APPLICATIONS OF DIFFERENTIATION

4

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4.2. Mean value theorem

4.3. Derivates and the shapes of Graphs

4.5. Optimization Problems

4.6. Newton's Method

4.7. Antiderivatives

4.1
Maximum and Minimum Values

In this section, we will learn:

How to find the maximum

and minimum values of a function.

Definition

MAXIMUM & MINIMUM VALUES

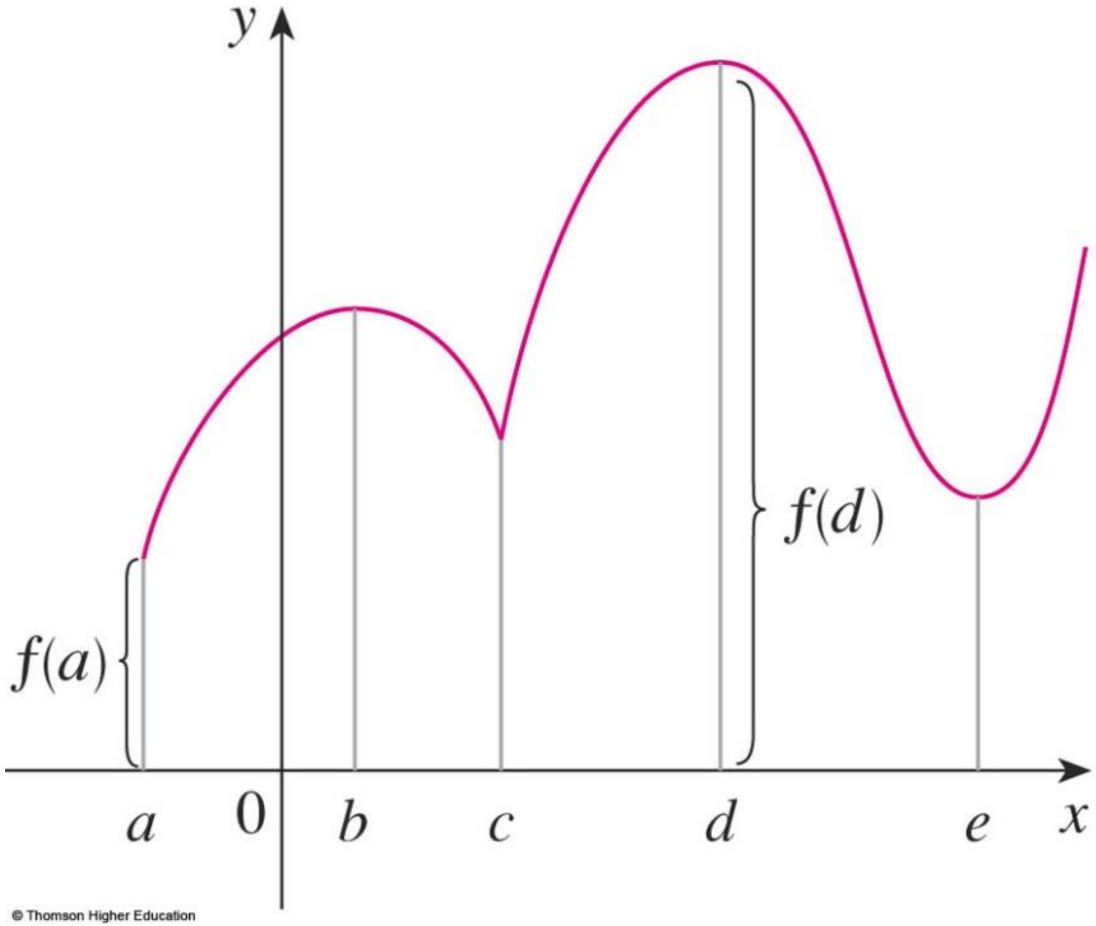
A function f has an **absolute maximum** (or **global maximum**) at c

if $f(c) \geq f(x)$, for all x in D,

where D is the domain of f.

The number $f(c)$ is called

the **maximum value of f on D.**



MAXIMUM & MINIMUM VALUES

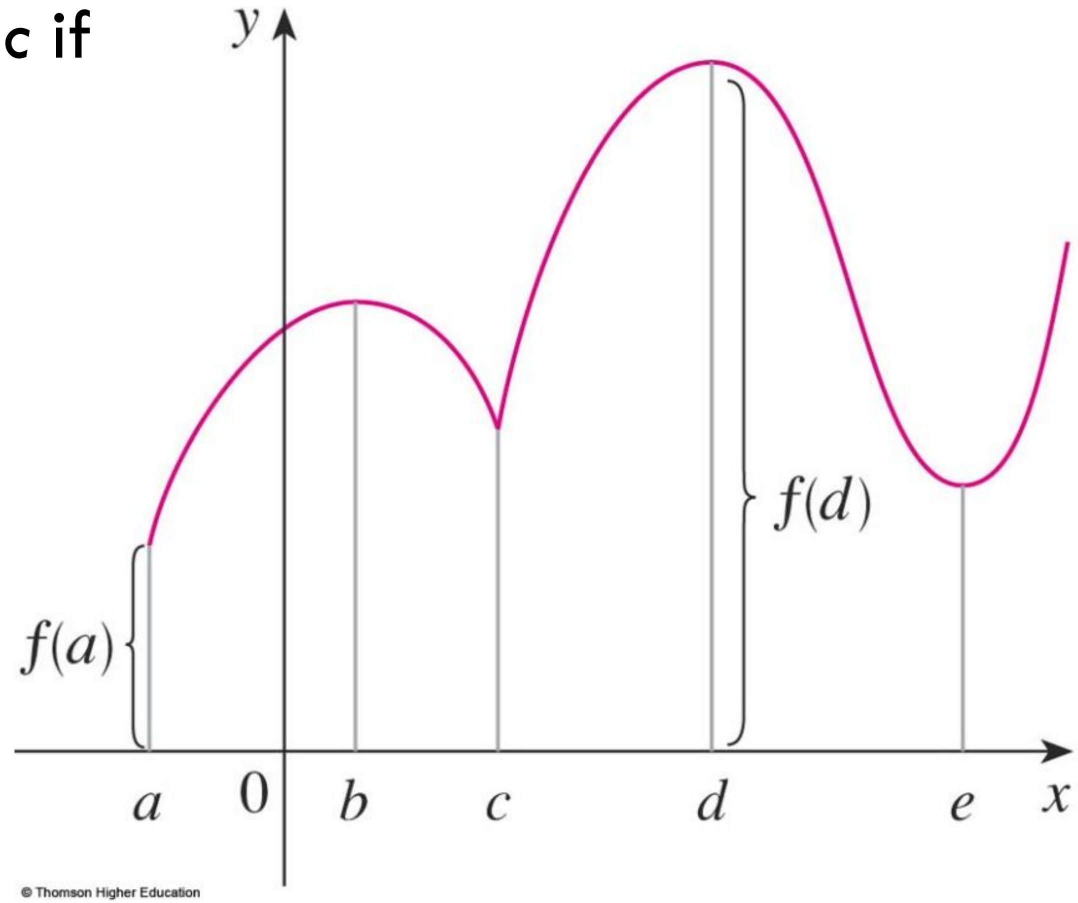
Similarly, *f* has an **absolute minimum** at c if

$$f(c) \leq f(x) \text{ for all } x \text{ in } D$$

where *D* is the domain of *f*.

- The maximum and minimum values of *f*

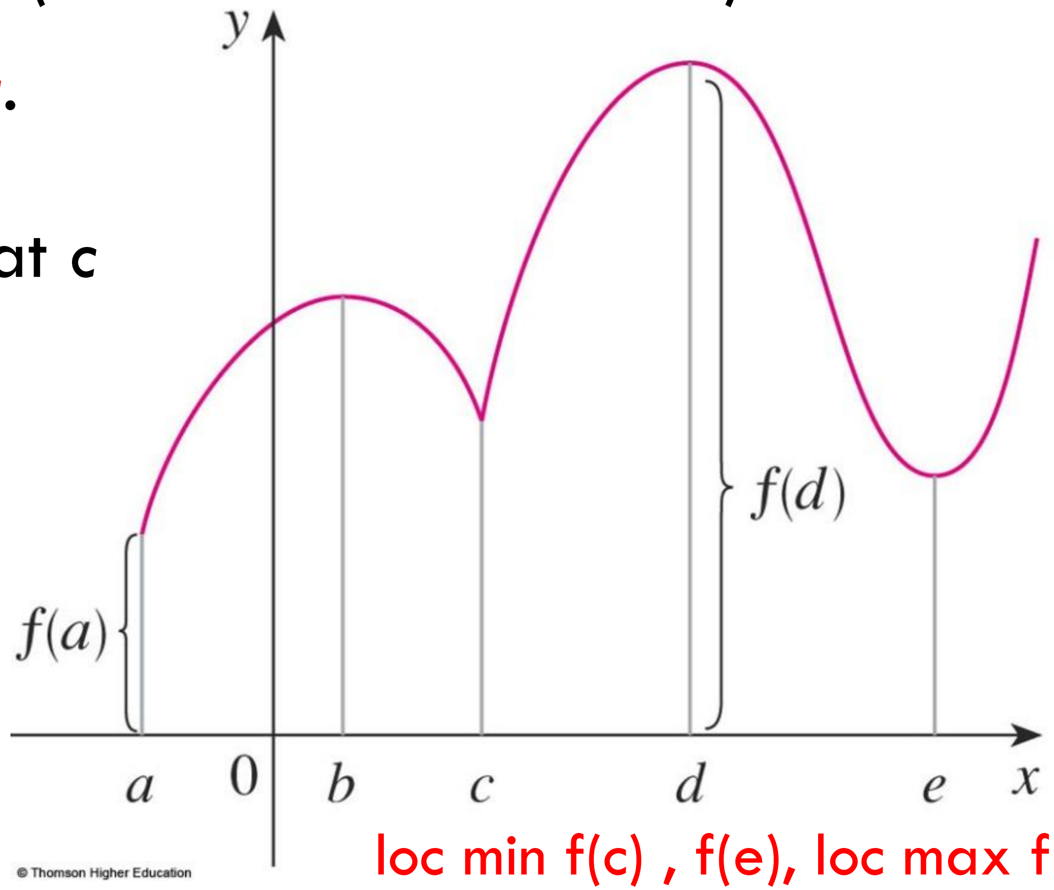
are called the **extreme values** of *f*.



Abs min f(a), abs max f(d)

MAXIMUM & MINIMUM VALUES

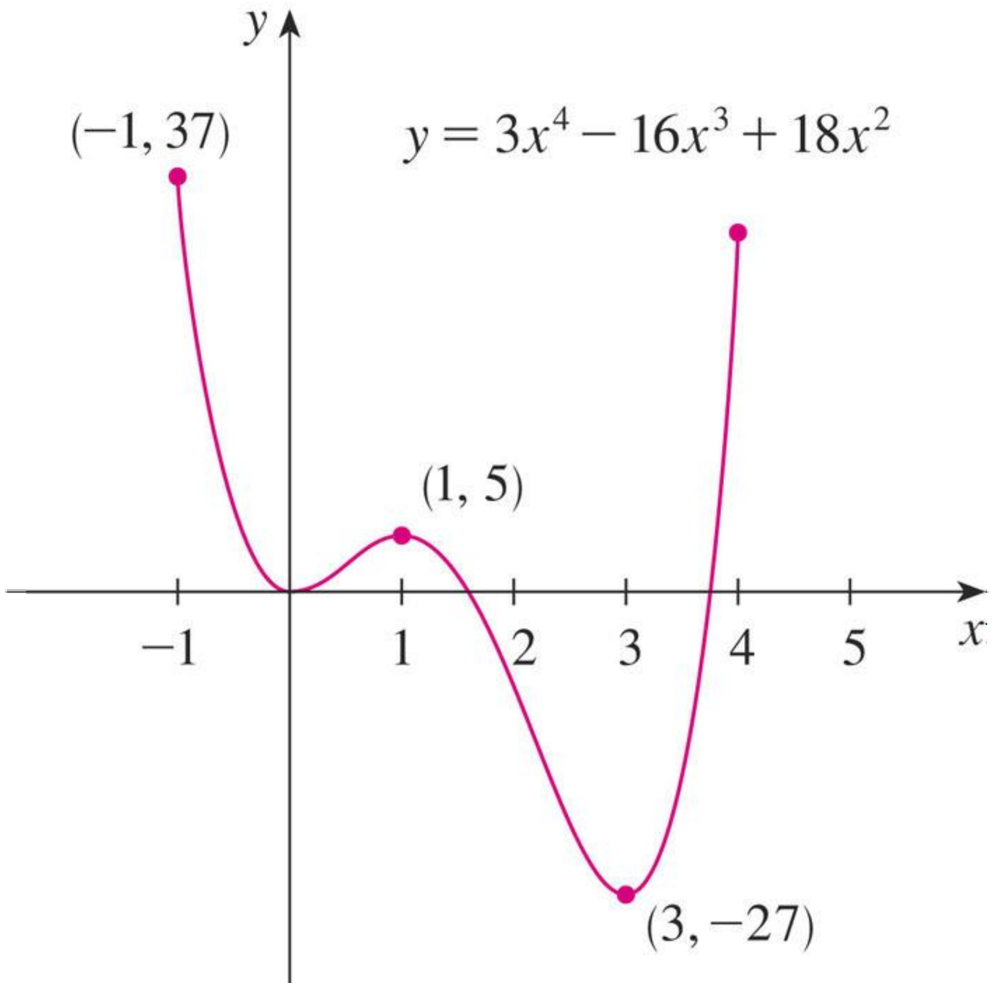
- A function f has a **local maximum** (or **relative maximum**) at c if $f(c) \geq f(x)$ **when x is near c.**
- Similarly, f has a **local minimum** at c if $f(c) \leq f(x)$ **when x is near c.**



loc min f(c) , f(e), loc max f(b), f(d)

Example.

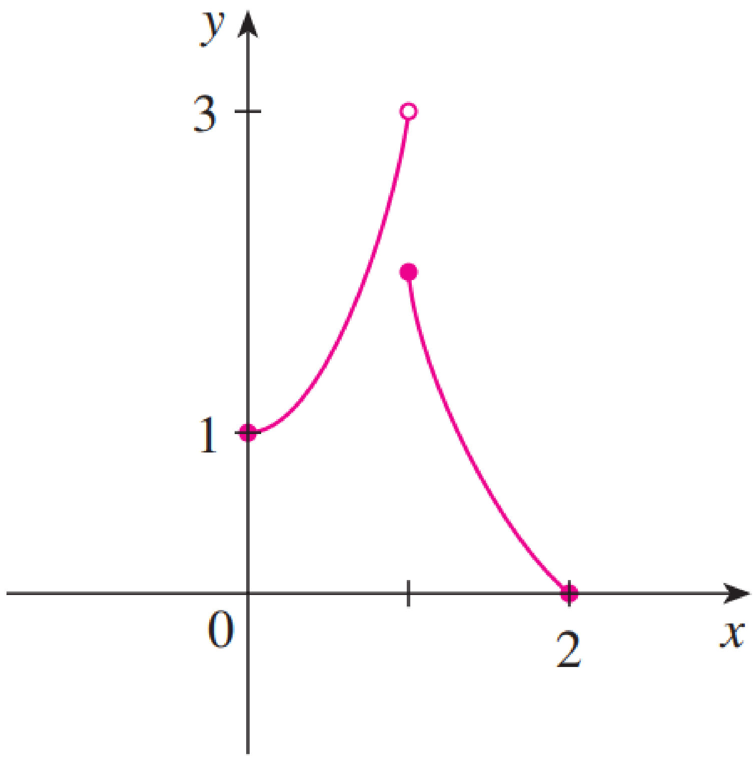
(1) Absolute (global) maximum — abs max
(2) Absolute (global) minimum — abs min
(3) Local (relative) maximum — loc max
(4) Local (relative) minimum — loc min



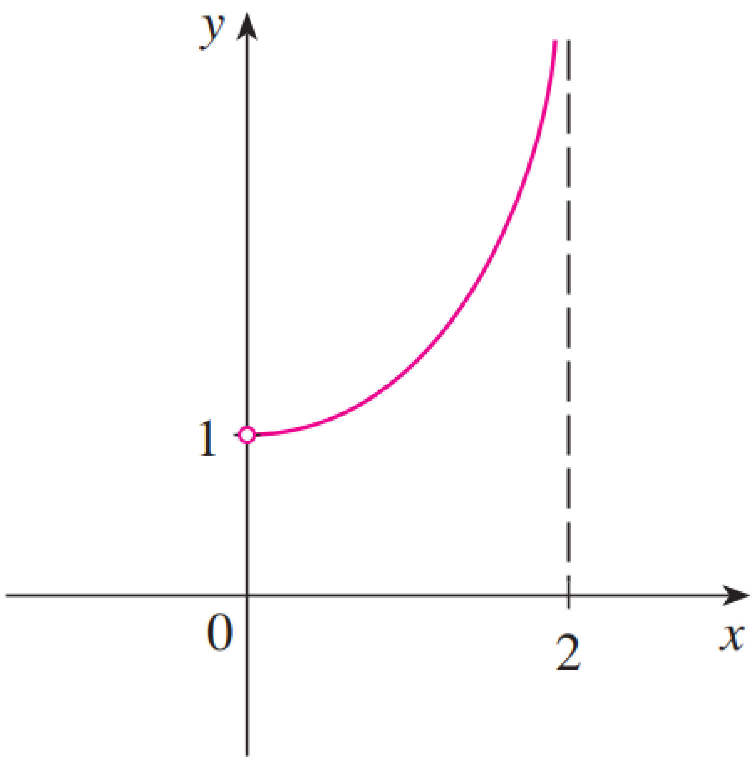
- ⋆ (A) $f(c) \geq f(x)$ for all x in D
- ⋆ (B) $f(c) \geq f(x)$ when x is near c
- ⋆ (C) $f(c) \leq f(x)$ for all x in D
- ⋆ (D) $f(c) \leq f(x)$ when x is near c

In the first figure, why isn't 3 the absolute maximum value?

In the second, does it have the absolute maximum and minimum value?



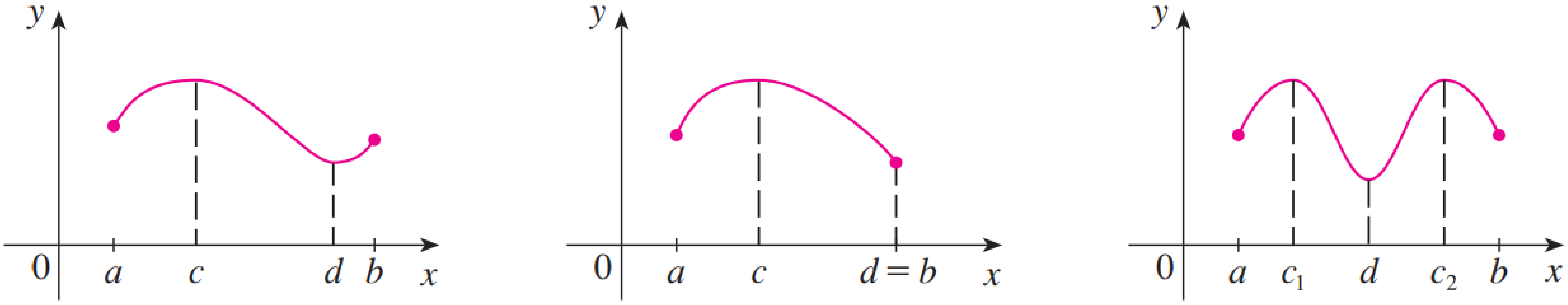
This function has minimum value f(2)=0, but no maximum value.



This continuous function g has no maximum or minimum.

EXTREME VALUE THEOREM

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

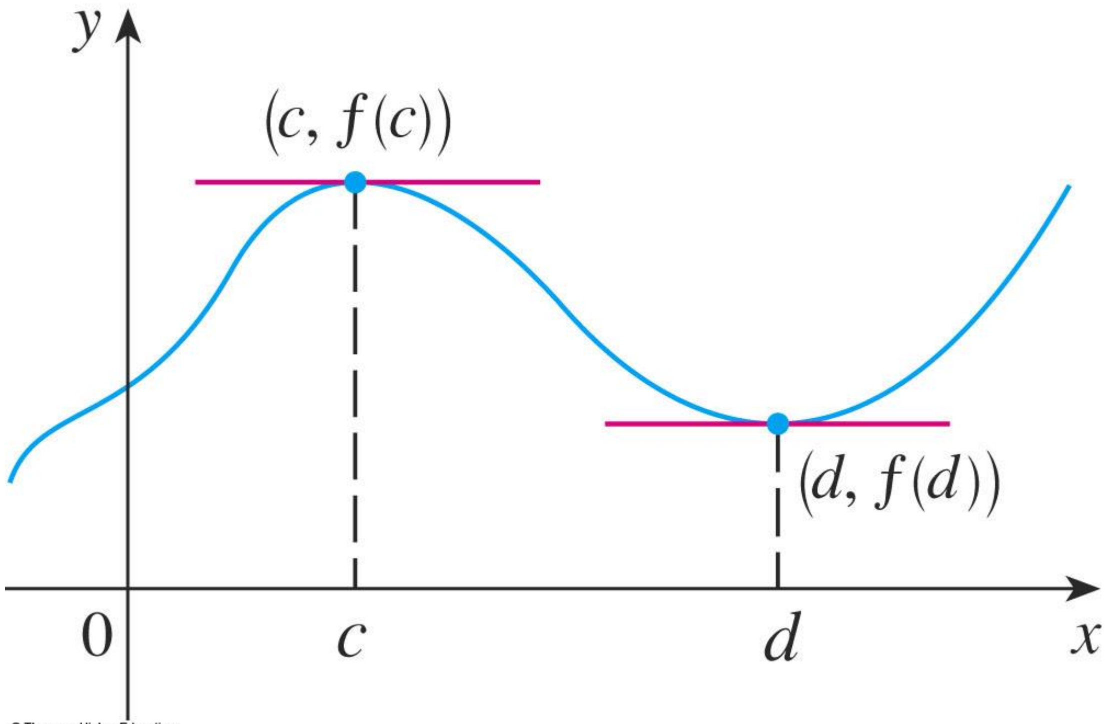


Three Continuous Functions on [a, b] which have abs max and abs min values

EXTREME VALUE THEOREM

The theorem does not tell us how to find these extreme values.

- We start by looking for local extreme values.

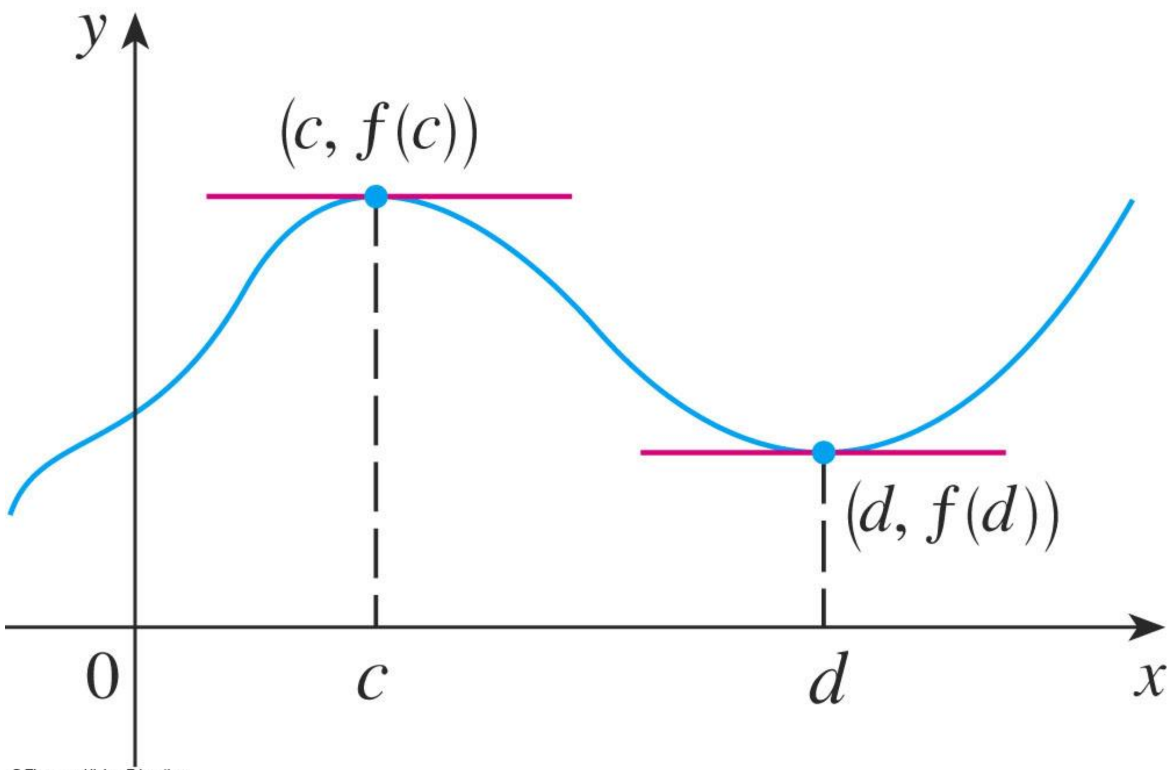


FERMAT'S THEOREM

If f *has* a **local maximum** or **minimum** at c, and

if $f'(c)$ **exists**,

then $f'(c) = 0$.



Example

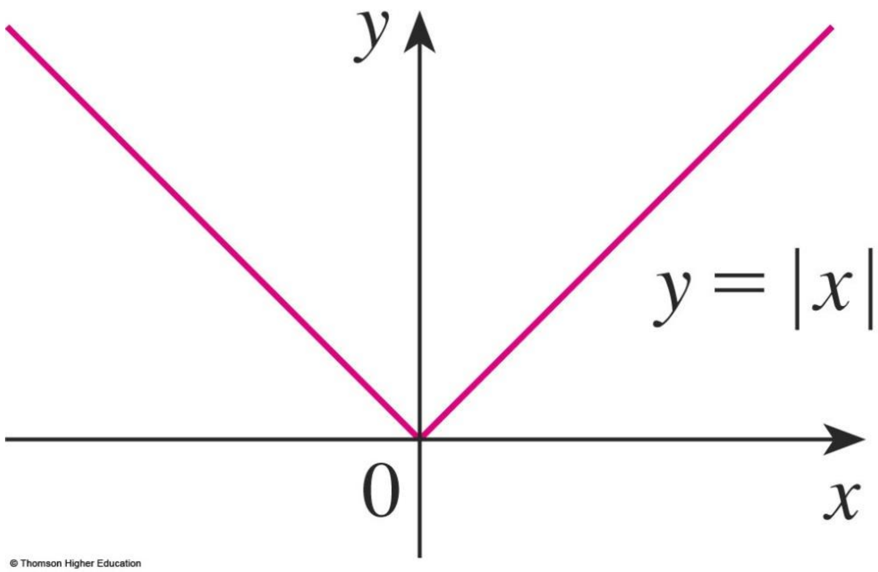
Is it true if say that

"$f'(c)=0$ if f has local extreme value at c?"

Answer: It is false, see next

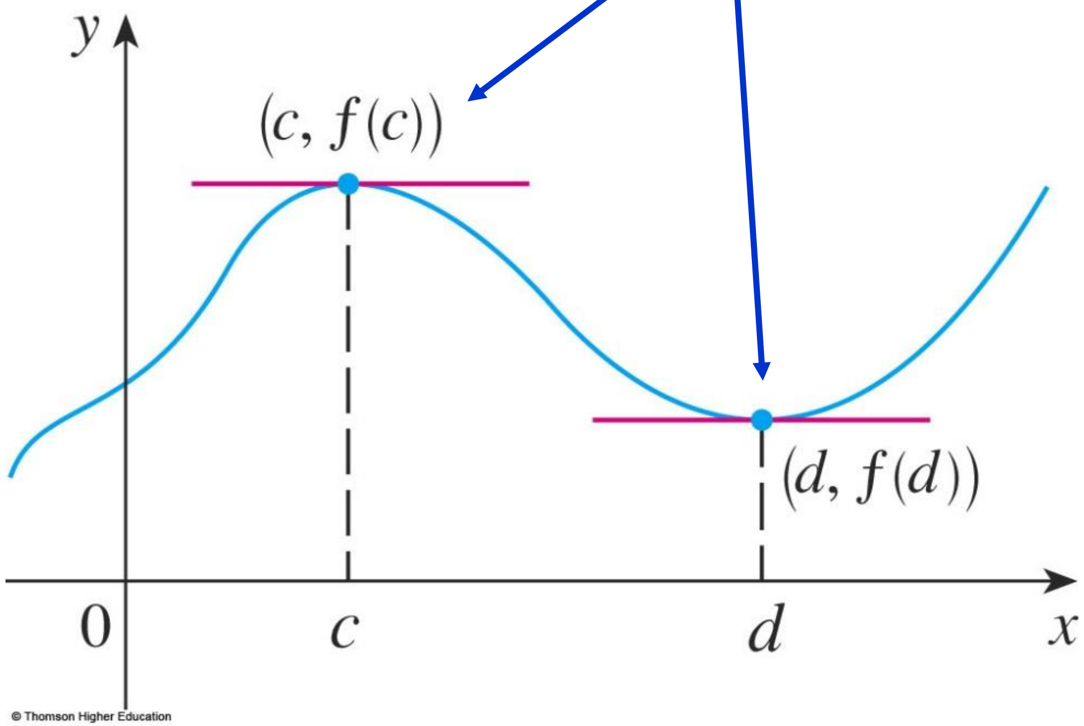
Example

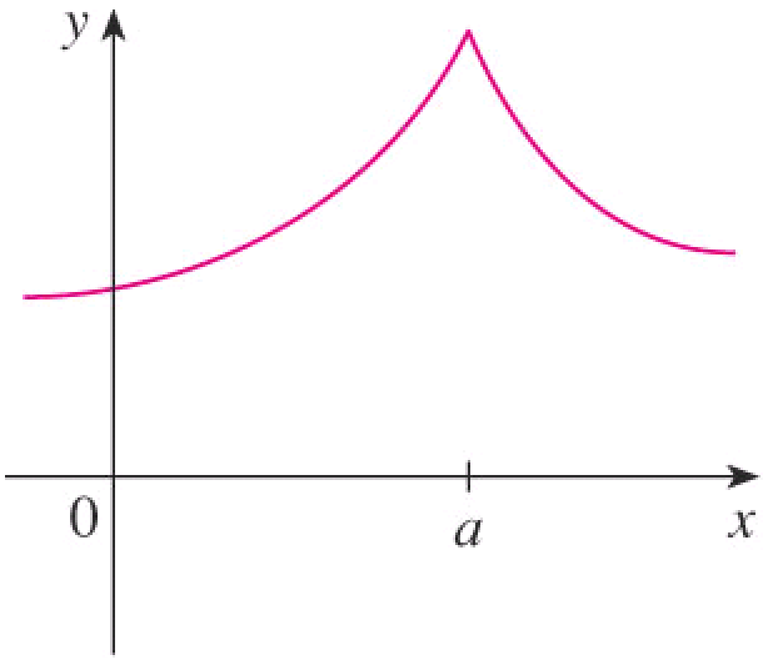
- The function $f(x) = |x|$ has its (local and absolute) minimum value at 0.
- $f'(0)$ does not exist.



CRITICAL NUMBERS

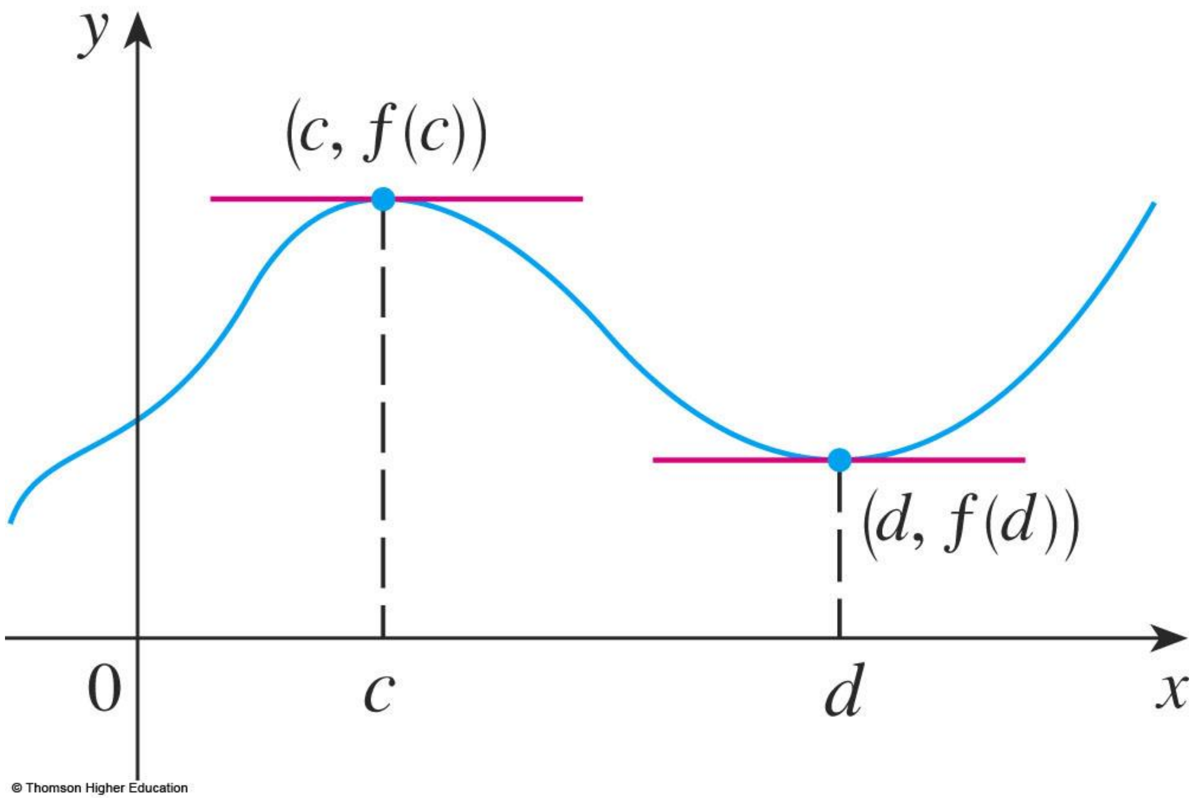
A **critical number (giá trị tới hạn)** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.





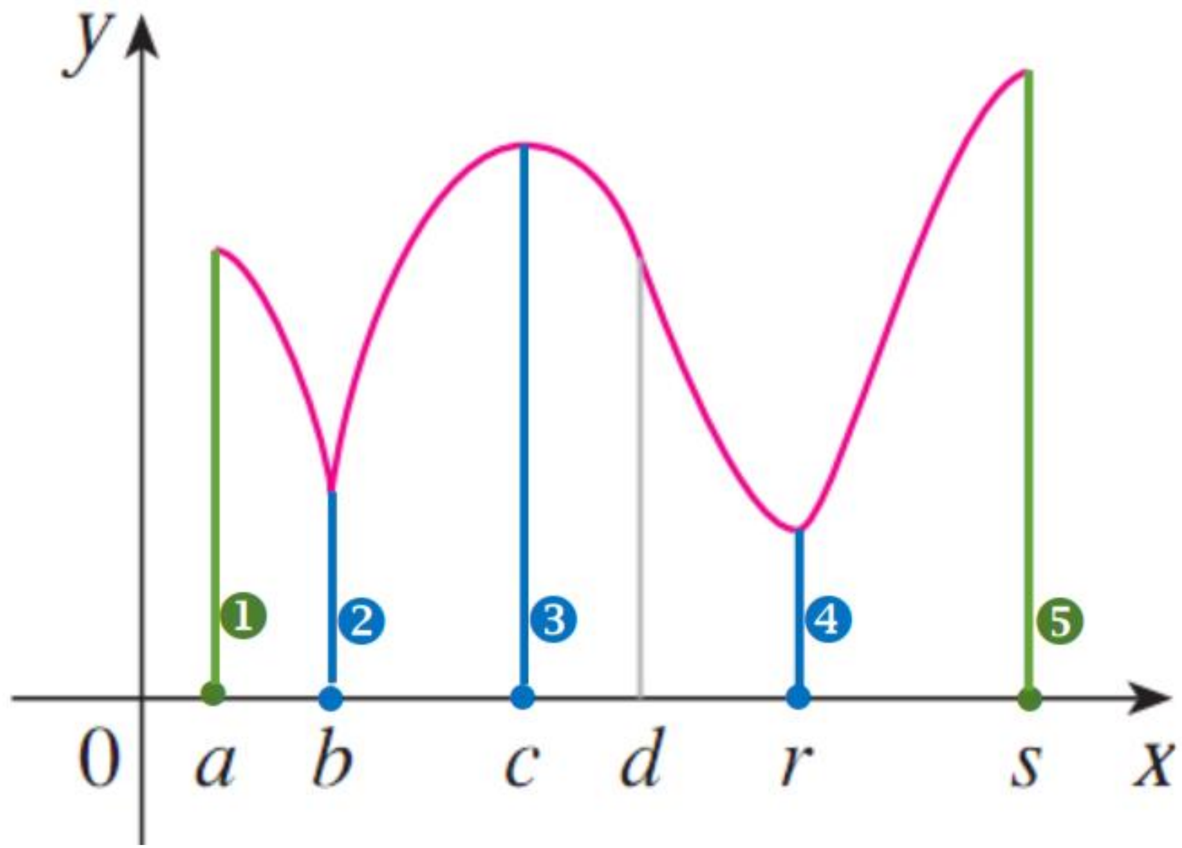
CRITICAL NUMBERS

If f has a local maximum or minimum at c, then c is a critical number of f.



How to find abs max and min?

A **few values** need to be considered



❶ & ❺: endpoints

❷, ❸ & ❹: critical numbers

CLOSED INTERVAL METHOD

To find the **absolute** maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest value from 1 and 2 is the absolute maximum value. The smallest is the absolute minimum value.

Example 1

Find the abs max and min values of $f(x) = x^2$ on $[-2, 1]$

- Critical number: $f'(x) = 2x = 0 \Longrightarrow x = 0$
- Check the values:
  - Critical point value: $f(0) = 0$
  - Endpoint values: $f(-2) = 4$, $f(1) = 1$

Abs min $f(0)$, abs max $f(-2)$

Example 2

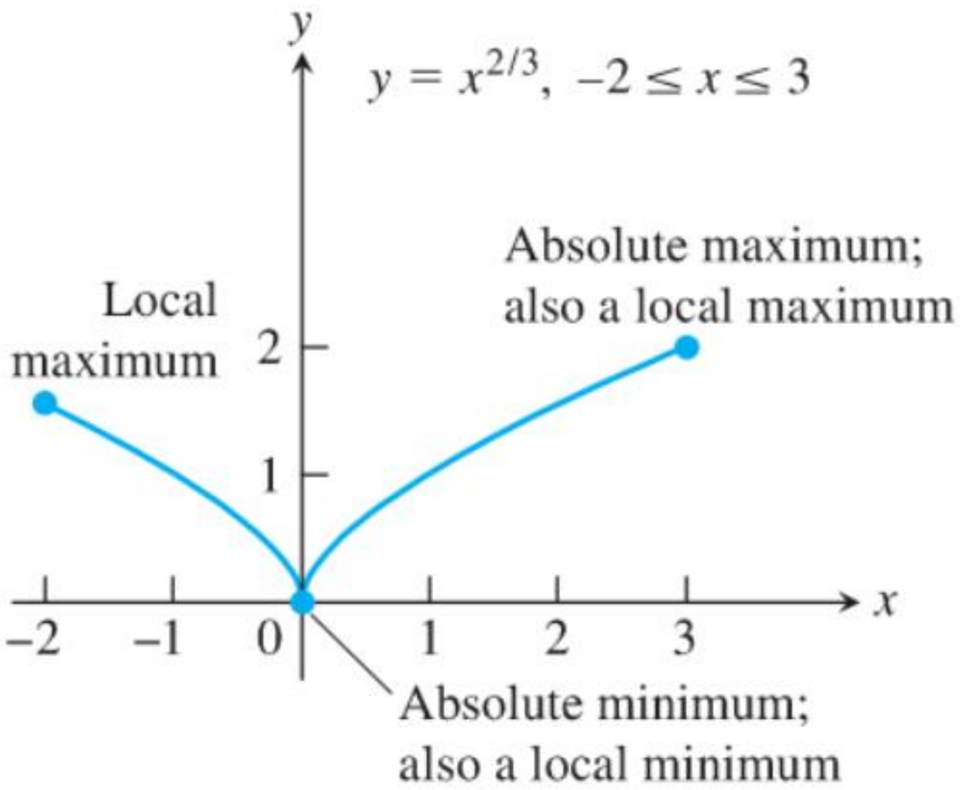
Find the absolute maximum and minimum values of $g(t) = 32t - t^4$ on $[-1, 1]$.

- ☆ Critical number: $g'(t) = 32 - 4t^3 = 0 \Rightarrow t = 2 > 1$, not in $[-1, 1]$
- ☆ The function's abs max and min values therefore occur at the endpoints:
  - $g(-1) = -33$ (abs min)
  - $g(1) = 31$ (abs max)

Example 3

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on [-2, 3].

- Critical point value: $f(0) = 0$ (abs min)
- Endpoint values: $f(-2) = 4^{1/3}$, $f(3) = 9^{1/3}$ (abs max)



Select the correct ones.

a. If $f'(c)=0$ then f has the local maximum or minimum at c.

b. If f has the absolute minimum value at c then $f'(c)=0$.

c. If f is continuous on (a,b) then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ for some c and d in (a,b).

d. All of the above.

e. None of the above.

Answer: e

4.2

The Mean Value Theorem

In this section, we will learn about:

The significance of the mean value theorem.

ROLLE'S THEOREM

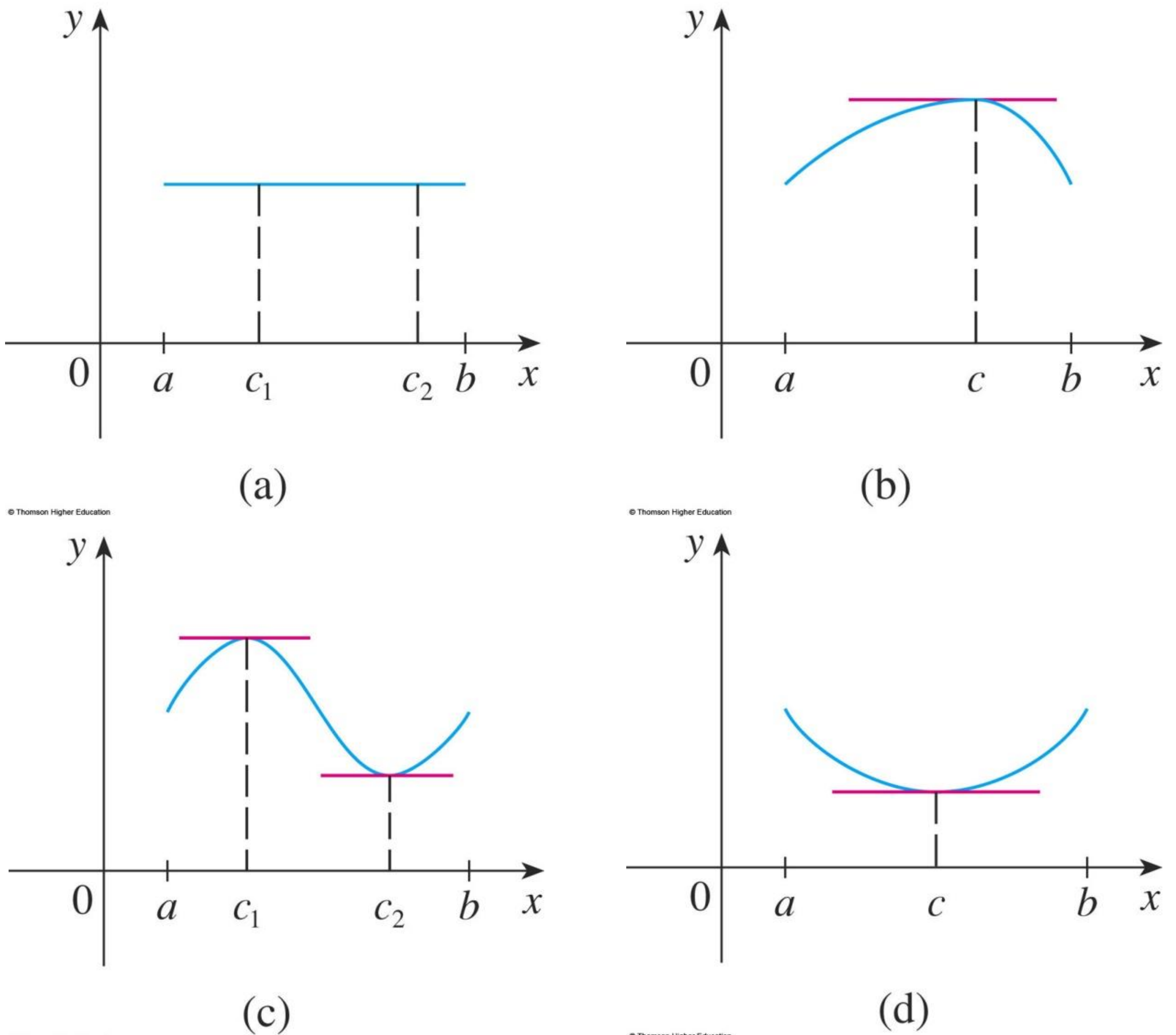
Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)
3. $f(a) = f(b)$

Then, there is a number c in (a, b) such that $f'(c) = 0$.

ROLLE'S THEOREM

The figures show the graphs of four such functions.



Example

Let $f(x)=x^3-2x^2+x-5$. Find the numbers c in the Rolle's theorem?

Answer: $f'(c)=0$ iff $c=1$ or $c= 1/3$

MEAN VALUE THEOREM

Equations 1 and 2

Let f be a function that fulfills two hypotheses:

1. f is continuous on the closed interval $[a, b]$.

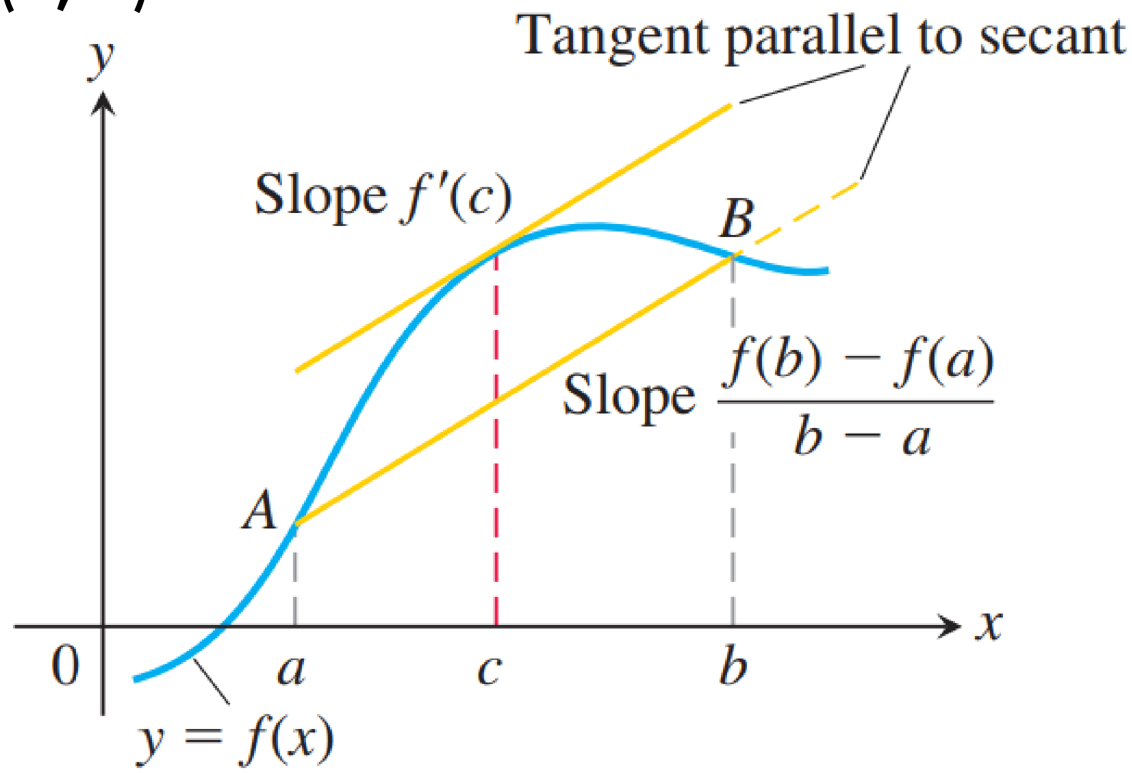
2. f is differentiable on the open interval (a, b).

Then, there is a number **c in (*a*, *b*)** such that

$$f\,'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

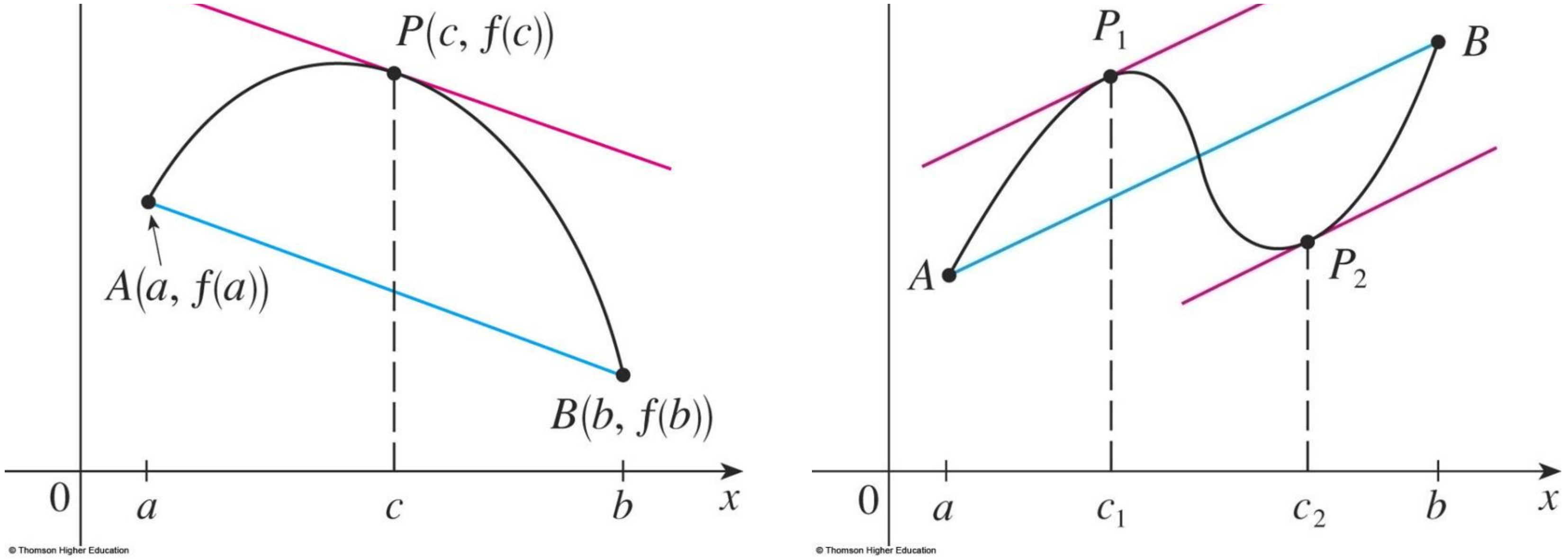
$$f(b) - f(a) = f\,'(c)(b - a)$$



MEAN VALUE THEOREM

f ′(c) is the slope of the tangent line at (c, *f*(c)).

- There is at least one point *P*(c, *f*(c)) on the graph where the slope of the tangent line is the same as the slope of the secant line *AB*.



Example

Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

We are given that f is differentiable - and therefore continuous - everywhere.

$\Rightarrow$ In particular, we can apply the Mean Value Theorem on the interval [0, 2].

There exists a number c such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

So, $f(2) = -3 + 2f'(c)$

- We are given that $f'(x) \leq 5 \ \forall x$

$\Rightarrow f'(c) \leq 5.$

$\Rightarrow 2\, f'(c) \leq 10.$

$\Rightarrow f(2) = -3 + 2\, f'(c) \leq -3 + 10 = 7$

- The largest possible value for $f(2)$ is 7.

MEAN VALUE THEOREM

- If $f\,'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

- If $f'\,(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b).

That is, $f(x) = g(x) + c$ where c is a constant.

APPLICATIONS OF DIFFERENTIATION

4.3

Derivatives and the Shapes of Graphs

In this section, we will learn:

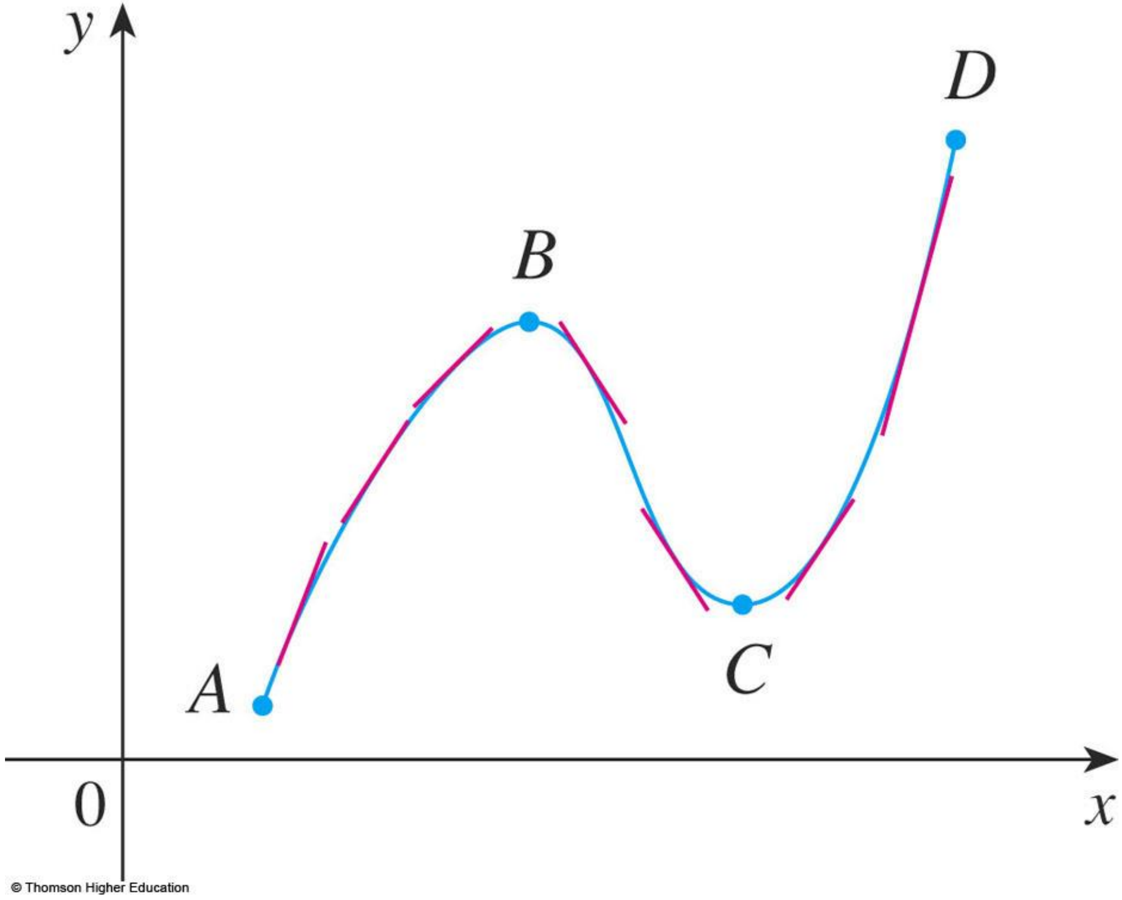
How the derivative of a function gives us the direction

in which the curve proceeds at each point.

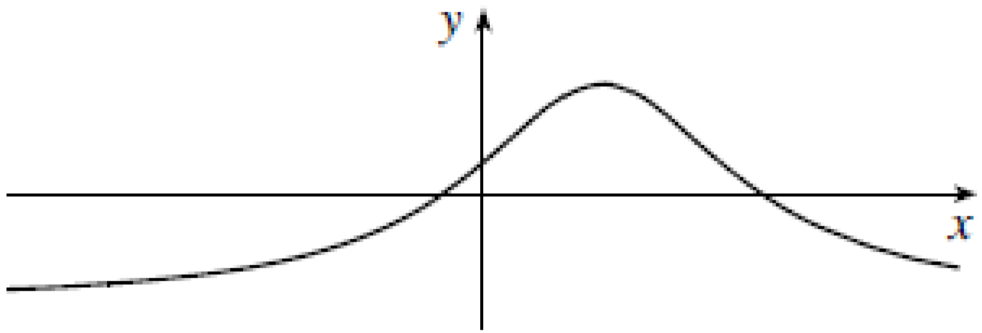
INCREASING/DECREASING TEST (I/D TEST)

a. If $f'(x) > 0$ on an interval, then f is increasing on that interval.

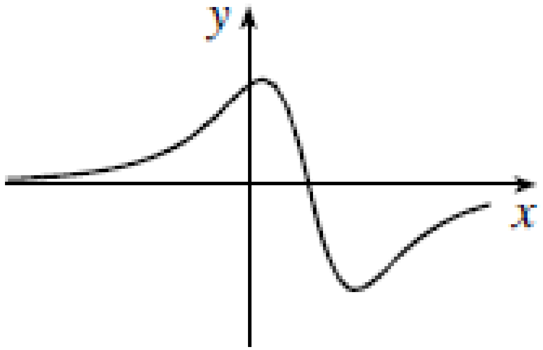
b. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.



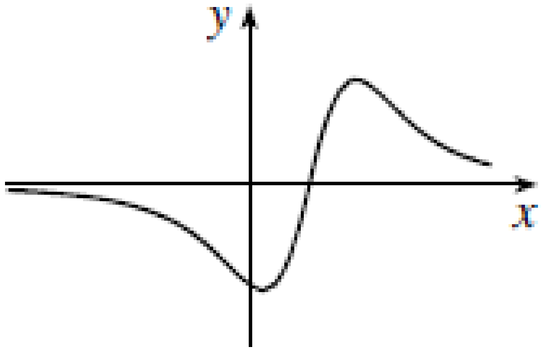
• **Drill Question:** The graph of f is shown below. Which of the following could be the graph of f'?



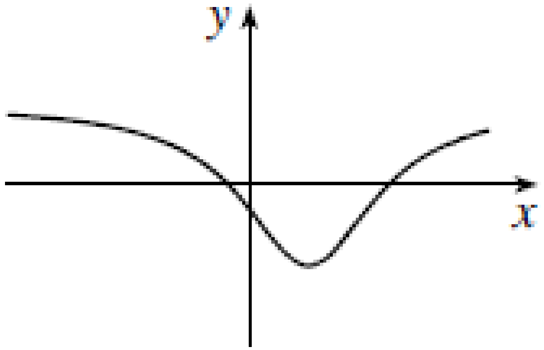
(A)



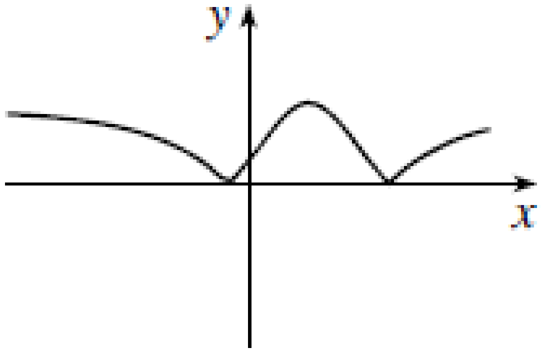
(B)



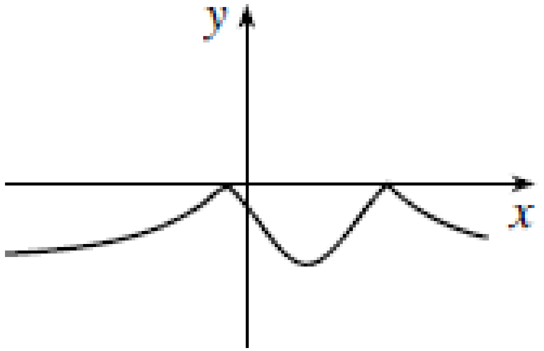
(C)



(D)



(E)

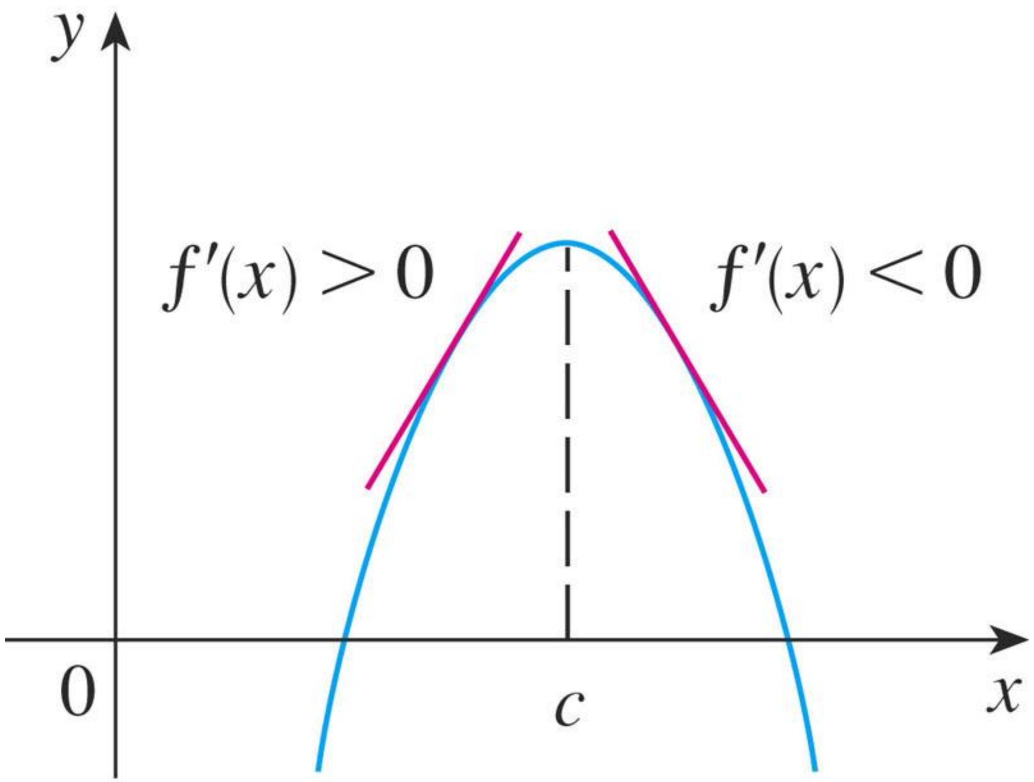


Answer: (A)

FIRST DERIVATIVE TEST

Suppose that c is a critical number of a continuous function *f*.

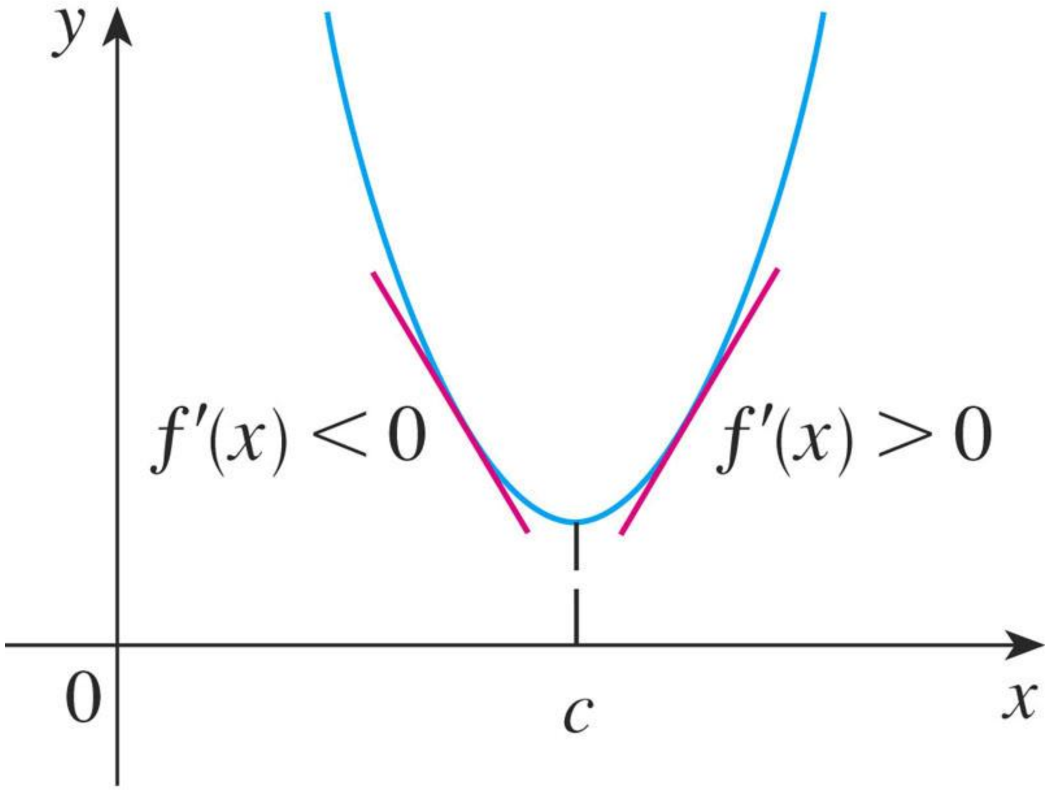
a. If *f'* changes from positive to negative at c, then *f* has a local maximum at c.



(a) Local maximum

FIRST DERIVATIVE TEST

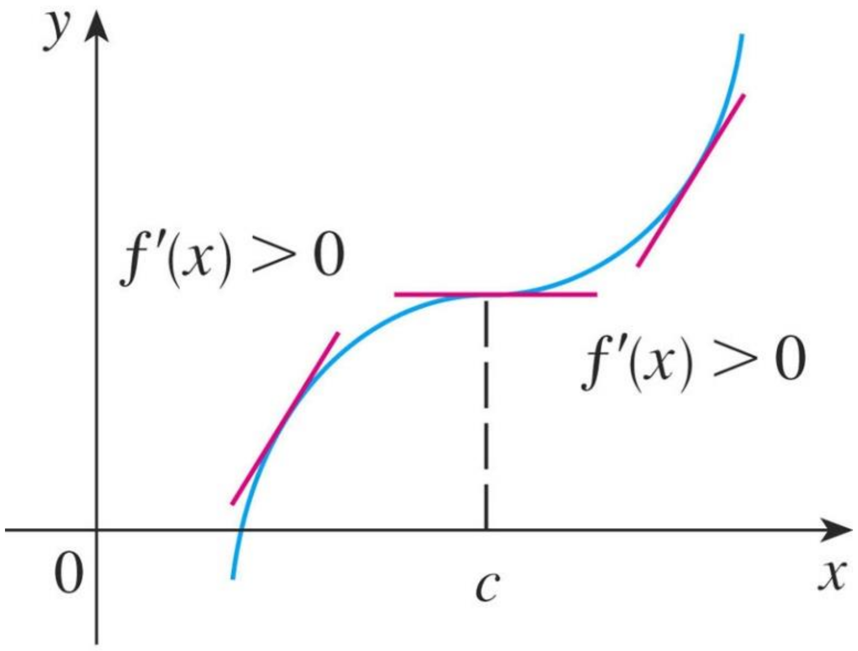
b. If *f'* changes from negative to positive at c, then *f* has a local minimum at c.



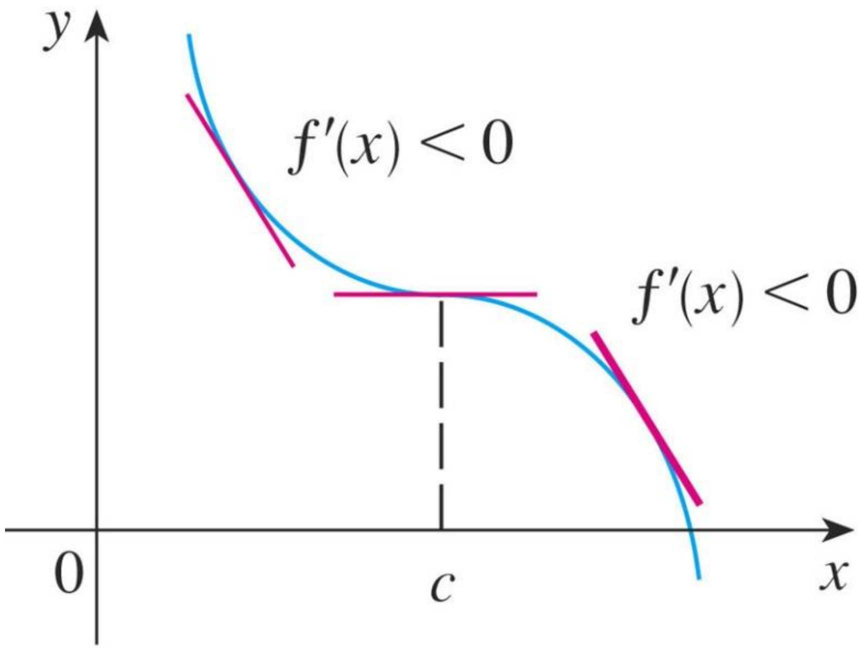
(b) Local minimum

FIRST DERIVATIVE TEST

c. If *f'* does not change sign at c then *f* has no local maximum or minimum at c.



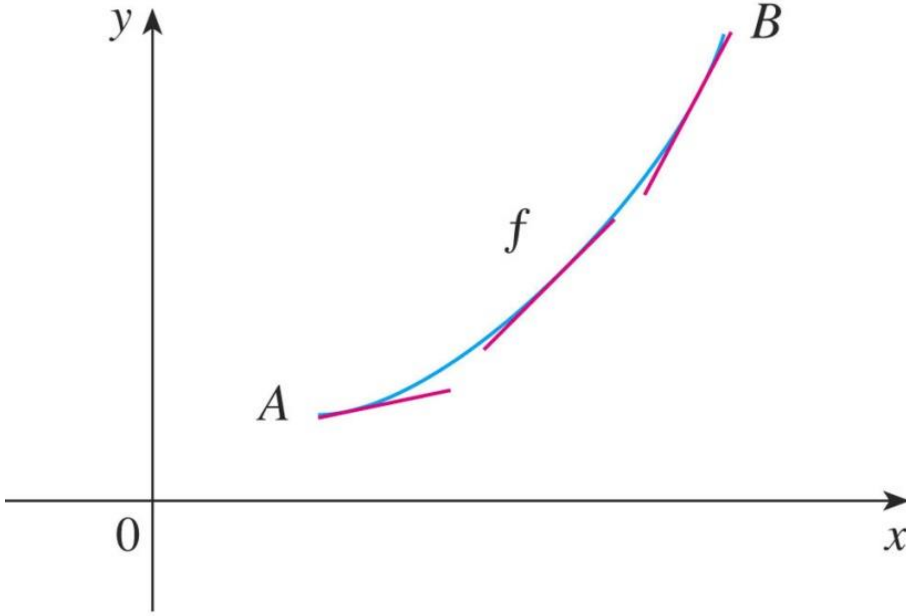
(c) No maximum or minimum



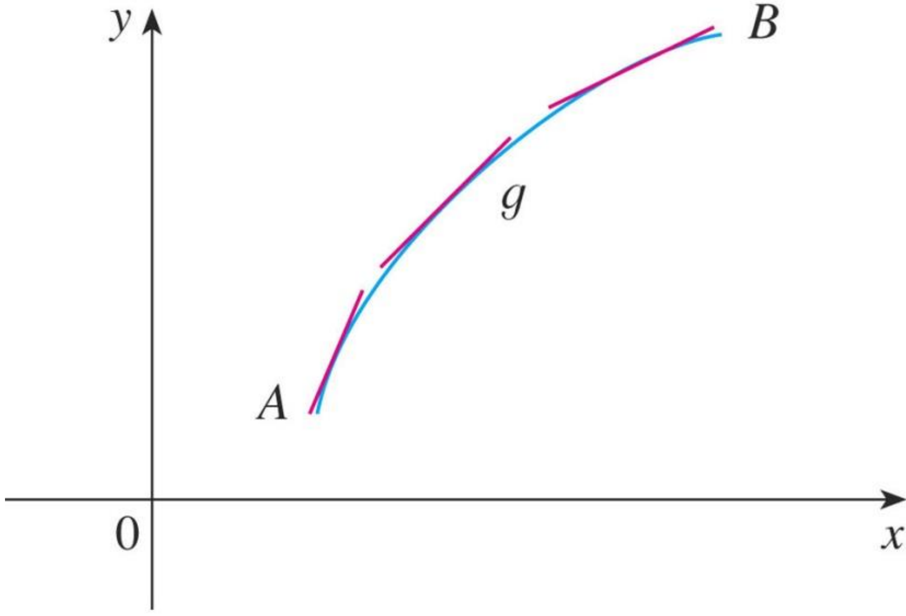
(d) No maximum or minimum

CONCAVE UPWARD/DOWNDWARD

- The curve lies above the tangents and f is called **concave upward** (lõm lên) on (a, b).
- The curve lies below the tangents and g is called **concave downward** (lõm xuống) on (a, b).



(a) Concave upward

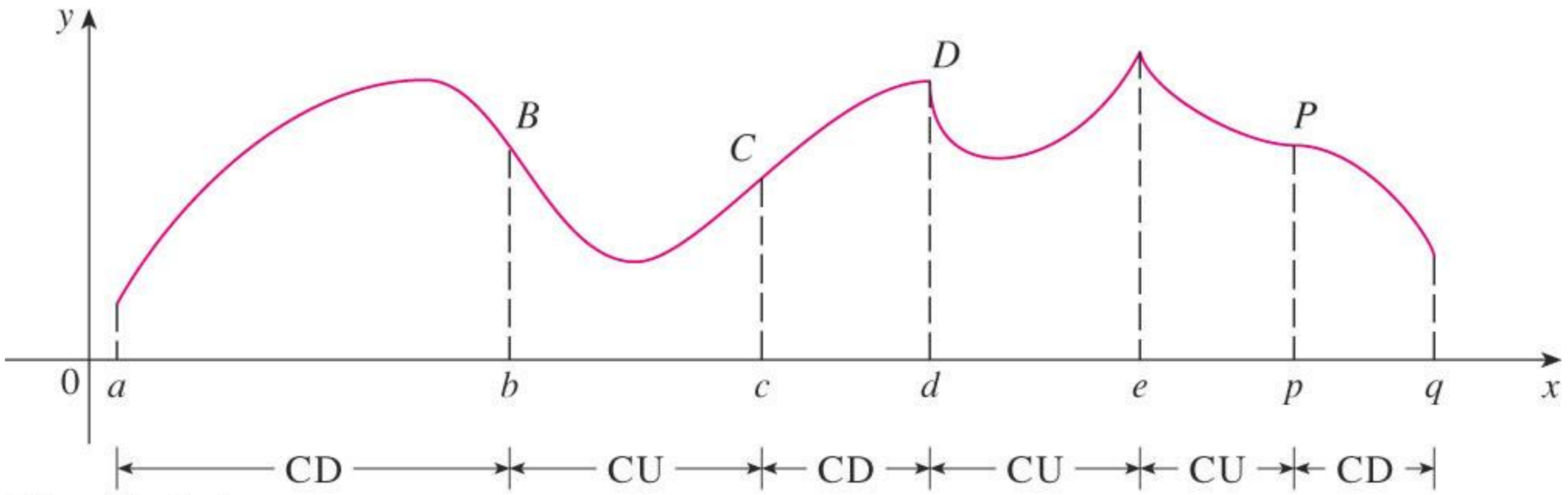


(b) Concave downward

CONCAVITY TEST

a. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

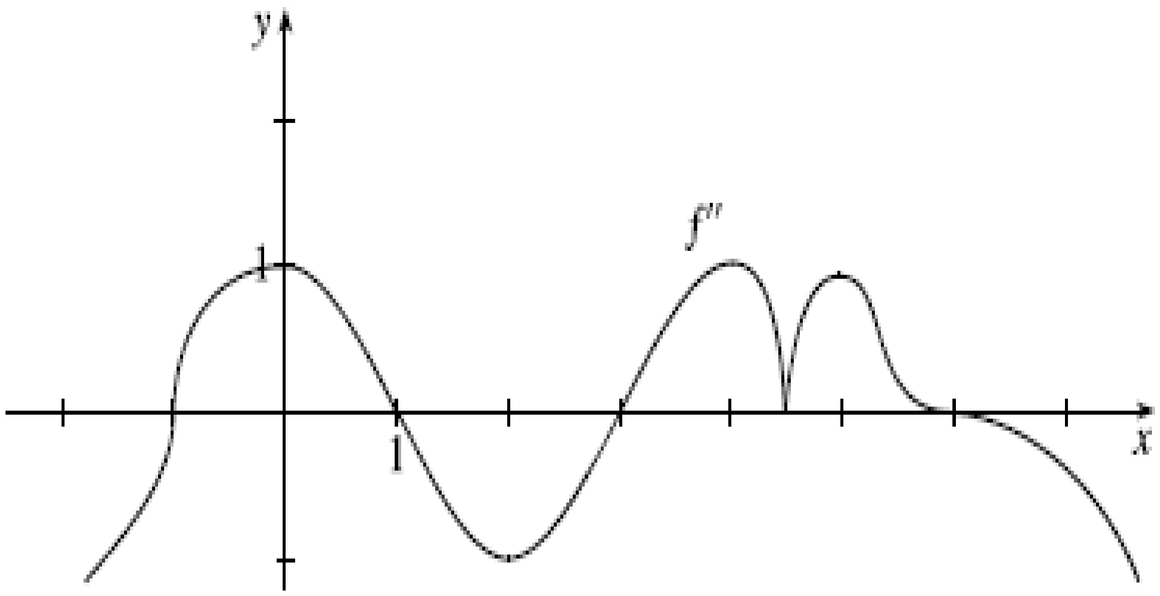
b. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.



INFLECTION POINT

A point P on a curve $y = f(x)$ is called an **inflection point (điểm uốn)** if f is continuous there and the curve changes from concave upward to concave downward (or from concave downward to concave upward at P).

• Given a graph of f'' as below, have the students indicate the points of inflection of f, and explain their reasoning.



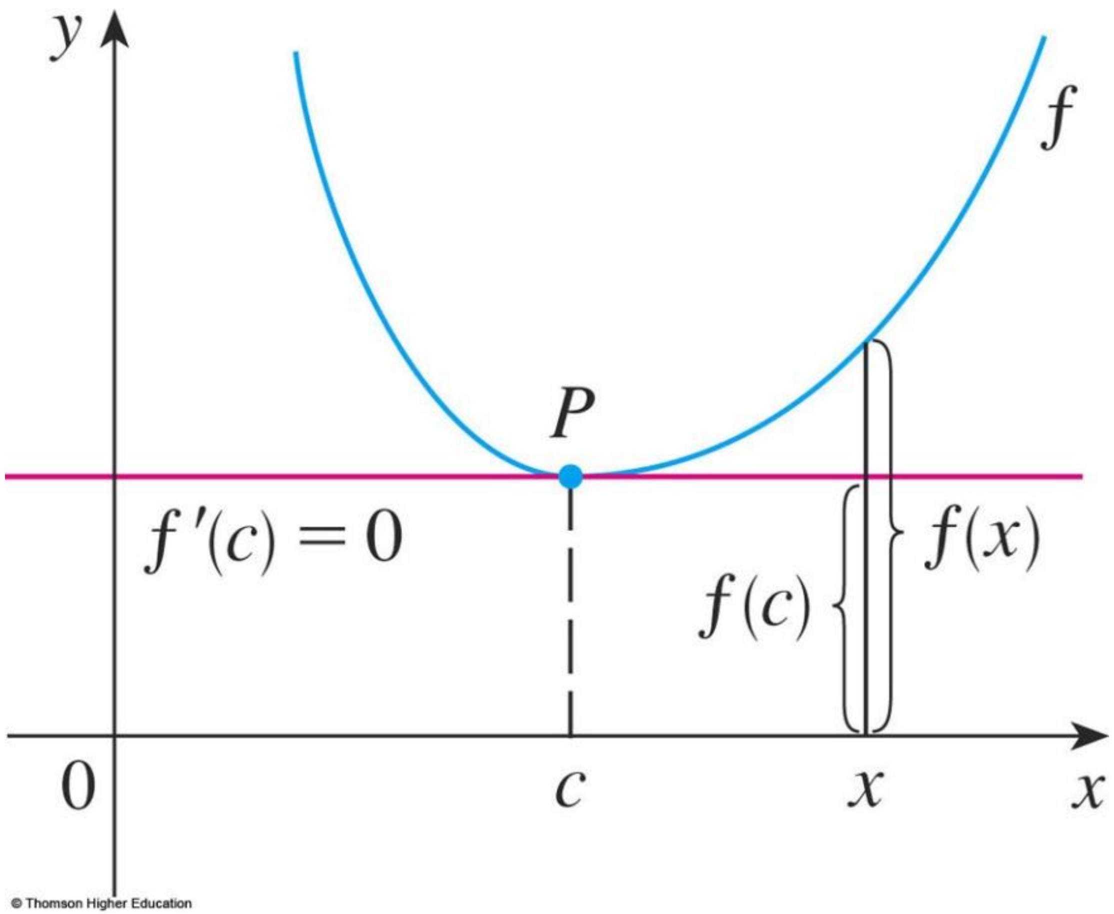
Answer: (-1,f(-1)),(1,f(1)),(3,f(3)),(6,f(6))

SECOND DERIVATIVE TEST

Suppose f'' is continuous near c.

a. If $f'(c) = 0$ and **$f''(c) > 0$,**
   then f has a **local minimum** at c.

b. If $f'(c) = 0$ and **$f''(c) < 0$,**
   then f has a **local maximum** at c.



Choose the correct one.

A If f has local extreme value at c then $f'(c)=0$.

B If $f'(c)=0$ then f has local extreme value at c.

C If $f''(3)=0$ then $(3,f(3))$ is an inflection point of f.

D There exists a function such that $f'(x)$ is nonzero for all x and $f(1)=f(0)$.

E None of the above

Answer: e

APPLICATIONS OF DIFFERENTIATION

4.5
Optimization Problems

In this section, we will learn:

How to solve problems involving

maximization and minimization of factors.

UNDERSTAND THE PROBLEM

Read the problem carefully until it is clearly understood.

- What is the **unknown**?
- What are the **given quantities**?
- What are the given **conditions**?

FIND THE ABSOLUTE MAX./MIN. VALUE OF *F*

Use the methods of Sections 4.1 and 4.3 to find the absolute maximum or minimum value of f.

- In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 4.1 can be used.

Example

1. Find two positive numbers such that the sum is 24 and the product is the largest?

2. Find two positive numbers such that the product is 36 and the sum is the smallest?

Answer: 12,12, and 6,6

Example

Find the point on the line $y = 2x$ -3 that is closest to the origin.

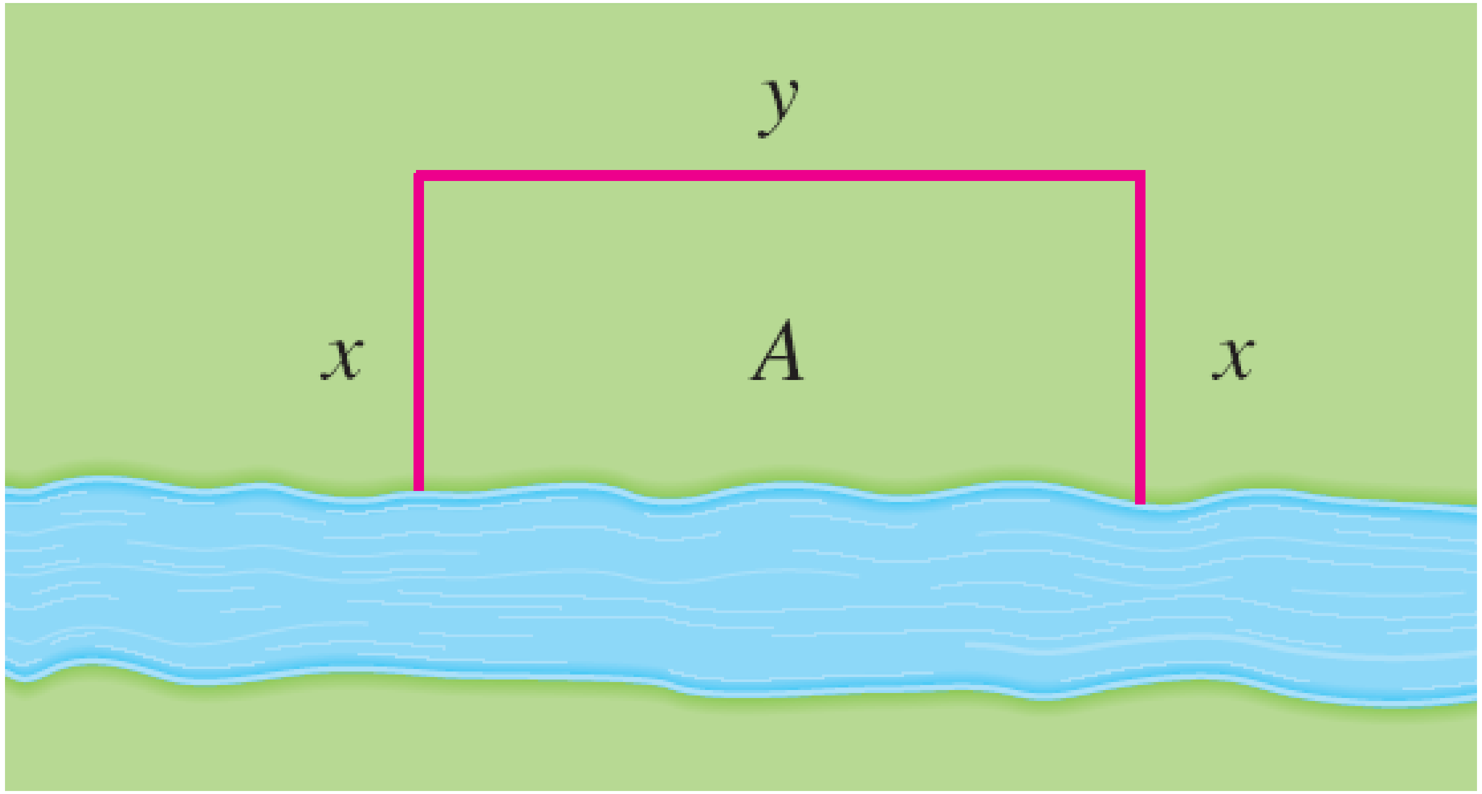
(x,2x-3) is in the line and the distance from it to the origin is …

We find the minimum of…

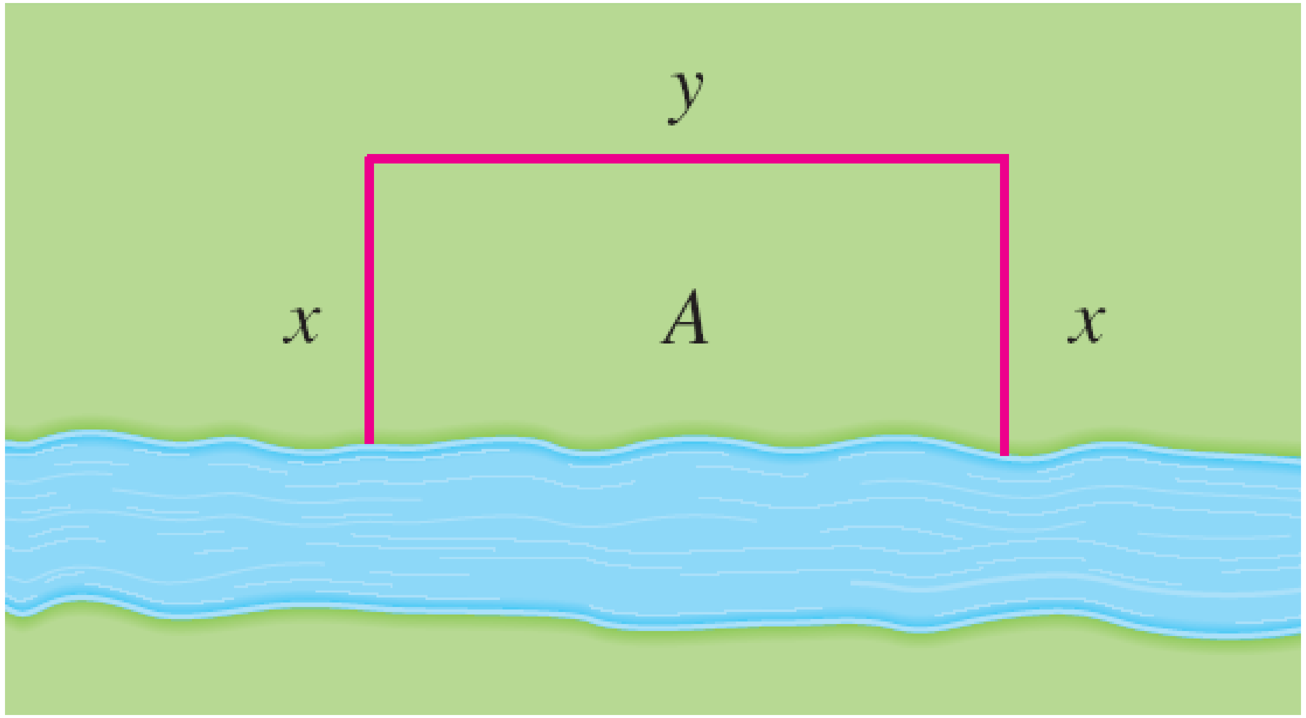
Example

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river.

- What are the dimensions of the field that has the **largest area** ?



This figure illustrates the general case. We wish to maximize the area A of the rectangle.



Then, we express A in terms of x and y: $A = xy$

$2x + y = 2400$

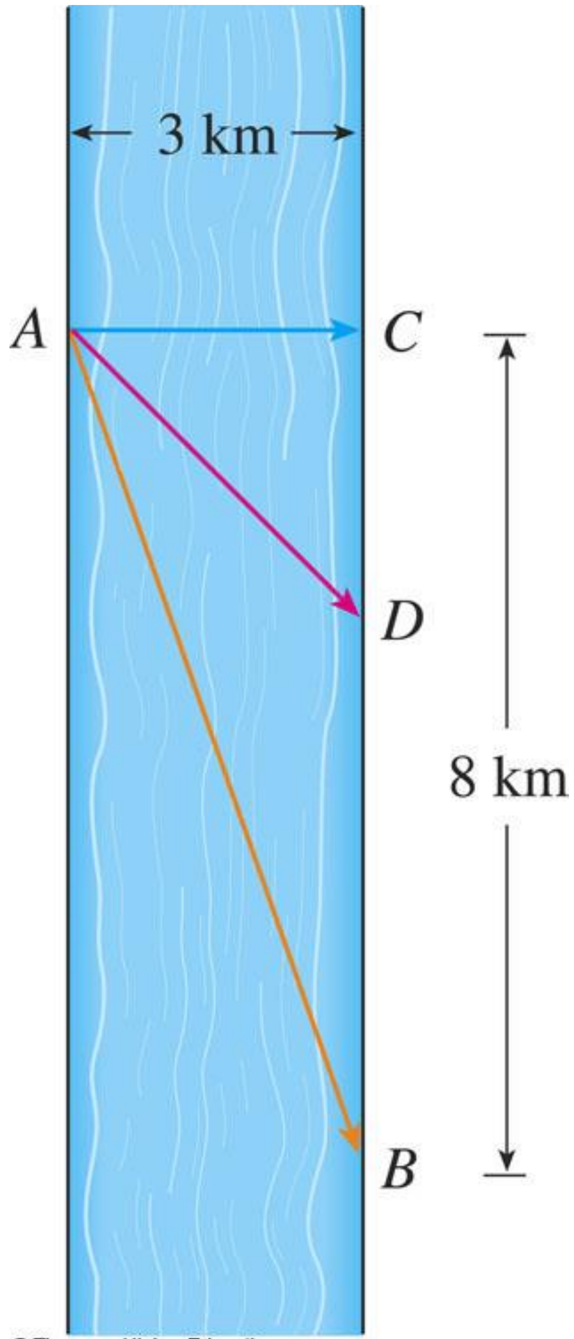
So, $A(x) = 2400x - 2x^2, 0 \leq x \leq 1200$

…

Example

A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B (8 km downstream on the opposite bank) as quickly as possible.

If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible?



We use the equation:

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

- Then, the rowing time is:

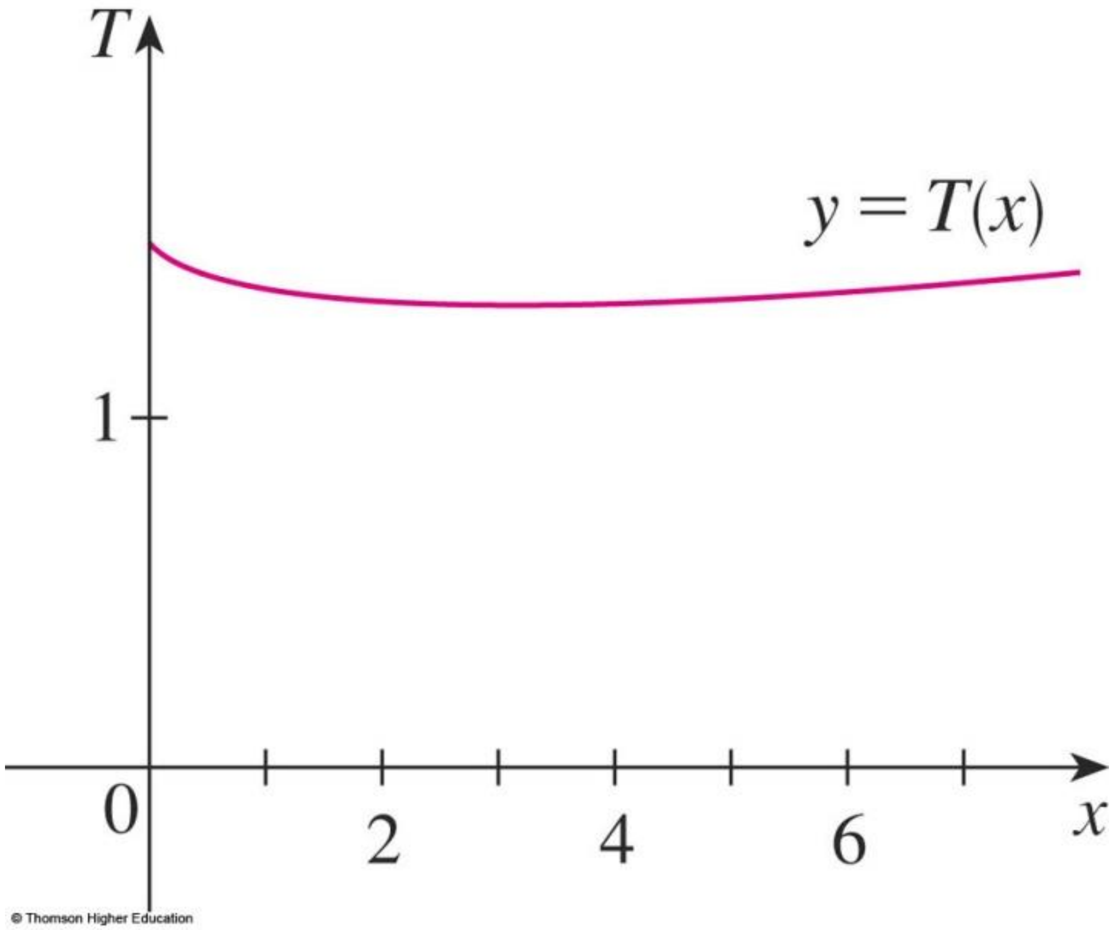
$$\frac{\sqrt{x^2+9}}{6}$$

- The running time is:

$$\frac{8-x}{8}$$

- So, the total time T as a function of x is:

$$T(x) = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}$$



Example

A rectangular storage container with an open top is to have a volume of 15 m^3 The length of its base is twice the width.

Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the **cost** of materials for the **cheapest** such container.

3.6

Newton's Method

In this section, we will learn:

How to solve high-degree equations

using Newton's method.

NUMERICAL ROOTFINDERS

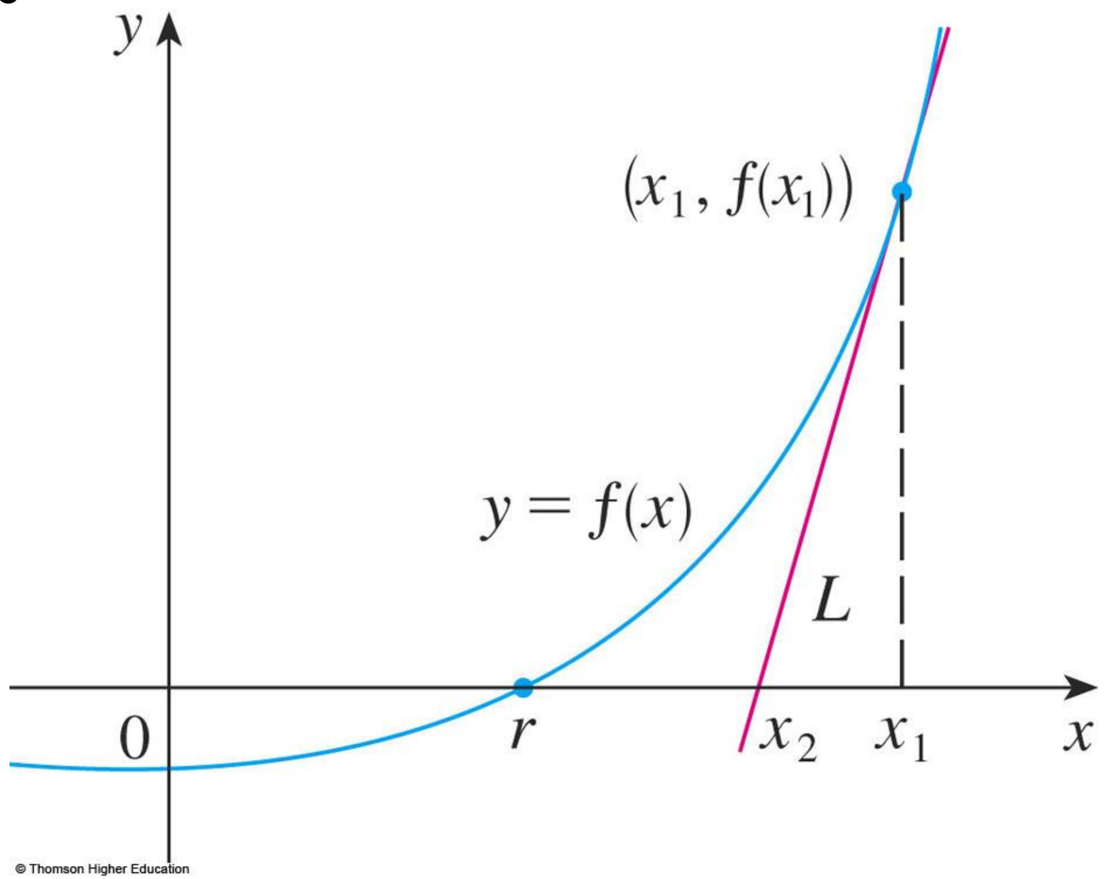
How do those numerical root finders work?

- They use a variety of methods.
- Most, though, make some use of Newton's method, also called the Newton-Raphson method.

NEWTON'S METHOD

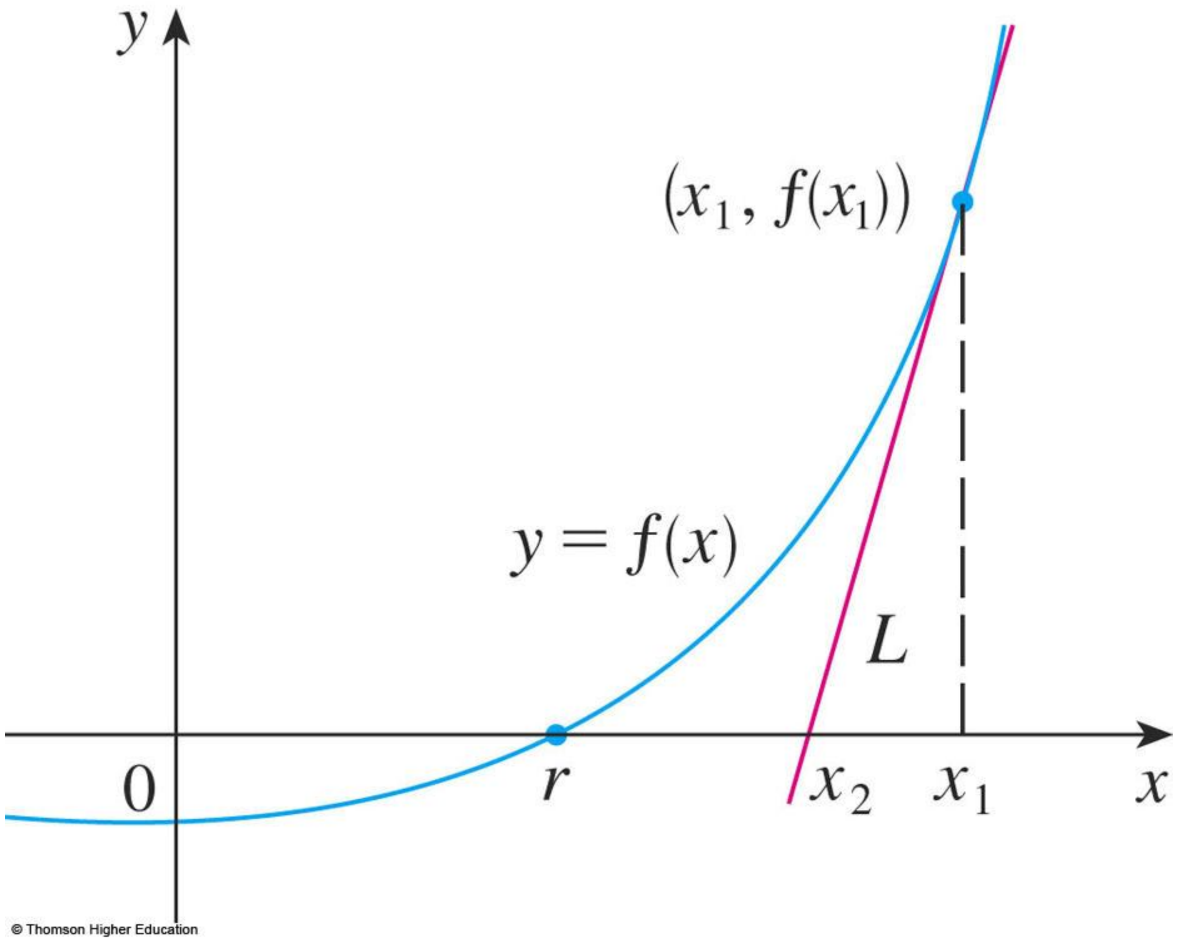
We start with a first approximation x_1, which is obtained by one of the following methods:

- Guessing
- A rough sketch of the graph of f
- A computer-generated graph of f



NEWTON'S METHOD

Consider the **tangent line *L*** to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the x-intercept of L, labeled $\mathbf{x_2}$.



SECOND APPROXIMATION

As the x-intercept of L is x_2, we set $y = 0$ and obtain:

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

If $f'(x_1) \neq 0$, we can solve this equation for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

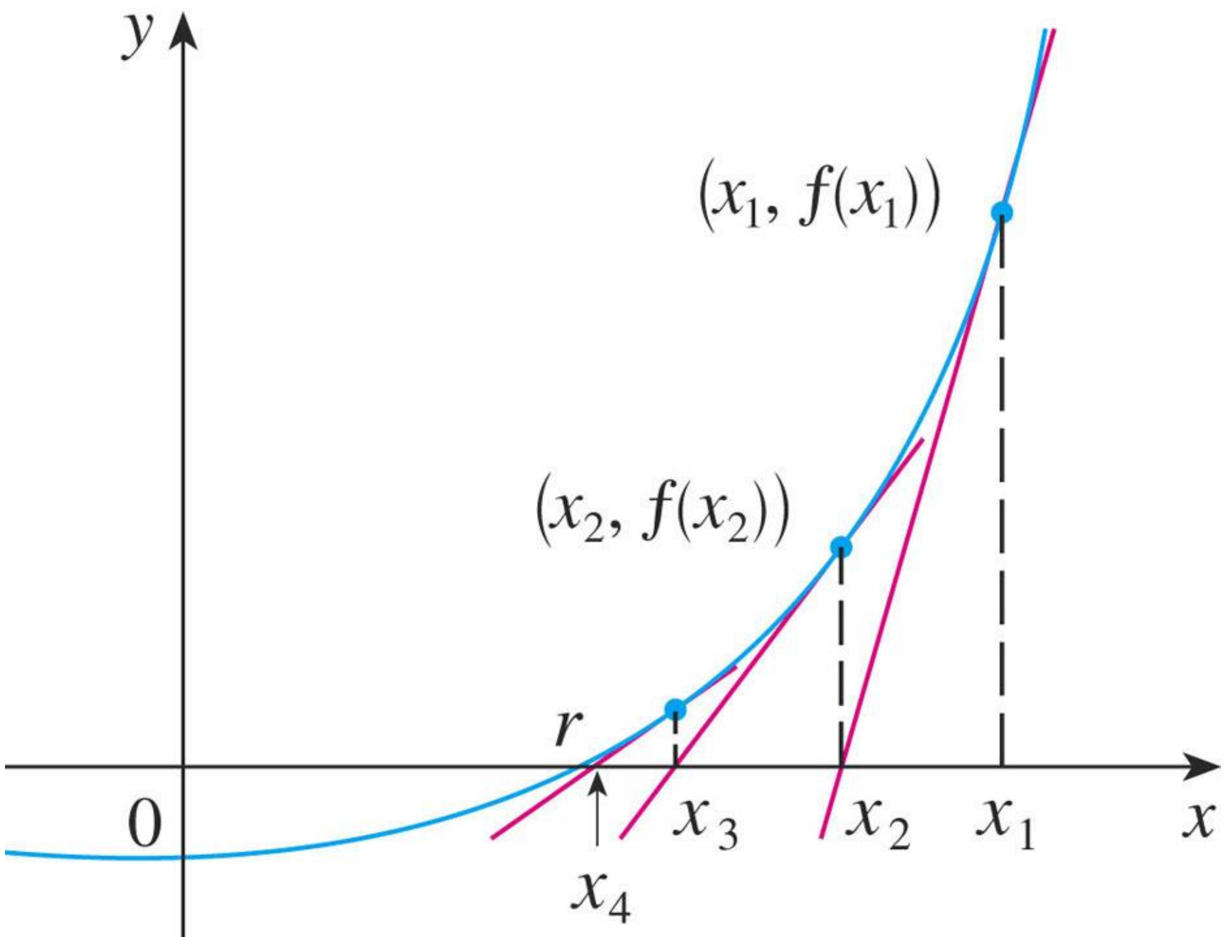
We use x_2 as a second approximation to r.

Equation/Formula 2

SUBSEQUENT APPROXIMATION

In general, if the *n*th approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by:

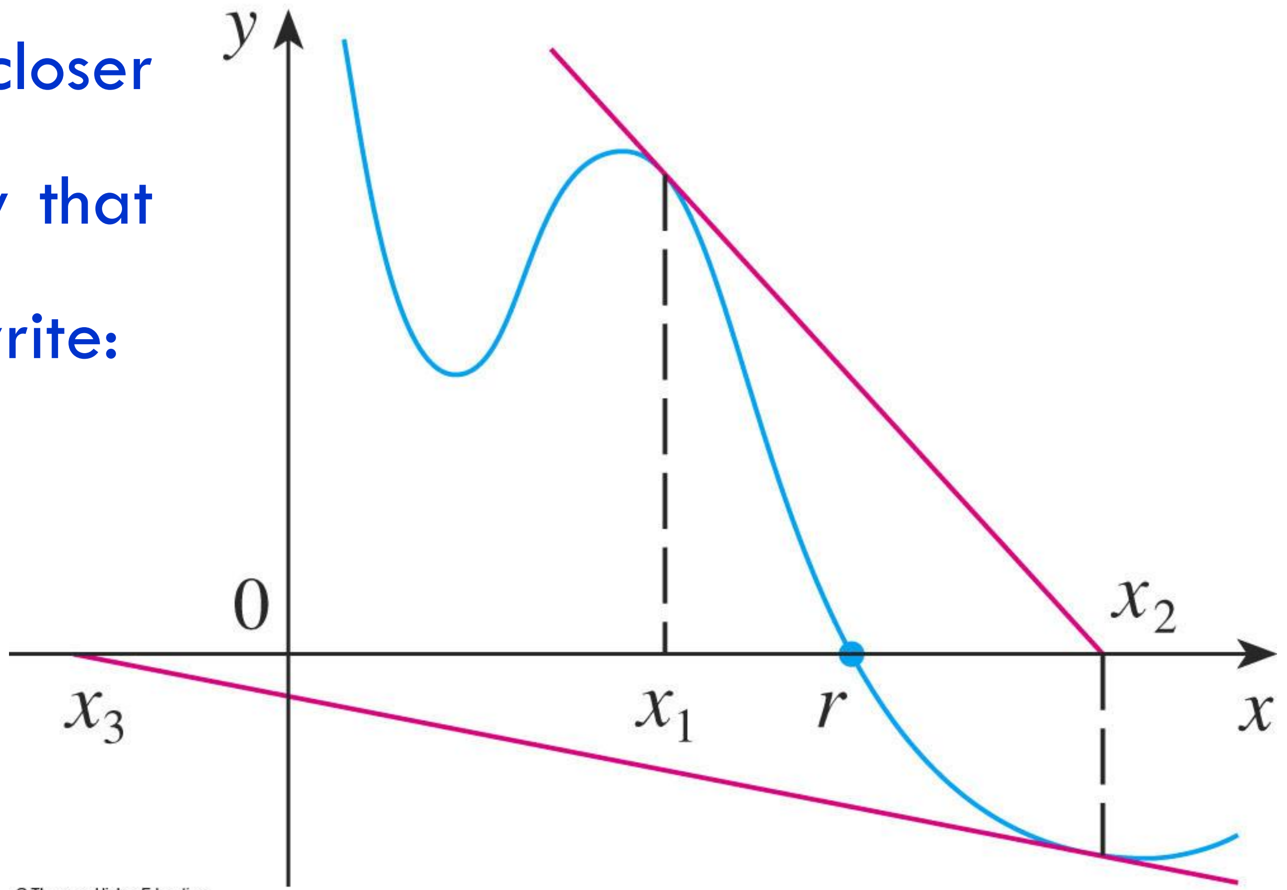
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



CONVERGENCE

If the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence converges to r and we write:

$$\boxed{\lim_{n\to\infty} x_n = r}$$



Example 2

Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.

- First, we observe that finding $\sqrt[6]{2}$ is equivalent to finding the positive root of the equation $x^6 - 2 = 0$
- So, we take $f(x) = x^6 - 2$
- Then, $f'(x) = 6x^5$

Example 2

So, Formula 2 (Newton's method) becomes:

$$x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5}$$

Example 2

Choosing $x_1 = 1$ as the initial approximation, we obtain:

$$x_2 \approx 1.16666667 \qquad x_3 \approx 1.12644368$$

$$x_4 \approx 1.12249707 \qquad x_5 \approx 1.12246205$$

$$x_6 \approx 1.12246205$$

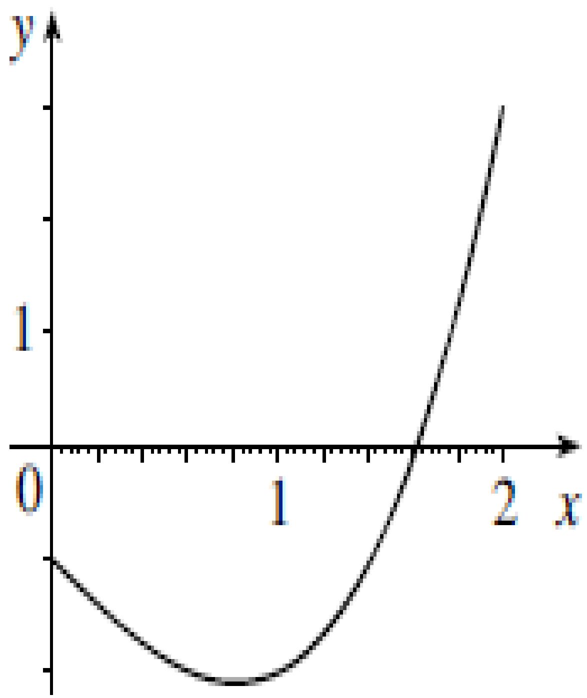
As x_5 and x_6 agree to eight decimal places, we conclude that

$$\sqrt[6]{2} \approx 1.12246205$$

to eight decimal places.

Drill Question: Use Newton's method to find x_4, the fourth approximation to the positive root of the equation $x^3 - 2x - 1 = 0$. Choose your own x_1.

Answer: The graph of $x^3 - 2x - 1$ gives 1.6 as a reasonable choice for x_1. $x_1 = 1.6 \Rightarrow x_2 \approx 1.6183099$, $x_3 \approx 1.6180341$, $x_4 \approx x_5 \approx 1.6180340$.



APPLICATIONS OF DIFFERENTIATION

4.7
Antiderivatives

In this section, we will learn about:

Antiderivatives and how they are useful

in solving certain scientific problems.

DEFINITION

A function F is called an **antiderivative** of f on an interval I if

$$F'(x) = f(x), \text{ for all } x \text{ in } I.$$

ANTIDERIVATIVES

If ***F* is an antiderivative of *f* on an interval *I***, the most general antiderivative of *f* on *I* is

$$F(x) + C$$

where C is an arbitrary constant.

Let $f(x) = \tan x$, and F(x) is an antiderivative of f(x). Evaluate f and tell whether F is increasing or decreasing at x= - 3 (rad)

ANTIDERIVATIVE FORMULA

Table 2

Here, we list some particular antiderivatives.

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec^2 x$	$\tan x$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$1/x$	$\ln \lvert x \rvert$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
$\cos x$	$\sin x$		

Example 6

RECTILINEAR MOTION

A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$.

Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm.

- Find its position function $s(t)$.

A particle moves along the x-axis so that its velocity at time t is given by $3 \sin 2t$.

Assuming it starts at the origin, where is it at $t = \pi$ seconds?

a. 0   b. 3/2   c. 1/2   d. - 1/2

Answer: a

Let $f(x)=4-3x$ for all $x>=2$. Select the correct one.

a. 2 is the local minimum value.

b. 2 is the local maximum value.

c. -2 is the local minimum value.

d.-2 is the maximum local value.

e. 2 is the absolute minimum value.

f. -2 is the absolute maximum value.

g.None of the above.

Answer: f

A piece of wire (dây kim loại) 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle (tam giác đều).

How should the wire be cut for the square so that the total area (of the square and the triangle) enclosed is a minimum? Round the result to the nearest hundredth.

Answer: If x is the length of the square then the side of the triangle is $1/3\ (10-4x)$
We find the minimum of $x^2+\ldots$

p. 148: 23 – 44

p. 155: 1 – 4 ; 23;

p. 162: 21 – 25

p. 176: 2 – 6 ; 16;36

p. 183: 5, 6, 9, 10

p.190: 30; 33 – 36

Thanks