Chapter 2
Basic Structures
Sets, Functions
Sequences, and Sums

Topics

- Sets
- Set operations
- Functions
- Sequences
- Summations

Basic discrete structures

- **Discrete math =**
  - study of the discrete structures used to represent discrete objects
- Many discrete structures are built using **sets**
  - **Sets = collection of objects**

Examples of discrete structures built with the help of sets:

- **Combinations**
- **Relations**
- **Graphs**

SETS

2.1- Sets

- **Definition**: A **set** is a (unordered) collection of objects. These objects are sometimes called **elements** or **members** of the set. (Cantor's naive definition)

- **Examples:**
  - **Vowels in the English alphabet**

    V = { a, e, i, o, u }
  - **First seven prime numbers**.

    X = { 2, 3, 5, 7, 11, 13, 17 }

2.1- Sets

Representing a set by:

1) Listing (enumerating) the members of the set.

2) Definition by property, using the set builder notation

{x| x has property P}.

Example:

- Even integers between 50 and 63.

1) E = {50, 52, 54, 56, 58, 60, 62}

2) E = {x| 50 <= x < 63, x is an even integer}

If enumeration of the members is hard we often use ellipses.

Example: a set of integers between 1 and 100

- **A= {1,2,3 …, 100}**

Important sets in discrete math

- **Natural numbers:**
  - **N** = {0,1,2,3, …}

- **Integers**
  - **Z** = {…, -2,-1,0,1,2, …}

- **Positive integers**
  - $\mathbf{Z}^+$ = {1,2, 3.…}

- **Rational numbers**
  - **Q** = $\{p/q \mid p \in Z, q \in Z, q \neq 0\}$

- **Real numbers**
  - **R**

Equality

Definition: Two sets are equal if and only if they have the same elements.

Example:

- **{1,2,3} = {3,1,2} = {1,2,1,3,2}**

Note: Duplicates don't contribute anything new to a set, so remove them. The order of the elements in a set doesn't contribute anything new.

Example: Are {1,2,3,4} and {1,2,2,4} equal?

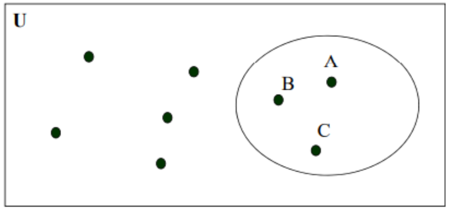
No!

Special sets

- **Special sets:**
  - **The universal set** is denoted by **U:** the set of all objects under the consideration.
  - **The empty set** is denoted as **∅ or { }.**

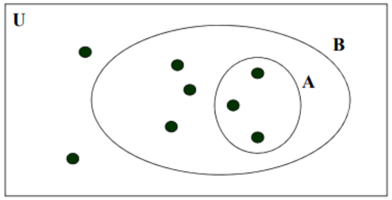
Venn diagrams

- A set can be visualized using **Venn Diagrams**:
  - V={ A, B, C }



A Subset

- **Definition**: A **set** A is said to be a **subset** of B **if and only if** every element of A is also an element of B. We use **A ⊆ B** to indicate **A is a subset of B**.



- Alternate way to define A is a subset of B:
  $\forall x\ (x \in A) \rightarrow (x \in B)$

Empty set/Subset properties

Theorem $\varnothing \subseteq S$

- **Empty set is a subset of any set.**

Proof:

- Recall the definition of a subset: all elements of a set A must be also elements of B: $\forall x\ (x \in A \rightarrow x \in B)$.
- We must show the following implication holds for any S
  $\forall x\ (x \in \varnothing \rightarrow x \in S)$
- Since the empty set does not contain any element, $x \in \varnothing$ is **always False**
- Then the implication is **always True.**

End of proof

Subset properties

Theorem: $S \subseteq S$

- **Any set S is a subset of itself**

Proof:

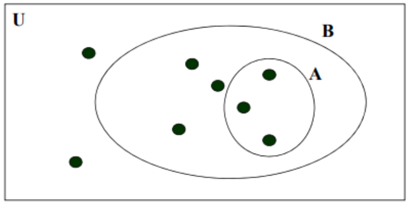
- the definition of a subset says: all elements of a set A must be also elements of B: $\forall x\ (x \in A \rightarrow x \in B)$.
- Applying this to S we get:
- $\forall x\ (x \in S \rightarrow x \in S)$ which is trivially **True**
- End of proof

Note on equivalence:

- Two sets are equal if each is a subset of the other set.

A proper subset

Definition: A **set** **A** is said to be a **proper subset** of B if and only if $A \subseteq B$ and $A \neq B$. We denote that A is a proper subset of B with the notation $A \subset B$.



Example: A={1,2,3} B ={1,2,3,4,5}

Is: $A \subset B$? Yes.

Cardinality

Definition: Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say S is a finite set and that n is the **cardinality of S**. The cardinality of S is denoted by **| S |**.

Examples:

- **V={1 2 3 4 5}**
  | V | = 5
- **A={1,2,3,4, …, 20}**
  |A| =20
- **$|\varnothing| = 0$**

Infinite set

Definition: A set is **infinite** if it is not finite.

Examples:

- The set of natural numbers is an infinite set.
- N = {1, 2, 3, ... }
- The set of reals is an infinite set.

Example.

Which statements are true?

$x \in \{x\}$ (T)

$x \subseteq \{x\}$ (F)

$\{a, b\} \subseteq \{a, b, \{a, b\}, c\}$ (T)

$\{a, b\} \in \{a, b, \{a, b\}, c\}$ (T)

$\emptyset \subseteq \{\emptyset\}$ (T)

$\emptyset \in \{\emptyset\}$ (T)

$\{a, b, c\} \subseteq \{a, b, c\}$ (T)

$\{a, b, c\} \in \{a, b, c\}$ (F)

Power set

Definition: Given a set S, the **power set** of S is the set of all subsets of S. The power set is denoted by **P(S).**

Examples:

- Assume an empty set $\varnothing$
- What is the power set of $\varnothing$? $P(\varnothing) = \{ \varnothing \}$
- What is the cardinality of $P(\varnothing)$? $| P(\varnothing) | = 1$.

- Assume set $\{1\}$
- $P(\{1\}) = \{ \varnothing, \{1\} \}$
- $|P(\{1\})| = 2$

Power set

- $P(\{1\}) = \{ \varnothing, \{1\} \}$
- $|P(\{1\})| = 2$

- Assume $\{1,2\}$
- $P(\{1,2\}) = \{ \varnothing, \{1\}, \{2\}, \{1,2\} \}$
- $|P(\{1,2\})| =4$

- Assume $\{1,2,3\}$
- $P(\{1,2,3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$
- $|P(\{1,2,3\} | = 8$

- **If S is a set with $|S| = n$ then $| P(S) | = ?$**

Power set

- $P(\{1\}) = \{ \varnothing, \{1\} \}$
- $|P(\{1\})| = 2$

- Assume $\{1,2\}$
- $P(\{1,2\}) = \{ \varnothing, \{1\}, \{2\}, \{1,2\} \}$
- $|P(\{1,2\})| =4$

- Assume $\{1,2,3\}$
- $P(\{1,2,3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$
- $|P(\{1,2,3\} | = 8$

- **If S is a set with $|S| = n$ then $| P(S) | = 2^n$**

N-tuple

- Sets are used to represent unordered collections.
- **Ordered-n tuples** are used to represent an ordered collection.

Definition: An **ordered n-tuple** (x1, x2, ..., xN) is the ordered collection that has x1 as its first element, x2 as its second element, ..., and xN as its N-th element, $N \geq 2$.

Cartesian product

Definition: Let S and T be sets. The **Cartesian product of S and T**, denoted by **S x T,** is the set of all ordered pairs (s,t), where $s \in S$ and $t \in T$. Hence,

- $S \times T = \{ (s,t) \mid s \in S \wedge t \in T \}$.

Examples:

- S = {1,2} and T = {a,b,c}
- S x T = { (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) }
- T x S = { (a,1), (a, 2), (b,1), (b,2), (c,1), (c,2) }
- Note: $S \times T \neq T \times S$ **!!!!**

Cardinality of the Cartesian product

- |S x T| = |S| * |T|.

Example:

- A= {John, Peter, Mike}
- B ={Jane, Ann, Laura}
- A x B= {(John, Jane),(John, Ann) , (John, Laura), (Peter, Jane), (Peter, Ann) , (Peter, Laura) , (Mike, Jane) , (Mike, Ann) , (Mike, Laura)}
- |A x B| = 9
- |A|=3, |B|=3 → |A| |B|= 9

Definition: A subset of the Cartesian product A x B is called a relation from the set A to the set B.

Exercises

1. List the members of these sets.
a) $\{x \mid x$ is a real number such that $x^2 = 1\}$
b) {x | x is a positive integer less than 12}
c) $\{x \mid x$ is the square of an integer and $x < 100\}$
d) $\{x \mid x$ is an integer such that $x^2 = 2\}$
2. For each of the following sets, determine whether 2 is an element of that set.
a) $\{x \in R \mid x$ is an integer greater than $1\}$
b) $\{x \in R \mid x$ is the square of an integer$\}$
c) {2,{2}}
d) {{2},{{2}}}
e) {{2},{2,{2}}}
f) {{{2}}}

Exercises

4. Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$, and $D = \{4, 6, 8\}$. Determine which of these sets are subsets of which other of these sets.
5. Determine whether these statements are true or false.

a) $\emptyset \in \{\emptyset\}$
b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$
c) $\{\emptyset\} \in \{\emptyset\}$
d) $\{\emptyset\} \in \{\{\emptyset\}\}$
e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$
f) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

Exercises

6. What is the cardinality of each of these sets?

a) $\{a\}$
b) $\{\{a\}\}$
c) $\{a, \{a\}\}$
d) $\{a, \{a\}, \{a, \{a\}\}\}$
e) $\emptyset$
f) $\{\emptyset\}$
g) $\{\emptyset, \{\emptyset\}\}$
h) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

7. Find the power set of each of these sets, where a and b are distinct elements.

a) $\{a\}$
b) $\{a, b\}$
c) $\{\emptyset, \{\emptyset\}\}$

8. How many elements does each of these sets have where a and b are distinct elements?

a) $P(\{a, b, \{a, b\}\})$
b) $P(\{\emptyset, a, \{a\}, \{\{a\}\}\})$
c) $P(P(\emptyset))$
d) $\emptyset$
e) $\{\emptyset, \{a\}\}$
f) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$
g) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Exercises

9. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find

a) $A \times B$.
b) $B \times A$.

10. Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find

a) $A \times B \times C$.
b) $C \times B \times A$.
c) $C \times A \times B$.
d) $B \times B \times B$.

11. Find A^2 if

a) $A = \{0, 1, 3\}$.
b) $A = \{1, 2, a, b\}$.

12. Find A^3 if

a) $A = \{a\}$.
b) $A = \{0, a\}$.

Quiz

Given $A=\{0,\emptyset\}$. Find the cardinality of $P(AxA)$.

Select one:

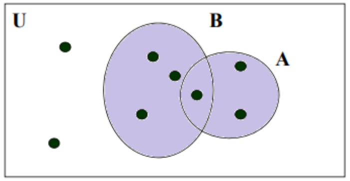
- a. 2
- b. $\{(0,\emptyset),(0,0),(\emptyset,\emptyset), (\emptyset,0)\}$
- c. 4
- d. 16

SET OPERATIONS

2.2- Set Operations

Definition: Let A and B be sets. The **union of A and B**, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

- Alternate: $A \cup B = \{ x \mid x \in A \vee x \in B \}$.

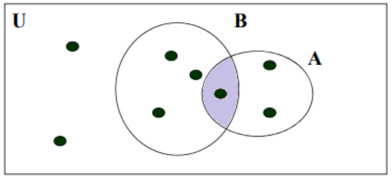


- **Example:**
- $A = \{1,2,3,6\}$ $B = \{2,4,6,9\}$
- $A \cup B = \{1,2,3,4,6,9\}$

2.2- Set Operations

Definition: Let A and B be sets. The **intersection of A and B**, denoted by $A \cap B$, is the set that contains those elements that are in both A and B.

- Alternate: $A \cap B = \{ x \mid x \in A \wedge x \in B \}$.



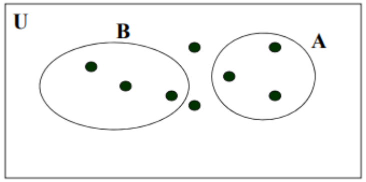
Example:

- $A = \{1,2,3,6\}$ $B = \{2, 4, 6, 9\}$
- $A \cap B = \{2, 6\}$

2.2- Set Operations

Definition: Two sets are called **disjoint** if their intersection is empty.

- Alternate: A and B are disjoint **if and only if** $A \cap B = \varnothing$.



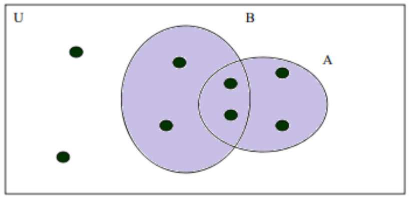
Example:

- $A=\{1,2,3,6\}$ $B=\{4,7,8\}$ Are these disjoint?
- Yes.
- $A \cap B = \varnothing$

2.2- Set Operations

Cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$



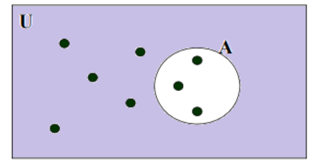
The generalization of this result to unions of an arbitrary number of sets is called the **principle of inclusion–exclusion**.

2.2- Set Operations

Definition: Let U be the **universal set**: the set of all objects under the consideration.

Definition: The **complement of the set A**, denoted by $\overline{A}$, is the complement of A with respect to U.

- Alternate: $\overline{A} = \{ x \mid x \notin A \}$



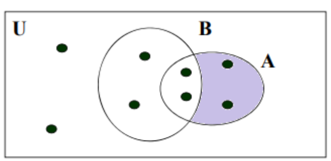
Example: U={1,2,3,4,5,6,7,8} A ={1,3,5,7}

- $\overline{A}$ = {2,4,6,8}

2.2- Set Operations

Definition: Let A and B be sets. The **difference of A and B**, denoted by **A - B**, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

- Alternate: $A - B = \{ x \mid x \in A \wedge x \notin B \}$.



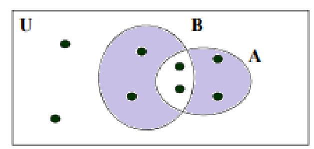
Example: A= {1,2,3,5,7} B = {1,5,6,8}

- A - B ={2,3,7}

2.2- Set Operations

Definition: Let A and B be subsets of a universal set U. The **symmetric difference** of A and B, denoted by $\mathbf{A \oplus B}$, is the set containing those elements in either A or B, but not in both A and B.

- Alternate: $\mathbf{A \oplus B} = \{x \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$



Example: A = {1, 2, 3, 5, 7}, B = {1, 5, 6, 8}

- $A \oplus B$ = {2, 3, 7, 6, 8}

Set Identities

Set Identities (analogous to logical equivalences)

- **Identity**
  - $A \cup \varnothing = A$
  - $A \cap U = A$
- **Domination**
  - $A \cup U = U$
  - $A \cap \varnothing = \varnothing$
- **Idempotent**
  - $A \cup A = A$
  - $A \cap A = A$

Set Identities

- **Double complement**
  - $\overline{\overline{A}} = A$
- **Commutative**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive**
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Set Identities

- **DeMorgan**
  - $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$
  - $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
- **Absorbtion Laws**
  - $A \cup (A \cap B) = A$
  - $A \cap (A \cup B) = A$
- **Complement Laws**
  - $A \cup \overline{A} = U$
  - $A \cap \overline{A} = \varnothing$

Set identities

- **Set identities can be proved using membership tables.**
- List each combination of sets that an element can belong to. Then show that for each such a combination the element either belongs or does not belong to both sets in the identity.
- Prove: $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

A	B	$\overline{A}$	$\overline{B}$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
1	1	0	0	0	0
1	0	0	1	0	0
0	1	1	0	0	0
0	0	1	1	1	1

Generalized unions and itersections

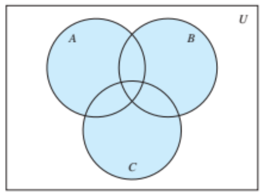
Definition: The **union of a collection of sets** is the set that contains those elements that are members of at least one set in the collection.

$$\bigcup_{i=1}^{n} A_i = \{A_1 \cup A_2 \cup \ldots \cup A_n\}$$

Example:

- Let $A_i = \{1,2,\ldots,i\}$ i =1,2,…,n
-

$$\bigcup_{i=1}^{n} A_i = \{1, 2, \ldots, n\}$$



(a) *A* U *B* U *C* is shaded.

Generalized unions and intersections

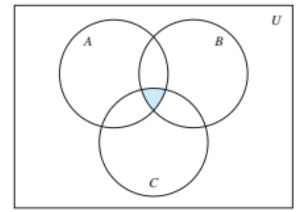
Definition: The **intersection of a collection of sets** is the set that contains those elements that are members of all sets in the collection.

$$\bigcap_{i=1}^{n} A_i = \{A_1 \cap A_2 \cap \ldots \cap A_n\}$$

Example:

- Let $A_i = \{1,2,\ldots,i\}$ i =1,2,…,n

$$\bigcap_{i=1}^{n} A_i = \{1\}$$



(b) *A* ∩ *B* ∩ *C* is shaded.

Computer representation of sets

- **How to represent sets in the computer?**
- **One solution: Data structures like a list**
- **A better solution:**
- Assign a bit in a bit string to each element in the universal set and set the bit to 1 if the element is present otherwise use 0

Example:

All possible elements: U={1 2 3 4 5}

- Assume A={2,5}
  - Computer representation: A = 01001
- Assume B={1,5}
  - Computer representation: B = 10001

Computer representation of sets

Example:

- A = 01001
- B = 10001

- The **union** is modeled with a bitwise **or**
- $A \vee B = 11001$
- The **intersection** is modeled with a bitwise **and**
- $A \wedge B = 00001$
- The **complement** is modeled with a bitwise **negation**
- $\overline{A}$ =10110

Exercises

1. Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets.

a) $A \cap B$ b) $A \cup B$ c) $A - B$ d) $B - A$

2. . Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of these sets in terms of A and B.

a) the set of sophomores taking discrete mathematics in your school

b) the set of sophomores at your school who are not taking discrete mathematics

c) the set of students at your school who either are sophomores or are taking discrete mathematics

d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

Exercises

3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cup B$. b) $A \cap B$. c) $A - B$. d) $B - A$.

4. Let A and B be sets. Show that

a) $(A \cap B) \subseteq A$. b) $A \subseteq (A \cup B)$.

c) $A - B \subseteq A$. d) $A \cap (B - A) = \emptyset$.

e) $A \cup (B - A) = A \cup B$.

5. Show that if A is a subset of a universal set U, then

a) $A \oplus A = \emptyset$. b) $A \oplus \emptyset = A$.

c) $A \oplus U = A$. d) $A \oplus A = U$

Exercises

6. Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of these sets with bit strings where the ith bit in the string is 1 if i is in the set and 0 otherwise.

a) $\{3, 4, 5\}$

b) $\{1, 3, 6, 10\}$

c) $\{2, 3, 4, 7, 8, 9\}$

7. Using the same universal set as in the last exercise, find the set specified by each of these bit strings.

a) 11 1100 1111

b) 01 0111 1000

c) 10 0000 0001

Exercises

7. Show how bitwise operations on bit strings can be used to find these combinations of $A = \{a, b, c, d, e\}$, $B = \{b, c, d, g, p, t, v\}$, $C = \{c, e, i, o, u, x, y, z\}$, and $D = \{d, e, h, i, n, o, t, u, x, y\}$.

a) $A \cup B$ b) $A \cap B$

c) $(A \cup D) \cap (B \cup C)$ d) $A \cup B \cup C \cup D$

Exercises

8. Which of the following sets is the power set of some other set?

A. {∅, {a}, {∅, a}}
B. {∅, {a}, {∅}, {∅, a}}
C. {∅, {a}, {∅}, {{∅}, a}}
D. {∅, {a, ∅}}

Exercises

8. Let U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}.

Given the subsets A = {1, 2, 3, 4, 8} and B = {0, 5, 6, 7, 9}. The bit string representing the subset A – B is

Select one:

A. 00 1110 0010

B. 01 1110 0110

C. 01 1110 0010

D. 00 1011 0010

Exercises

8. Let A and B be sets. Assume that $A \times B = \emptyset$

Choose the best answer:

A. $A = \emptyset$

B. $(A = \emptyset) \wedge (B = \emptyset)$

C. $(A = \emptyset) \oplus (B = \emptyset)$

D. $(A = \emptyset) \vee (B = \emptyset)$

E. None of the choices is correct

Exercises

9. Given that the bit string representations for the subsets A and B are 11 0000 1100 and 11 0011 0010. What is the bit string representation for the set $A \oplus B$:

A. 11 0000 1101

B. 00 0011 1110

C. 00 1100 0001

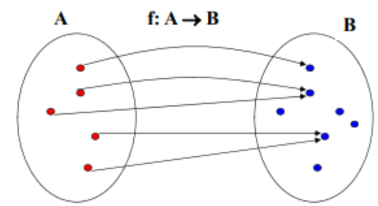
D. 11 0011 1110

E. None of the choices is correct

FUNCTIONS

2.3. Functions

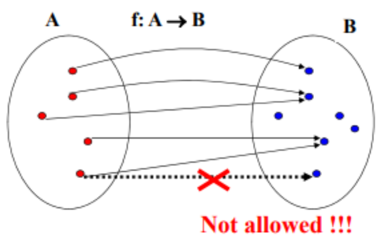
- **Definition**: Let A and B be two sets. A **function from A to B**, denoted **$f : A \to B$**, is an assignment of exactly one element of B to each element of A. We write f(a) = b to denote the assignment of b to an element a of A by the function f.



Remark. Functions are sometimes also called **mappings** or **transformations**.

2.3. Functions

- **Definition**: Let A and B be two sets. A **function from A to B**, denoted **$f : A \to B$**, is an assignment of exactly one element of B to each element of A. We write f(a) = b to denote the assignment of b to an element a of A by the function f.



Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using ‘standard’ functions)

Example1:

- Let A = {1,2,3} and B = {a,b,c}
- Assume f is defined as:
  - $1 \to c$
  - $2 \to a$
  - $3 \to c$
- Is f a function ?

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  - $1 \to c$
  - $2 \to a$
  - $3 \to c$
- Is f a function ?
- **Yes.** since f(1)=c, f(2)=a, f(3)=c. each element of A is assigned an element from B

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example 2:

- Let A = {1,2,3} and B = {a,b,c}
- Assume g is defined as
  - $1 \to c$
  - $1 \to b$
  - $2 \to a$
  - $3 \to c$
- Is g a function?

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- Assume g is defined as
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  - $1 \to b$
  - $2 \to a$
  - $3 \to c$
- Is g a function?
- **No.** g(1) = is assigned both c and b.

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example 3:

- A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}
- Define h: $A \to B$ as:
  - h(x) = x mod 3.
  - (the result is the remainder after the division by 3)
- Assignments:
- $0 \to 0$ $3 \to 0$
- $1 \to 1$ $4 \to 1$
- $2 \to 2$ …

Important sets

Definitions: Let f be a function from A to B.

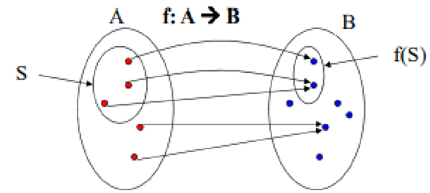
- We say that A is the **domain** of f and B is the **codomain** of f.
- If f(a) = b, **b is the image of a** and **a is a pre-image of b**.
- **The range of f** is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

Example: Let A = {1,2,3} and B = {a,b,c}

- Assume f is defined as: $1 \to c$, $2 \to a$, $3 \to c$
- What is the image of 1?
- $1 \to c$ c is the image of 1
- What is the pre-image of a?
- $2 \to a$ 2 is a pre-image of a.
- Domain of f ? {1,2,3}
- Codomain of f ? {a,b,c}
- Range of f ? {a,c}

Image of a subset

Definition: Let f be a function from set A to set B and let S be a subset of A. The image of S is a subset of B that consists of the images of the elements of S. We denote the image of S by f(S), so that $f(S) = \{ f(s) \mid s \in S \}$.



Example:

- Let A = {1,2,3} and B = {a,b,c} and f: $1 \to c$, $2 \to a$, $3 \to c$
- Let S = {1,3} then image f(S) = **?**

Image of a subset

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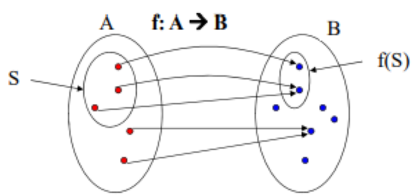

Example:

- Let A = {1,2,3} and B = {a,b,c} and f: $1 \to c$, $2 \to a$, $3 \to c$
- Let S = {1,3} then image f(S) = **?**

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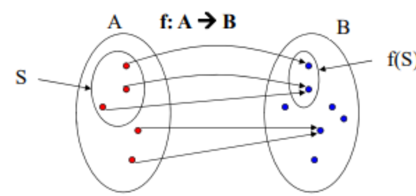


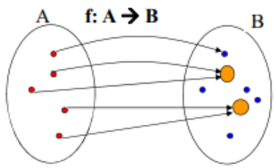
Example:

- Let A = {1,2,3} and B = {a,b,c} and f: $1 \to c$, $2 \to a$, $3 \to c$
- Let S = {1,3} then image f(S) = {c}.

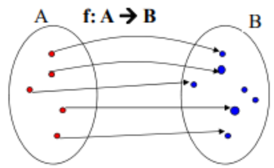
Injective function

Definition: A function f is said to be **one-to-one, or injective**, if and only if f(x) = f(y) implies x = y for all x, y in the domain of f. A function is said to be an **injection if it is one-to-one**.

Alternative: A function is one-to-one if and only if f(x) ≠ f(y), whenever x ≠ y. This is the contrapositive of the definition.



Not injective function



Injective function

Injective function

Example 1: Let A = {1,2,3} and B = {a,b,c}

- Define f as
  - 1 → c
  - 2 → a
  - 3 → c
- Is f one to one? **No**, it is not one-to-one since f(1) = f(3) = c, and 1 ≠ 3.

Example 2: Let g : Z → Z, where g(x) = 2x - 1.

- Is g one-to-one (why?)
- **Yes.**
- Suppose g(a) = g(b), i.e., 2a - 1 = 2b - 1 => 2a = 2b

`` => a = b.

Injective function

Example 1: Let A = {1,2,3} and B = {a,b,c}

- Define f as
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- Is f one to one? **No**, it is not one-to-one since f(1) = f(3) = c, and 1 ≠ 3.

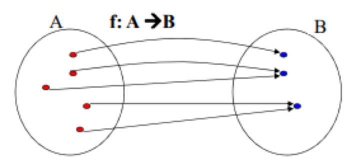
Example 2: Let g : Z → Z, where g(x) = 2x - 1.

- Is g one-to-one (why?)

Surjective function

Definition: A function f from A to B is called **onto**, or **surjective**, if and only if for every b ∈ B there is an element a ∈ A such that f(a) = b.

Alternative: all co-domain elements are covered



Surjective function

Example 1: Let A = {1,2,3} and B = {a,b,c}

- Define f as
  - $1 \to c$
  - $2 \to a$
  - $3 \to c$

- Is f an onto?
- **No.** f is not onto, since $b \in B$ has no pre-image.

Example 2: A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}

- Define h: $A \to B$ as $h(x) = x \bmod 3$.

- Is h an onto function?

Surjective function

Example 1: Let A = {1,2,3} and B = {a,b,c}

- Define f as
  - $1 \to c$
  - $2 \to a$
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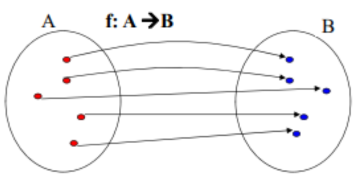
Example 2: A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}

- Define h: $A \to B$ as $h(x) = x \bmod 3$.

- Is h an onto function?
- **Yes.** h is onto since a pre-image of 0 is 6, a pre-image of 1 is 4, a pre-image of 2 is 8.

Bijective functions

Definition: A function f is called **a bijection** if it is **both one-to-one and onto**.



Bijective functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c}
  - Define f as
    - $1 \to c$
    - $2 \to a$
    - $3 \to b$
- Is f a bijection?
- **?**

Bijective functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c}
  - Define f as
    - $1 \rightarrow c$
    - $2 \rightarrow a$
    - $3 \rightarrow b$
- Is f a bijection?
- **Yes.** It is both one-to-one and onto.

Bijective functions

Theorem: Let f be a function f: A →A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Proof:

→ **A is finite** and **f is one-to-one (injective)**

- Is f an onto function (surjection)?
- **Yes.** Every element points to exactly one element. Injection assures they are different. So we have |A| different elements A points to. Since f: A → A the co-domain is covered thus the function is also a surjection (and a bijection)

← **A is finite and f is an onto function**

- Is the function one-to-one?

Bijective functions

Theorem: Let f be a function f: A →A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Proof:

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- **Yes.** Every element points to exactly one element. Injection assures they are different. So we have |A| different elements A points to. Since f: A → A the co-domain is covered thus the function is also a surjection (and a bijection)

← **A is finite and f is an onto function**

- Is the function one-to-one?
- **Yes.** Every element maps to exactly one element and all elements in A are covered. Thus the mapping must be one-toone

Bijective functions

Theorem: Let f be a function f: A →A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Please note the above is not true when A is an infinite set.

- **Example:**
  - $f : Z \rightarrow Z$, where $f(z) = 2 * z$.
  - f is one-to-one but not onto.
    - $1 \rightarrow \mathbf{2}$
    - $2 \rightarrow \mathbf{4}$
    - $3 \rightarrow \mathbf{6}$
  - 3 has no pre-image.

Functions on real numbers

Definition: Let f1 and f2 be functions from A to **R** (reals). Then f1 + f2 and f1 * f2 are also functions from A to **R** defined by

- (f1 + f2)(x) = f1(x) + f2(x)
- (f1 * f2)(x) = f1(x) * f2(x).

Examples:

- **Assume**
  - f1(x) = x - 1
  - f2(x) = x^3 + 1

then

- (f1 + f2)(x) = x^3 + x
- (f1 * f2)(x) = x^4 - x^3 + x - 1.

Increasing and decreasing functions

Definition: A function f whose domain and codomain are subsets of real numbers is **strictly increasing** if f(x) > f(y) whenever x > y and x and y are in the domain of f. Similarly, f is called **strictly decreasing** if f(x) < f(y) whenever x > y and x and y are in the domain of f.

Example:

- Let g : **R** → **R**, where g(x) = 2x - 1. Is it increasing ?

Increasing and decreasing functions

Definition: A function f whose domain and codomain are subsets of real numbers is **strictly increasing** if f(x) > f(y) whenever x > y and x and y are in the domain of f. Similarly, f is called **strictly decreasing** if f(x) < f(y) whenever x > y and x and y are in the domain of f.

Example:

- Let g : **R** → **R**, where g(x) = 2x - 1. Is it increasing ?
- **Proof .**

  For x>y holds 2x > 2y and subsequently 2x-1 > 2y-1

  Thus g is strictly increasing.

Increasing and decreasing functions

Definition: A function f whose domain and codomain are subsets of real numbers is **strictly increasing** if f(x) > f(y) whenever x > y and x and y are in the domain of f. Similarly, f is called **strictly decreasing** if f(x) < f(y) whenever x > y and x and y are in the domain of f.

Note: Strictly increasing and strictly decreasing functions are one-to-one.

Why?

One-to-one function: A function is one-to-one if and only if f(x) ≠ f(y), whenever x ≠ y.

Identity function

Definition: Let A be a set. The **identity function** on A is the function $i_A: A \to A$ where $i_A(x) = x$.

Example:

- Let A = {1,2,3}

Then:

- $i_A(1) = ?$

Identity function

Definition: Let A be a set. The **identity function** on A is the function $i_A: A \to A$ where $i_A(x) = x$.

Example:

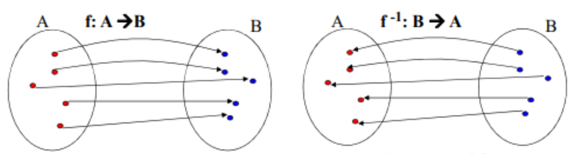
- Let A = {1,2,3}

Then:

- $i_A(1) = 1$
- $i_A(2) = 2$
- $i_A(3) = 3$.

Inverse functions

Definition: Let f be a **bijection** from set A to set B. The **inverse function of f** is the function that assigns to an element b from B the unique element a in A such that f(a) = b. The inverse function of f is denoted by $\mathbf{f^{-1}}$. Hence, $f^{-1}(b) = a$, when f(a) = b. If the inverse function of f exists, f is called **invertible.**

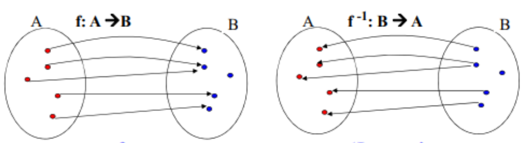


f is bijective **Inverse of f**

Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. **Why?**

Assume f is not one-to-one:

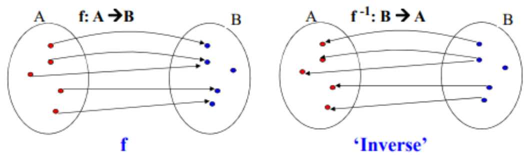


f **'Inverse'**

Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. **Why?**

Assume f is not one-to-one:

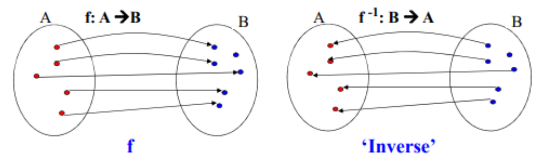


Inverse is not a function. One element of B is mapped to two different elements.

Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. **Why?**

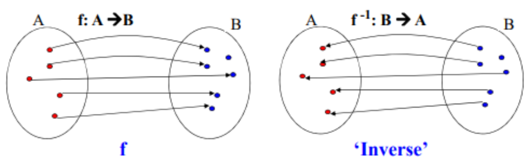
Assume f is not onto:



Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. **Why?**

Assume f is not onto:



Inverse is not a function. One element of B is not assigned any value in B.

Inverse functions

Example 1:

- Let A = {1,2,3} and i_A be the identity function
- $i_A(1) = 1$ $\qquad$ $i_A^{-1}(1) = 1$
- $i_A(2) = 2$ $\qquad$ $i_A^{-1}(2) = 2$
- $i_A(3) = 3$ $\qquad$ $i_A^{-1}(3) = 3$
- Therefore, the inverse function of i_A is i_A.

Inverse functions

Example 2:

- Let $g : \mathbf{R} \rightarrow \mathbf{R}$, where $g(x) = 2x - 1$.
- What is the inverse function g^{-1} ?

Inverse functions

Example 2:

- Let $g : \mathbf{R} \rightarrow \mathbf{R}$, where $g(x) = 2x - 1$.
- What is the inverse function g^{-1} ?

Approach to determine the inverse:

$y = 2x - 1 \Rightarrow y + 1 = 2x$

$\Rightarrow (y+1)/2 = x$

- Define $\mathbf{g^{-1}(y) = x = (y+1)/2}$

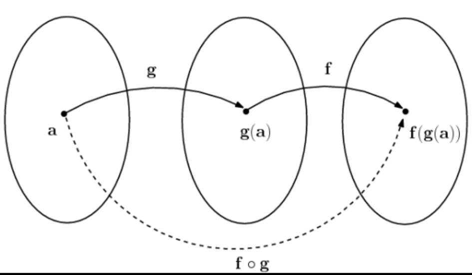
Test the correctness of inverse:

- $g(3) = 2*3 - 1 = 5$
- $g^{-1}(5) = (5+1)/2 = 3$
- $g(10) = 2*10 - 1 = 19$
- $g^{-1}(19) = (19+1)/2 = 10$.

Composition of functions

Definition. Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The **composition of the functions** f and g, denoted by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a))$$



Composition of functions

Example 1. Let A = {1, 2, 3} and B = {a, b, c, d}

g: $A \rightarrow A$,

$1 \rightarrow 3$

$2 \rightarrow 1$

$3 \rightarrow 2$

f: $A \rightarrow B$

$1 \rightarrow b$

$2 \rightarrow a$

$3 \rightarrow d$

$f \circ g$: $A \rightarrow B$

$1 \rightarrow d$

$2 \rightarrow b$

$3 \rightarrow a$

Composition of functions

Example 2. Let f and g be two functions from $\mathbb{Z}$ to $\mathbb{Z}$, where $f(x) = 2x$ and $g(x) = x^2$

$f \circ g: \mathbb{Z} \to \mathbb{Z}$

$(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2$

$g \circ f: \mathbb{Z} \to \mathbb{Z}$

$(g \circ f)(x) = ?$

Composition of functions

Example 2. Let f and g be two functions from $\mathbb{Z}$ to $\mathbb{Z}$, where $f(x) = 2x$ and $g(x) = x^2$

$f \circ g: \mathbb{Z} \to \mathbb{Z}$

$(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2$

$g \circ f: \mathbb{Z} \to \mathbb{Z}$

$(g \circ f)(x) = g(f(x)) = g(2x) = 4x^2$

Note that the order of the function composition matters

Floor and ceiling functions

Definitions:

- The **floor function** assigns a real number x the largest integer that is less than or equal to x. The floor function is denoted by $\lfloor x \rfloor$.
- The **ceiling function** assigns to the real number x the smallest integer that is greater than or equal to x. The ceiling function is denoted by $\lceil x \rceil$.

Other important functions:

- Factorials: $n! = n(n-1)$ such that $1! = 1$

Remark. For all real numbers x we have

$$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$$

Floor and ceiling functions

Example 1. For all real numbers x we have

$$\left\lfloor \frac{1}{2} \right\rfloor = 0,\ \left\lceil \frac{1}{2} \right\rceil = 1,\ \left\lfloor -\frac{1}{2} \right\rfloor = -1,\ \left\lceil -\frac{1}{2} \right\rceil = 0,\ \lfloor 3.1 \rfloor = 3,\ \lceil 3.1 \rceil = 4,$$
$$\lfloor 7 \rfloor = 7,\ \lceil 7 \rceil = 7$$

Example 2. Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?

Solution. To determine the number of bytes needed, we determine the smallest integer that is at least as large as the quotient when 100 is divided by 8, the number of bits in a byte. Consequently, $\lceil 100/8 \rceil = \lceil 12.5 \rceil = 13$ bytes are required.

Floor and ceiling functions

Example 3. In asynchronous transfer mode (ATM) (a communications protocol used on backbone networks), data are organized into cells of 53 bytes. How many ATM cells can be transmitted in 1 minute over a connection that transmits data at the rate of 500 kilobits per second?
Solution. In 1 minute, this connection can transmit $500{,}000 \cdot 60 = 30{,}000{,}000$ bits. Each ATM cell is 53 bytes long, which means that it is $53 \cdot 8 = 424$ bits long. To determine the number of cells that can be transmitted in 1 minute, we determine the largest integer not exceeding the quotient when 30,000,000 is divided by 424. Consequently, $\lfloor 30{,}000{,}000/424 \rfloor = 70{,}754$ ATM cells can be transmitted in 1 minute over a 500 kilobit per second connection.

Floor and ceiling functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x+1$
(2)	$x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

Exercises

1. Which of the following function f: R → R is NOT one-to-one?
A. $f(x) = 3x + 21$
B. $f(x) = \lfloor 2x \rfloor$
C. $f(x) = 2x^3 + 5$
2. Let f, g: N → N be two functions defined respectively by
$f(n) = 2n$ and $g(n) = 3n - 1$
The value of the composition $g \circ f$ at n = 2 is
A. 5 B. 10 C. 11 D. 4 E. 20

Exercises

3. Why is f not a function from R to R if
a) $f(x) = 1/x$?
b) $f(x) = \sqrt{x}$?
c) $f(x) = \pm\sqrt{(x^2+1)}$?

4. Determine whether f is a function from **Z** to **R** if
a) $f(n) = \pm n$.
b) $f(n) = \sqrt{n^2+1}$.
c) $f(n) = 1/(n^2-4)$.

Exercises

5. Determine whether each of these functions from {a, b, c, d} to itself is one-to-one.

a) f (a) = b, f (b) = a, f (c) = c, f (d) = d

b) f (a) = b, f (b) = b, f (c) = d, f (d) = c

c) f (a) = d, f (b) = b, f (c) = c, f (d) = d

6. Which functions in Exercise 5 are onto?

7. Determine whether each of these functions is a bijection from R to R.

a) $f(x) = -3x + 4$

b) $f(x) = -3x^2 + 7$

c) $f(x) = (x + 1)/(x + 2)$

d) $f(x) = x^5 + 1$

Exercises

8. Find these values.

a) $\lfloor 1.1 \rfloor$	b) $\lceil 1.1 \rceil$
c) $\lfloor -0.1 \rfloor$	d) $\lceil -0.1 \rceil$
e) $\lceil 2.99 \rceil$	f) $\lceil -2.99 \rceil$
g) $\lfloor 1/2 + \lceil 1/2 \rceil \rfloor$	h) $\lceil \lfloor 1/2 \rfloor + \lceil 1/2 \rceil + 1/2 \rceil$

9. Find these values.

a) $\lceil 3/4 \rceil$	e) $\lceil 3 \rceil$
b) $\lceil -3/4 \rceil$	f) $\lfloor -1 \rfloor$
c) $\lfloor 7/8 \rfloor$	g) $\lfloor 1/2 + \lceil 3/2 \rceil \rfloor$
d) $\lfloor -7/8 \rfloor$	h) $\lfloor 1/2 \cdot \lfloor 5/2 \rfloor \rfloor$

Exercises

10. Let S = {−1, 0, 2, 4, 7}. Find f(S) if

a) $f(x) = 1$.

b) $f(x) = 2x + 1$.

c) $f(x) = \lceil x/5 \rceil$.

d) $f(x) = \lfloor (x^2 + 1)/3 \rfloor$.

11. Let $f(x) = \lfloor x^2/3 \rfloor$. Find f(S) if

a) S = {−2, −1, 0, 1, 2, 3}.

b) S = {0, 1, 2, 3, 4, 5}.

12. Let f be the function from R to R defined by $f(x) = x^2$. Find

a) $f^{-1}(\{1\})$.

b) $f^{-1}(\{x \mid 0 < x < 1\})$.

c) $f^{-1}(\{x \mid x > 4\})$

13. Determine whether the function $f : Z \times Z \to Z$ is onto if

a) $f(m, n) = m + n$.

b) $f(m, n) = m^2 + n^2$.

c) $f(m, n) = m$

Exercises

14. How many bytes are required to encode n bits of data where n equals

a) 4? b) 10? c) 500? d) 3000?

15. Data are transmitted over a particular Ethernet network in blocks of 1500 octets (blocks of 8 bits). How many blocks are required to transmit the following amounts of data over this Ethernet network? (Note that a byte is a synonym for an octet, a kilobyte is 1000 bytes, and a megabyte is 1,000,000 bytes.)

a) 150 kilobytes of data

b) 1.544 megabytes of data

Exercises

16. Let f and g be functions from {1, 2, 3, 4} to {a, b, c, d} and from {a, b, c, d} to {1, 2, 3, 4}, respectively, with f (1) = d, f (2) = c, f (3) = a, and f (4) = b, and g(a) = 2, g(b) = 1, g(c) = 3, and g(d) = 2.

a) Is f one-to-one? Is g one-to-one?

b) Is f onto? Is g onto?

c) Does either f or g have an inverse? If so, find this inverse.

Exercises

Let f be floor function and g be ceiling function.
Which of the following is true ?

Select one:

- a. f(-3.1)= - 3
- b. g(-4.5) = - 4
- c. g(7) = 8
- d. f(5.3) = 6

Exercises

Study relations in the set of real numbers R:

(i) f: $R \to R$; $f(x)= (x+1)/(x^2 + 3)$

(ii) f:$R \to R$; $f(x) = x/(2x^2 - 6x - 1)$

Select correct statement(s)

Select one:

- a. (i) is not a function, (ii) is not a function
- b. (i) is a function, (ii) is a function
- c. (i) is not a function, (ii) is a function
- d. (i) is a function, (ii) is not a function

Exercises

If f: $Z \to N$; $f(x) = (2 - x)^2$.
Which of the following statements is true?

(i) f is one-to-one

(ii) f is onto

Select one:

- a. (i)
- b. Both
- c. (ii)
- d. None

SEQUENCES

2.4- Sequences

Definition: A **sequence is a function** from a subset of the set of integers (typically the set {0,1,2,...} or the set {1,2,3,...} to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.

Notation: $\{a_n\}$ is used to represent the sequence (note {} is the same notation used for sets, so be careful). $\{a_n\}$ represents the ordered list $a_1, a_2, a_3, \ldots$.

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \ldots \end{array}$$

$\{a_n\}$

2.4- Sequences

Examples:

- (1) $a_n = n^2$, where $n = 1,2,3\ldots$
  - What are the elements of the sequence?

    1, 4, 9, 16, 25, ...
- (2) $a_n = (-1)^n$, where n=0,1,2,3,...
  - Elements of the sequence?

    1, -1, 1, -1, 1, ...
- 3) $a_n = 2^n$, where n=0,1,2,3,...
  - Elements of the sequence?

    1, 2, 4, 8, 16, 32, ...

Arithmetic progression

Definition: An **arithmetic progression** is a sequence of the form

$$a, a + d, a + 2d, \ldots, a + nd$$

where a is the *initial term* and d is *common difference*, such that both belong to R.

Remark. An arithmetic progression is a discrete analogue of the linear function f (x) = dx + a.

Example:

- $s_n = -1 + 4n$ for $n = 0, 1, 2, 3, \ldots$
- members: $-1, 3, 7, 11, \ldots$

Geometric progression

Definition: An **geometric progression** is a sequence of the form

$$a, ar, ar^2, \ldots, ar^k$$

where a is the *initial term* and r is *common ratio*, such that both belong to R.

Remark. An geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$.

Example:

- $a_n = (1/2)^n$ for n = 0, 1, 2, 3, …
- members: 1, 1/2, 1/4, 1/8, …

Sequences

- Given a sequence finding a rule for generating the sequence is not always straightforward.

Example:

- Assume the sequence: 1, 3, 5, 7, 9, …
- What is the formula for the sequence?
- Each term is obtained by adding 2 to the previous term.

1, 1 + 2 = 3, 3 + 2 = 5, 5 + 2 = 7

- It suggests an arithmetic progression: a + nd

with a = 1 and d = 2

$$a_n = 1 + 2n$$

Sequences

- Given a sequence finding a rule for generating the sequence is not always straightforward.

Example:

- Assume the sequence: 1, 1/3, 1/9, 1/27, …
- What is the formula for the sequence?
- The denominators are powers of 3.

1, 1/3 = 1/3, 1/(3 * 3) = 1/9, 1/(3*3*3) = 1/27

- This suggests a geometric progression: ar^k

with a = 1 and r = 1/3

$$a_n = (1/3)^n$$

Recursively defined sequences

The n-th element of the sequence $\{a_n\}$ is defined recursively in terms of the previous elements of the sequence and the initial elements of the sequence.

Example.

- $a_n = a_{n-1} + 2$ assuming $a_0 = 1$;
- $a_0 = 1$;
- $a_1 = 3$;
- $a_2 = 5$;
- $a_3 = 7$;
- Can you write a_n non-recursively using n?
- $a_n = 1 + 2n$

Fibonacci sequence

Recursively defined sequence, where $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for n = 2, 3, …

- $f_2 = 1$
- $f_3 = 2$
- $f_4 = 3$
- $f_5 = 5$

Some Useful Sequences

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100,… .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000,… .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000,… .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024,… .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049,… .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800,… .

SUMMATIONS

Summations

Summation of the terms of a sequence:

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \dots + a_n$$

The variable j is referred to as the index of summation.

- m is the *lower limit* and
- n is the *upper limit* of the summation.

Summations

Example:

- **1) Sum the first 7 terms of {n²} where n=1,2,3, … .**
-
$$\sum_{j=1}^{7} a_j = \sum_{j=1}^{7} j^2 = 1 + 4 + 16 + 25 + 36 + 49 = 140$$

- **2) What is the value of**
$$\sum_{k=4}^{8} a_j = \sum_{k=4}^{8} (-1)^j = 1 + (-1) + 1 + (-1) + 1 = 1$$

Arithmetic series

Definition: The sum of the terms of the arithmetic progression a, a+d,a+2d, …, a+nd is called an **arithmetic series**.

Theorem: The sum of the terms of the arithmetic progression a, a+d,a+2d, …, a+nd is

$$S = \sum_{j=1}^{n}(a + jd) = na + d\sum_{j=1}^{n} j = na + d\frac{n(n+1)}{2}$$

- **Why?**

Arithmetic series

Theorem: The sum of the terms of the arithmetic progression a, a+d,a+2d, …, a+nd is

$$S = \sum_{j=1}^{n}(a + jd) = na + d\sum_{j=1}^{n} j = na + d\frac{n(n+1)}{2}$$

Proof:

$$S = \sum_{j=1}^{n}(a + jd) = \sum_{j=1}^{n} a + \sum_{j=1}^{n} jd = na + d\sum_{j=1}^{n} j$$

$$\sum_{j=1}^{n} j = 1+2+3+4+\dots+(n-2)+(n-1)+n$$

Arithmetic series

Theorem: The sum of the terms of the arithmetic progression a, a+d,a+2d, …, a+nd is

$$S = \sum_{j=1}^{n}(a + jd) = na + d\sum_{j=1}^{n} j = na + d\frac{n(n+1)}{2}$$

Proof:

$$S = \sum_{j=1}^{n}(a + jd) = \sum_{j=1}^{n} a + \sum_{j=1}^{n} jd = na + d\sum_{j=1}^{n} j$$

$$\sum_{j=1}^{n} j = 1+2+3+4+\dots+(n-2)+(n-1)+n$$

$$\underbrace{n+1 \qquad n+1 \quad \dots \quad n+1}_{\frac{n}{2}*(n+1)}$$

Arithmetic series

Example:

$$S=\sum_{j=1}^{5}(2+j3)=$$
$$=\sum_{j=1}^{5}2+\sum_{j=1}^{5}j3=$$
$$=2\sum_{j=1}^{5}1+3\sum_{j=1}^{5}j=$$
$$=2*5+3\sum_{j=1}^{5}j=$$
$$=10+3\frac{(5+1)}{2}*5=$$
$$=10+45=55$$

Arithmetic series

Example 2:

$$S=\sum_{j=3}^{5}(2+j3)=$$
$$=\left[\sum_{j=1}^{5}(2+j3)\right]-\left[\sum_{j=1}^{2}(2+j3)\right] \Leftarrow \textbf{Trick}$$
$$=\left[2\sum_{j=1}^{5}1+3\sum_{j=1}^{5}j\right]-\left[2\sum_{j=1}^{2}1+3\sum_{j=1}^{2}j\right]$$
$$=55-13=42$$

Double summations

Example:

$$S=\sum_{i=1}^{4}\sum_{j=1}^{2}(2i-j)=$$
$$=\sum_{i=1}^{4}\left[\sum_{j=1}^{2}2i-\sum_{j=1}^{2}j\right]=$$
$$=\sum_{i=1}^{4}\left[2i\sum_{j=1}^{2}1-\sum_{j=1}^{2}j\right]=$$
$$=\sum_{i=1}^{4}\left[2i*2-\sum_{j=1}^{2}j\right]=$$
$$=\sum_{i=1}^{4}[2i*2-3]=$$
$$=\sum_{i=1}^{4}4i-\sum_{i=1}^{4}3=$$
$$=4\sum_{i=1}^{4}i-3\sum_{i=1}^{4}1= \quad 4*10-3*4=28$$

Geometric series

Definition: The sum of the terms of a geometric progression a, ar, ar^2, ..., ar^k is called **a geometric series**.

Theorem: The sum of the terms of a geometric progression a, ar, ar^2, ..., ar^n is

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1. \end{cases}$$

Geometric series

Theorem: The sum of the terms of a geometric progression a, ar, ar^2, ..., ar^n is $S = \sum_{j=0}^{n}(ar^j) = a\sum_{j=0}^{n} r^j = a\left[\frac{r^{n+1}-1}{r-1}\right]$

Proof:

$$S = \sum_{j=0}^{n} ar^j = a + ar + ar^2 + ar^3 + \ldots + ar^n$$

- multiply S by r

$$rS = r\sum_{j=0}^{n} ar^j = ar + ar^2 + ar^3 + \ldots + ar^{n+1}$$

- Substract $rS - S = \left[ar + ar^2 + ar^3 + \ldots + ar^{n+1}\right] - \left[a + ar + ar^2 .. + ar^n\right]$

$$= ar^{n+1} - a$$

$$\Rightarrow \quad S = \frac{ar^{n+1}-a}{r-1} = a\left[\frac{r^{n+1}-1}{r-1}\right]$$

Geometric series

Example:

$$S = \sum_{j=0}^{3} 2(5)^j =$$

General formula:

$$S = \sum_{j=0}^{n}(ar^j) = a\sum_{j=0}^{n} r^j = a\left[\frac{r^{n+1}-1}{r-1}\right]$$

$$S = \sum_{j=0}^{3} 2(5)^j = 2 * \frac{5^4-1}{5-1} =$$

$$= 2 * \frac{625-1}{4} = 2 * \frac{624}{4} = 2 * 156 = 312$$

Infinite geometric series

- **Infinite geometric series can be computed in the closed form for x<1**
- **How?**

$$\sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} \sum_{n=0}^{k} x^n = \lim_{k\to\infty} \frac{x^{k+1}-1}{x-1} = -\frac{1}{x-1} = \frac{1}{1-x}$$

- **Thus:**

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Some Useful Summation Formulae

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \|x\| < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty}, kx^{k-1}, \|x\| < 1$	$\frac{1}{(1-x)^2}$

Exercises

1. Find these terms of the sequence $\{a_n\}$, where $a_n = 2 \cdot (-3)n + 5n$.

a) a_0 b) a_1 c) a_4 d) a_5

2. What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where an equals

a) $(-2)^n$? b) 3?

c) $7 + 4^n$? d) $2^n + (-2)^n$?

3. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = 6a_{n-1}$, $a_0 = 2$

b) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$

Exercises

4. What are the values of these sums?

a) $\sum_{k=1}^{5}(k+1)$ **b)** $\sum_{j=0}^{4}(-2)^j$

c) $\sum_{i=1}^{10} 3$ **d)** $\sum_{j=0}^{8}(2^{j+1} - 2^j)$

5. What are the values of these sums, where S = {1, 3, 5, 7}?

a) $\sum_{j \in S} j$ **b)** $\sum_{j \in S} j^2$

c) $\sum_{j \in S} (1/j)$ **d)** $\sum_{j \in S} 1$

Exercises

6. Compute each of these double sums?

a) $\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j)$ **b)** $\sum_{i=0}^{2}\sum_{j=0}^{3}(2i+3j)$

c) $\sum_{i=1}^{3}\sum_{j=0}^{2} i$ **d)** $\sum_{i=0}^{2}\sum_{j=1}^{3} ij$

Exercises

Let $a_n = - a_{n-2}$ for all n >1. If $a_0 = 3$ and $a_1 = 5$, find a_7.

Select one:

- a. 3
- b. 7
- c. -5
- d. -3

Exercises

Suppose a_n is defined recursively by: $a_0=3$, $a_{n+1}=3.a_n$, $n>0$. What is a_n ?

Select one:

- a. $a_n=3^n$
- b. $a_n=3^{n+1}$
- c. $a_n=3n$
- d. $a_n=3n+3$

Exercises

Find $f(2)$ and $f(3)$ if
$f(n) = f(n - 1)\times f(n - 2) + 1$, and $f(0) = 1$, $f(1) = 4$

Select one:

- a. f(2) = 36, f(3) = 60
- b. f(2) = 30, f(3) = 66
- c. f(2) = 5, f(3) = 21
- d. f(2) = 15, f(3) = 20

Exercises

Study the following sequences:

$a_n = 3n -2$, n = 1, 2, 3, ...

$b_n = b_{n-1} + 3$ for n>1 and $b_1 = 1$

Select true statements.

Select one or more:

- a. $b_3 = 7$
- b. $b_3 = 9$
- c. $a_n = b_n$ for all n > 0
- d. We can't compute b_n for all n > 0

CARDINALITY OF SETS

Cardinality

Recall: The cardinality of a finite set is defined by the number of elements in the set.

Definition: The sets A and B have **the same cardinality** if there is a one-to-one correspondence between elements in A and B. In other words if there is a bijection from A to B. Recall bijection is one-to-one and onto.

Example: Assume $A = \{a,b,c\}$ and $B = \{\alpha,\beta,\gamma\}$ and function f defined as:

- $a \to \alpha$
- $b \to \beta$
- $c \to \gamma$

F defines a bijection. Therefore A and B have the same cardinality, i.e. $|A| = |B| = 3$.

Cardinality

Definition: A set that is either finite or has the same cardinality as the set of positive integers Z^+ is called **countable.** A set that is not countable is called **uncountable.**

Why these are called countable?

- **The elements of the set can be enumerated and listed.**

Countable sets

Example:

- Assume $A = \{0, 2, 4, 6, \ldots\}$ set of even numbers. Is it countable?
- Using the definition: Is there a bijective function $f: Z^+ \to A$

$Z+ = \{1, 2, 3, 4, \ldots\}$

- Define a function f: $x \to 2x - 2$ (an arithmetic progression)
  - $1 \to 2(1)-2 = 0$
  - $2 \to 2(2)-2 = 2$
  - $3 \to 2(3)-2 = 4$ ...
- one-to-one (why?) $2x-2 = 2y-2 \Rightarrow 2x = 2y \Rightarrow x = y$.
- onto (why?) $\forall a \in A$, $(a+2)/2$ is the pre-image in Z^+.
- Therefore $|A| = |Z^+|$.

Countable sets

Theorem:

- **The set of integers Z is countable.**

Solution:

Can list a sequence:

$0, 1, -1, 2, -2, 3, -3, \ldots\ldots\ldots$

Or can define a bijection from Z^+ to **Z**:

- When n is even: $f(n) = n/2$
- When n is odd: $f(n) = -(n-1)/2$

Countable sets

Definition:

- A *rational number* can be expressed as the ratio of two integers p and q such that $q \neq 0$.
  - ¾ is a rational number
  - $\sqrt{2}$ is not a rational number.

Theorem:

- **The positive rational numbers are countable.**

Solution:

The positive rational numbers are countable since they can be arranged in a sequence:

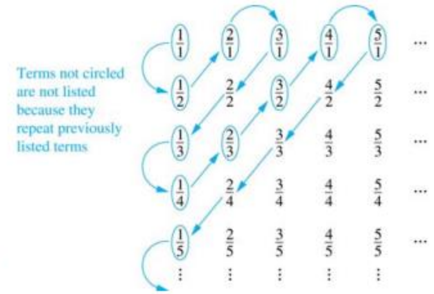
$$r_1, r_2, r_3, \ldots$$

Countable sets

Theorem:

- **The positive rational numbers are countable.**

First row $q = 1$.
Second row $q = 2$.
etc.



Constructing the List

First list p/q with $p + q = 2$.
Next list p/q with $p + q = 3$

And so on.

Uncountable set

Theorem: The set of real numbers (R) is an uncountable set.

Proof by a contradiction.

1) Assume that the real numbers are countable.
2) Then every subset of the reals is countable, in particular, the interval from 0 to 1 is countable. This implies the elements of this set can be listed say r1, r2, r3, ... where

- $r1 = 0.d_{11}d_{12}d_{13}d_{14} \ldots$
- $r2 = 0.d_{21}d_{22}d_{23}d_{24} \ldots$
- $r3 = 0.d_{31}d_{32}d_{33}d_{34} \ldots\ldots$
- where the $d_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}$.

Uncountable set

Proof cont.

3) Want to show that not all reals in the interval between 0 and 1 are in this list.

- Form a new number called
  - $r = 0.d_1d_2d_3d_4 \ldots$ where

$$d_i = \begin{cases} 2, & \text{if } d_{ii} \neq 2 \\ 3 & \text{if } d_{ii} = 2 \end{cases}$$

- Example: suppose r1 = 0.75243... d1 = 2
  r2 = 0.524310... d2 = 3
  r3 = 0.131257... d3 = 2
  r4 = 0.9363633... d4 = 2
  ... ...
  rt = 0.23222222... dt = 3

Uncountable set

- $r = 0.d_1d_2d_3d_4 \ldots$ where

$$d_i = \begin{cases} 2, \text{ if } d_{ii} \neq 2 \\ 3 \text{ if } d_{ii} = 2 \end{cases}$$

- **Claim:** r is different than each member in the list.
- Is each expansion unique? Yes, if we exclude an infinite string of 9s.
-
- Example: $.0285\bar{0} = .0284\bar{9}$
- Therefore r and r_i differ in the i-th decimal place for all i.

Examples p.159, 160

sets	countable	uncountable	cardinality
{a, b, ..., z}, {x| x^5 -3x^2 – 11 = 0}, ...	✓	×	$< \infty$
{0, 2, 4, ..., }	✓	×	$\aleph_0$
N, Z^+, Z, Q, Z×Z, ...	✓	×	$\aleph_0$
{x| 0 < x < 1}, R,...	×	✓	$2^{\aleph_0}$

Summary

- Sets
- Set operations
- Functions
- Sequences
- Summations

Thanks