Chapter 3
Derivertive

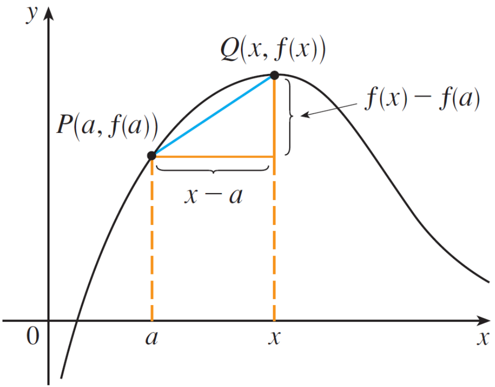
Contents

WHY STUDY DERIVATIVES?

- ⋆ How one quantity changes in relation to another quantity?
- ⋆ Derivative = Rates of change: occur in all the sciences.
- ⋆ Velocity, density, current, power, and temperature gradient in physics;
- ⋆ rate of reaction in chemistry;
- ⋆ rate of growth and blood velocity in biology;
- ⋆ marginal cost and marginal profit in economics;
- ⋆ rate of heat flow in geology;
- ⋆ rate of improvement of performance in psychology;
- ⋆ rate of spread of a rumor in sociology (analyzing innovations or fads or fashions)
- ⋆ these are all special cases of a single mathematical concept, the derivative.
  $\Rightarrow$ The power of mathematics lies in its abstractness.

- A curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$,
- We consider a nearby point $Q(x, f(x))$, where $x \neq a$
- The **slope of the secant line** PQ :

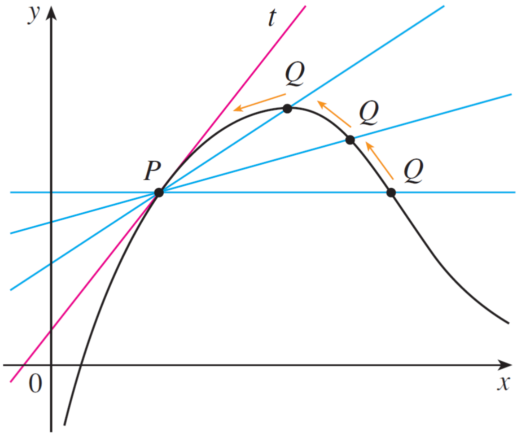
$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$



The tangent problem

- ⋆ We let Q approach P along the curve C by letting x approach a,
- ⋆ If m_{PQ} approaches a number m, then we define the tangent t to be the line through P with slope m.

$$m = \lim_{Q \to P} m_{PQ} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



The tangent problem

Definition. The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

If $h = x - a$, then $x = a + h$ and so the slope of the tangent line in definition becomes

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Example.

- ⋆ Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.
- ⋆ Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

The velocity problem

Investigate the example of a falling ball.

- ⋆ Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450m above the ground. Find the velocity of the ball after 5 seconds.
- ⋆ If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the following equation.

$$s(t) = 0.5gt^2 = 4.9t^2$$

The velocity problem

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$
$$= \frac{s(5.1) - s(5)}{0.1} = 49.49 \text{ m/s}$$

Thus, the (instantaneous) velocity after 5 s is: $v = 49$ m/s

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

The velocity problem

We define the **velocity** (or instantaneous velocity) $v(a)$ **at time** $t = a$ to be the limit of these average velocities:

$$v(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

Definition. The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.
or

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If we replace a in above formular by a variable x, we obtain

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

OTHER NOTATIONS

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

$$\left.\frac{dy}{dx}\right|_{x=a} \quad \text{or} \quad \left.\frac{dy}{dx}\right]_{x=a}$$

Example.

1. Show that a constant function $f(x) = k$ has derivertive $f'(x) = 0$.

2. Find $f'(x)$ if $f(x) = \sin x$

3. Find $f'(0)$ if $f(x) = \begin{cases} \dfrac{\sqrt{1+2x}-1}{x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$

4. Find $f'(0)$ if $f(x) = \sqrt[3]{x}$

Differentiable

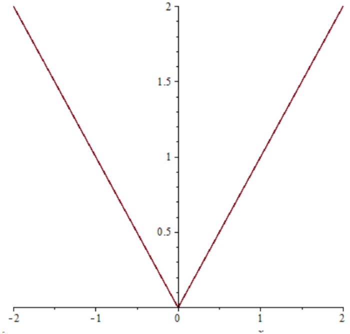
Definition. A function f is differentiable at a if $f'(a)$ exists. It is **differentiable** on an open interval (a, b) [or (a, ∞) or $(-\infty$, a) or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Theorem. If f is differentiable at a, then f continuous at a.

Differentiable

- ⋆ f' (a) exists $\Longrightarrow f$ is differentiable at a
- ⋆ Differentiable at a $\Longrightarrow f$ continuous at a (ALWAYS TRUE)
- ⋆ Continuous at a $\Longrightarrow f$ differentiable ? (NOT ALWAYS TRUE)

Example. $y = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$



REVIEW

- $\star$ Derivative $=$ (instantaneous) rate of change ($=$ rate)
- $\star$ Slope of tangent line at $a = f'(a)$
- $\star$ Velocity of particle at $t = v(t) = s'(t)$, where $s(t)$ is the position function.
- $\star$ Acceleration of particle at $t = a(t) = v'(t)$
- $\star$ Differentiable at $x = a \Longrightarrow$ continuous at $x = a$

Differentiation formulas

- ⋆ $\frac{d}{dx}(k) = 0$
- ⋆ $\frac{d}{dx}(x^n) = nx^{n-1}, n \in \mathbb{R}$
- ⋆ $\frac{d}{dx}(a^x) = a^x \ln a$
- ⋆ $\frac{d}{dx}(e^x) = e^x$
- ⋆ $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$
- ⋆ $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- ⋆ $\frac{d}{dx}\sin x = \cos x$
- ⋆ $\frac{d}{dx}\cos x = -\sin x$
- ⋆ $\frac{d}{dx}\sec x = \sec x \tan x, \quad (\sec x = 1/\cos x)$
- ⋆ $\frac{d}{dx}\tan x = \sec^2 x$
- ⋆ $\frac{d}{dx}\csc x = -\csc x \cot x, \quad (\csc x = 1/\sin x)$
- ⋆ $\frac{d}{dx}\cot x = -\csc^2 x$

Differentiation rules

Constant multiple	$[cf(x)]' = cf'(x)$
Sum and difference rule	$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
Linearity rule	$[af(x) + bg(x)]' = af'(x) + bg'(x)$
Product rule	$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$
Quotient rule	$\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$

Example. Find equations of the tangent line and normal line to the curve

$$y = \frac{\sqrt{x}}{1+x^2}$$

at the point $(1, 1/2)$.

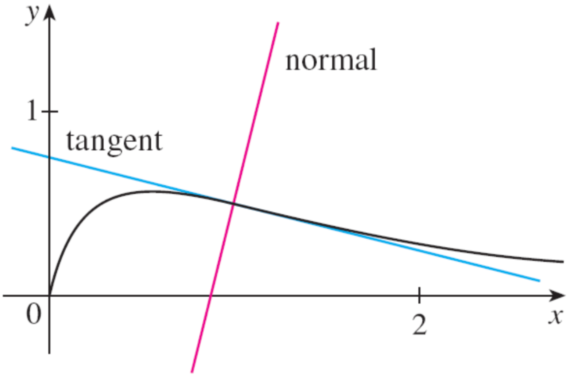
REMARK: Tangent line (Δ) equation at $M(x, y) \in (C) : y = f(x)$

$$(\Delta) : y = f'(x_0)(x - x_0) + f(x_0)$$

The **normal line** is defined as the line that is perpendicular to the tangent line at the point of tangency.

Tangent line of the graph of this curve at $(1, 1/2)$ is:

$$y = -\frac{1}{4}x + \frac{3}{4}$$



Example. Differentiate the functions given in Problems $1-5$.

1. $f(x) = \sin x + \cos x$
2. $g(t) = t^2 + \cos t + \cos \dfrac{\pi}{4}$
3. $h(x) = 2x^3 \sin x - 3x \cos x$
4. $p(x) = x^2 \cos x$
5. $q(x) = \dfrac{\sin x}{x}$.

Leibniz notation

First derivative:	y'	$f'(x)$	$\dfrac{dy}{dx}$ or $\dfrac{d}{dx}f(x)$
Second derivative	y''	$f''(x)$	$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$ or $\dfrac{d^2}{dx^2}f(x)$
Third derivative	y'''	$f'''(x)$	$\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right) = \dfrac{d^3y}{dx^3}$ or $\dfrac{d^3}{dx}f(x)$
Fourth derivative	$y^{(4)}$	$f^{(4)}(x)$	$\dfrac{d^4y}{dx^4}$ or $\dfrac{d^4}{dx^4}f(x)$
...	...	...	
n th derivative	$y^{(n)}$	$f^{(n)}(x)$	$\dfrac{d^ny}{dx^n}$ or $\dfrac{d^n}{dx^n}f(x)$

Example. In Problems $1-4$, find f', f'', f''', and $f^{(4)}$.

1. $f(x) = x^5 - 5x^3 + x + 12$
2. $f(x) = \dfrac{1}{4}x^8 - \dfrac{1}{2}x^6 - x^2 + 2$
3. $f(x) = \dfrac{-2}{x^2}$
4. $f(x) = \dfrac{4}{\sqrt{x}}$

Chain rule

If g is differentiable at x and f is differentiable at $g(x)$, then the **composite function** $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and f' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = u(x)$ are both differentiable functions, then

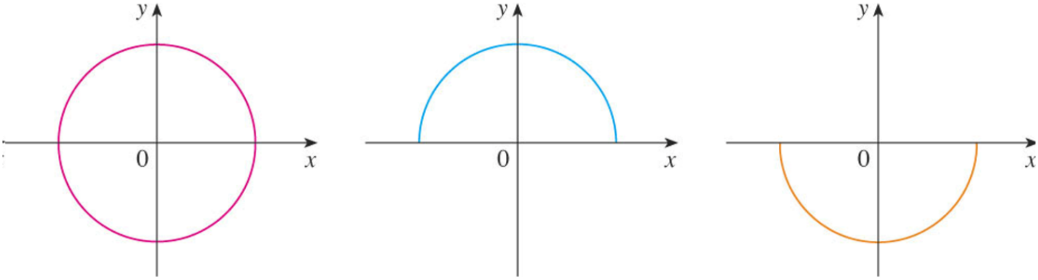
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example. Find the derivertive of the function

1. $y = \sin 4x$
2. $y = \sqrt{4 + 3x}$
3. $y = (1 - x^2)^{10}$
4. $y = \tan(\sin x)$

Example. The graphs of f and g are the upper and lower semicircles of the circle

$$x^2 + y^2 = 25$$



(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

To find the derivative of the implicit form:

- **Step 1:** Differentiate both sides of the equation with respect to x. Remember that y is really a function of x for part of the curve and use the chain rule.
- **Step 2:** Solve the differentiated equation algebraically for $\dfrac{dy}{dx}$

Example. Let $y = f(x)$ is a differentiable function of x such that

$$x^2 + y^2 = 25$$

a) Find $\dfrac{dy}{dx}$

b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

Solution.

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = 0$$

Remembering that y is a function of x and using the Chain Rule, we have:

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dy}\left(y^2\right)\frac{dy}{dx} = 2y\frac{dy}{dx}$$
$$2x + 2y\frac{dy}{dx} = 0$$

Then, we solve this equation for $\dfrac{dy}{dx}$: $\dfrac{dy}{dx} = -\dfrac{x}{y}$
At the point $(3, 4)$ we have $x = 3$ and $y = 4$.
So, $\dfrac{dy}{dx} = -\dfrac{3}{4}$
Thus, an equation of the tangent to the circle at $(3, 4)$ is:

$$y = -\frac{3}{4}(x-3) + 4 \text{ or } 3x + 4y = 25$$

Example. Find $\dfrac{dy}{dx}$ by implicit differentiation in Problems 1-10.

1. $x^2y + 2y^3 = 3x + 2y$
2. $x^2 + y = x^3 + y^3$
3. $xy = 25$
4. $xy(2x + 3y) = 2$
5. $\dfrac{1}{y} + \dfrac{1}{x} = 1$
6. $\tan \dfrac{x}{y} = y$
7. $\cos xy = 1 - x^2$
8. $e^{xy} + 1 = x^2$
9. $\ln(xy) = e^{2x}$
10. $e^{xy} + \ln y^2 = x$

Logarithmic differentiation is especially valuable as a means for handling complicated product or quotient functions and exponential functions where variables appear in both the base and the exponent.

Example.

- ⋆ Find $\dfrac{dy}{dx}$, where $y = (x+1)^{2x}$.
- ⋆ Find the derivative of $y = \dfrac{e^{2x}(2x-1)^6}{(x^3+5)^2(4-7x)}$ if $y > 0$

The derivative can be interpreted as a rate of change, which leads to a wide variety of applications. Viewed as rates of change, derivatives may represent such quantities as the speed of a moving object, the rate at which a population grows, a manufacturer's marginal cost, the rate of inflation, or the rate at which natural resources are being depleted.

- ⋆ Average rate of change $= \dfrac{\text{CHANGE IN } y}{\text{CHANGE IN } x} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$
- ⋆ Instaneous rate change $= \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$
- ⋆ Relative rate of change $= \dfrac{f'(x)}{f(x)}$

Example. Let $f(x) = x^2 - 4x + 7$

1. Find the average rate of change of f with respect to x between $x = 3$ and 5
2. Find the instantaneous rate of change of f at $x = 3$.

An object that moves along a straight line with position $s(t)$ has velocity $v(t) = \dfrac{ds}{dt}$ and acceleration $a(t) = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$ when these derivatives exist. The speed of the object is $|v(t)|$.

Example. Assume that the position at time t of an object moving along a line is given by

$$s(t) = 3t^3 - 40.5t^2 + 162t$$

for t on $[0, 8]$.

1. Find the initial position, velocity, and acceleration for the object and discuss the motion.
2. Compute the total distance traveled.

Example. A particle moving on the $x-$axis has position

$$x(t) = 2t^3 + 3t^2 - 36t + 40$$

after an elapsed time of t seconds.

1. Find the velocity of the particle at time t.
2. Find the acceleration at time t.
3. What is the total distance traveled by the particle during the first 3 seconds?

FORMULA FOR THE HEIGHT OF A PROJECTILE

$$h(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$$

- v_0 is the initial velocity.
- s_0 is the initial height.
- g is the acceleration due to gravity ($32 ft/s^2$ or $9.8 m/s^2$)

Example.

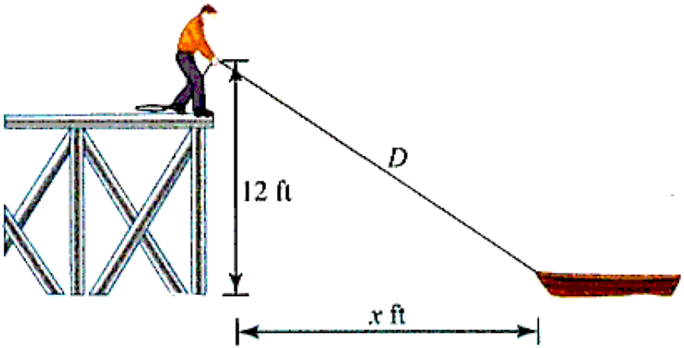
Suppose a person standing at the top of the Tower of Pisa (176ft high) throws a ball directly upward with an initial speed of $96\text{ft}/\text{s}$.

1. Find the ball's height, its velocity, and its acceleration at time t.
2. When does the ball hit the ground and what is its impact velocity?
3. How far does the ball travel during its flight?

SOLVING RELATED RATE PROBLEMS

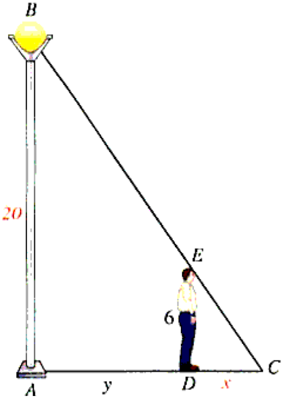
- ⋆ **Step 1** Draw a figure, if appropriate, and assign variables to the quantities that vary. Be careful not to label a quantity with a number unless it never changes in the problem.
- ⋆ **Step 2** Find a formula or equation that relates the variables. Eliminate unnecessary variables.
- ⋆ **Step 3** Differentiate the equations. You will usually differentiate implicitly with respect to time.
- ⋆ **Step 4** Substitute specific numerical values and solve algebraically for any required rate.

Example.



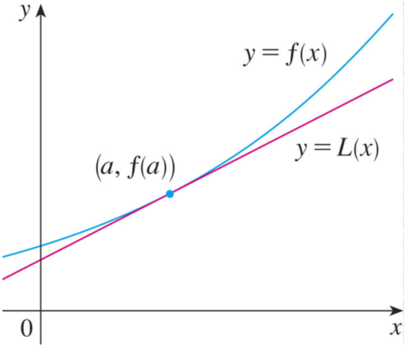
A person is standing at the end of a pier 12ft above the water and is pulling in a rope attached to a rowboat at the waterline at the rate of 6ft of rope per minute, as shown in Figure. How fast is the boat moving in the water when it is 16ft from the pier?

Example.



A person $6\mathrm{ft}$ tall is walking away from a street light $20\mathrm{ft}$ high at the rate of $7\mathrm{ft/s}$. At what rate is the length of the person's shadow increasing?

Linear Approximations



We use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when **x is near** a

$$\boxed{L(x) = y = f(a) + f'(a)(x - a)}$$

The approximation

$$\boxed{f(x) \approx f(a) + f'(a)(x-a) = L(x)}$$

is called the linear approximation of f at a.

$$|x-a| \leq \varepsilon, \forall \varepsilon \geq 0$$

Example. Determine the linear approximation for $f(x) = \sqrt[3]{x}$ at $x = 8$. Use the linear approximation to approximate the value of $\sqrt[3]{8.05}$ and $\sqrt[3]{25}$.

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}}, \quad f(8) = 2 \quad f'(8) = \frac{1}{12}$$

The linear approximation is then,

$$L(x) = 2 + \frac{1}{12}(x-8) = \frac{1}{12}x + \frac{4}{3}.$$

Furthermore,

$$\begin{array}{ll} L(8.05) = 2.00416667 & \sqrt[3]{8.05} = 2.00415802 \\ L(25) = 3.41666667 & \sqrt[3]{25} = 2.92401774 \end{array}$$

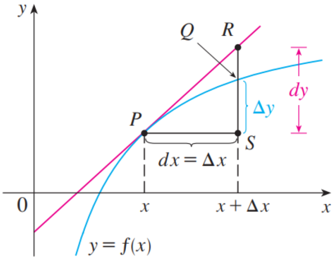
So, at $x = 8.05$ this linear approximation does a very good job of approximating the actual value. However, at $x = 25$ it doesn't do such a good job.

Differential (Vi phân)

Given a function $y = f(x)$ we call dy and dx differentials and the relationship between them is given by,

$$\boxed{dy = f'(x)dx}$$

Differentials provide us with a way of estimating the amount a function changes as a result of a small change in input values.



Example. Compute the differential for each of the following.

1. $y = t^3 - 4t^2 + 7t$
2. $y = x^2 \sin(2x)$

Example. The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

Solution:

- If the radius of the sphere is r, then its volume is $V = \frac{4}{3}\pi r^3$.
- This can be approximated by the differential

$$\begin{aligned} dV &= 4\pi r^2 dr \\ &= 4\pi 21^2 0.05 \approx 277 \text{ cm}^3. \end{aligned}$$

- The maximum error in the calculated volume is about 277 cm^3.

Prob 1. Find an equation of the tangent line to the curve at the given point

a) $y = \dfrac{x-1}{x-2}$, $(3, 2)$

b) $y = \dfrac{2x}{x^2+1}$, $(0, 0)$

c) $y = 3 - 2x + x^2$, $\quad x = 1$

d) $y = \dfrac{3-2x}{x-1}$, $\quad y = -1$

Prob 2. Find all the values of x where the tangent line to the graph of the function is horizontal $f(x) = x^3 + 4x^2 - 11x + 11$

a) 1

b) $\dfrac{11}{3}; -1$

c) $\dfrac{-11}{3}; 1$

d) $\dfrac{-11}{3}; \dfrac{11}{3}; 1$

Prob 3. Find the derivative of the following function $y = \dfrac{x}{\sin x}$

a) $\dfrac{\sin x - x\cos x}{\sin^2 x}$

b) $\dfrac{\sin x + x\cos x}{\sin^2 x}$

c) $\dfrac{1}{\cos x}$

d) $\dfrac{\sin x - x\cos x}{\sin x}$

Prob 4. Find the derivative of the following function $y = \dfrac{e^{x^2}}{\sin x}$

a) $\dfrac{e^{x^2}(2x\sin x + \cos x)}{\sin^2 x}$

b) $\dfrac{e^{x^2}(2x + \cos x)}{\sin^2 x}$

c) $\dfrac{e^{x^2}(2x\sin x - \cos x)}{\sin^2 x}$

d) $\dfrac{e^{x^2}(2x\sin x - \cos x)}{\sin^2 x}$

Prob 5. Find the derivative of the following function $y = xe^{-2x}$

a) $(1+2x)e^{-2x}$ b) $(1-2x)e^{-2x}$ c) e^{-2x} d) $-2e^{-2x}$

Prob 6. Find y''

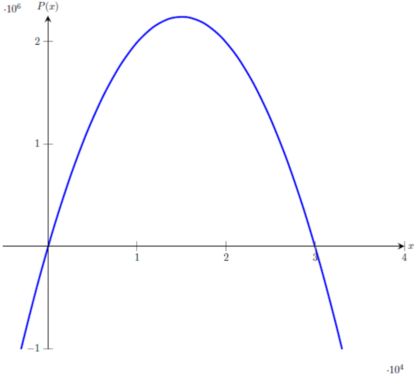
a) $y = xe^{3x-1}$ b) $y = \sqrt[3]{2x+1}$ c) $y = e^{-x}\cos x$

Prob 7. The position in feet of race car along a straight track after t seconds is modeled by the function $s(t) = 8t^2 - \frac{1}{16}t^3$

a) Find the average velocity of the vehicle over the time interval $[4, 4.1]$

b) Find the instantaneous velocity of the vehicle at $t = 4$ seconds.

Prob 8. A toy company can sell x electronic gaming systems at a price of $p = -0.01x + 400$ dollars per gaming system. The cost of manufacturing x systems is given by $C(x) = 100x + 10000$ dollars. Find the rate of change of profit when 10000 game produced. Should the toy company increase or decrease production?(profit=x.p-c(x))



Prob 9. A table of values for f, f', g and g' is given

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

a) If $h(x) = f(g(x))$, find $h'(1)$

b) If $H(x) = g(f(x))$, find $H'(1)$

c) If $F(x) = f(f(x))$, find $F'(2)$

d) If $G(x) = g(g(x))$, find $G'(3)$.

Prob 10. If $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 7, f'(1) = 4$, find $h'(1)$.

Prob 11. Find f' in terms of g'

a) $f(x) = g(\sin 2x)$

b) $f(x) = g\left(e^{1-3x}\right)$.

Prob 12. Find $\dfrac{dy}{dt}$ for:

a) $y = x^3 + x + 2, \dfrac{dx}{dt} = 2$ and $x = 1$.

b) $y = \ln x, \dfrac{dx}{dt} = 1$ and $x = e^2$

Prob 13. Find y' by implicit differentiation

a) $x^4 + y^4 = 16x + y$

b) $\sqrt{x} + \sqrt{y} = 4$.

c) $x^3 + xy = y^2$

Prob 14. Find the linearization $L(x)$ of the function at a

a) $f(x) = \dfrac{1}{\sqrt{2+x}}, \quad a = 2$

b) $f(x) = \sqrt[3]{5-x}, \quad a = -3$.

Prob 16. Compute the differential of the function $y = e^{2x^2}$

a) $dy = 4xe^{2x^2}dx$

b) $dy = 4e^{2x^2}dx$

c) $dy = 2xe^{2x^2}dx$

d) $dy = e^{2x^2}dx$

Prob 17. Compute the differential of the function $y = \ln(\sin x)$

a) $dy = \cot x dx$

b) $dy = \sin x dx$

c) $dy = \tan x dx$

d) $\cos x \ln(\sin x) dx$

Prob 18. Let $y = \dfrac{x-1}{x+1}$. Compute the value of $dy(1)$

a) $dy(1) = 2dx$

b) $dy(1) = \dfrac{1}{4}dx$

c) $dy(1) = \dfrac{1}{2}dx$

d) $dy(1) = dx$

Prob 19. Let $y = \dfrac{e^x}{e^x+1}$. Compute $dy(\ln 2)$

a) $dy(\ln 2) = \dfrac{1}{2}dx$

b) $dy(\ln 2) = \dfrac{4}{9}dx$

c) $dy(\ln 2) = \dfrac{2}{9}dx$

d) $dy(\ln 2) = 0dx$

Thank you for your attention.