Chapter 1
FUNCTIONS AND GRAPHS

LEARNING OBJECTIVES

- ⋆ Determine the domain, range, and zeros of a function.
- ⋆ Analyze the symmetry of the graph of a function.
- ⋆ Create new functions from old functions.

WHY STUDY THIS CHAPTER?

- ⋆ Calculus is the mathematics that describes changes in functions.
- ⋆ The fundamental objects that we deal with in calculus are functions.
- ⋆ Functions are used as mathematical models of real-world phenomena.

Definition. A **function** f is a rule that assigns to each element x in a set D *exactly one* element, called $f(x)$, in a set E.

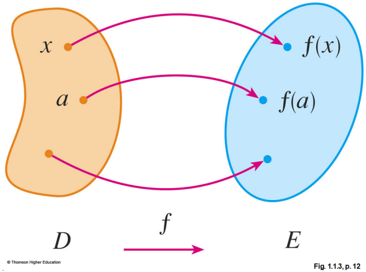
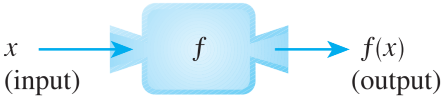
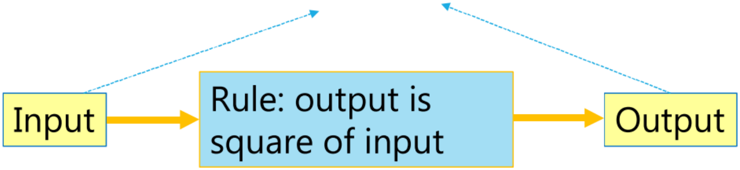


Fig. 1.1.3, p. 12



- ⋆ The set D is called the **domain** of the function f
- ⋆ The **range of f** is the set of all possible values of $f(x)$ as x varies throughout the domain

$$f(x) = x^2$$



Prob 1. Find the domain of each function:

a) $f(x) = \sqrt{x+2}$

b) $f(x) = \dfrac{1}{x^2 - x}$

c) $f(x) = \ln(x+1) - \dfrac{x}{\sqrt{x-1}}$

Prob 2. Find the range of each function:

a) $f(x) = \sqrt{x-1}$

b) $f(x) = x^2 - 2x$

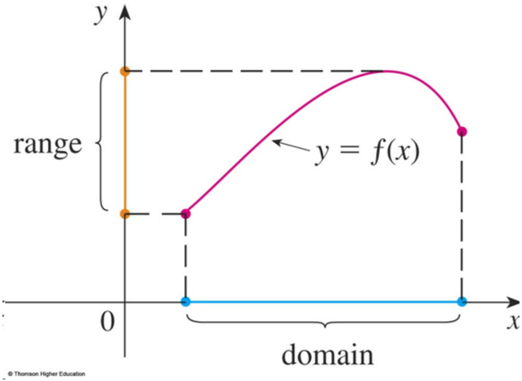
c) $f(x) = 2x^2 - 1$

d) $f(x) = \sin x$

The **graph** of f is the **set** of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

The graph of f also allows us to picture:

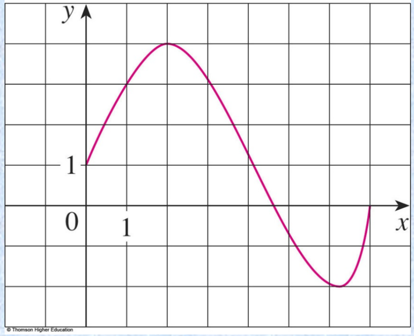
- ⋆ The **domain of f** on the $x-$axis
- ⋆ Its **range** on the $y-$axis



Example. The graph of a function f is shown.

a) Find the values of $f(1)$ and $f(5)$.

b) What is the domain and range of f ?



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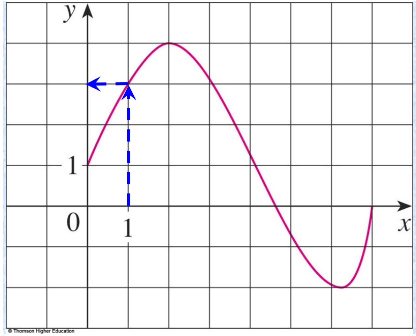

⋆ $f(1) = 3$

Example. The graph of a function f is shown.

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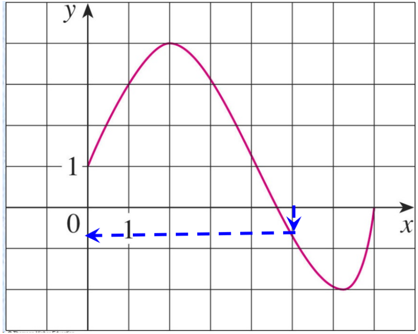


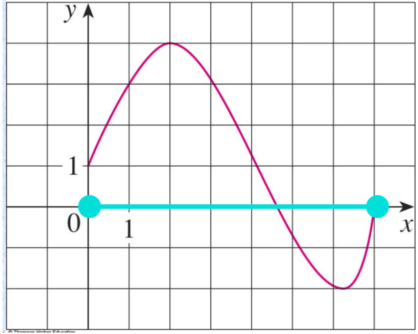
- $\star$ $f(1) = 3$
- $\star$ $f(5) \approx -0.7$

Example. The graph of a function f is shown.

a) Find the values of $f(1)$ and $f(5)$.

b) What is the domain and range of f ?

- $f(1) = 3$
- $f(5) \approx -0.7$
- $D = [0, 7]$



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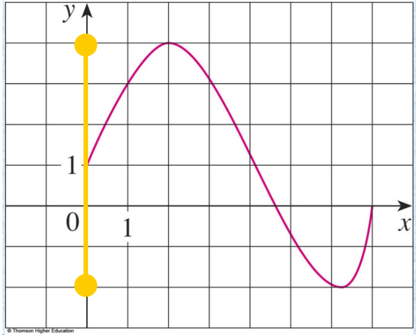


- $f(1) = 3$
- $f(5) \approx -0.7$
- $D = [0, 7]$
- Range$(f) = [-2, 4]$

Definition. If $f(a) = 0$, then a is called a **zero of f**.

Example. Find all zeros of $f(x) = x^3 - 3x^2 + 2x$.

Solution.

$x^3 - 3x^2 + 2x = 0$

$\Leftrightarrow x = 0, x = 1, x = 2.$

$\Rightarrow$ Zeros of f are: $0, 1, 2$

Prob 3. Find all zeros, if any, of the functions.

a) $f(x) = \sqrt{8x - 1}$

b) $g(x) = \dfrac{3}{x - 4}$

c) $h(x) = 4|x - 5|$

There are **four possible ways** to represent a function:

- ⋆ Algebraically (by an explicit formula)
- ⋆ Visually (by a graph)
- ⋆ Numerically (by a table of values)
- ⋆ Verbally (by a description in words)

EXAMPLE

The human population of the world P depends on the time t.

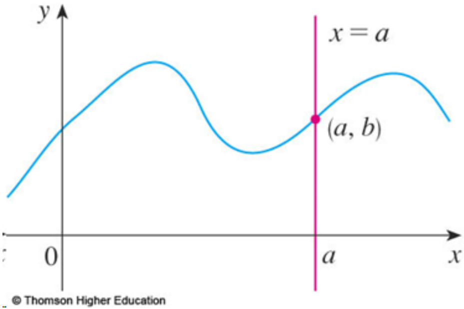
- The table gives estimates of the world population P(t) at time t, for certain years.
- However, for each value of the time t, there is a corresponding value of P, and we say that P is a function of t.

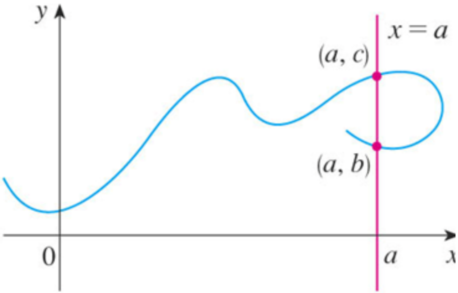
Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

"When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running".

Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

A curve in the $xy-$plane is the graph of a function of x if and only if **no vertical line** intersects the curve **more than once**.



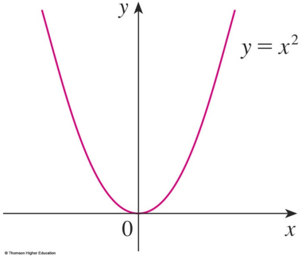


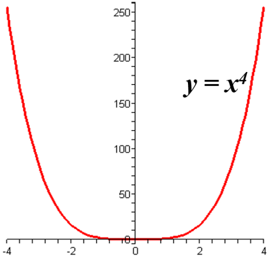
Definition. If a function f satisfies:

$$f(-x) = f(x), \text{ for all } x \text{ in } D$$

then f is called an **even function**.

The geometric significance of an even function is that its graph is **symmetric with respect to the $y-$axis**.





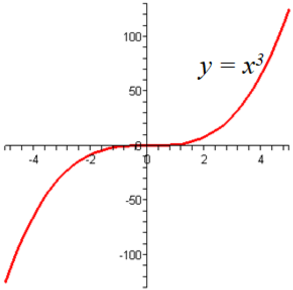
SYMMETRY: ODD FUNCTION

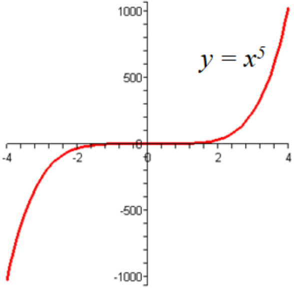
Definition. If a function f satisfies:

$$f(-x) = -f(x), \text{ for all } x \text{ in } D$$

then f is called an **odd function**.

The graph of an odd function is **symmetric about the origin**.





Prob 4. Let f is an **odd function**. If $(-3, 5)$ is in the graph of f then which point is also in the graph of f?

a) $(3, 5)$ b) $(-3, -5)$ c) $(3, -5)$ d) All of the others

Prob 5. Suppose f is an odd function and g is an even function. What can we say about the function $f.g$ defined by $(f.g)(x) = f(x)g(x)$? Prove your result.

Prob 6. Determine whether is even, odd, or neither:

a) $f(x) = x^2 - 3$

b) $g(x) = x^3 + 4x$

c) $h(x) = \dfrac{3x}{x^2 + 4}$

d) $k(x) = x^4 - x$

e) $p(x) = \sqrt{3x - 1}$

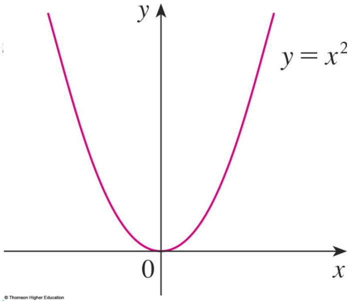
INCREASING AND DECREASING FUNCTIONS

A function f is called **increasing on an interval I** if:

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

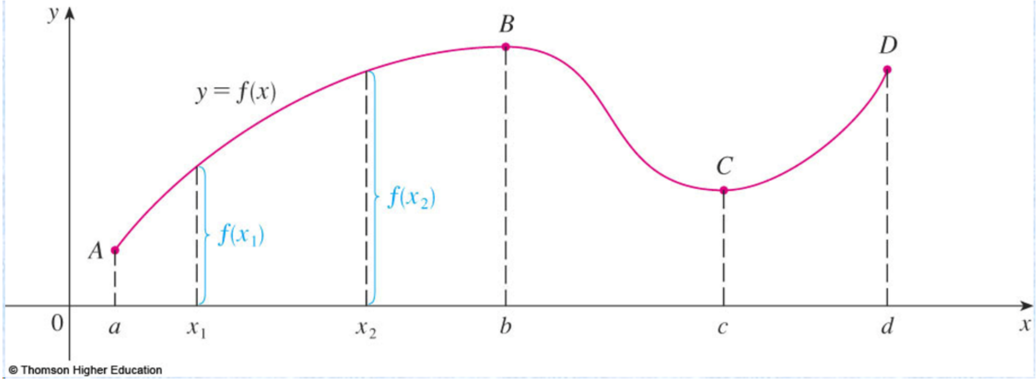
It is called **decreasing on I** if:

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$



EXAMPLE

The function f is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$.



1. If f is a function then $f(x+2) = f(x) + f(2)$

   a) True b) False

2. If $f(s) = f(t)$ then $s = t$

   a) True b) False

3. Let f be a function. We can find s and t such that $s = t$ and $f(s)$ is not equal to $f(t)$

   a) True b) False

Combining Functions with Mathematical Operators

Given two functions f and g, we can define four new functions:

1. $(f+g)(x) = f(x) + g(x)$
2. $(f-g)(x) = f(x) - g(x)$
3. $(f \cdot g)(x) = f(x)g(x)$
4. $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ for $g(x) \neq 0$.

Example. Given the functions $f(x) = 2x - 3$ and $g(x) = x^2 - 1$, find each of the following functions and state its domain.

a) $(f + g)(x)$ b) $(f - g)(x)$ c) $(f \cdot g)(x)$ d) $\left(\dfrac{f}{g}\right)(x)$

Solution.

a) $(f + g)(x) = f(x) + g(x) = x^2 + 2x - 4$. The domain $D = (-\infty, \infty)$.

b) $(f - g)(x) = f(x) - g(x) = -x^2 + 2x - 2$. The domain $D = (-\infty, \infty)$.

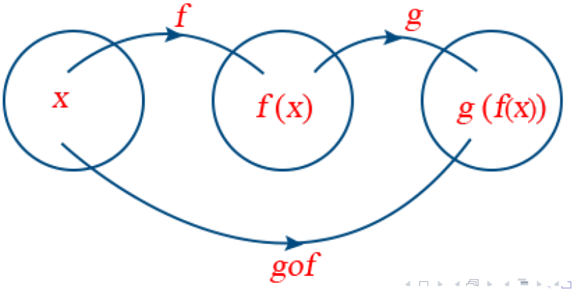
c) $(f \cdot g)(x) = f(x)g(x) = 2x^3 - 3x^2 - 2x + 3$. The domain $D = (-\infty, \infty)$.

d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{2x - 3}{x^2 - 1} = \dfrac{2x - 3}{(x + 1)(x - 1)}$.
The domain $D = \{x \in \mathbb{R} \mid x \neq \pm 1\}$.

Function composition

Definition. Consider the functions $f : A \to B$, and $g : D \to E$. If B is a subset of D, then the composite function $(g \circ f)(x)$ is the function with domain A such that

$$(g \circ f)(x) = g(f(x)).$$



Example. Consider the functions $f(x) = x^2 + 1$ and $g(x) = \dfrac{1}{x}$.

a) Find $(g \circ f)(x)$ and state its domain and range.

b) Evaluate $(g \circ f)(4), (g \circ f)(-1/2)$.

c) Find $(f \circ g)(x)$ and state its domain and range.

d) Evaluate $(f \circ g)(4), (f \circ g)(-1/2)$

Solution.

(1) $(g \circ f)(x) = g(f(x)) = g\left(x^2+1\right) = \dfrac{1}{x^2+1}$.

– Since $x^2+1 \neq 0$ for all real numbers x, the domain of $(g \circ f)(x)$ is the set of all real numbers.

– Since $0 < \dfrac{1}{x^2+1} \leq 1$, the range is, at most, the interval $(0,1]$.

(2) $(g \circ f)(4) = \dfrac{1}{17}$, and $(g \circ f)(-1/2) = \dfrac{4}{5}$. 3

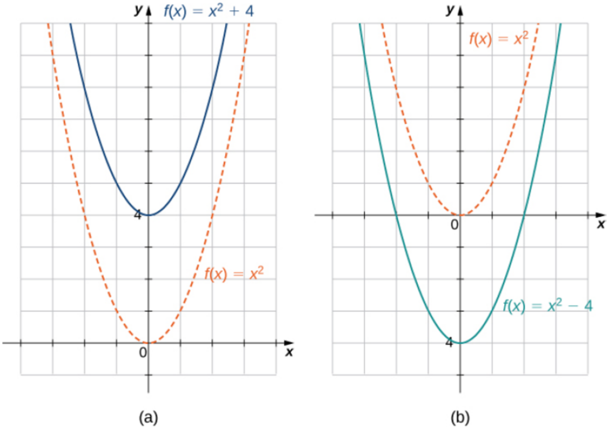
$(f \circ g)(x) = f(g(x)) = f\left(\dfrac{1}{x}\right) = \dfrac{1}{x^2} + 1$.

– The domain of $f \circ g$ is the set of all real numbers x such that $x \neq 0$.

– The range of $f \circ g$ is the set $\{y \mid y > 1\}$.

(4) $(f \circ g)(4) = \dfrac{17}{16}$, and $(f \circ g)\left(-\dfrac{1}{2}\right) = 5$.

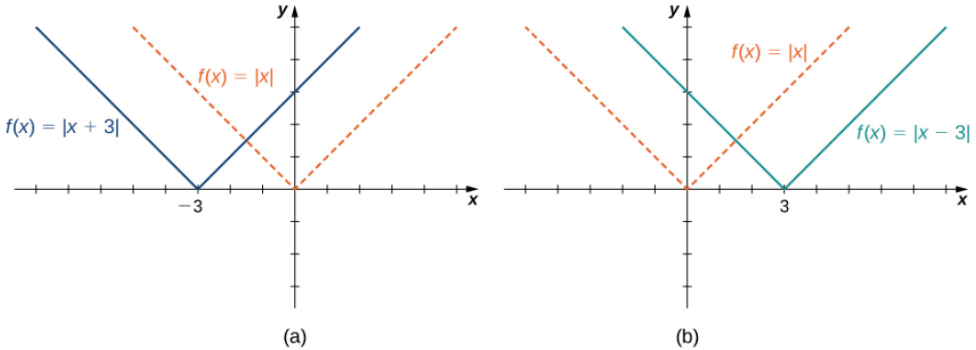
A vertical shift



(a) For $c > 0$, the graph of $y = f(x) + c$ is a **vertical shift up c units** of the graph of $y = f(x)$.

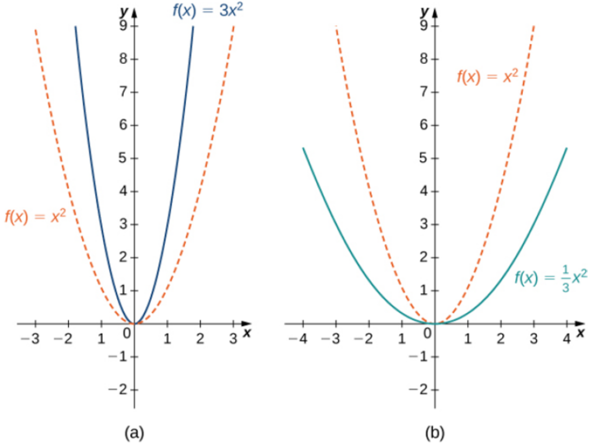
(b) For $c > 0$, the graph of $y = f(x) - c$ is a **vertical shift down c units** of the graph of $y = f(x)$.

A horizontal shift



(a) For $c > 0$, the graph of $y = f(x + c)$ is a **horizontal shift left c units** of the graph of $y = f(x)$.
(b) For $c > 0$, the graph of $y = f(x - c)$ is a **horizontal shift right c units** of the graph of $y = f(x)$.

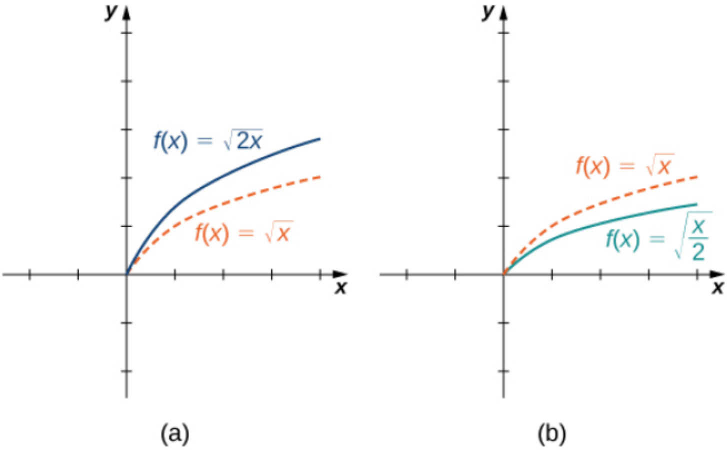
Vertical stretch(giãn) and compression (co)



(a) If $c > 1$, the graph of $y = cf(x)$ is a vertical stretch of the graph of $y = f(x)$.

(b) If $0 < c < 1$, the graph of $y = cf(x)$ is a vertical compression of the graph of $y = f(x)$

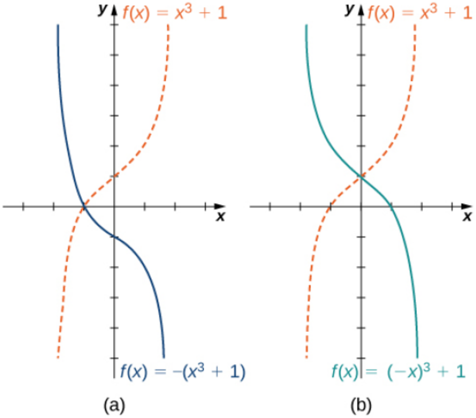
Horizontal stretch (giãn) and compression (co)



(a) If $c > 1$, the graph of $y = f(cx)$ is a horizontal compression of the graph of $y = f(x)$.

(b) If $0 < c < 1$, the graph of $y = f(cx)$ is a horizontal stretch of the graph of $y = f(x)$.

Reflections



(a) The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x -axis.
(b) The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected about the y-axis.

Summary

Transformation of f $\quad (c > 0)$	Effect on the graph of f
$f(x) + c$	Vertical shift up c units
$f(x) - c$	Vertical shift down c units
$f(x + c)$	Shift left by c units
$f(x - c)$	Shift right by c units
$cf(x)$	Vertical stretch if $c > 1$; vertical compression if $0 < c < 1$
$f(cx)$	Horizontal stretch if $0 < c < 1$; horizontal compression if $c > 1$
$-f(x)$	Reflection about the $x-$axis
$f(-x)$	Reflection about the $y-$axis

Prob 7. Let $f(x) = \sqrt{x+1}$ and $g(x) = \sqrt{5+x}$. Find $g \circ f$?

Prob 8. Use the table to evaluate the expression $(f \circ g)(2)$:

x	1	2	3	4	5	6
$f(x)$	3	4	2	0	1	2
$g(x)$	3	6	1	3	4	0

Prob 9. Let $f(x) = \dfrac{x^2 + x + 1}{x}$. Find

a) $f\left(x + \dfrac{1}{x}\right)$

b) $f(2x - 1)$

Prob 10. Explain how the following graphs are obtained from the graph of $f(x)$

a) $f(x - 4)$

b) $f(x) + 3$

c) $f(x - 2) - 3$