Chapter 1 Matrix Algebra

CONTENTS

OBJECTIVES

- ⋆ Practicing some matrix operations
- ⋆ Find matrix inverse
- ⋆ Using matrix transformations to reveal the geometrical meaning of matrix multiplication and inverse.
- ⋆ The relationship of matrix algebra to linear equations

DEFINITIONS

Definition. An $m \times n$ matrix (or a matrix of size $m \times n$) is a rectangular array of numbers with m rows and n columns

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m\times n} = (a_{ij})_{m\times n}$$

- ⋆ A is denoted simply as $A = (aij)$
- ⋆ $m \times n$ size of the matrix A
- ⋆ The $(i, j)-$entry of A (denoted by a_{ij}) lies in row i and column j

Example.

$$A = \begin{pmatrix} 7 & -3 & 1/2 \\ 3 & -5 & 0 \end{pmatrix}$$

(1,3)-entry
a[1,3] = 1/2
$a_{13} = 1/2$

A is a 2×3 matrix (or a matrix of size 2×3)// 2 rows, 3 columns

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & -1 \end{bmatrix}$$

3×3 matrix,a square matrix

$$C = \begin{bmatrix} 2.00 \\ -3.00 \\ 1.75 \end{bmatrix}$$

3×1 matrix, column matrix

ZERO MATRIX

Definition. The **zero matrix** (ma trận không) is a matrix in which all elements are zero, denoted by $0_{m \times n}$ (or 0).

Example.

$$0_{3\times 4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

SQUARE MATRIX

Definition. An $n \times n$ matrix is called a square matrix (ma trận vuông) of size m.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

$M_n(\mathbb{R})$: the set of all n square matrices on $\mathbb{R}$.

The elements $a_{11}; a_{22}; \ldots; a_{nn}$ are called the **main diagonal** (đường chéo chính) of A.

Example. $A = \begin{pmatrix} 2 & -3 & 1 \\ 0 & 4 & 1 \\ 2 & -2 & -3 \end{pmatrix}$

TRIANGULAR MATRICES

⋆ An **upper triangular** matrix (ma trận tam giác trên) is a square matrix in which all entries **below** the **main diagonal** are zero.

$$M = \begin{pmatrix} -1 & 0 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & -5 \end{pmatrix}, N = \begin{pmatrix} -2 & 3 \\ 0 & 1 \end{pmatrix}$$

⋆ A **lower triangular** matrix (ma trận tam giác dưới) is a square matrix in which all entries **above** the **main diagonal** are zero.

$$P = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 1 & 0 \end{pmatrix}, Q = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -4 & 2 & 0 & 0 \\ 0 & -1 & 0 & 5 \end{pmatrix}$$

⋆ Matrix A is called **triangular** if it is upper or lower triangular

DIAGONAL MATRIX

A **diagonal matrix** (ma trận đường chéo) is a square matrix with zero entries except possibly on the main diagonal.

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} = \operatorname{diag}(a_{11}, a_{22}, \dots, a_{nn})$$

Example.

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

IDENTITY MATRIX

The square matrix

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is known as the **identity matrix** (ma trận đơn vị) of dimension n.

Example. $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ldots$

EQUAL MATRIX

Two matrices $A = (a_{ij})$ and $B = (b_{ij})$.

$$A = B \Leftrightarrow \begin{cases} A, B \text{ have the } \textbf{same size} \\ a_{ij} = b_{ij}, \forall i, j \end{cases}$$

Example. Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$, discuss the possibility that $A = B, B = C, A = C$.

Example. Find x, y, z, t to

$$\begin{pmatrix} 1 & x-1 & y \\ z & -3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -3 & 1-2t \end{pmatrix} \Leftrightarrow \begin{cases} x = 1 \\ y = -1 \\ z = 2 \\ t = -2 \end{cases}$$

TRANSPOSE

- ⋆ If $A = (a_{ij})$ is any $m \times n$ matrix, the transpose of A, written A^T, is an $n \times m$ matrix defined by $A^T = (a_{ji})$.
- ⋆ The **row** i of A is the **column** i of A^T
- ⋆ The **column** j of A is the **row** j of A^T

Example.

$$A = \begin{pmatrix} 1 & -1 & 4 & 5 \\ 6 & -8 & 0 & 1 \\ 0 & 4 & -3 & 6 \end{pmatrix} \Longrightarrow A^T = \begin{pmatrix} 1 & 6 & 0 \\ -1 & -8 & 4 \\ 4 & 0 & -3 \\ 5 & 1 & 6 \end{pmatrix}$$

SYMMETRIC MATRIX - SKEW SYMMETRIC MATRIX

- ⋆ If $A^T = A$, then A is a **symmetric matrix** (ma trận đối xứng).
- ⋆ If $A^T = -A$, then A is a **skew symmetric matrix** (ma trận phản xạ).

Example.

- ⋆ $A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 4 & 5 \\ -2 & 5 & 6 \end{pmatrix}$ is a symmetric matrix.
- ⋆ $B = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & -3 \\ -1 & 3 & 0 \end{pmatrix}$ is a skew symmetric matrix.

SCALAR MULTIPLICATION (PHÉP NHÂN VÔ HƯỚNG)

- $\star$ If $A = [a_{i,j}]$, that is
$$kA = [ka_{i,j}]$$
- $\star$ $kA = 0 \to$ (either $k = 0$ or $A = 0$)
- $\star$ $(k = 0$ or $A = 0) \to kA = 0$

Example. Let $A = \begin{pmatrix} 3 & 4 & 1 \\ 0 & 1 & -3 \end{pmatrix}$. Then

- $\star$ $2A = \begin{pmatrix} 6 & 8 & 2 \\ 0 & 2 & -6 \end{pmatrix}$.
- $\star$ $-A = \begin{pmatrix} -3 & -4 & -1 \\ 0 & -1 & 3 \end{pmatrix}$.

MATRIX ADDITION OF SAME SIZE MATRICES

If $A = (a_{ij})$ and $B = (b_{ij})$, this take the form

$$A \pm B = (a_{ij} \pm b_{ij})_{m \times n}$$

Example.

- $\star$ $$\begin{pmatrix} -1 & 0 & 2 \\ 2 & 3 & -4 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 2 \\ 5 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 \\ 7 & 0 & -3 \end{pmatrix}$$

- $\star$ $$\begin{pmatrix} -1 & 0 & 2 \\ 2 & 3 & -4 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 2 \\ 5 & -3 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ -3 & 6 & -5 \end{pmatrix}$$

Example.

$$5\begin{pmatrix} 1 & -2 \\ 4 & 0 \\ -2 & 4 \end{pmatrix} - 2\begin{pmatrix} 9 & 2 & -1 \\ -8 & 8 & 4 \end{pmatrix}^T = \begin{pmatrix} 5 & -10 \\ 20 & 0 \\ -10 & 20 \end{pmatrix} - \begin{pmatrix} 18 & -16 \\ 4 & 16 \\ -2 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} -13 & 6 \\ 16 & -16 \\ -8 & 12 \end{pmatrix}$$

PROPERTIES

Let A, B and C denote arbitrary $n \times m$ matrices where n and m are fixed. Let k and p denote arbitrary real numbers. Then

1. $A + B = B + A$ (commutative law: giao hoán)
2. $A + (B + C) = (A + B) + C$ (associative law: kết hợp)
3. There is an $n \times m$ matrix 0 , such that $0 + A = A$ for each A.
4. For each A there is an $n \times m$ matrix, $-A$, such that $A + (-A) = 0$
5. $k(A + B) = kA + kB$.
6. $(k + p)A = kA + pA$.
7. $(kp)A = k(pA)$.
8. $1A = A$.

DOT PRODUCT (TÍCH VÔ HƯỚNG)

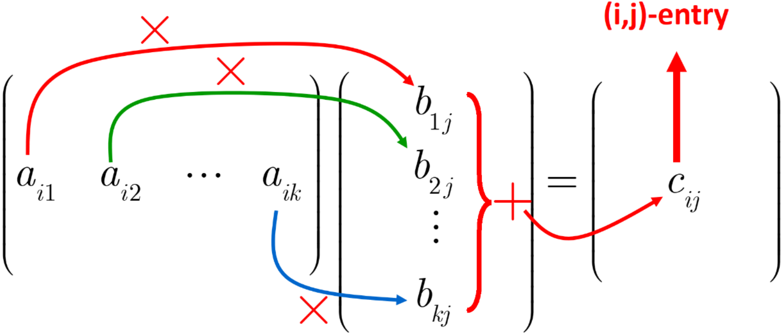
Example.

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 1 \times 3 + 2 \times 2 + 3 \times 1 = 10$$

MATRIX MULTIPLICATION

Suppose $A = [a_{ij}]$ is an $m \times k$ matrix and $B = [b_{ij}]$ is an $k \times n$ matrix, then the product $AB = [c_{ij}]$ is an $m \times n$ matrix whose the (i, j)-entry is the dot product of row i of A and column j of B

$$c_{ij} = (\text{row } i \text{ of } A).(\text{column } j \text{ of } B) = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{ik}b_{kj}$$



Example. Give $A = \begin{pmatrix} 2 & -1 & 4 \\ 4 & 1 & 0 \end{pmatrix}$; $B = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \\ 2 & 4 & 3 \end{pmatrix}$ Find AB và BA?

$$AB = \begin{pmatrix} 2 & -1 & 4 \\ 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \\ 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 7 & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}$$

$$c_{11} = \begin{pmatrix} 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = 2 \times 1 + (-1) \times 3 + 4 \times 2 = 7$$

Example. Give $A = \begin{pmatrix} 2 & -1 & 4 \\ 4 & 1 & 0 \end{pmatrix}$; $B = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \\ 2 & 4 & 3 \end{pmatrix}$ Find AB và BA?

$$AB = \begin{pmatrix} 2 & -1 & 4 \\ 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \\ 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 12 & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}$$

$$c_{12} = \begin{pmatrix} 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix} = 2 \times (-2) + (-1) \times 0 + 4 \times 4 = 12$$

Example. Give $A = \begin{pmatrix} 2 & -1 & 4 \\ 4 & 1 & 0 \end{pmatrix}$; $B = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \\ 2 & 4 & 3 \end{pmatrix}$ Find AB và BA?

$$AB = \begin{pmatrix} 2 & -1 & 4 \\ 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \\ 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 12 & 15 \\ 7 & -8 & 9 \end{pmatrix},$$

$$c_{23} = \begin{pmatrix} 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 4 \times 2 + 1 \times 1 + 0 \times 3 = 9$$

Example. Compute AB if $A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & 9 \\ 7 & 2 \\ 6 & 1 \end{pmatrix}$.

Solution.

$$
\begin{aligned}
C = AB = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{pmatrix} \begin{pmatrix} 8 & 9 \\ 7 & 2 \\ 6 & 1 \end{pmatrix} &= \begin{pmatrix} 2\cdot 8 + 3\cdot 7 + 5\cdot 6 & 2\cdot 9 + 3\cdot 2 + 5\cdot 1 \\ 1\cdot 8 + 4\cdot 7 + 7\cdot 6 & 1\cdot 9 + 4\cdot 2 + 7\cdot 1 \\ 0\cdot 8 + 1\cdot 7 + 8\cdot 6 & 0\cdot 9 + 1\cdot 2 + 8\cdot 1 \end{pmatrix} \\
&= \begin{pmatrix} 67 & 29 \\ 78 & 24 \\ 55 & 10 \end{pmatrix}
\end{aligned}
$$

Does BA exist?

THEOREM

Assume that a is any scalar, and that A, B, and C are matrices of sizes such that the indicated matrix products are defined. Then:

1. $IA = A$ and $AI = A$ where I denotes an identity matrix.
2. $A(BC) = (AB)C$.
3. $A(B + C) = AB + AC$
4. $(B + C)A = BA + CA$
5. $a(AB) = (aA)B = A(aB)$.
6. $(AB)^T = B^T A^T$.

KTH POWER OF A

For a square matrix A and positive integer k, the k th power of A is defined by multiplying this matrix by itself repeatedly; that is

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}}$$

where there are k copies of the matrix A.

Example. For $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$. Calculate A^2 and A^3

Solution.

$$A^2 = AA = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}$$

ROW-ECHELON MATRIX

$$\begin{pmatrix} 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A **row-echelon matrix** has 3 properties

- ⋆ All the **zero rows** are at the bottom
- ⋆ The first nonzero entry from the left in each nonzero row is a 1, called the **leading 1** for that row
- ⋆ Each leading 1 is **to the right** of all leading 1's in the rows above it

ROW-ECHELON MATRIX

The row-echelon matrix has the "staircase" form

leading ones

$$\begin{pmatrix} 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(for any choice in *-position)

ROW-ECHELON MATRIX

$$\begin{pmatrix} 1 & * & * \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 1 & * & * \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A REDUCED ROW-ECHELON MATRIX

A **reduced row-echelon matrix** (ma trận bậc thang theo dòng thu gọn) has the properties

- ⋆ It is a row-echelon matrix
- ⋆ Each leading 1 is the **only nonzero** entry in its column

$$\begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

REDUCED ROW- ECHELON MATRIX

$$\begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example. Find x, y so that the matrix

$$\begin{pmatrix} 1 & x & y \\ 0 & x & 1 \end{pmatrix}$$

a) is row-echelon matrix

b) is reduced row-echelon matrix

ELEMENTARY ROW OPERATIONS

The three elementary row operations are:

- ⋆ (Row Swap) Exchange any two rows: $d_i \leftrightarrow d_j$
- ⋆ (Scalar Multiplication) Multiply any row by a constant: $d_i := \alpha d_i, \alpha \neq 0$
- ⋆ (Row Sum) Add a multiple of one row to another row: $d_i := d_i + \beta d_j, \beta \neq 0$

Example. Use elementary row operations to transform A to B.

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ 3 & -1 & 2 \end{pmatrix}; B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -7/5 \\ 0 & 0 & 0 \end{pmatrix}.$$

GAUSSIAN ALGORITHM

Theorem. Every matrix can be brought to (reduced) row-echelon form by a series of elementary row operations

- ⋆ **Step 1.** If all row are **zeros**, stop
- ⋆ **Step 2.** Otherwise, find the **first column** from the left containing a **nonzero entry** (call it **a**) and **move** the row containing a to the **top position**
- ⋆ **Step 3.** **Multiply** that row by **1/a** to creat the **leading 1**
- ⋆ **Step 4.** By subtracting multiples of that row from the rows below it, make each entry below the leading 1 **zero**
- ⋆ **Step 5.** Repeat step 1-4 on the matrix consisting of the remaining rows

Example. Carry the matrix

$$A = \begin{pmatrix} 2 & 6 & -2 & 2 \\ -2 & -3 & 11 & 4 \\ 3 & 11 & 3 & 0 \end{pmatrix}$$

- $\star$ to row-echelon matrix
- $\star$ to reduced row-echelon matrix

Solution.

$$\begin{pmatrix} 2 & 6 & -2 & 2 \\ -2 & -3 & 11 & 4 \\ 3 & 11 & 3 & 0 \end{pmatrix} \xrightarrow{r_1 \to \frac{1}{2} r_1} \begin{pmatrix} 1 & 3 & -1 & 1 \\ -2 & -3 & 11 & 4 \\ 3 & 11 & 3 & 0 \end{pmatrix} \xrightarrow[r_3 \to r_3 - 3r_1]{r_2 \to r_2 + 2r_1} \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & 3 & 9 & 6 \\ 0 & 2 & 6 & -3 \end{pmatrix}$$

$$\xrightarrow{r_2 \to \frac{1}{3} r_2} \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 2 & 6 & -3 \end{pmatrix} \xrightarrow{r_3 \to r_3 - 2r_2} \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

$$\xrightarrow{r_3 \to -\frac{1}{7} r_3} \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

row-echelon matrix

$$\begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_2 \to r_2 - 2r_3]{r_1 \to r_1 - 2r_3} \begin{pmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_1 \to r_1 - 3r_2} \begin{pmatrix} 1 & 0 & -10 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

reduced row-echelon matrix

RANK OF THE MATRIX

- ⋆ The **reduced row-echelon form** of a matrix A is **uniquely determined** by A, but the row-echelon form of A is not unique
- ⋆ The number r of leading 1's is the same in each of the different row-echelon matrices
- ⋆ As r depends only on A and not on the row-echelon forms, it is called **the rank of the matrix A**, and written $\text{rank}(A) = r$

Example. If the a matrix A has the row-echelon matrix is

$$\begin{pmatrix} 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

then $\text{rank}(A) = 4$

Example. Compute the rank of

$$A = \begin{pmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{pmatrix}$$

Solution. The reduction of A to row - echelon form is

$$A = \begin{pmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 4 \\ 0 & -1 & 5 & -8 \\ 0 & 1 & -5 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -5 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Because this row - echelon matrix has two leading 1 's, $\mathrm{rank}(A) = 2$.

Exercise. Determine the rank of matrix

$$B = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -3 \\ 3 & -1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 9 \\ 3 & 6 & 9 & 11 \\ 4 & 8 & 14 & 10 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 5 & 1 & 3 \\ 5 & 12 & 2 & 7 \end{pmatrix} \quad E = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$$

INVERTIBLE MATRIX

If A is a square matrix, a matrix B is called an **inverse** (nghich đảo) of A if and only if

$$AB = I \quad \text{and} \quad BA = I$$

A matrix A that has an inverse is called an **invertible matrix** (ma trận khả nghịch).

The inverse of A is denoted by A^{-1}.

$$AA^{-1} = A^{-1}A = I_n$$

Example. Show that $B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ is an inverse of $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Solution. Compute AB and BA

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad BA = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence $AB = I = BA$, so B is indeed an inverse of A.

THE INVERSE OF AN 2×2 MATRIX

Consider the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The **determinant** of A is $\det(A) = ad - bc$

The **adjugate matrix** (ma trận liên hợp) of A is defined by $\operatorname{adj} A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example. $A = \begin{pmatrix} 3 & 2 \\ -1 & -1 \end{pmatrix} \Longrightarrow A^{-1} = \frac{1}{-1} \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}$

Exercise. Find the inverse of the following matrices

a) $A = \begin{pmatrix} 4 & 3 \\ 6 & 5 \end{pmatrix}$

b) $B = \begin{pmatrix} -2 & 1 \\ 7 & 5 \end{pmatrix}$

c) $C = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$

d) $D = \begin{pmatrix} a & 2 \\ 1 & a \end{pmatrix}$

THEOREM

All the following matrices are square matrices of the same size.

1. I is invertible and $I^{-1} = I$.
2. If A is invertible, so A^{-1}, and $(A^{-1})^{-1} = A$.
3. If A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.
4. If $A_1, A_2, \ldots, A_k$ are all invertible, so is their product $A_1A_2\ldots A_k$, and $(A_1A_2\ldots A_k)^{-1} = A_k^{-1}\ldots A_2^{-1}A_1^{-1}$.
5. If A is invertible, so is A^k for any $k \geq 1$, and $\left(A^k\right)^{-1} = (A^{-1})^k$.
6. If A is invertible and $a \neq 0$ is a number, then aA is invertible and $(aA)^{-1} = \dfrac{1}{a}A^{-1}$.
7. If A is invertible, so is its transpose A^T, and $\left(A^T\right)^{-1} = (A^{-1})^T$.

FIND AN INVERSE MATRIX BY USING ELEMENTARY ROW OPERATIONS

$$(A \mid I_n) \xrightarrow{\varphi_1} (A_1 \mid B_1) \to \ldots \xrightarrow{\varphi_p} (I_n \mid A^{-1})$$

Example. Find the inverse of the following matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{pmatrix}$$

$$(A \mid I_3) = \left(\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[r_3 \to r_3 + 6r_1]{r_2 \to r_2 - r_1} \left(\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[r_3 \to r_3 + 4r_2]{r_1 \to r_1 + r_2} \left(\begin{array}{rrr|rrr} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right)$$

$$\xrightarrow{r_3 \to -r_3} \left(\begin{array}{rrr|rrr} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right)$$

$$\xrightarrow[r_2 \to r_2 + r_3]{r_1 \to r_1 + r_3} \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix}$$

MATRIX EQUATIONS

Let $A, A' \in M_n(\mathbb{R})$ be invertible and $B \in M_{n\times p}(\mathbb{R}), C \in M_{m\times n}(\mathbb{R})$, $D \in M_n(\mathbb{R})$. Then

1. $AX = B \iff X = A^{-1}B$
2. $XA = C \iff X = CA^{-1}$
3. $AXA' = D \iff X = A^{-1}DA'^{-1}$

Example.

1. Solve the equation $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} X = \begin{pmatrix} -2 & 3 \\ 2 & 5 \end{pmatrix}$
2. Solve the equation $X \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 2 & 5 \end{pmatrix}$
3. Solve the equation $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} X \begin{pmatrix} 3 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}$

LINEAR TRANSFORMATION

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** (biến đổi tuyến tính) if it satisfies the following two conditions for all vectors x and y in $\mathbb{R}^n$ and all scalars a:

1. $T(x+y) = T(x) + T(y)$
2. $T(ax) = aT(x)$

Theorem. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then for each $k = 1, 2, \ldots$

$$T\left(a_1x_1 + a_2x_2 + \cdots + a_kx_k\right) = a_1T\left(x_1\right) + a_2T\left(x_2\right) + \ldots + a_kT\left(x_k\right)$$

for all scalars a_i and all vectors x_i in $\mathbb{R}^n$.

Example. If $T:\mathbb{R}^2\to\mathbb{R}^2$ is a linear transformation, $T\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\-3\end{pmatrix}$ and $T\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}5\\1\end{pmatrix}$, find $T\begin{pmatrix}4\\3\end{pmatrix}$.

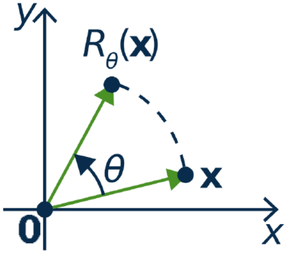
Solution. Write $z=\begin{pmatrix}4\\3\end{pmatrix}, x=\begin{pmatrix}1\\1\end{pmatrix}$, and $y=\begin{pmatrix}1\\-2\end{pmatrix}$.

- ⋆ We want to find numbers a and b such that $z=ax+by$.
- ⋆ The solution is, $a=\frac{11}{3}$ and $b=\frac{1}{3}$, so $z=\frac{11}{3}x+\frac{1}{3}y$.

Thus

$$T(z)=T\left(\frac{11}{3}x+\frac{1}{3}y\right)=\frac{11}{3}T(x)+\frac{1}{3}T(y)=\frac{11}{3}\begin{pmatrix}2\\-3\end{pmatrix}+\frac{1}{3}\begin{pmatrix}5\\1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}27\\-32\end{pmatrix}$$

ROTATIONS



Let

$$R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

denote the transformation that rotates any vector counterclockwise about the origin through an angle of θ.

Theorem. The rotation $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation, and is induced by the matrix

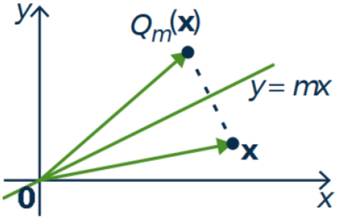
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Example. We denote by $R_\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ counterclockwise rotation about the origin through an angle of π. We see that

$$R_\pi \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$$

so R_π is a matrix transformation

REFLECTION IN THE LINE



Let

$$Q_m : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

denote **reflection in the line** $y = mx$.

Theorem. The transformation $Q_m : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, reflection in the line $y = mx$, is a linear transformation and is induced by the matrix

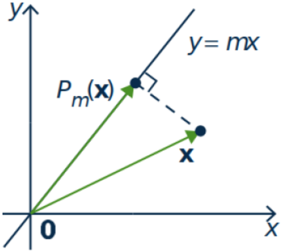
$$\frac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$$

Example. We denote by $Q_m : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ reflection in the line $y = x$. We see that

$$Q_m \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

so Q_m is a matrix transformation

PROJECTION ON THE LINE



Let

$$P_m : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

denote **projection on the line $y = mx$**.

Theorem. The transformation $P_m : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, projection on the line $y = mx$, is a linear transformation and is induced by the matrix

$$\frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$$

Example. Let $P_m : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be projection on the line $x - 2y = 0$. Find $P_m \begin{pmatrix} -5 \\ 6 \end{pmatrix}$.

Solution. We have

$$P_m \begin{pmatrix} -5 \\ 6 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.4 \\ 0.4 & 0.2 \end{pmatrix} \begin{pmatrix} -5 \\ 6 \end{pmatrix} = \begin{pmatrix} -1.6 \\ -0.8 \end{pmatrix}$$

Thank you for your attention.

Prob 1. Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 & -4 \\ -1 & 2 & 1 \end{bmatrix}$. Compute the matrix

a) $2A - B^T$ b) AB c) BA d) AC

e) CC^T. f) C^TC g) A^3 h) B^2A^T

Prob 2. Suppose that A and B are $n \times n$ matrices. Simplify the expression

a) $(A+B)^2 - (A-B)^2$

b) $A(BC - CD) + A(C - B)D - AB(C - D)$

Prob 3. Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 8 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 5 & 2 & 1 \\ 1 & 8 & 0 & -6 \\ 1 & 4 & 3 & 7 \end{bmatrix}$. Compute AB

Prob 4. Find the inverse of each of the following matrices.

a) $\begin{bmatrix} 1 & 5 \\ 2 & -1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -1 & 2 \\ -5 & 7 & -11 \\ -2 & 3 & -5 \end{bmatrix}$

Prob 5. Given $A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix}$. Find a matrix X such that

a) $AX = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ b) $AX = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ c) $XA = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

Prob 6. Find A when

a) $(3A)^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ b) $(I + 2A)^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

c) $(A^{-1} - 2I)^T = -2 \begin{bmatrix} 1 & 4 \\ 3 & 11 \end{bmatrix}$

Prob 7. Find A^{-1} if

a) $A^2 - 6A + 5I = 0$ b) $A^2 + 3A - I = 0$ c) $A^4 = I$

Prob 8. Compute $\begin{bmatrix} -1 & 3 \\ 0 & 1 \end{bmatrix}^{101}$

Prob 9. Solve for X

a) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} X = \begin{bmatrix} 1 & -1 \\ 3 & 3 \end{bmatrix}$

b) $ABXC = B^T$

c) $AX^TBC = B$ where A, B and C are $n \times n$ invertible matrices.

Prob 10. Find the $(2,1)$-entry of the product

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 5 & 1 \\ 4 & -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 2 \\ 5 & 1 & 0 \\ 0 & 4 & 3 \end{bmatrix}$$

Prob 11. Compute the rank of each of the following matrices.

a) $\begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & -1 & 3 \\ -1 & 4 & 5 & -2 \\ 1 & 6 & 3 & 4 \end{bmatrix}$

Prob 12. Determine the values of m such that the rank of the matrix is 2.

a) $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 5 \\ 1 & 2 & m \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 1 & 5 \\ -3 & 6 & 1 & m \end{bmatrix}$

Prob 13. Find all values of m for which the matrix is invertible $\begin{bmatrix} m & 1 & 3 \\ 1 & 3 & 2 \\ 1 & 4 & 5 \end{bmatrix}$

Prob 14. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and assume that $T(1,2) = (-1,1)$ and $T(0,3) = (-3,3)$.

a) Compute $T(11,-5)$ b) Compute $T(1,11)$ c) Find the matrix of T

Prob 15. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that the matrix of T is $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. Find $T(3,2)$