2+4x+5}$

a) $\pi/2$ b) $\pi/3$ c) $2\pi/3$ d) $\pi/4$

Prob 19. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

a) $\displaystyle\int_{1}^{\infty} \frac{dx}{(3x+1)^2}$ b) $\displaystyle\int_{-\infty}^{0} \frac{dx}{2x-5}$ c) $\displaystyle\int_{0}^{4} \frac{1}{\sqrt{4-x}}dx$

d) $\displaystyle\int_{0}^{1} \frac{dx}{4x-1}$ e) $\displaystyle\int_{3}^{4} \frac{dx}{\sqrt{x-3}}$