Chapter 2. Systems of Linear Equations

OUTLINE

1. Definition
2. Solutions of a system of linear equations
3. Solving systems of linear equations
4. Kronecker-Capelli Theorem

DEFINITION

A general system of m **linear equations** with n **unknowns** can be written as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where $x_1, x_2, \ldots, x_n$ are the unknowns, $a_{i,j} \in \mathbb{R}$ are the coefficients; $b_i \in \mathbb{R}$ are the constant terms;

- A solution of a linear system is a tuple $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement.
- The solution set of the system is the set of all the solutions.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{m\times 1}$$

coefficient matrix

constant matrix

DEFINITION

A **system of homogeneous linear equations** (hệ phương trình tuyến tính thuần *nhât*)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

always has a solution $u = (0, 0, \ldots, 0)$. It is called a **trivial solution** (nghiệm tầm thường).

DEFINITION

The **augmented matrix** (ma trận bổ sung) of the general linear system is

$$\overline{A} = (A \mid B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

Example. Give the systems of linear equations

$$\begin{cases} x_1 + x_2 + x_3 = 2 \\ 2x_1 + 3x_2 + x_3 = 3 \\ x_1 - x_2 - 2x_3 = -6 \end{cases}$$

Find **augmented matrix**.

$$\overline{A} = (A \mid B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{array}\right)$$

Example. Give the **augmented matrix**

$$\overline{A} = \left(\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Find the systems of linear equations and it's solution.

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ \qquad\;\; x_2 - x_3 = 1 \\ \qquad\qquad\quad x_3 = 2 \end{cases} \Longrightarrow \begin{cases} x_1 = 1 \\ x_2 = 3 \\ x_3 = 2 \end{cases}$$

DEFINITION

- ⋆ A system that has no solution is called **inconsistent** (không tương thích/ không nhất quán)
- ⋆ A system that has at least one solution is called **consistent** (tương thích/ nhất quán)

Inconsistent (không tương thích)	Consistent (tương thích)	
No solutions (vô nghiệm)	Unique solution (nghiệm duy nhất)	Infinitely many solutions (vô số nghiệm)

Example.

$$\begin{cases} x+2y=1 \\ x+2y=3 \end{cases}$$

no solution

Inconsistent

$$\begin{cases} x+y-z=1 \\ x+y+z=3 \end{cases}$$

(0,2,1), (2,0,1)   (t,2-t,1)

Consistent

(infinitely many solutions)

(t,2-t,1) is called a **general solution** and given in **parametric form**, t is **parameter** **(t is arbitrary)**

THEOREM

Any system of linear equations has one of the following exclusive conclusions.

1. No solution
2. Unique solution
3. Infinitely many solutions

GAUSSIAN ELIMINATION

1. Perform elementary row operations to put the **augmented matrix** into the **row-echelon form**

2. Solve the equation of the kth row for x_k, then substitute back into the equation of the $k-1$ row to obtain a solution for x_{k-1}, etc.

Note: If you see the row

$$\left(\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & \dots & 0 & 0 & \neq 0 \end{array}\right)$$

then the systems of linear equations have no solutions.

Example. Solve the linear equations

$$\begin{cases} x_1 - 2x_2 + 4x_3 = 12 \\ 2x_1 - x_2 + 5x_3 = 18 \\ -x_1 + 3x_2 - 3x_3 = -8 \end{cases} \qquad (*)$$

The **augmented matrix** (ma trận bổ sung) of the linear system is

$$\overline{A} = (A \mid B) = \left(\begin{array}{ccc|c} 1 & -2 & 4 & 12 \\ 2 & -1 & 5 & 18 \\ -1 & 3 & -3 & -8 \end{array}\right)$$

$$\overline{A} = (A \mid B) = \left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 2 & -1 & 5 & 18 \\ -1 & 3 & -3 & -8 \end{array}\right) \xrightarrow[r_3 \to r_3 + r_1]{r_2 \to r_2 + (-2) r_1} \left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 3 & -3 & -6 \\ 0 & 1 & 1 & 4 \end{array}\right)$$

$$\xrightarrow{r_2 \to \left(\frac{1}{3}\right) r_2} \left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & 4 \end{array}\right) \xrightarrow{r_3 \to r_3 - r_2} \left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 6 \end{array}\right)$$

$$\xrightarrow{r_3 \to \left(\frac{1}{3}\right) r_3} \left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$(*) \Longleftrightarrow \begin{cases} x_1 - 2x_2 + 4x_3 = 12 \\ \qquad\quad x_2 - x_3 = -2 \\ \qquad\qquad\quad x_3 = 3 \end{cases} \Longleftrightarrow \begin{cases} x_1 = 2 \\ x_2 = 1 \\ x_3 = 3 \end{cases}$$

Exercise. Solve the linear equations

a) $$\begin{cases} 2x_1 + 3x_2 + 4x_3 = 3200 \\ x_1 + 2x_2 + x_3 = 1700 \\ x_1 + x_2 + 2x_3 = 1300 \end{cases}$$

b) $$\begin{cases} x + y + 2z = 1 \\ 2x + y + 3z = 2 \\ 3x + 2y + 5z = 3 \end{cases}$$

c) $$\begin{cases} 5x_1 - 2x_2 + 5x_3 - 3x_4 = 3 \\ 4x_1 + x_2 + 3x_3 - 2x_4 = 1 \\ 2x_1 + 7x_2 - x_3 = -1 \end{cases}$$

HOMOGENEOUS EQUATIONS

- ⋆ The system is called **homogeneous** (thuần nhất) if the constant matrix has all the entry are zeros
- ⋆ Note that every homogeneous system **has at least one solution** $(0, 0, \ldots, 0)$, called **trivial solution** (nghiệm tầm thường)
- ⋆ If a homogeneous system of linear equations has **nontrivial solution** (nghiệm không tầm thường) then it has **infinite family of solutions** (vô số nghiệm)

Example. Show that the following homogeneouss system has nontrivial solutions.

$$\begin{cases} x_1 - x_2 + 2x_3 + x_4 = 0 \\ 2x_1 + 2x_2 - x_4 = 0 \\ 3x_1 + x_2 + 2x_3 + x_4 = 0 \end{cases}$$

HOMOGENEOUS EQUATIONS

Definition. If a homogeneous system of linear equations has **more variables than equations**, then it has nontrivial solution (in fact, infinitely many)

Note that the converse of theorem 1 is not true

THEOREM

Theorem. Kronecker-Capelli Theorem

If $\overline{A} = [A \mid B]$ is the augmented matrix of the system with n unknowns, then $\mathrm{rank}(\overline{A}) = \mathrm{rank}(A)$ or $\mathrm{rank}(\overline{A}) = \mathrm{rank}(A) + 1$. Furthermore,

- ⋆ if $\mathrm{rank}(\overline{A}) = \mathrm{rank}(A) + 1$ then the system has no solution.
- ⋆ if $\mathrm{rank}(\overline{A}) = \mathrm{rank}(A) = n$ then the system has a unique solution.
- ⋆ if $\mathrm{rank}(\overline{A}) = \mathrm{rank}(A) < n$ then the system has infinitely many solutions with $n - \mathrm{rank}(A)$ free parameters.

Example. Solve and argue the system of linear equations by parameter m

$$\begin{cases} x - 2y + z = 3 \\ 3x - 5y + z = m \\ -x + y + z = -1 \end{cases}$$

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SUMMARY

System of	Inconsistent (no solutions)	Consistent	
		Unique solution (exactly one solution)	Infinitely many solutions
linear equations	yes	yes	yes
linear equations that has more variables than equations	yes	no	yes
homogeneous linear equations	no	yes	yes
homogeneous linear equations that has more variables than equations	no	no	yes

Thank you for your attention.