DISCRETE MATHEMATICS AND ITS APPLICATIONS

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Course introduction

Course name: Discrete Mathematics (MAD101)

Textbook: Discrete Mathematics and Its applications, Kenneth H.Rosen, McGraw Hill

Topics:

1. Logic and Proofs
2. Sets, Functions, Sequences, and Sums
3. Algorithms and the Integers
4. Induction and Recursion
5. Counting
6. Advanced counting techniques
7. Relations
8. Graphs
9. Trees

Chapter 1. THE FUNCTIONS: LOGIC AND PROOFS

Topics

1.1 Propositional Logic
1.2 Propositional Equivalences
1.3 Predicates and Quantifiers
1.4 Nested Quantifiers
1.5 Rules of Inference

1.1 Propositional Logic

Proposition

A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is either **true or false**, but **NOT both**.

Example.
– Pitt is located in the Oakland section of Pittsburgh.
• (T)
– 5 + 2 = 8.
• (F)
– It is raining today.
• (either T or F)

Proposition

Example.
– How are you?
• a question is not a proposition
– x + 5 = 3
• since x is not specified, neither true nor false
– 2 is a prime number.
• (T)
– She is very talented.
• since she is not specified, neither true nor false
– There are other life forms on other planets in the universe.
• either T or F

Proposition

Example. Which of the following sentences are propositions?
a) Great!
b) Tokyo is the capital of Japan
c) What time is it?
d) It is now 3pm
e) 1+7=9
f) x+1=3
g) x+y=z

Truth table

- **Remark:** A proposition can be true (**True/T/1**) or false (**False/F/0**)
- **Truth table:** displays the relationship between truth values (T or F) of different propositions.

p
True/ T / 1
False / F / 0

Rows: all possible values of elementary propositions:

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Exercise

Which of these sentences are propositions? What are the truth values of those that are propositions?
a) Boston is the capital of Massachusetts.
b) Miami is the capital of Florida.
c) 2 + 3 = 5.
d) 5 + 7 = 10.
e) x + 2 = 11.
f) Answer this question.

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Composite statements

- New propositions, called **compound propositions**, are formed from existing propositions using **logical operators/connectives**.
- Logical operators:
  - Negation (¬)
  - Conjunction (∧)
  - Disjunction (∨)
  - Exclusive or (⊕)
  - Implication (→)
  - Biconditional (↔)

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Negation

Let p be a proposition.
The **negation** of p is the statement “It is not the case that p”.
Notation: **¬p** or $\bar{p}$

p	¬p
1	0
0	1

p	$\neg p$
T	F
F	T

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Negation

Example. The negation of the proposition "Lisa's PC runs Linux" is "Lisa's PC does not run Linux".

Exercise. What is the negation of each these propositions?

a) Linda is younger than Sanjay.

b) There is no pollution in New Jersey.

c) 2 + 1 = 3

d) Mei has an MP3 player.

Conjunction

Let p and q be propositions.
The **conjunction** of p and q is the proposition "p **and** q"
Notation: $p \wedge q$

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Conjunction

Example.

p: "Rebecca's PC has more than 16 GB free hard disk space"

q: "The processor in Rebecca's PC runs faster than 1 GHz"

$p \wedge q$: "Rebecca's PC has more than 16 GB free hard disk space, and the processor in Rebecca's PC runs faster than 1 GHz". It can be expressed simply as "Rebecca's PC has more than 16 GB free hard disk space, and its processor runs faster than 1 GHz."

Disjunction/Inclusive Or

Let p and q be propositions.
The **disjunction** of p and q is the proposition "p **or** q"
Notation: $p \vee q$

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Disjunction

Example.

p: “Rebecca’s PC has more than 16 GB free hard disk space”

q: “The processor in Rebecca’s PC runs faster than 1 GHz”

p ∨ q: “Rebecca’s PC has at least 16 GB free hard disk space, or the processor in Rebecca’s PC runs faster than 1 GHz”.

Exclusive Or (XOR)

Let p and q be propositions.
The **exclusive or** of p and q is the proposition that is **true** when **exactly one of p and q is true** and is false otherwise.
Notation: **p ⊕ q** or ***p* XOR *q***

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Exclusive Or (XOR)

Example.

p: A student can have a salad with dinner.

q: A student can have soup with dinner.

p ⊕ q: A student can have soup or a salad, but not both, with dinner.

Conditional statements/Implication

Let p and q be propositions.
The **conditional statement** of p and q is the proposition “**if** p, **then** q”.
Notation: $p \rightarrow q$

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Conditional statements

- Implication: $p \to q$

p: hypothesis/antecedent/premise
q: conclusion/consequence

- There are several ways to express

"if p, then q"	"p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q whenever p"
"q when p"	"q is necessary for p"
"a necessary condition for p is q"	"q follows from p"
"q unless $\neg p$"	"q provided that p"

Conditional statements

Implication: p → q can be associated with 3 variants:

- **Converse**: q → p
- **Contrapositive**: ¬q → ¬p
- **Inverse**: ¬p → ¬q

Conditional statements

Example.
p: Maria learns discrete mathematics.
q: Maria will find a good job.
p → q: If Maria learns discrete mathematics, then she will find a good job./Maria will find a good job when she learns discrete mathematics./For Maria to get a good job, it is sufficient for her to learn discrete mathematics./Maria will find a good job unless she does not learn discrete mathematics.

Conditional statements

Example.
What is the value of the variable x after the statement

"if 2 + 2 = 4 then x := x + 1"

if x = 0 before this statement is encountered? (The symbol := stands for assignment. The statement x := x + 1 means the assignment of the value of x + 1 to x.)

Conditional statements

Example. Find the contrapositive, the converse, and the inverse of the conditional statement "The home team wins whenever it is raining."
Hint.
p: It is raining
q: The home team wins

Biconditionals/Bi-implications

Let p and q be propositions.
The **biconditional statement** of p and q is the proposition "**p if and only if q**".
Notation: **p ↔ q**

p	q	p ↔ q
0	0	1
0	1	0
1	0	0
1	1	1

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditionals

Some other common ways to express:

"p is necessary and sufficient for q"
"if p then q, and conversely"
"p iff q." "p exactly when q."

Example.
p: "You can take the flight" and q: "You buy a ticket"
p ↔ q: "You can take the flight if and only if you buy a ticket"

Precedence of Logical Operators

Operator	Precedence
()	1 (from inner to outer)
¬	2
∧	3
∨	4
→	5
↔	6

Example. ¬ p ∨ q ∧ r means (¬ p) ∨ (q ∧ r)

Constructing the truth table

Example: Construct a truth table for (p → q) ∧ (¬p ↔ q)

• Simpler if we decompose the sentence to elementary and intermediate propositions

p	q	¬p	p → q	¬p ↔ q	(p → q) ∧ (¬p ↔ q)
T	T				
T	F				
F	T				
F	F				

Constructing the truth table

Example: Construct a truth table for (p → q) ∧ (¬p ↔ q)

p	q			q	(p → q) ∧ (¬p ↔ q)
T	T				
T	F				
F	T				
F	F				

Rows: all possible combinations of values for elementary propositions: **2^n values**

Constructing the truth table

Example: Construct a tru ↔ q)

Auxiliary compound propositions and their values

p	q	¬p	p → q	¬p ↔ q	(p → q) ∧ (¬p ↔ q)
T	T				
T	F				
F	T				
F	F				

Typically the target (unknown) compound proposition and its values

Constructing the truth table

Example: Construct a truth table for (p → q) ∧ (¬p ↔ q)

p	q	¬p	p → q	¬p ↔ q	(p → q) ∧ (¬p ↔ q)
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Logic and Bit Operations

- Computers represent information using bits.
- A **bit** is a symbol with **two** possible **values**, namely, **0** and **1**.
- A bit can be used to represent a truth value, that is 1 represents true/T and 0 represents false/F.
- A variable is called a **Boolean variable** if **its value** is either **true or false**. Consequently, a Boolean variable can be represented using a bit.

Logic and Bit operations

- Computer bit operations correspond to the logical connectives. We will also use the notation **OR, AND,** and **XOR** for the operators **∨, ∧,** and ⊕ in various programming languages.
- **A bit string** is a sequence of zero or more bits. The **length** of this string is **the number of bits** in the string.

Example. 101010011 is a bit string of length nine.

Logic and Bit operations

- We can extend bit operations to bit strings.
- We define the **bitwise OR**, **bitwise AND**, and **bitwise XOR** of two strings of the **same length** to be the strings that have as their bits the **OR**, **AND**, and **XOR** of the corresponding bits in the two strings, respectively. We use the symbols ∨, ∧, and ⊕ to represent the bitwise OR, bitwise AND, and bitwise XOR operations, respectively.

Example. Find the bitwise OR, bitwise AND, and bitwise XOR of the bit strings 01 1011 0110 and 11 0001 1101.

Exercises

1. Which of these sentences are propositions? What are the truth values of those that are propositions?

a) Boston is the capital of Massachusetts.

b) Miami is the capital of Florida.

c) 2 + 3 = 5.

d) 5 + 7 = 10.

e) x + 2 = 11.

f) Answer this question

Exercises

2. What is the negation of each of these propositions?

a) Steve has more than 100 GB free disk space on his laptop.

b) Zach blocks e-mails and texts from Jennifer.

c) $7 \cdot 11 \cdot 13 = 999$.

d) Diane rode her bicycle 100 miles on Sunday.

Exercises

3. Let p and q be the propositions:

p: I bought a lottery ticket this week.

q: I won the million dollar jackpot.

Express each of these propositions as an English sentence.

a) $\neg p$ b) $p \vee q$ c) $p \to q$

d) $p \wedge q$ e) $p \leftrightarrow q$ f) $\neg p \to \neg q$

g) $\neg p \wedge \neg q$ h) $\neg p \vee (p \wedge q)$

Exercises

4. Let p and q be the propositions "Swimming at the New Jersey shore is allowed" and "Sharks have been spotted near the shore," respectively. Express each of these compound propositions as an English sentence.

a) $\neg q$ b) $p \wedge q$ c) $\neg p \vee q$

d) $p \to \neg q$ e) $\neg q \to p$ f) $\neg p \to \neg q$

g) $p \leftrightarrow \neg q$ h) $\neg p \wedge (p \vee \neg q)$

Exercises

5. Let p and q be the propositions

p: It is below freezing.

q: It is snowing.

Write these propositions using p and q and logical connectives (including negations).

a) It is below freezing and snowing.

b) It is below freezing but not snowing.

c) It is not below freezing and it is not snowing.

d) It is either snowing or below freezing (or both).

e) If it is below freezing, it is also snowing.

f) Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.

g) That it is below freezing is necessary and sufficient for it to be snowing.

6. Let p, q, and r be the propositions

p: You get an A on the final exam.

r: You get an A in this class.

q: You do every exercise in this book.

Write these propositions using p, q, and r and logical connectives (including negations).

a) You get an A in this class, but you do not do every exercise in this book.

b) You get an A on the final, you do every exercise in this book, and you get an A in this class.

c) To get an A in this class, it is necessary for you to get an A on the final.

d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.

f) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

Exercises

7. Determine whether each of these conditional statements is true or false.

a) If 1 + 1 = 2, then 2 + 2 = 5.

b) If 1 + 1 = 3, then 2 + 2 = 4.

c) If 1 + 1 = 3, then 2 + 2 = 5.

d) If monkeys can fly, then 1 + 1 = 3.

Exercises

8. Determine whether these biconditionals are true or false.

a) 2 + 2 = 4 if and only if 1 + 1 = 2.

b) 1 + 1 = 2 if and only if 2 + 3 = 4.

c) 1 + 1 = 3 if and only if monkeys can fly.

d) 0 > 1 if and only if 2 > 1.

Exercises

9. For each of these sentences, determine whether an inclusive or, or an exclusive or, is intended. Explain your answer.

a) Coffee or tea comes with dinner.

b) A password must have at least three digits or be at least eight characters long.

c) The prerequisite for the course is a course in number theory or a course in cryptography.

d) You can pay using U.S. dollars or euros

Exercises

10. Write each of these statements in the form "if p, then q" in English.

a) It is necessary to wash the boss's car to get promoted.

b) Winds from the south imply a spring thaw.

c) A sufficient condition for the warranty to be good is that you bought the computer less than a year ago.

d) Willy gets caught whenever he cheats.

e) You can access the website only if you pay a subscription fee.

f) Getting elected follows from knowing the right people.

g) Carol gets seasick whenever she is on a boat.

Exercises

11. Write each of these statements in the form "p if and only if q" in English.

a) If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside.

b) For you to win the contest it is necessary and sufficient that you have the only winning ticket.

c) You get promoted only if you have connections, and you have connections only if you get promoted.

d) If you watch television your mind will decay, and conversely.

e) The trains run late on exactly those days when I take it.

Exercises

12. State the converse, contrapositive, and inverse of each of these conditional statements.

a) If it snows today, I will ski tomorrow.

b) I come to class whenever there is going to be a quiz.

c) A positive integer is a prime only if it has no divisors other than 1 and itself

Exercises

13. Let p = "You drive at more than 70km/h" and q = "You got a speeding ticket". Translate the sentence into a logical expression "If you do not drive at more than 70km/h then you will not get a speeding ticket".

A. $\neg p \wedge \neg q$ B. $\neg p \rightarrow \neg q$

C. $\neg q \rightarrow \neg p$ D. $p \rightarrow \neg q$

Exercises

14. Construct a truth table for each of these compound propositions.

a) $p \rightarrow \neg p$

b) $p \rightarrow (\neg q \vee r)$

c) $\neg p \rightarrow (q \rightarrow r)$

d) $(p \rightarrow q) \wedge (\neg p \rightarrow r)$

e) $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$

f) $(p \vee q) \rightarrow (p \oplus q)$

g) $(p \vee q) \oplus (p \wedge q)$

h) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$

Exercises

15. What is the value of x after each of these statements is encountered in a computer program, if x = 1 before the statement is reached?

a) if x + 2 = 3 then x := x + 1

b) if (x + 1 = 3) OR (2x + 2 = 3) then x := x + 1

c) if (2x + 3 = 5) AND (3x + 4 = 7) then x := x + 1

d) if (x + 1 = 2) XOR (x + 2 = 3) then x := x + 1

e) if x < 2 then x := x + 1

Exercises

16. Find the bitwise OR, bitwise AND, and bitwise XOR of each of these pairs of bit strings.

a) 101 1110, 010 0001

b) 1111 0000, 1010 1010

c) 00 0111 0001, 10 0100 1000

d) 11 1111 1111, 00 0000 0000

Exercises

17. Evaluate each of these expressions.

a) 1 1000 ∧ (0 1011 ∨ 1 1011)

b) (0 1111 ∧ 1 0101) ∨ 0 1000

c) (0 1010 ⊕ 1 1011) ⊕ 0 1000

d) (1 1011 ∨ 0 1010) ∧ (1 0001 ∨ 1 1011)

Applications of Propositional Logic

- **Translating sentences in natural language (such as English) into logical expressions** is an essential part of specifying both hardware and software systems. System and software engineers take requirements in natural language and produce precise and unambiguous specifications that can be used as the basis for system development.

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Applications of Propositional Logic

Example. How can this English sentence be translated into a logical expression?

"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Solution. Let q, r, and s represent "You can ride the roller coaster," "You are under 4 feet tall," and "You are older than 16 years old," respectively. Then the sentence can be translated to

$$(r \land \neg s) \rightarrow \neg q$$

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Applications of Propositional Logic

Example. A woman told her daughter, An, that she would eat ice cream or cakes if she did well in the final exam. However, An understand that she can eat both when she gets good results.

A mother should say that if you gets a good final results, you can choose ice cream or cakes, but not both.

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Applications of Propositional Logic

- Logical connectives are used extensively in **searches of large collections of information**, such as indexes of Web pages. Because these searches employ techniques from propositional logic, they are called **Boolean searches**.
- In Boolean searches, the connective AND is used to match records that contain both of two search terms, the connective OR is used to match one or both of two search terms, and the connective NOT (sometimes written as AND NOT) is used to exclude a particular search term.

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Applications of Propositional Logic

Example. using Boolean searching to find Web pages about universities in New Mexico, we can look for pages matching NEW AND MEXICO AND UNIVERSITIES. The results of this search will include those pages that contain the three words NEW, MEXICO, and UNIVERSITIES. This will include all of the pages of interest, together with others such as a page about new universities in Mexico. (Note that Google, and many other search engines, do require the use of "AND" because such search engines use all search terms by default.) Most search engines support the use of quotation marks to search for specific phrases. So, it may be more effective to search for pages matching "NEW MEXICO" AND UNIVERSITIES.

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Applications of Propositional Logic

- **Logic in programming**

Example.

```
int n, i, flag = 0;
printf("Enter a positive integer: ");
scanf("%d", &n);

// 0 and 1 are not prime numbers
// change flag to 1 for non-prime number
if (n == 0 || n == 1)
  flag = 1;

for (i = 2; i <= n / 2; ++i) {

  // if n is divisible by i, then n is not prime
  // change flag to 1 for non-prime number
  if (n % i == 0) {
    flag = 1;
    break;
  }
}
```

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1.2 Propositional Equivalences

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Tautology and Contradiction

- A **tautology (T)** is a compound proposition that is **always true**.
- A **contradiction (F)** is a compound proposition that is **always false**.
- A **contingency** is a compound proposition that is **neither a tautology nor a contradiction**

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

P	Q	¬Q	P ∧Q	(P ∧ Q) ∨ (¬Q)
T	T	F	T	T
T	F	T	F	T
F	T	F	F	F
F	F	T	F	T

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Logical Equivalences

- The compound propositions p and q are called **logically equivalent** if **$p \leftrightarrow q$ is a tautology**.
- Notation: $p \equiv q$

Remark.

- The symbol $\equiv$ is not a logical connective, and $p \equiv q$ is not a compound proposition but rather is the statement that $p \leftrightarrow q$ is a tautology. The symbol $\Leftrightarrow$ is sometimes used instead of $\equiv$ to denote logical equivalence.
- Two methods for proving logical equivalences: truth table and other logical equivalences.

Logical Equivalences

Example. Show that $p \to q$ and $\neg p \vee q$ are logically equivalent. (This is known as the conditional disjunction equivalence.)

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Logical Equivalences

Example. Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is the distributive law of disjunction over conjunction.

Logical Equivalences

Equivalence		Name
$p \wedge T \equiv p$	$p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$	$p \wedge F \equiv F$	Domination Laws
$p \vee p \equiv p$	$p \wedge p \equiv p$	Idempotent Laws
$\neg(\neg p) \equiv p$		Double Negation Laws
$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$	Commutative Laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$		Associative Laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$		Distributive Laws

Logical Equivalences

Equivalence		Name
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$		De Morgan Laws
$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$	Absorption Laws
$p \vee \neg p \equiv T$	$p \wedge \neg p \equiv F$	Negation Laws

Logical Equivalences

Equivalences	Equivalences
$p \to q \equiv \neg p \vee q$	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \to q \equiv \neg q \to \neg p$	$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \vee q \equiv \neg p \to q$	$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$p \wedge q \equiv \neg(p \to \neg q)$	$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$
$\neg(p \to q) \equiv p \wedge \neg q$	
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$	
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$	
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$	
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$	

Logical Equivalences

Example: Show $(p \wedge q) \to p$ is a tautology

In other words $((p \wedge q) \to p \Leftrightarrow T)$

Proof via truth table:

p	q	$p \wedge q$	$(p \wedge q) \to p$
T	T	T	**T**
T	F	F	**T**
F	T	F	**T**
F	F	F	**T**

Logical Equivalences

Example: Show $(p \wedge q) \to p$ is a tautology

In other words $((p \wedge q) \to p \Leftrightarrow T)$

- Proof: (we must show $(p \wedge q) \to p \Leftrightarrow T$)

$(p \wedge q) \to p \Leftrightarrow \neg(p \wedge q) \vee p$		Useful
	$\Leftrightarrow [\neg p \vee \neg q] \vee p$	DeMorgan
	$\Leftrightarrow [\neg q \vee \neg p] \vee p$	Commutative
	$\Leftrightarrow \neg q \vee [\neg p \vee p]$	Associative
	$\Leftrightarrow \neg q \vee [T]$	Useful
	$\Leftrightarrow T$	Domination

Logical Equivalences

Example. Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

Solution.

Method 1. Truth table

Method 2.

$\neg(p \rightarrow q)$	$\equiv \neg(\neg p \vee q)$	by the conditional-disjunction equivalence
	$\equiv \neg(\neg p) \wedge \neg q$	by the second De Morgan law
	$\equiv p \wedge \neg q$	by the double negation law

Logical Equivalences

Example. Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution.

Method 1. Truth table

Method 2.

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) \\
&\equiv T \vee T \\
&\equiv T
\end{aligned}
$$

Exercises

1. Use truth tables to verify these equivalences.

a) $p \wedge T \equiv p$ b) $p \vee F \equiv p$

c) $p \wedge F \equiv F$ d) $p \vee T \equiv T$

e) $p \vee p \equiv p$ f) $p \wedge p \equiv p$

2. Show that each of these conditional statements is a tautology by using truth tables.

a) $(p \wedge q) \rightarrow p$ b) $p \rightarrow (p \vee q)$

c) $\neg p \rightarrow (p \rightarrow q)$ d) $[\neg p \wedge (p \vee q)] \rightarrow q$

e) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

Exercises

3. Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent.
4. Show that $\neg(p \oplus q)$ and $p \leftrightarrow q$ are logically equivalent.
5. Show that $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology.
6. Show that $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ are not logically equivalent.
7. Show that $(p \rightarrow q) \rightarrow (r \rightarrow s)$ and $(p \rightarrow r) \rightarrow (q \rightarrow s)$ are not logically equivalent.

Exercises

8. Use De Morgan's laws to find the negation of each of the following statements.

a) Jan is rich and happy.

b) Carlos will bicycle or run tomorrow.

c) Mei walks or takes the bus to class.

d) Ibrahim is smart and hard working.

Exercises

9. Let p and q be propositions. Which statements are correct?

i. $p \vee \neg q$ is a tautology.

ii. $p \wedge \neg q$ is a tautology.

iii. $p \vee \neg q$ is a contradiction.

Iv. $p \wedge \neg q$ is a contradiction.

A. ii and iii

B. iii and iv

C. Iv

D. i and iv

E. None of the other choices

Exercises

10. How many rows appear in a truth table for the compound proposition $(p \wedge q \wedge \neg r) \vee (\neg p \leftrightarrow s) \vee (s \rightarrow q)$?

A. 32

B. 8

C. 64

D. 4

E. 16

1.3 Predicates and Quantifiers

Limitations of the propositional logic

Propositional logic: the world is described in terms of elementary propositions and their logical combinations

Elementary statements:

- **Typically refer to objects, their properties and relations. But these are not explicitly represented** in the propositional logic
  - **Example:**
    - "John is a UPitt student."

      John $\longrightarrow$ **a Upitt student**

      object $\longrightarrow$ a property
    - Objects and properties are hidden in the statement, it is not possible to reason about them

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Limitations of the propositional logic

(1) Statements that must be repeated for many objects

- In propositional logic these must be exhaustively enumerated

- **Example:**
  - If John is a CS UPitt graduate then John has passed cs441

  Translation:
  - John is a CS UPitt graduate $\rightarrow$ John has passed cs441

  Similar statements can be written for other Upitt graduates:
  - Ann is a CS Upitt graduate $\rightarrow$ Ann has passed cs441
  - Ken is a CS Upitt graduate $\rightarrow$ Ken has passed cs441
  - …
- **What is a more natural solution to express the above knowledge?**

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Limitations of the propositional logic

(1) Statements that must be repeated for many objects

- **Example:**
  - If John is a CS UPitt graduate then John has passed cs441

  Translation:
  - John is a CS UPitt graduate $\rightarrow$ John has passed cs441

  Similar statements can be written for other Upitt graduates:
  - Ann is a CS Upitt graduate $\rightarrow$ Ann has passed cs441
  - Ken is a CS Upitt graduate $\rightarrow$ Ken has passed cs441
  - …
- **Solution:** make statements with **variables**
  - If x is a CS Upitt graduate then x has passed cs441
  - x is a CS UPitt graduate $\rightarrow$ x has passed cs441

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Limitations of the propositional logic

(2) Statements that define the property of the group of objects

- **Example:**
  - All new cars must be registered.
  - Some of the CS graduates graduate with honors.

- **Solution:** make statements with **quantifiers**
  - **Universal quantifier** –the property is satisfied by all members of the group
  - **Existential quantifier** – at least one member of the group satisfy the property

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Predicates

A **propositional function** is a sentence that contains **one or more variables**.

A **predicate** refers to a **property** that the subject of the statement can have

Predicates

Example.

Symbol	Statement	Predicate	Propositions
P(x)	x > 5	> 5	P(6), P(0), P(-1.2), …
Q(x, y)	x – y = 3	= 3	Q(0, 0), Q(4, 1), …

Predicates

Example. Let Q(x, y) denote the statement "x = y + 3". What are the truth values of the propositions Q(1, 2) and Q(3, 0)?

Solution: To obtain Q(1, 2), set x = 1 and y = 2 in the statement Q(x, y). Hence, Q(1, 2) is the statement "1 = 2 + 3", which is false. The statement Q(3, 0) is the proposition "3 = 0 + 3", which is true.

Preconditions and Postconditions

- Predicates are also used to establish the correctness of computer programs, that is, to show that computer programs always produce the desired output when given valid input.
- The statements that describe **valid input** are known as **preconditions**.
- The conditions that the **output** should satisfy when the program has run are known as **postconditions**.

Preconditions and Postconditions

Example. Consider the following program, designed to interchange the values of two variables x and y.

temp := x

x := y

y := temp

The predicate P(x, y): "x = a and y = b," where a and b are the values of x and y before we run the program. Because we want to verify that the program swaps the values of x and y for all input values, for the postcondition we can use Q(x, y): "x = b and y = a".

To verify that the program always does what it is supposed to do, suppose that the precondition P(x, y) holds. That is, we suppose that the statement "x = a and y = b" is true. This means that x = a and y = b. The first step of the program, temp := x, assigns the value of x to the variable temp, so after this step we know that x = a, temp = a, and y = b. After the second step of the program, x := y, we know that x = b, temp = a, and y = b. Finally, after the third step, we know that x = b, temp = a, and y = a. Consequently, after this program is run, the postcondition Q(x, y) holds, that is, the statement "x = b and y = a" is true.

Quiz

Study the following computer code segment:

x:= 5

y:= 6

If (1+1=0) OR (2+2=1) then x:=x+1

If (1+1=2) XOR (1+2=3) then y=y+1

What are values of x and y after the codes execute?

Select one:

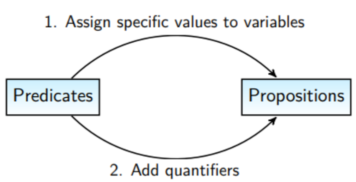
- a. 5; 7
- b. 6; 6
- c. 6; 8
- d. 5; 6

Predicates to Propositions

There are two methods to obtain propositions from predicates:

1. Assign specific values to variables

2. Add quantifiers (all, some, many, none, and few), called **quantification**



Quantifiers

Quantifiers are words that refer to quantities such as "all" or "some" and they tell for how many elements a given predicate is true.

- Introduced into logic by logicians Charles Sanders Pierce and Gottlob Frege during late 19th century
- Two types of quantifiers:
  1. Universal quantifier (∀)
  2. Existential quantifier (∃)

Universal quantifier (∀)

- Let P(x) be a propositional function and D be the domain of x. The **universal quanlification** of P(x) is a statement "P(x) for all values of x in the domain". Notation: ∀xP(x)
- Forms:
  - "P(x) is true for all values of x"
  - "For all x, P(x)"
  - "For each/every x, P(x)"
  - "Given any x, P(x)"
- It is true if P(x) is true for each x in D; It is false if P(x) is false for at least one x in D
- A **counterexample** to the universal statement is the value of x for which P(x) is false

Universal quantifier (∀)

Example.

Statements	Domain	Truth value
$\forall x \in D, x^2 \geq x$	D = {1, 2, 3}	True
$\forall x \in \mathbb{R}, x^2 \geq x$	$\mathbb{R}$	False (counterexample: x=0.1)

Existential quantifier (∃)

- Let P(x) be a propositional function and D be the domain of x. The **existential quanlification** of P(x) is a statement "There exists an element x in the domain such that P(x)". Notation: ∃xP(x)
- Forms:
  - "There exists/We can find an x such that P(x)"
  - "For some x, P(x)"
  - "There is at least one x such that P(x)"
  - "There is some x such that P(x)"
- It is true if P(x) is true for at least one x in D; It is false if P(x) is false for all x in D
- A **counterproof** to the existential statement is the proof to show that P(x) is true is for no x

Existential quantifier (∃)

Example.

Statements	Domain	Truth value
$\exists x \in D, x^2 \geq x$	D = {1, 2, 3}	True
$\exists x \in \mathbb{R}, x^2 \geq x$	$\mathbb{R}$	True
$\exists x \in \mathbb{Z}, x + 1 \leq x$	$\mathbb{Z}$	False

Uniqueness quantifier (∃!)

- Let P(x) be a propositional function and D be the domain of x. The **uniqueness quanlification** of P(x) is a statement "There exists a unique x such that P(x)". Notation: ∃!xP(x)
- Forms:
  - "There exists a unique x such that P(x)"
  - "There is exactly one x such that P(x)"
  - "There is one and only one x such that P(x)"
- It is true if P(x) is true for a unique number x in D; It is false if P(x) is false for at least x in D

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Existential quantifier (∃)

Example.

Statements	Domain	Truth value
∃!x ∈ D, x – 1 = 0	D = {1, 2, 3}	True

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Exercises.

1. Let P(x) denote the statement "x ≤ 4." What are these truth values?
a) P(0) b) P(4) c) P(6)
2. Let P(x) be the statement "The word x contains the letter a." What are these truth values?
a) P(orange) b) P(lemon) c) P(true) d) P(false)
3. Determine the truth value of each of these statements if the domain consists of all integers.
a) ∀n(n + 1 > n) b) ∃n(2n = 3n)
c) ∃n(n = −n) d) ∀n(3n ≤ 4n)

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Exercises.

4. A(x): "x = 1" B(x): "x > 2" C(x): "x < 2" D(x): "x < 5" E: "x > 5"

Universe of discourse is {1, 2, 3}. Which of the following proposition(s) is/are True or False?

a) ∀x(C(x) → A(x))

b) ∀x(C(x) ∨ B(x))

c) ∃x(D(x) → A(x))

d) ∃xE(x)

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Quantifiers over finite domains

When the domain of a quantifier is finite, that is, when all its elements can be listed, quantified statements can be expressed using propositional logic. In particular, when the elements of the domain are $x_1, x_2, \ldots, x_n$, where n is a positive integer,

- the universal quantification $\forall x P(x)$ is the same as the conjunction

$$P(x_1) \land P(x_2) \land \cdots \land P(x_n),$$

because this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.

- the existential quantification $\exists x P(x)$ is the same as the disjunction

$$P(x_1) \lor P(x_2) \lor \cdots \lor P(x_n),$$

because this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

Connection between qualification and looping

It is sometimes helpful to think in terms of looping and searching when determining the truth value of a quantification. Suppose that there are n objects in the domain for the variable x. To determine whether $\forall x P(x)$ is true, we can loop through all n values of x to see whether P(x) is always true. If we encounter a value x for which P(x) is false, then we have shown that $\forall x P(x)$ is false. Otherwise, $\forall x P(x)$ is true. To see whether $\exists x P(x)$ is true, we loop through the n values of x searching for a value for which P(x) is true. If we find one, then $\exists x P(x)$ is true. If we never find such an x, then we have determined that $\exists x P(x)$ is false. (Note that this searching procedure does not apply if there are infinitely many values in the domain. However, it is still a useful way of thinking about the truth values of quantifications.)

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have **higher precedence** than all logical operators from propositional calculus.

Example. $\forall x P(x) \lor Q(x)$ means $(\forall x P(x)) \lor Q(x)$

Logical Equivalences Involving Quantifiers

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Example.

$\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$

Negating Quantified Expressions

$\neg(\forall x P(x)) \equiv \exists x(\neg P(x))$
$\neg(\exists x Q(x)) \equiv \forall x(\neg Q(x))$

Rule: *to negate a quantifier:*
- move the negation to the inside;
- switch **∃ to ∀**, **∀ to ∃**.

Negating Quantified Expressions

Example.
- ∀ primes p, p is odd

Negation: ∃ primes p, p is even
- $\forall x(P(x) \rightarrow Q(x))$

Negation: $\exists x(P(x) \wedge \neg Q(x))$

Negating Quantified Expressions

Example. What are the negations of the statements $\forall x(x2 > x)$ and $\exists x(x2 = 2)$?

Translating from English into Logical Expressions

Example. Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

C(x): "x has studied calculus" S(x): "person x is in this class"

Q(x, y): "Student x has studied subject y"

Method 1. Domain for x consists of the students in the class

$$\forall x C(x).$$

Method 2. Domain for x consists of all people

$$\forall x(S(x) \rightarrow C(x))$$

or $\forall x(S(x) \rightarrow Q(x, \text{calculus}))$

Exercises

1. State the value of x after the statement if P(x) then x := 1 is executed, where P(x) is the statement "x > 1," if the value of x when this statement is reached is

A. x = 0 B. x = 1 C. x = 2

2. Let P(x) be the statement "x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these quantifications in English.

a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\exists x \neg P(x)$ d) $\forall x \neg P(x)$

Exercises

3. Translate these statements into English, where C(x) is "x is a comedian" and F(x) is "x is funny" and the domain consists of all people.

a) $\forall x(C(x) \rightarrow F(x))$ b) $\forall x(C(x) \land F(x))$

c) $\exists x(C(x) \rightarrow F(x))$ d) $\exists x(C(x) \land F(x))$

Exercises

4. Let P(x): "x can speak Russian", Q(x): "x knows the computer language C++." Express each of these sentences in terms of P(x), Q(x), quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

a) There is a student at your school who can speak Russian and who knows C++.

b) There is a student at your school who can speak Russian but who doesn't know C++.

c) Every student at your school either can speak Russian or knows C++.

d) No student at your school can speak Russian or knows C++.

Exercises

5. Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")

a) All dogs have fleas.

b) There is a horse that can add.

c) Every koala can climb.

d) No monkey can speak French.

e) There exists a pig that can swim and catch fish.

Exercises

6. Express the negation of each of these statements in terms of quantifiers without using the negation symbol.

a) $\forall x(x > 1)$

b) $\forall x(x \le 2)$

c) $\exists x(x \ge 4)$

d) $\exists x(x < 0)$

e) $\forall x((x < -1) \vee (x > 2))$

f) $\exists x((x < 4) \vee (x > 7))$

Exercises

7. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

a) $\forall x(x^2 \ge x)$

b) $\forall x(x > 0 \vee x < 0)$

c) $\forall x(x = 1)$

d) $\forall x(x^2 \ne x)$

e) $\forall x(x^2 \ne 2)$

f) $\forall x(|x| > 0)$

1.4 Nested Quantifiers

Nested Quantifiers

$\forall x \forall y P(x; y)$ = For all x and for all y, P(x; y) is true

$\forall x \exists y P(x; y)$ = For all x there is y such that P(x; y) is true

$\exists x \forall y P(x; y)$ = There exists x such that for all y, P(x; y) is true

$\exists x \exists y P(x; y)$ = There exist x and y such that P(x; y) is true

Remark. The order of quantifiers are important when multiple quantifiers are involved

Example. Determine the truth values of the following propositions on the set of real numbers.

a) $\forall x \forall y(x + y = 1)$

b) $\forall x \exists y(x + y = 1)$

c) $\exists x \forall y(x + y = 1)$

d) $\exists x \exists y(x + y = 1)$

Ví dụ

- *Mệnh đề "$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + 2y < 1$" đúng hay sai?* *Mệnh đề sai vì tồn tại $x_0 = 0, y_0 = 1 \in \mathbb{R}$ mà $x_0 + 2y_0 \geq 1$.*
- *Mệnh đề "$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + 2y < 1$" đúng hay sai?* *Mệnh đề đúng vì với mỗi $x = a \in \mathbb{R}$, tồn tại $y_a \in \mathbb{R}$ như*
- *Mệnh đề "$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + 2y < 1$" đúng hay sai?* *Mệnh đề sai vì không thể có $x = a \in \mathbb{R}$ để bất đẳng thức $a + 2y < 1$ được thỏa với mọi $y \in \mathbb{R}$ (chẳng hạn, $y = \frac{-a}{2} + 2$ không thể thỏa bất đẳng thức này.)*
- *Mệnh đề $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x + 2y < 1$ đúng hay sai? Mệnh đề đúng vì tồn tại $x_0 = 0, y_0 = 0 \in \mathbb{R}$ thỏa $x_0 + 2y_0 < 1$.*

Nested Quantifiers

Example. Let Q(x, y) denote "x + y = 0." What are the truth values of the quantifications ∃y∀xQ(x, y) and ∀x∃yQ(x, y), where the domain for all variables consists of all real numbers?

Solution.

∃y∀xQ(x, y) = "There is a real number y such that for every real number x, Q(x, y). There is no real number y such that x + y = 0 for all real numbers x so that the statement ∃y∀xQ(x, y) is false.

∀x∃yQ(x, y) = "For every real number x there is a real number y such that Q(x, y)". Given a real number x, there is a real number y such that x + y = 0; namely, y = −x. Hence, the statement ∀x∃yQ(x, y) is true.

Nested Quantifiers

Remark. The order of quantifiers are important when multiple quantifiers are involved.

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Translate Logical Expressions into Sentences

Example 1. ∀x∀y[(x > 0) ∧ (y > 0) → (xy > 0)] where x; y are real numbers.

Example 2. Let x, y represent students in a university, and

C(x) = "x has a laptop" F(x; y) = "x and y are friends"

Translate the logical expression ∀x[C(x) ∨ ∃y(C(y) ∧ F (x; y))].

Translating English Sentences into Logical Expressions

Example 1. Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Hint. F(x) to represent "x is female," P(x) to represent "x is a parent," and M(x, y) to represent "x is the mother of y".

Negating Nested Quantifiers

$\neg(\forall x \forall y P(x; y)) = \exists x \exists y \neg P(x; y)$

$\neg(\forall x \exists y P(x; y)) = \exists x \forall y \neg P(x; y)$

$\neg(\exists x \forall y P(x; y)) = \forall x \exists y \neg P(x; y)$

$\neg(\exists x \exists y P(x; y)) = \forall x \forall y \neg P(x; y)$

Negating Nested Quantifiers

Example.

- $\neg\forall x \exists y(xy = 1) \equiv \exists x \neg\exists y(xy = 1)$

  $\equiv \exists x \forall y \neg(xy = 1)$

  $\equiv \exists x \forall y(xy \neq 1)$
- $\neg\forall \epsilon > 0\ \exists \delta > 0\ \forall x(0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)$

$\equiv \exists \epsilon > 0\ \neg\exists \delta > 0\ \forall x(0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)$

$\equiv \exists \epsilon > 0\ \forall \delta > 0\ \neg\forall x(0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)$

$\equiv \exists \epsilon > 0\ \forall \delta > 0\ \exists x\ \neg(0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)$

$\equiv \exists \epsilon > 0\ \forall \delta > 0\ \exists x(0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon)$

Exercises

1. Let Q(x, y) be the statement "x has sent an e-mail message to y", where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

a) $\exists x \exists y Q(x, y)$ c) $\forall x \forall y Q(x, y)$

b) $\forall x \exists y Q(x, y)$ d) $\exists y \forall x Q(x, y)$

2. Translate these statements into English, where the domain for each variable consists of all real numbers.

a) $\forall x \exists y(x < y)$

b) $\forall x \forall y(((x \geq 0) \wedge (y \geq 0)) \rightarrow (xy \geq 0))$

c) $\forall x \forall y \exists z(xy = z)$

Exercises

3. Let $T(x, y)$ mean that student x likes cuisine y, where the domain for x consists of all students at your school and the domain for y consists of all cuisines. Express each of these statements by a simple English sentence.

a) $\neg T(\text{Abdallah Hussein, Japanese})$

b) $\exists x T(x, \text{Korean}) \wedge \forall x T(x, \text{Mexican})$

c) $\exists y(T(\text{Monique Arsenault}, y) \vee T(\text{Jay Johnson}, y))$

4. Let $Q(x, y)$ be the statement "Student x has been a contestant on quiz show y." Express each of these sentences in terms of $Q(x, y)$, quantifiers, and logical connectives, where the domain for x consists of all students at your school and for y consists of all quiz shows on television.

a) There is a student at your school who has been a contestant on a television quiz show.

b) No student at your school has ever been a contestant on a television quiz show.

c) There is a student at your school who has been a contestant on Jeopardy! and on Wheel of Fortune.

d) Every television quiz show has had a student from your school as a contestant.

e) At least two students from your school have been contestants on Jeopardy!

Exercises

5. Express each of these statements using mathematical and logical operators, predicates, and quantifiers, where the domain consists of all integers.

a) The sum of two negative integers is negative.

b) The difference of two positive integers is not necessarily positive.

c) The sum of the squares of two integers is greater than or equal to the square of their sum.

d) The absolute value of the product of two integers is the product of their absolute values.

Exercises

6. Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

a) $\neg\exists y\exists x P(x, y)$

b) $\neg\forall x\exists y P(x, y)$

c) $\neg\exists y(Q(y) \wedge \forall x\neg R(x, y))$

d) $\neg\exists y(\exists x R(x, y) \vee \forall x S(x, y))$

e) $\neg\exists y(\forall x\exists z T(x, y, z) \vee \exists x\forall z U(x, y, z))$

Exercises

7. Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")

a) No one has lost more than one thousand dollars playing the lottery.

b) There is a student in this class who has chatted with exactly one other student.

c) No student in this class has sent e-mail to exactly two other students in this class.

d) Some student has solved every exercise in this book.

e) No student has solved at least one exercise in every section of this book.

Exercises

8. Let $Q(x, y)$ be the statement "$x + y = x - y$." If the domain for both variables consists of all integers, what are the truth values?

a) $Q(1, 1)$ b) $Q(2, 0)$

c) $\forall y Q(1, y)$ d) $\exists x Q(x, 2)$

e) $\exists x \exists y Q(x, y)$ f) $\forall x \exists y Q(x, y)$

g) $\exists y \forall x Q(x, y)$ h) $\forall y \exists x Q(x, y)$

i) $\forall x \forall y Q(x, y)$

Exercises

9. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

a) $\forall x \forall y(x^2 = y^2 \rightarrow x = y)$

b) $\forall x \exists y(y^2 = x)$

c) $\forall x \forall y(xy \geq x)$

Exercises

10. Suppose $L(x, y)$ is the statement "Student x solves question y correctly". Translate the statement into logical expression "For any question in the exam, there are at least two students who solve it correctly and at least one student who does not".

A. $\forall y(\exists x(L(x, y) \wedge \exists x' L(x', y) \wedge \exists x'' \neg L(x'', y)))$

B. $\forall y(\exists x(L(x, y) \wedge \exists x' \neg L(x', y))$

C. $\forall y(\exists x(L(x, y) \wedge \exists x'((x \neq x') \wedge L(x', y)) \wedge \exists x'' \neg L(x'', y))$

Exercises

11. Let F(x) = "x is a student of the Business department",
J(x) = "x knows the computer language Java",
R(x) = "x can speak Russian".
The domain consists of all students of the university.
Express the sentence into a logical expression "Some student of the Business department either can speak Russian or known Java".

A. $\forall x(F(x) \to (J(x) \vee R(x)))$
B. $\forall x(F(x) \to (J(x) \wedge R(x)))$
C. $\exists x(F(x) \to (J(x) \vee R(x)))$
D. $\exists x(F(x) \to (J(x) \wedge R(x)))$
E. None of the other choices

1.5 Rules of Inference

Need for inference?

Why do we study propositional and predicate logic?

- We want to use them to solve problems.
- To solve a problem by using logic, we often need to start from some "premises" and obtain a certain "conclusion using inference rules.

Example: Computer scientists often need to verify the correctness of a program.

- One possible approach is to prove the program is correct.
- So one can start from the program and the semantics of the used programming language (i.e., the premise), and use logic inference to obtain a conclusion that the program does the right job.

Arguments, argument forms and their validity

- An **argument** in propositional logic is sequence of propositions. All but the final proposition are called **premises** and the final proposition is called the **conclusion**. An argument is **valid** if the truth of all its premises implies that the conclusion is *true*.
- An **argument form** in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is valid if no matter which propositions are substituted for the propositional variables in its premises, if the premises are all true, then the conclusion is true.

In other words, an argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion q is valid if and only if

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$$

is a tautology.

Arguments, argument forms and their validity

Example. Consider the following argument

"If you have a current password, then you can log onto the network."

"You have a current password."

Therefore,

"You can log onto the network."

Let p be "You have a current password" and q be "You can log onto the network".

The form of the above argument is:

$$\begin{array}{l} p \to q \\ \underline{p \qquad} \\ \therefore q \end{array}$$

The argument is valid since ((p → q) ∧ p) → q is a tautology.

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Rules of Inference

Inference rule	Tautology	Name
$\begin{array}{l} p \\ \underline{p \to q} \\ \therefore q \end{array}$	(p ∧ (p → q)) → q	Modus ponens
$\begin{array}{l} \neg q \\ \underline{p \to q} \\ \therefore \neg p \end{array}$	(¬q ∧ (p → q)) → ¬p	Modus tollens
$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$	((p → q) ∧ (q → r)) → (p → r)	Hypothetical syllogism
$\begin{array}{l} p \vee q \\ \underline{\neg p \qquad} \\ \therefore q \end{array}$	((p ∨ q) ∧ ¬p) → q	Disjunctive syllogism

Rules of Inference

Inference rule	Tautology	Name
$\begin{array}{l} \underline{p \qquad} \\ \therefore p \vee q \end{array}$	p → (p ∨ q)	Addition
$\begin{array}{l} \underline{p \wedge q} \\ \therefore p \end{array}$	(p ∧ q) → p	Simplification
$\begin{array}{l} p \\ \underline{q \qquad} \\ \therefore p \wedge q \end{array}$	((p) ∧ (q)) → (p ∧ q)	Conjunction
$\begin{array}{l} p \vee q \\ \underline{\neg p \vee r} \\ \therefore q \vee r \end{array}$	((p ∨ q) ∧ (¬p ∨ r)) → (q ∨ r)	Resolution

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Rules of Inference

Example. Which rule of inference is used in each argument below?

a) Alice is a Math major. Therefore, Alice is either a Math major or a CSI major.

b) Jerry is a Math major and a CSI major. Therefore, Jerry is a Math major.

c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.

d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.

e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

f) I go swimming or eat an ice cream. I did not go swimming. Therefore, I eat an ice cream.

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Using Rules of Inference to Build Arguments

Example. Show that the hypotheses:

- It is not sunny this afternoon and it is colder than yesterday.
- We will go swimming only if it is sunny.
- If we do not go swimming, then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

lead to the conclusion:

- We will be home by the sunset.

Main steps:

1. Translate the statements into propositional logic.
2. Write a formal proof, a sequence of steps that state hypotheses or apply inference rules to previous steps.

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Show that the hypotheses:

- It is not sunny this afternoon and it is colder than yesterday. $\neg s \wedge c$
- We will go swimming only if it is sunny. $w \rightarrow s$
- If we do not go swimming, then we will take a canoe trip. $\neg w \rightarrow t$
- If we take a canoe trip, then we will be home by sunset. $t \rightarrow h$

lead to the conclusion:

- We will be home by the sunset. h

Step	Reason
1. $\neg s \wedge c$	hypothesis
2. $\neg s$	simplification
3. $w \rightarrow s$	hypothesis
4. $\neg w$	modus tollens of 2 and 3
5. $\neg w \rightarrow t$	hypothesis
6. t	modus ponens of 4 and 5
7. $t \rightarrow h$	hypothesis
8. h	modus ponens of 6 and 7

Where:
s: "it is sunny this afternoon"
c: "it is colder than yesterday"
w: "we will go swimming"
t: "we will take a canoe trip.
h: "we will be home by the sunset."

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Fallacies

- Fallacy = misconception resulting from incorrect argument.
- **Fallacy of affirming the conclusion**

  Based on $((p \rightarrow q) \wedge q) \rightarrow p$ which is NOT A TAUTOLOGY.

Example 1. If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics. Therefore, you did every problem in this book.

- **Fallacy of denying the hypothesis**

  Based on $((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$ which is NOT A TAUTOLOGY.

Example 2. If you do every problem in this book, then you will learn discrete mathematics. You do not every problem in this book. Therefore, you did not learn discrete mathematics.

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Rules of Inference for Quantified Statements

Rule of Inference	Name
$\dfrac{\forall x P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x P(x)}$	Existencial generalization

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Rules of Inference for Quantified Statements

Example. Show that the premises: "A student in this class has not read the book", and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book".

Solution.

A(x) be "x is in this class", B(x) be "x has read the book" and P(x) be "x passed the first exam".

Hypotheses/Premises: $\exists x(A(x) \land \neg B(x))$ and $\forall x(A(x) \rightarrow P(x))$

Conclusion: $\exists x(P(x) \land \neg B(x))$

Rules of Inference for Quantified Statements

Hypotheses: $\exists x(A(x) \land \neg B(x))$ and $\forall x(A(x) \rightarrow P(x))$.

Conclusion: $\exists x(P(x) \land \neg B(x))$.

Step	Reason
1. $\exists x(A(x) \land \neg B(x))$	Hypothesis
2. $A(a) \land \neg B(a)$	Existencial instantiation from (1)
3. $A(a)$	Simplification from (2)
4. $\forall x(A(x) \rightarrow P(x))$	Hypothesis
5. $A(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \land \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \land \neg B(x))$	Existential generalization from (8)

Combining Rules of Inference for Propositions and Quantified Statements

These inference rules are frequently used and combine propositions and quantified statements:

Universal Modus Ponens

$\forall x(P(x) \rightarrow Q(x))$

P(a), where a is a particular element in the domain

$\therefore Q(a)$

Universal Modus Tollens

$\forall x(P(x) \rightarrow Q(x))$

$\neg Q(a)$, where a is a particular element in the domain

$\therefore \neg P(a)$

Exercises

1. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If Socrates is human, then Socrates is mortal.

Socrates is human.

$\therefore$ Socrates is mortal.

Exercises

2. Determine whether each of these arguments is valid. If an argument is correct, what rule of inference is being used? If it is not, what logical error occurs?

a) If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$.

b) If n is a real number with $n > 3$, then $n^2 > 9$. Suppose that $n^2 \leq 9$. Then $n \leq 3$.

c) If n is a real number with $n > 2$, then $n^2 > 4$. Suppose that $n \leq 2$. Then $n^2 \leq 4$.

Exercises

3. Determine whether these are valid arguments.

a) If x is a positive real number, then x^2 is a positive real number. Therefore, if a^2 is positive, where a is a real number, then a is a positive real number.

b) If $x^2 \neq 0$, where x is a real number, then $x \neq 0$. Let a be a real number with $a^2 \neq 0$; then $a \neq 0$.

Exercises

4. Given the hypotheses:

- "If you send me an email, I will finish writing the program"
- "If you do not send email then I will go to bed early"
- "If I go to bed early then I will go jogging tomorrow morning"

Show that these hypotheses lead to the conclusion: "If I do not finish writing the program then I will go jogging tomorrow morning".

Quiz

Which argument is valid?

A. If I get a scholarship then I will be enrolled in FPT University. I am enrolled in FPT university. Therefore, I must have received the scholarship.

B. If Binh finishes the class in the top 3, then Binh will visit Singapore. If going to Singapore, Binh will visit NUS. Therefore, if Binh finishes the class in the top 3, Binh will visit NUS.

C. If I choose SE major then I will have to take Discrete Mathematics. I am not an SE major. Thus, I do not have to take Discrete Mathematics.

Introduction to Proofs

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Some terminology

- **Theorem**: a statement that can be shown to be true (sometimes referred to as **facts** or **results**). Less important theorems are often called **propositions**.
- A **lemma** is a less important theorem, used as an auxiliary result to prove a more important theorem.
- A **corollary** is a theorem proven as an easy consequence of a theorem.
- A **conjecture** is a statement that is being proposed as a true statement. If later proven, it becomes a theorem, but it may be false.
- **Axiom** (or **postulates**) are statements that we assume to be true (algebraic axioms specify rules for arithmetic like commutative laws).
- A **proof** is a valid argument that establishes the truth of a theorem. The statements used in a proof include axioms, hypotheses (or premises), and previously proven theorems. Rules of inference, together with definition of terms, are used to draw conclusions from other assertions, tying together the steps of a proof

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Proof Methods

3 methods of showing statements of the type $p \to q$ are true:

1. **Direct proofs**: Assume p is true; the last step establishes q is true.
2. **Proof by Contraposition**: Uses a direct proof of the contrapositive of $p \to q$, which is $\neg q \to \neg p$. That is, assume $\neg q$ is true; the last step established $\neg p$ is true.
3. **Proof by Contradiction**: To prove that P is true, we assume $\neg P$ is true and reach a contradiction, that is that $(r \wedge \neg r)$ is true for some proposition r. In particular, to prove $(p \to q)$, we assume $(p \to q)$ is false, and get as a consequence a contradiction. Assuming that $(p \to q)$ is false = $(\neg p \vee q)$ is false = $(p \wedge \neg q)$ is true.

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