Chapter 2
LIMITS

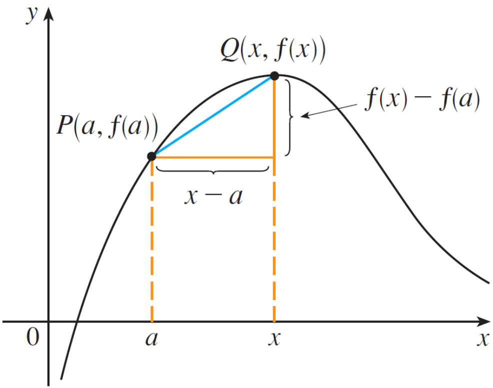
WHEN NEED THE CONCEPT OF LIMIT?

- ⋆ Continuity of functions
- ⋆ Defining derivatives // velocity, tangent line, acceleration, rate of change (next chapter)
- ⋆ Defining integral // calculating areas, distance, volume, length (quantities) (later chapters)
- ⋆ Defining sum of series (later chapters)

THE TANGENT PROBLEM

- ⋆ A curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$,
- ⋆ We consider a nearby point $Q(x, f(x))$, where $x \neq a$
- ⋆ The **slope of the secant line** PQ :

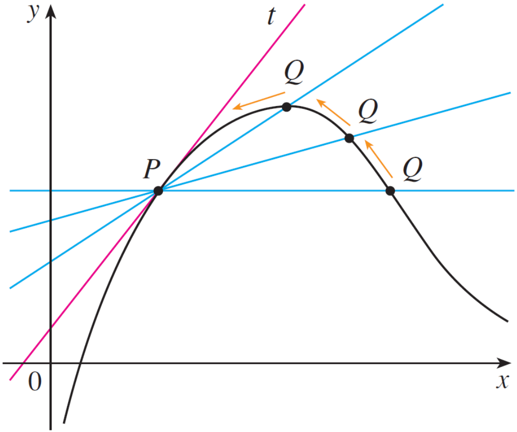
$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$



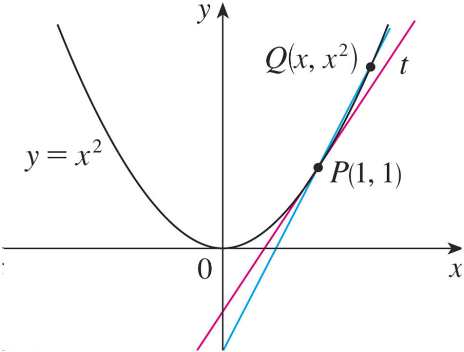
THE TANGENT PROBLEM

- ⋆ We let Q approach P along the curve C by letting x approach a,
- ⋆ If m_{PQ} approaches a number m, then we define the tangent t to be the line through P with slope m.

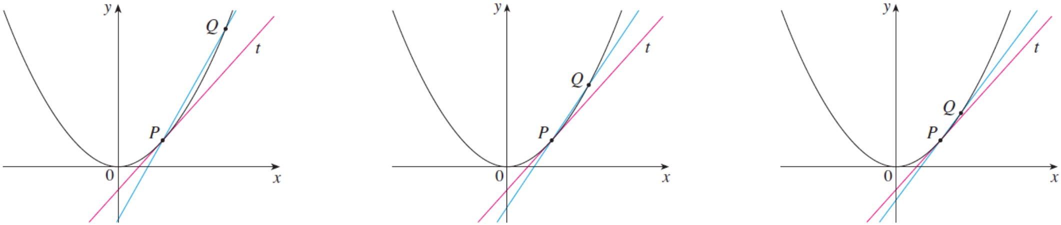
$$m = \lim_{Q \to P} m_{PQ} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



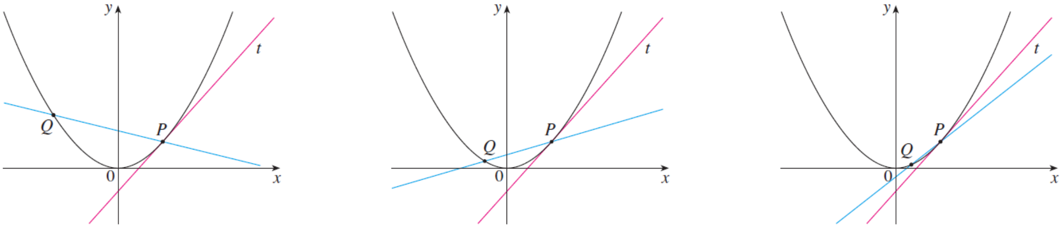
Example. Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.



$$m = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{x^2 - 1}{x - 1}$$



Q approaches P from the right



Q approaches P from the left

FIGURE 3

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

- $m = \lim\limits_{x \to 1} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1} = 2$
- The tangent line $y = 2(x-1) + 1 = 2x - 1$

THE VELOCITY PROBLEM

Investigate the example of a falling ball.

- ⋆ Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450m above the ground. Find the velocity of the ball after 5 seconds.
- ⋆ If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the following equation.

$$s(t) = 0.5gt^2 = 4.9t^2$$

THE VELOCITY PROBLEM

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{s(5.1) - s(5)}{0.1} = 49.49 \text{ m/s}$$

Thus, the (instantaneous) velocity after 5 s is: $v = 49$ m/s

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

THE AREA PROBLEM

We begin by attempting to solve the area problem:
Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

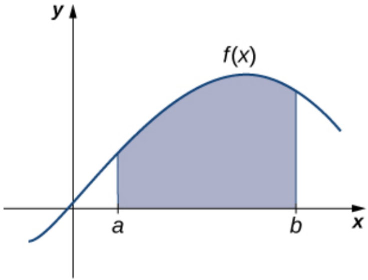


Figure 2.8 The Area Problem: How do we find the area of the shaded region?

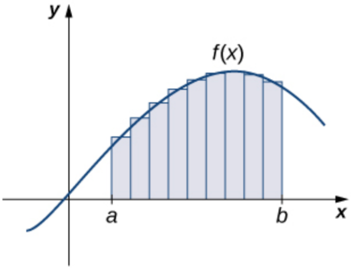


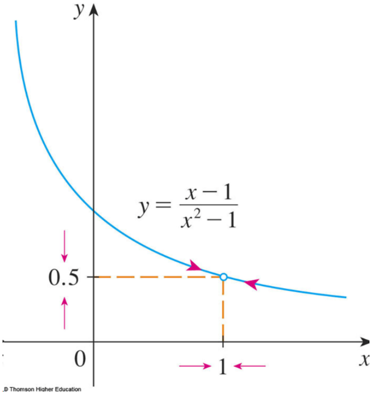
Figure 2.9 The area of the region under the curve is approximated by summing the areas of thin rectangles.

THE LIMIT OF A FUNCTION

In general, we write

$$\lim_{x \to a} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a **but not equal to a**.



Example. Given $f(x) = \dfrac{x-1}{x^2-1}$. What happens to $f(x)$ as x gets closer and closer to **1 from the left**?

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$$\lim_{x\to 1^-} \frac{x-1}{x^2-1} = 0.5$$

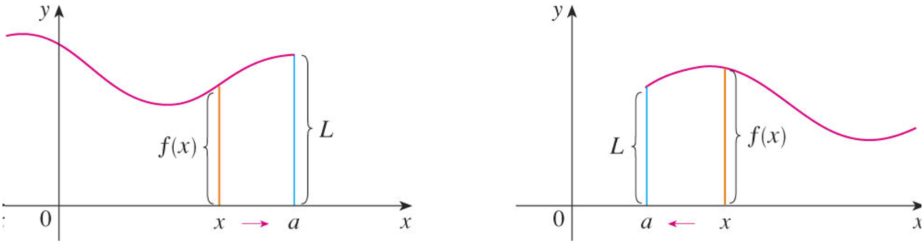
x goes to 1, $x < 1$

ONE-SIDED LIMITS

We write

$$\lim_{x \to a^-} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to L by taking x **to be sufficiently close to** a **and** x **less than** a.



(a) $\lim_{x \to a^-} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$

ONE–SIDED LIMITS

Similarly, "*the right-hand limit of $f(x)$ as x approaches a is equal to L* "and we write

$$\lim_{x \to a^+} f(x) = L$$

Theorem $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

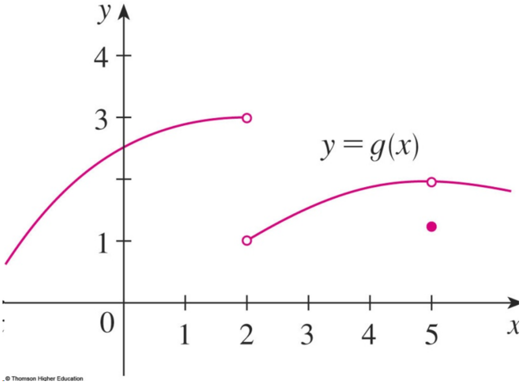
FINDING LIMITS

- ⋆ Using graphs
- ⋆ Using table of values of $f(x)$
- ⋆ Using limit laws
- ⋆ Using analytic technique
- ⋆ And more: L'hospital's rule (ignored)

Using graphs

Example. Using the given graph to find the values of limits.

- ⋆ $\lim\limits_{x\to 2^-} g(x)$
- ⋆ $\lim\limits_{x\to 2^+} g(x)$
- ⋆ $\lim\limits_{x\to 2} g(x)$
- ⋆ $\lim\limits_{x\to 5^-} g(x)$
- ⋆ $\lim\limits_{x\to 5^+} g(x)$
- ⋆ $\lim\limits_{x\to 5} g(x)$



Using table of values of $f(x)$

Example. Guess the value of $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

x	$\dfrac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983

From the table at the left we guess that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

THE LIMIT LAWS

Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

THE LIMIT LAWS

6. $\lim\limits_{x\to a}[f(x)]^n = \left[\lim\limits_{x\to a} f(x)\right]^n$

7. $\lim\limits_{x\to a} c = c$

8. $\lim\limits_{x\to a} x = a$

9. $\lim\limits_{x\to a} x^n = a^n$

10. $\lim\limits_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$

11. $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$

Example. If $f(x) = \begin{cases} 3x - 4, x \neq 0 \\ 10, x = 0, \end{cases}$ then find $\lim_{x \to 0} f(x) = -4$.

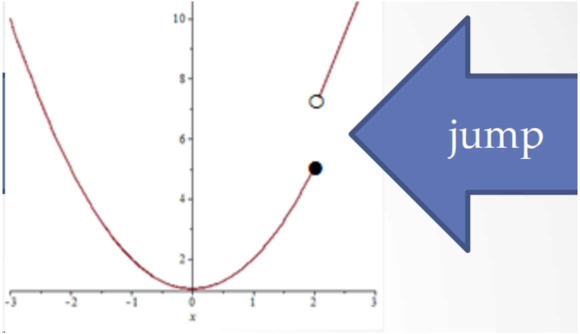
It does not matter that $f(0) = 10$.
For $x \neq 0$, and thus for all x near 0, $f(x) = 3x - 4$ and therefore

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (3x - 4) = -4$$

USING THE LIMIT LAWS

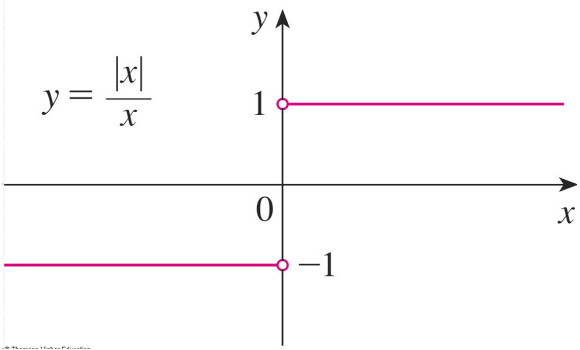
Theorem $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

Example. If $f(x) = \begin{cases} x^2 + 1, x \le 2 \\ 5x - 3, x > 2 \end{cases}$ then find $\lim\limits_{x \to 2} f(x)$



The graph **jumps** at $x = 2$

Example. Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.



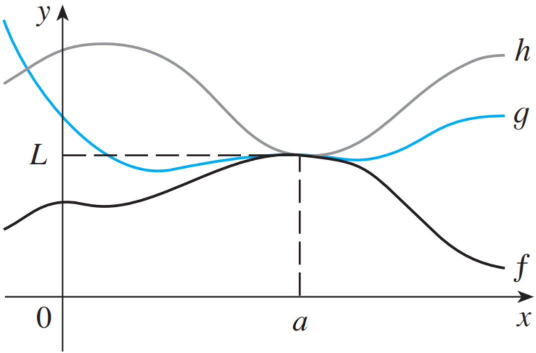
PROPERTIES OF LIMITS

Theorem If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$$

SQUEEZE THEOREM

The Squeeze Theorem (the Sandwich Theorem or the Pinching Theorem) states that, if $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$. Then $\lim_{x \to a} g(x) = L$



Example. Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$

- ⋆ Note that we cannot use

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x}$$

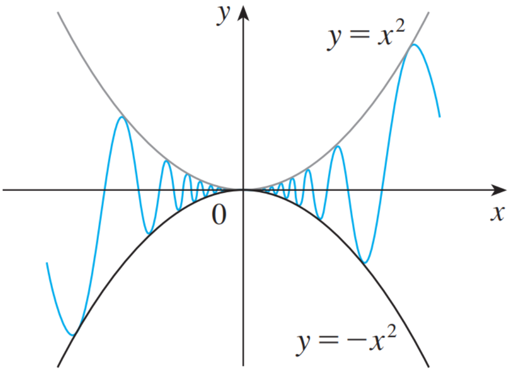
- ⋆ This is because $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

However, since $-1 \leq \sin\dfrac{1}{x} \leq 1$ we have:

$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$$

Taking $f(x) = -x^2$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain:

$$\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$$



ANALYTIC TECHNIQUE

Example. Find the value of the limit

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

- ⋆ $\dfrac{x^2 - 9}{x - 3} = \dfrac{(x-3)(x+3)}{x-3} = x + 3$

- ⋆ $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3} = \lim\limits_{x \to 3}(x+3) = 6$ Using limit laws

ANALYTIC TECHNIQUE

Example. Find the value of the limit

a) $\lim_{x \to -3} \dfrac{x^2 + x - 6}{x + 3}$

b) $\lim_{h \to 0} \dfrac{\sqrt{9 + h^2} - 3}{h^2}$

L'Hospital's Rule

Suppose that we have one of the following cases,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0} \quad \text{OR} \quad \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\pm\infty}{\pm\infty}$$

where a can be any real number, infinity or negative infinity. In these cases we have,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example. Evaluate the following limits

a) $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$

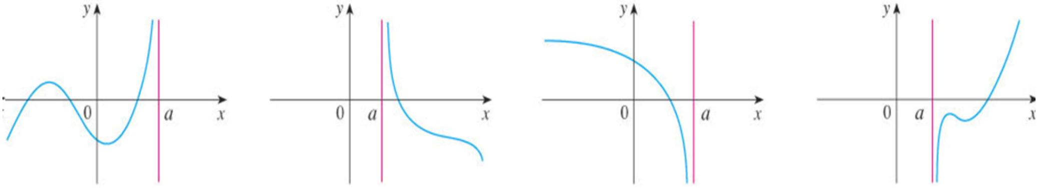
b) $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^2}$

Vertical asymptotes

Definition. The line $x = a$ is called the **vertical asymptote** of $f(x)$ if we have one of the following:

$$\lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$



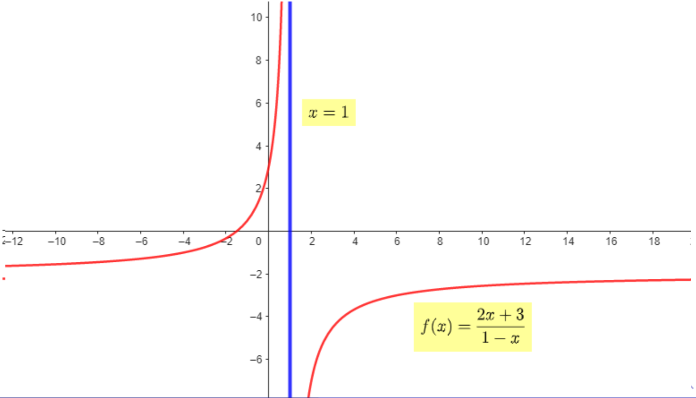
(a) $\lim_{x\to a^-} f(x) = \infty$ (b) $\lim_{x\to a^+} f(x) = \infty$ (c) $\lim_{x\to a^-} f(x) = -\infty$ (d) $\lim_{x\to a^+} f(x) = -\infty$

Example. Find vertical asymptotes $f(x) = \dfrac{2x+3}{1-x}$

Solution.

$$\lim_{x\to 1^-} \frac{2x+3}{1-x} = \infty = \frac{\neq 0}{0}$$

Vertical asymptote of f : $x = 1$



Find vertical Asymptotes $(x = a)$

Common Method: Find $x = a$ such that: $f(x)$ tends to $\dfrac{\neq 0}{0}$

Example. Find vertical asymptotes

a) $f(x) = \dfrac{x-2}{x+5}$

b) $f(x) = \dfrac{x-2}{x^2+5x+6}$

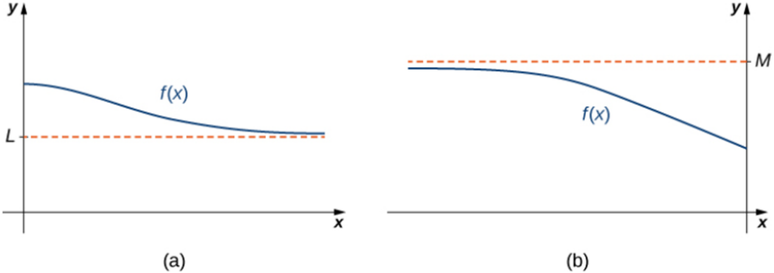
c) $f(x) = \dfrac{x-2}{x^2-5x+6}$

d) $f(x) = \dfrac{x-2}{x^2+6}$

Horizontal asymptotes

Definition. The line $y = L$ is called the **horizontal asymptote** of $f(x)$ if we have one of the following:

$$\lim_{x\to\infty} f(x) = L \qquad \lim_{x\to-\infty} f(x) = L$$



(a) As $x \to \infty$, the values of f are getting arbitrarily close to L. The line $y = L$ is a horizontal asymptote of f.

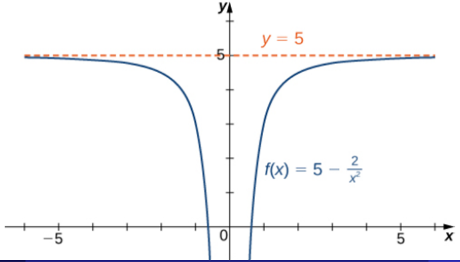
(b) As $x \to -\infty$, the values of f are getting arbitrarily close to M. The line $y = M$ is a horizontal asymptote of f.

Example. Let $f(x) = 5 - \dfrac{2}{x^2}$. Determine the horizontal asymptote(s) for f.

Solution. We have

$$\lim_{x\to+\infty}\left(5 - \frac{2}{x^2}\right) = 5 \quad \text{and} \quad \lim_{x\to-\infty}\left(5 - \frac{2}{x^2}\right) = 5.$$

Therefore, $f(x) = 5 - \dfrac{2}{x^2}$ has a horizontal symptote of $y = 5$ and f approaches this asymptote as $x \to \pm\infty$ as shown in the following graph.



Example. Find the asymptotes of the function

$$f(x) = \frac{x^3 - 1}{x^3 + x^2 - 2}$$

Solution.

- ⋆ $\displaystyle \lim_{x\to\infty} \frac{x^3 - 1}{x^3 + x^2 - 2} = \lim_{x\to\infty} \frac{1 - \frac{1}{x^3}}{1 + \dfrac{1}{x} - \dfrac{2}{x^3}} = 1 \Longrightarrow y = 1$ is horizontal asymptote
- ⋆ $\displaystyle \frac{x^3 - 1}{x^3 + x^2 - 2} = \frac{(x-1)\left(x^2 + x + 1\right)}{(x-1)\left(x^2 + 2x + 2\right)}$
- ⋆ $\displaystyle \lim_{x\to 1} \frac{x^3 + 1}{x^3 + x^2 - 2} = \frac{3}{5}$

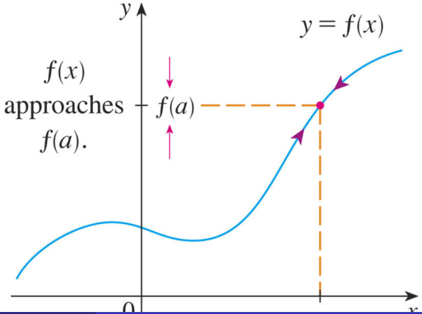
CONTINUITY - Definition

Definition. A function f is continuous at a number a if:

$$\lim_{x \to a} f(x) = f(a)$$

Notice that:

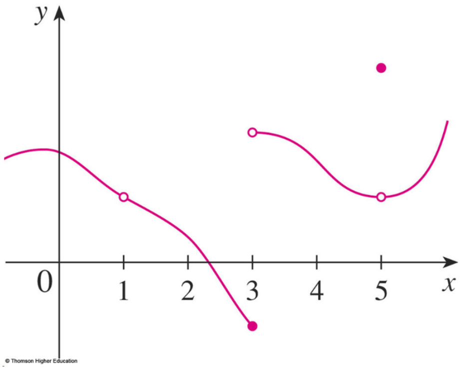
- ⋆ $f(a)$ is defined - that is, a is in the domain of f
- ⋆ $\lim_{x \to a} f(x)$ exists.
- ⋆ $\lim_{x \to a} f(x) = f(a)$



CONTINUITY - Definition

If f is defined near a - that is, f is defined on an open interval containing a, except perhaps at a - we say that f is **discontinuous** at a if f **is not continuous at** a.

The figure shows the graph of a function f. At which numbers is f discontinuous? Why?

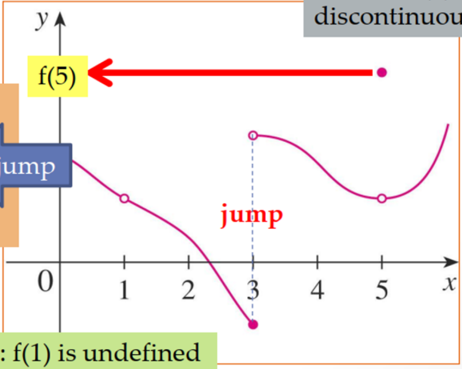


Discontinuous at a = NOT continuous at *a*.

At $x = 5$: $f(5) \neq \lim_{x\to 5} f(x)$ ➔ discontinuous at $x = 5$

At $x = 3$: large change in $f(x)$ for small change in x. In fact **left side limit ≠ right side limit**

➔ Discontinuous at $x = 3$



At $x = 1$: $f(1)$ is undefined

➔ discontinuous at $x = 1$

CONTINUITY - Definition

Definition. If $f(x)$ is discontinuous at a, then

1. f has a removable discontinuity at a if $\lim\limits_{x\to a} f(x)$ exists.
   (**Note:** When we state that $\lim\limits_{x\to a} f(x)$ exists, we mean that $\lim\limits_{x\to a} f(x) = L$, where L is a real number.)
2. f has a jump discontinuity at a if $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist, but $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$.
   (**Note:** When we state that $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an infinite discontinuity at a if $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$.

CONTINUITY - Definition

Definition. A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a)$$

CONTINUITY - Definition

A function f is **continuous on an interval** if it is continuous at every number in the interval.

If f is defined only on one side of an endpoint of the interval, we understand 'continuous at the endpoint' to mean 'continuous from the right' or 'continuous from the left.'

CONTINUITY - Theorem

If f and g are continuous at a; and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

CONTINUITY - Theorem

The following types of functions are continuous at every number in their domains:

- ⋆ Polynomials // Hàm đa thức
- ⋆ Rational functions // Hàm hữu tỷ
- ⋆ Root functions // Hàm căn thức
- ⋆ Trigonometric functions// Hàm lượng giác

CONTINUITY - Theorem

If f is continuous at b and $\lim_{x\to a} g(x) = b$ then

$$\lim_{x\to a} f(g(x)) = f(b)$$

In other words,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

If x is close to a, then $g(x)$ is close to b; and, since f is continuous at b, if $g(x)$ is close to b, then $f(g(x))$ is close to $f(b)$.

CONTINUITY - Theorem

If g is continuous at a and f is continuous at $g(a)$, then the composite function $(f \circ g)(x) = f(g(x))$ is continuous at a.

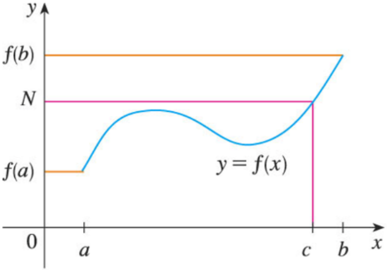
"This theorem is often expressed informally by saying "a continuous function of a continuous function is a continuous function."

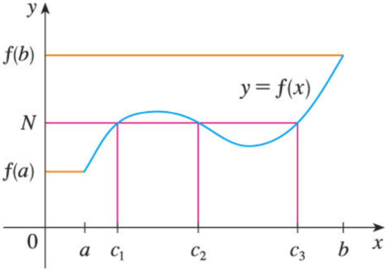
INTERMEDIATE VALUE THEOREM

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where

$$f(a) \neq f(b)$$

Then, there exists a number c in (a, b) such that $f(c) = N$.





$f(a) < N < f(b)$

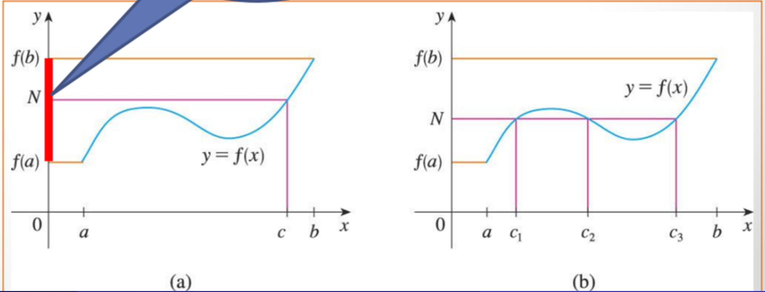
f is continuous on [a, b]

$\Rightarrow$

f(c) = N

for some c in (a, b)

Values of f(x) change continuously from f(a) to f(b)



Example. Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

Solution.

- ⋆ Let $f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$
- ⋆ We are looking for a solution of the given equation that is, a number c between 1 and 2 such that $f(c) = 0$.
- ⋆ Therefore, we take $a = 1, b = 2$, and $N = 0$ in the theorem.
- ⋆ We have, $f(1) = -1 < 0$ and $f(2) = 12 > 0$. Thus $f(1) < N < f(2)$.
- ⋆ So, the Intermediate Value Theorem says there is a number c in $[1, 2]$ such that $f(c) = 0$. In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in the interval $[1, 2]$.

Example. Check whether there is a solution to the equation $x^5 - 2x^3 - 2$ between the interval $[0; 2]$.

Example. If $f(1) > 0$ and $f(3) < 0$ then there exists a number c between 1 and 3 such that $f(c) = 0$

a) True

b) False

Example. Which is the equation expressing the fact that " f *is continuous at 2* "?

a) $\lim\limits_{x \to 2} f(x) = 2$

b) $\lim\limits_{x \to \infty} f(x) = 2$

c) $\lim\limits_{x \to 0} f(x) = \infty$

d) $\lim\limits_{x \to 2} f(x) = 0$

e) $\lim\limits_{x \to 2} f(x) = f(2)$

Thank you for your attention.

Problems

Prob 1. Evaluate the limit $\lim\limits_{x\to 0}\left(\dfrac{1}{x\sqrt{1+x}}-\dfrac{1}{x}\right)$

a) $+\infty$ b) 0 c) 1 d) $-\dfrac{1}{2}$

Prob 2. Evaluate the limit $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}$

a) 1 b) $1/2$ c) $3/2$ d) $2/3$

Prob 3. Evaluate the limit $\lim\limits_{x\to +\infty}(\sqrt{1+x}-\sqrt{x})$

a) $+\infty$ b) 0 c) 1 d) $1/2$

Prob 4. Evaluate the limit $\lim\limits_{x \to -\infty} \left(\sqrt{x^2+x+1}+x\right)$

a) $+\infty$ b) 0 c) $1/2$ d) $-1/2$

Prob 5. Evaluate the limit $\lim\limits_{x \to -\infty} \left(\sqrt{x^2+x}+\sqrt{x^2-1}\right)$

a) $1/2$ b) $-\infty$ c) $-1/2$ d) $+\infty$

Prob 6. Evaluate the limit $\lim\limits_{x \to -\infty} \left(\sqrt[3]{x^3+1}+x\right)$

a) 1 b) 0 c) -1 d) $-\infty$

Prob 7. Evaluate the limit $\lim\limits_{x\to\pi/2}\left(\frac{\pi}{2}-x\right)\tan x$

a) π b) $\frac{1}{\pi}$ c) 0 d) 1

Prob 8. Evaluate the limit $\lim\limits_{x\to 0}\frac{\ln(1-2x\sin x)}{1-\cos 2x}$

a) -1 b) $\frac{-1}{4}$ c) 1 d) $\frac{1}{2}$

Prob 9. Find the vertical and horizontal asymptotes for each rational function:

a) $f(x) = \dfrac{x^2 - 2x}{x + 4}$

b) $f(x) = \dfrac{x^2 + 3x - 7}{x^2 - 5x - 6}$

c) $f(x) = \dfrac{x - 7}{4x}$

d) $f(x) = \dfrac{x + 12}{5x^2 - 10x}$

e) $f(x) = \dfrac{3x^2 - 9}{x^2 + 7x + 12}$

Prob 10. Determine where the function $f(x)$ is continuous

a) $f(x) = \dfrac{2x^2 + x - 1}{x - 2}$

b) $f(x) = \dfrac{x - 9}{\sqrt{4x^2 + 4x + 1}}$

c) $f(x) = \ln(2x + 5)$

Prob 11. Let $g(x) = \begin{cases} \dfrac{x}{2} + 3 & \text{if } x > 2 \\ x - k & \text{if } x \leq 2 \end{cases}$. Find k such that g is continuous at $x = 2$

a) -2 b) 2 c) 4 d) -5

Prob 12. Let $f(x) = \begin{cases} (x-2)^3 & \text{if } x \leq -2 \\ (x+4)^3 & \text{if } x > -2 \end{cases}$. Find the points at which the funtion f is discontinuous.

a) -2 b) 2 c) 4

d) 0 e) -4 f) None of others