Chapter 2: Probability

LEARNING OBJECTIVES

1. Sample Spaces and Events
2. Interpretations of Probability
3. Addition Rules
4. Conditional Probability
5. Multiplication and Total Probability Rules
6. Independence
7. Bayes' Theorem

Sample spaces and events

Definition — Random experiment

- An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a **random experiment.**
- The set of all possible outcomes of a random experiment is called the **sample space** of the experiment. The sample space is denoted as S.
- An **event** is a subset of the sample space of a random experiment.

Example

Random experiment: Roll a die

Sample space: $S = \{1, 2, 3, 4, 5, 6\}$

Event: $E_1 = \{\text{Die is even}\} = \{2, 4, 6\}$

$E_2 = \{\text{Die is odd}\} = \{1, 3, 5\}$

Tree Diagrams

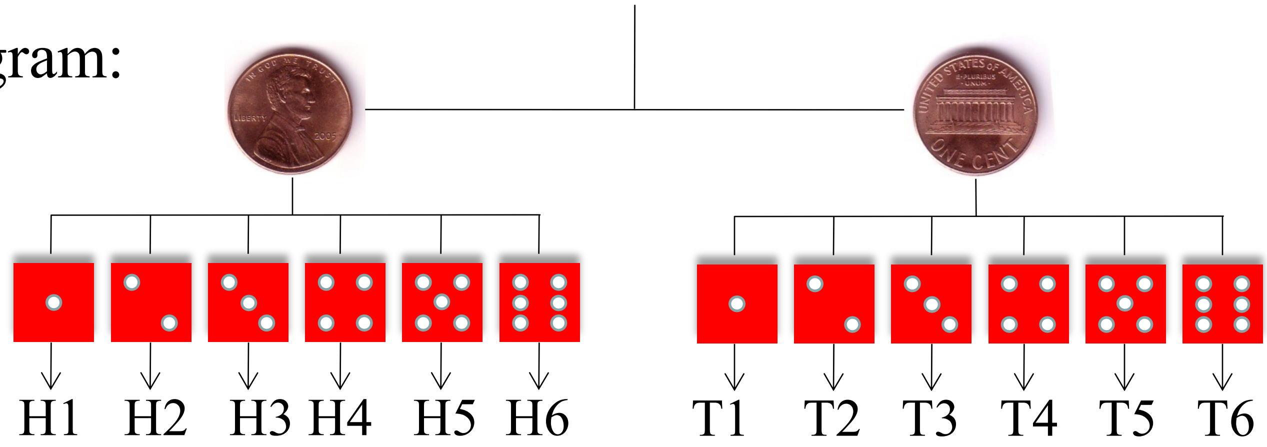
Sample spaces can also be described graphically with **tree diagrams.**

- When a sample space can be constructed in several steps or stages, we can represent each of the n_1 ways of completing the first step as a branch of a tree.
- Each of the ways of completing the second step can be represented as n_2 branches starting from the ends of the original branches, and so forth.

Example

A probability experiment consists of tossing a coin and then rolling a six-sided die. Describe the sample space.

Tree diagram:

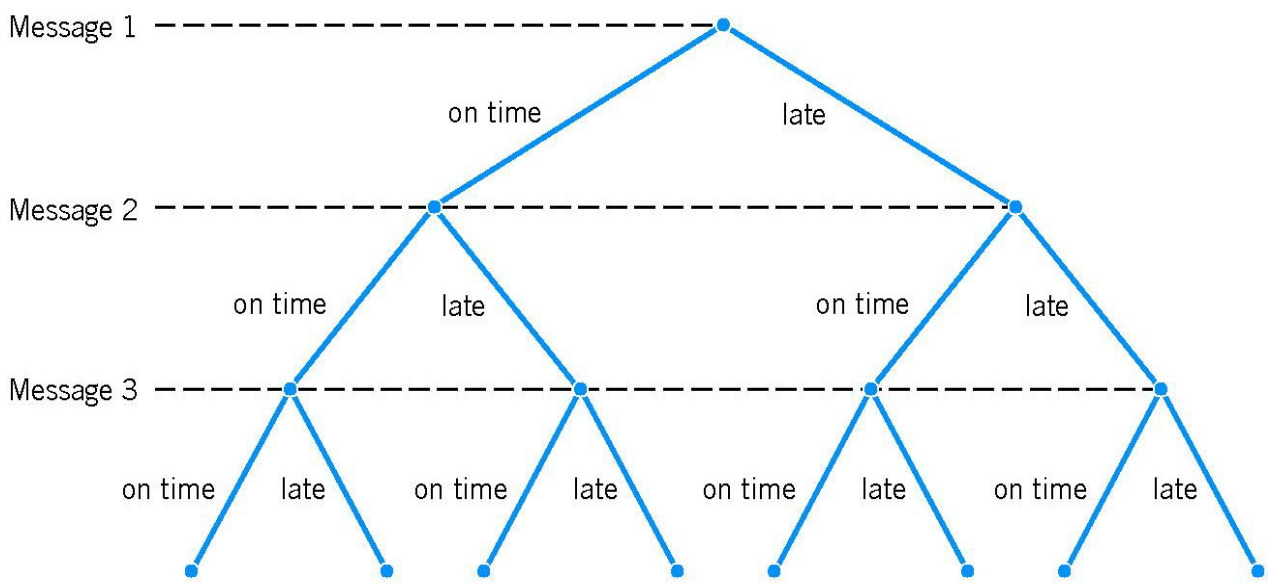


The sample space has 12 outcomes:

S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}

Example

Each message in a digital communication system is classified as to whether it is received within the time specified by the system design. If three messages are classified, use a tree diagram to represent the sample space of possible outcomes.



Basic Set Operations

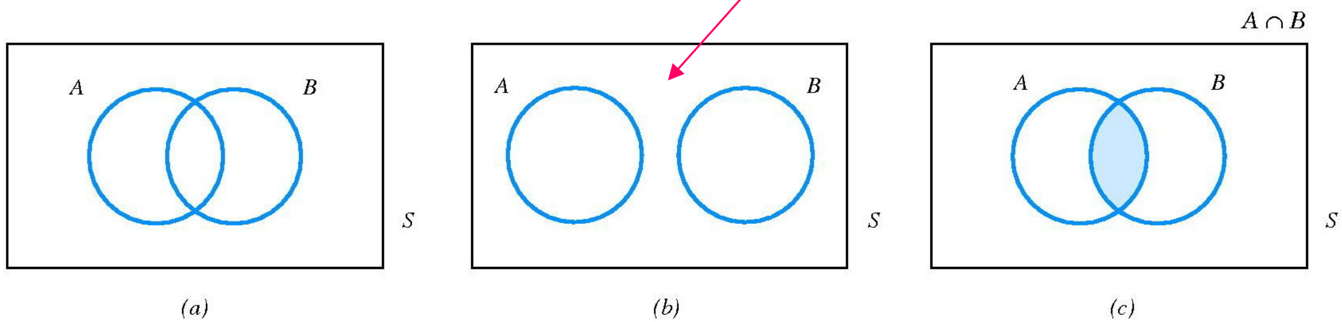
The **union** of two events is the event that consists of all outcomes that are contained in either of the two events. We denote the union as $E_1 \cup E_2$.

The **intersection** of two events is the event that consists of all outcomes that are contained in both of the two events. We denote the intersection as $E_1 \cap E_2$.

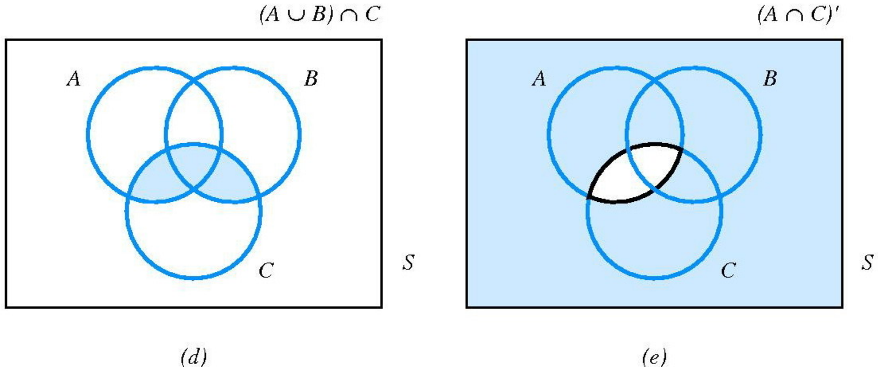
The **complement** of an event in a sample space is the set of outcomes in the sample space that are not in the event. We denote the component of the event E as E'.

Venn Diagrams

mutually exclusive



Sample space S with events A and B



Important properties:

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

$$A = (A \cap B) \cup (A \cap B')$$

Introduction

There are three approaches to assessing the probability of an uncertain event:

1. **a priori classical probability**: the probability of an event is based on prior knowledge of the process involved.

2. **empirical classical probability**: the probability of an event is based on observed data.

3. **subjective probability**: the probability of an event is determined by an individual, based on that person's past experience, personal opinion, and/or analysis of a particular situation.

Equally Likely Outcomes

Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$.

1. **a priori classical probability**

$$\text{Probability of Occurrence} = \frac{X}{T} = \frac{\text{number of ways the event can occur}}{\text{total number of possible outcomes}}$$

2. **empirical classical probability**

$$\text{Probability of Occurrence} = \frac{\text{number of favorable outcomes observed}}{\text{total number of outcomes observed}}$$

Interpretations of Probability

Example priori classical probability

Find the probability of selecting a face card (Jack, Queen, or King) from a standard deck of 52 cards.

$$\text{Probability of Face Card} = \frac{X}{T} = \frac{\text{number of face cards}}{\text{total number of cards}}$$

$$\frac{X}{T} = \frac{12 \text{ face cards}}{52 \text{ total cards}} = \frac{3}{13}$$

Example empirical classical probability

Find the probability of selecting a male taking statistics from the population described in the following table:

	Taking Stats	Not Taking Stats	Total
Male	84	145	229
Female	76	134	210
Total	160	279	439

$$\text{Probability of Male Taking Stats} = \frac{\text{number of males taking stats}}{\text{total number of people}}$$

$$= \frac{84}{439} = 0.191$$

Axioms of Probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

If S is the sample space and E is any event in a random experiment,

(1) $P(S) = 1$

(2) $0 \leq P(E) \leq 1$

(3) For two events E_1 and E_2 with $E_1 \cap E_2 = \varnothing$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

The special addition rule

1. If A and B are mutually exclusive events,

$$P(A \cup B) = P(A) + P(B)$$

2. A collection of events, $E_1, E_2, \ldots, E_k$ is said to be **mutually exclusive** if for all pairs,

$$E_i \cap E_j = \varnothing$$

For a collection of mutually exclusive events,

$$P(E_1 \cup E_2 \cup \ldots \cup E_k) = P(E_1) + P(E_2) + \ldots + P(E_k)$$

The general addition rule

1. Two events: *A* and *B* are any events,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

2. Three events:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Example

Find the probability of selecting a male or a statistics student from the population described in the following table:

	Taking Stats	Not Taking Stats	Total
Male	84	145	229
Female	76	134	210
Total	160	279	439

$$P(\text{Male or Stats}) = P(M) + P(S) - P(\text{Male and Stats})$$
$$= 229/439 + 160/439 - 84/439 = 305/439$$

Conditional Probability

To introduce conditional probability, consider an example involving manufactured parts.

Let D denote the event that a part is defective and let F denote the event that a part has a surface flaw.

Then, we denote the probability of D given, or assuming, that a part has a surface flaw as $P(D|F)$. This notation is read as the **conditional probability** of D given F, and it is interpreted as the probability that a part is defective, given that the part has a surface flaw.

Conditional Probability

Example

Find the probability of selecting a male and taking statistics student from the population described in the following table:

	Taking Stats	Not Taking Stats	Total
Male	84	145	229
Female	76	134	210
Total	160	279	439

P(Male and Stats) = P(Male).P(Stats)??

Conditional Probability

Example

Find the probability of selecting a male and taking statistics student from the population described in the following table:

	Taking Stats	Not Taking Stats	Total
Male	84	145	229
Female	76	134	210
Total	160	279	439

P(Male and Stats) = P(Male).P(Stats|Male)

= 229/439 . (84/229) = 84/439

Check: P(Stats). P(Male|Stats) = ?

Example 2-22(page 42)

Table 2-3 Parts Classified

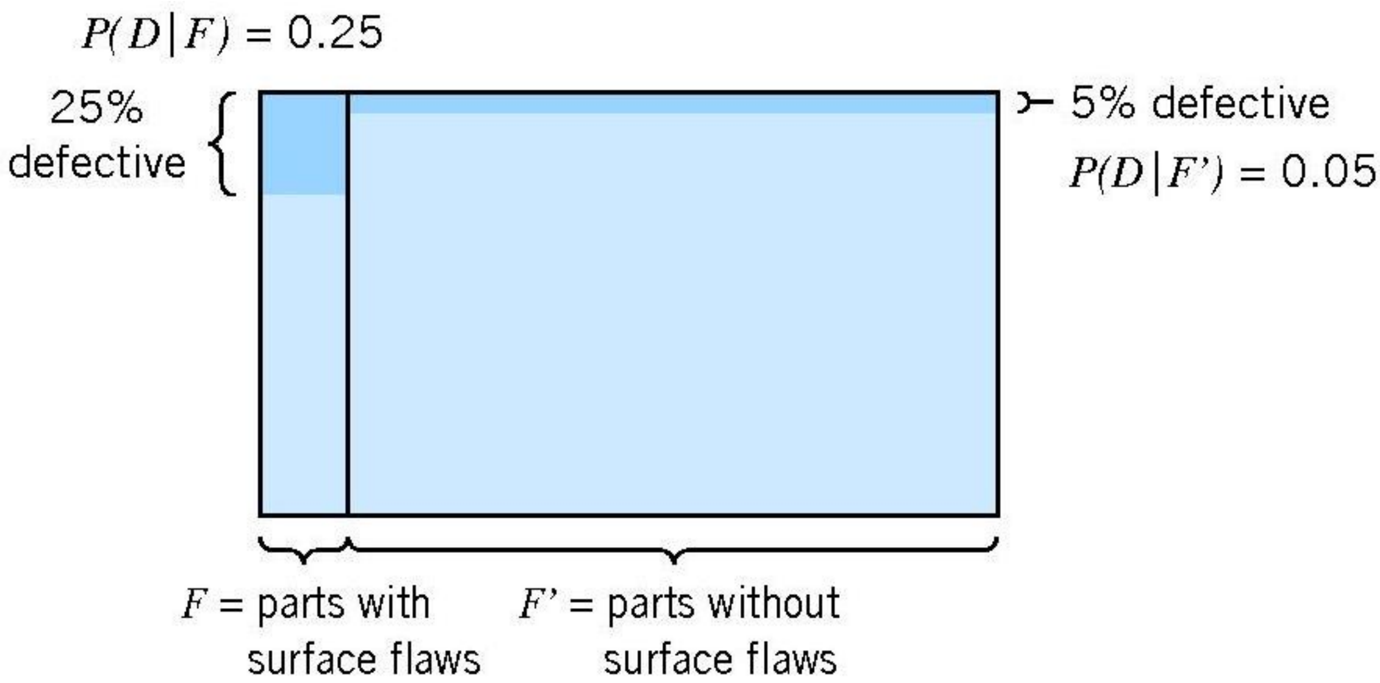
		Surface Flaws		
		Yes (event F)	No	Total
Defective	Yes (event D)	10	18	38
	No	30	342	362
	Total	40	360	400

Find P(D/F) =? P(D/F') =?

$$P(D|F) = 10/40 = 0.25$$

$$P(D|F') = 18/360 = 0.05$$

Conditional Probability



Conditional Probability

Definition — **Conditional Probability**

The conditional probability of an event B given an event A, denoted as $P(B|A)$, is

$$P(B|A) = P(B \cap A)/P(A)$$

for $P(A) > 0$.

Special case: all outcomes are equally likely

$$P(A \cap B)/P(A) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } A}$$

Example 2-23 (page 43)

$$P(D|F) = P(D \cap F)/P(F) = \frac{10}{400} \bigg/ \frac{40}{400} = \frac{10}{40}$$

Note that in this example all four of the following probabilities are different:

$$P(F) = 40/400 \qquad P(F|D) = 10/28$$
$$P(D) = 28/400 \qquad P(D|F) = 10/40$$

Here, $P(D)$ and $P(D|F)$ are probabilities of the same event, but they are computed under two different states of knowledge. Similarly, $P(F)$ and $P(F|D)$ are computed under two different states of knowledge.

Conditional Probability

Example

Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.

What is the probability that a car has a CD player, given that it has AC ?

	CD	No CD	Total
AC	0.2	0.5	0.7
No AC	0.2	0.1	0.3
Total	0.4	0.6	1.0

$$P(CD \mid AC) = \frac{P(CD \text{ and } AC)}{P(AC)}$$

$$= \frac{0.2}{0.7} = .2857$$

Multiplication rules

Multiplication Rule

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

Example

The probability that an automobile battery subject to high engine compartment temperature suffers low charging current is 0.7. The probability that a battery is subject to high engine compartment temperature is 0.05.

The probability that a battery is subject to low charging current and high engine compartment temperature is

$$P(C \cap T) = P(C|T)P(T) = 0.7 \times 0.05 = 0.035$$

C={a battery suffers low charging current}

T ={a battery is subject to high engine compartment temperature}

Multiplication rules

Remak:

1. $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

2. $P(A \mid B) = P(A$ given that B, A if $B) = P(A \cap B) / P(B)$

Example

A jar contains 3 white balls, 5 black balls.

Find P(1 white & 1 black) =?

Example

Give 3 white balls, 5 black balls.
Find P(1 white & 1 black) =?

Ans: With replacement: 15/32
Without replacement: 15/28

Exercise

2. If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. We assume that an aircraft is present with probability 0.05. What is the probability of false alarm (a false indication of aircraft presence), and the probability of missed detecation (nothing registers, even though an aircraft is present)?

Answer: 0.095 & 0.0005

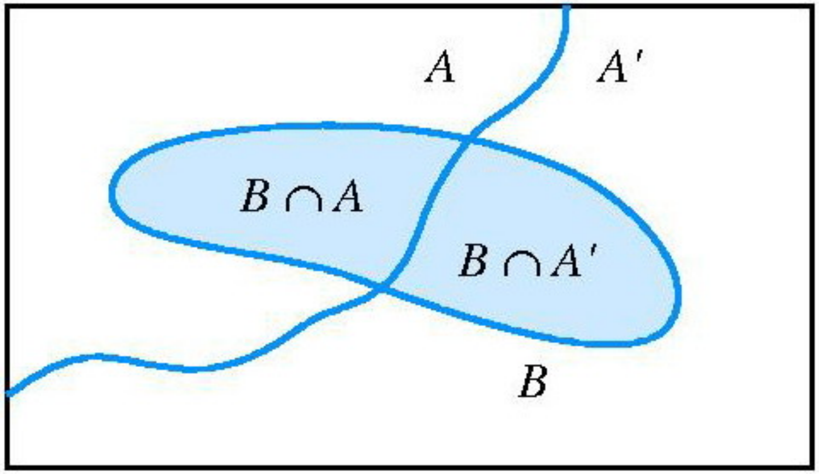
Exercise

3. You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against a randomly chosen opponent. What is the probability of winning?

Answer: 0.375

Partition of an event

Figure 2-15
Partitioning an event into two mutually exclusive subsets.



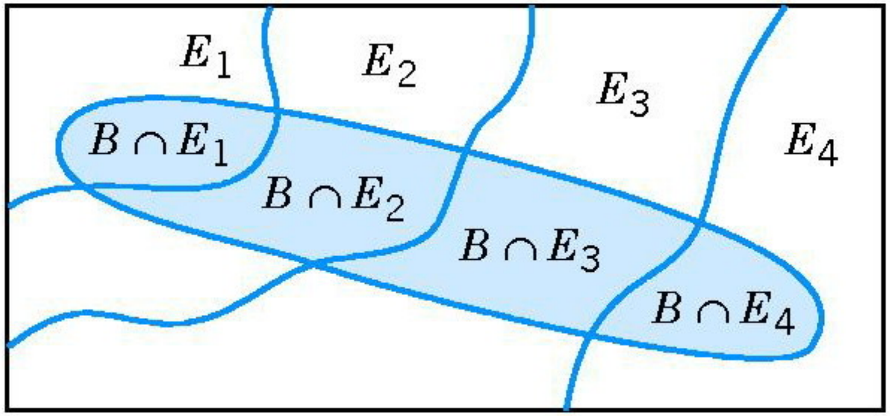


Figure 2-16
Partitioning an event into several mutually exclusive subsets.

$$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$$

Total Probability Rule

Total Probability Rule: two events

$$P(B) = P(BA) + P(BA') = P(B|A)P(A) + P(B|A')P(A')$$

Total Probability Rule: multiple events

$$\boxed{E_1 \cup E_2 \cup \ldots E_k = S}$$

Assume $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive sets. Then

$$\begin{aligned} P(B) &= P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k) \\ &= P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k) \end{aligned}$$

Example 2-27 (page 48)

Consider the contamination discussion at the start of this section. The information is summarized here

Probability of Failure	Level of Contamination	Probability of Level
0.1	High	0.2
0.005	Not High	0.8

Let F denote the event that the product fails, and let H denote the event that the chip is exposed to high levels of contamination. The requested probability is $P(F)$, and the information provided can be represented as

$$P(F|H) = 0.10 \quad \text{and} \quad P(F|H') = 0.005$$
$$P(H) = 0.20 \quad \text{and} \quad P(H') = 0.80$$

From Equation 2-11,

$$P(F) = 0.10(0.20) + 0.005(0.80) = 0.024$$

which can be interpreted as just the weighted average of the two probabilities of failure.

Independence

Definition

Two events is called independent if any one of the following equivalent statements is true:

(1) $P(A|B) = P(A)$

(2) $P(A \cap B) = P(A)P(B)$

(3) $P(B|A) = P(B)$

Proposition: If A and B are independent events, then so are events A and B', events A' and B, and events A' and B'.

Exercise

Let P(A) = 0.1; P(B) = 0.2. Which of the followings are true?

a) If A and B are independent, they are mutually exclusive.

b) If A and B are disjoint, they are independent.

c) If $P(A \cup B) = 0.28$, A and B are independent,

d) If $P(A \cup B) = 0.3$, A and B are mutually exclusive.

Example 2-31 (page 52)

A day's production of 850 manufactured parts contains 50 parts that do not meet customer requirements. Two parts are selected at random, without replacement, from the batch. Let A ={the first part is defective}, and let B ={the second part is defective}.

We suspect that these two events are not independent because knowledge that the first part is defective suggests that it is less likely that the second part selected is defective.

$$P(B|A) = 49/849$$

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|A')P(A') \\ &= (49/849)(50/850) + (50/849)(800/850) \\ &= 50/850 \end{aligned}$$

the two events are not independent, as we suspected.

Definition

The events $E_1, E_2, \ldots, E_n$ are independent if and only if for any subset of these events $E_{i_1}, E_{i_2}, \ldots, E_{i_k}$

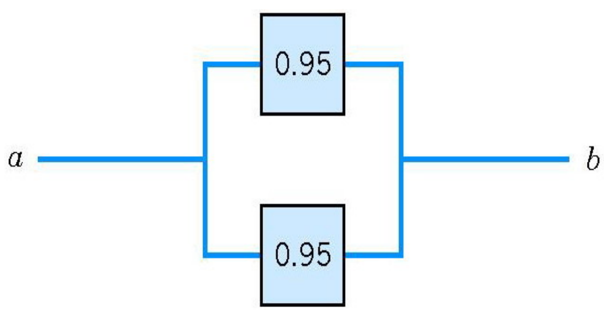
$$P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}) = P(E_{i_1})P(E_{i_2})\ldots P(E_{i_k})$$

Exercise

Two coins are tossed. Let A denote the event "at most one head on the two tosses," and let B denote the event "one head and one tail in both tosses." Are A and B independent events?

Example 2-34 (page 53)

The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?



$$P(T \cup B) = 1 - P[(T \cup B)'] = 1 - P(T' \cap B')$$

$$P(T' \cap B') = P(T')P(B') = (1-0.95)^2 = 0.05^2$$

$$P(T \cup B) = 1 - 0.05^2 = 0.9975$$

Exercise

A test for a certain rare disease is assumed to be correct 95% of the time. If a person has the disease, the test results are positive with probability 0.95 and if the person does not have the disease, the test results are negative with probability 0.95. A random person drawn from a certain population has probability 0.001 of having the disease. Given that the person just tested positive, what is the probability of having the disease?

Answer: 0.0187

Exercise

A test for a certain rare disease is assumed to be correct 95% of the time. If a person has the disease, the test results are positive with probability 0.95 and if the person does not have the disease, the test results are negative with probability 0.95. A random person drawn from a certain population has probability 0.001 of having the disease. Given that the person just tested positive, what is the probability of having the disease?

Sol:

P(Disease| Positive)= $\frac{\boldsymbol{P(Positive|Disease).P(Disease)}}{\boldsymbol{P(Positive)}}$

Bayes' Theorem

If $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive events and B is any event,

$$P(E_1 \mid B) = \frac{P(B \mid E_1)P(E_1)}{P(B \mid E_1)P(E_1) + P(B \mid E_2)P(E_2) + \ldots + P(B \mid E_k)P(E_k)}$$

$$\text{for } P(B) > 0$$

In special case:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} \quad \text{for } P(B) > 0$$

Example

In a state where cars have to be tested for the emission of pollutants, 25% of all cars emit excessive amount of pollutants. When tested, 99% of all cars that emit excessive amount of pollutants will fail, but 17% of all cars that do not emit excessive amount of pollutants will also fail. What is the probability that a car that fails the test actually emits excessive amounts of pollutants?

Let A denote the event that a car fails the test and B the event that it emits excessive amounts of pollutants:

$$P(B) = 0.25,\ P(A|B) = 0.99 \text{ and } P(A|B') = 0.17.$$

We have to find $P(B|A)$?

Bayes' Theorem

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}$$

where

$$P(A) = P(AB) + P(AB') = P(A|B)P(B) + P(A|B')P(B')$$

$$= 0.375$$

Substitution into the formula for $P(B|A)$ yields

$$P(B|A) = 0.66$$

Example

Two events A and B are such that $P[A \cap B] = 0.15$, $P[A \cup B] = 0.65$, and $P[A|B] = 0.5$. Find $P[B|A]$.

Solution $P[A \cup B] = P[A] + P[B] - P[A \cap B] \Rightarrow 0.65 = P[A] + P[B] - 0.15$. This means that $P[A] + P[B] = 0.65 + 0.15 = 0.80$. Also, $P[A \cap B] = P[B] \times P[A|B]$. This then means that

$$P[B] = \frac{P[A \cap B]}{P[A|B]} = \frac{0.15}{0.50} = 0.30$$

Thus, $P[A] = 0.80 - 0.30 = 0.50$. Since $P[A \cap B] = P[A] \times P[B|A]$, we have that

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{0.15}{0.50} = 0.30$$

Definition

A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment.

A random variable is denoted by an uppercase letter such as X. After an experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as x 70 milliamperes.

Random Variables

Definition

A **discrete** random variable is a random variable with a finite (or countably infinite) range.

A **continuous** random variable is a random variable with an interval (either finite or infinite) of real numbers for its range.

Summary

We have studied:

1. Sample Spaces and Events
2. Interpretations of Probability
3. Addition Rules
4. Conditional Probability
5. Multiplication and Total Probability Rules
6. Independence
7. Bayes' Theorem