An identity involving the imaginary error function $\mathrm{erfi}(x)$

Nicolás Quesada and Aaron Goldberg

The imaginary error function $\mathrm{erfi}(x)$ is customarily defined as $\mathrm{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{t^2}$. It is related to the error function by $\mathrm{erfi}(x)= \mathrm{erf}(\mathrm{i}x)/\mathrm{i}$, and can be also defined as

$$\frac{1}{\pi} \fint_{-\infty}^{\infty} dy\ \frac{e^{ay^2+2by}}{y} = \frac{2}{\pi} \int_0^{\infty} dy\ e^{ay^2} \frac{\sinh(2by)}{y} = \mathrm{erfi}\left(\frac{b}{\sqrt{-a}}\right) \text{ for } a<0. \tag{1}$$

In this note we investigate a property of a function related to the imaginary error function.

We can define two functions

$$\phi_0(x) \equiv \frac{\exp\left(-x^2\right)}{\sqrt[4]{\pi/2}}, \quad \phi_1(x) \equiv \mathcal{N}\left(\alpha\right)\phi_0\left(x\right)\mathrm{erfi}\left(\alpha x\right), \quad \mathcal{N}\left(\alpha\right) = \sqrt{\frac{\pi}{2\sin^{-1}\left(\frac{\alpha^2}{2-\alpha^2}\right)}}, \tag{2}$$

whose $\mathcal{L}^2$-orthogonality is seen from their respective parities.

We here show that ϕ_1 is, like ϕ_0, $\mathcal{L}^2$-normalized; to this end consider the integral

$$I = \int_{-\infty}^{\infty} dx\ \phi_1^2\left(x\right) = \mathcal{N}^2 \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx \exp\left(-2x^2\right)\mathrm{erfi}^2\left(\alpha x\right). \tag{3}$$

Note that this integral is, to the best of our knowledge, not currently tabulated in standard computer algebra systems (CAS) like Mathematica or Maple or the Wolfram Functions Site. We use Eq. (1) to write

$$I = \frac{\mathcal{N}^2}{\pi^2}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx \exp\left(-2x^2\right) \fint_{-\infty}^{\infty} \frac{du}{u} \exp\left(-\frac{u^2}{\alpha^2}+2ux\right) \fint_{-\infty}^{\infty} \frac{dv}{v} \exp\left(-\frac{v^2}{\alpha^2}+2vx\right) \tag{4a}$$

$$= \frac{\mathcal{N}^2}{\pi^2}\sqrt{\frac{2}{\pi}} \fint_{-\infty}^{\infty} \frac{du}{u} \fint_{-\infty}^{\infty} \frac{dv}{v} \exp\left(-\frac{u^2}{\alpha^2}-\frac{v^2}{\alpha^2}\right) \int_{-\infty}^{\infty} dx \exp\left(-2x^2+2ux+2vx\right) \tag{4b}$$

$$= \frac{\mathcal{N}^2}{\pi^2} \fint_{-\infty}^{\infty} \frac{du}{u} \exp\left(-\beta u^2\right) \fint_{-\infty}^{\infty} \frac{dv}{v} \exp\left(-\beta v^2 + uv\right), \tag{4c}$$

where we have liberally switched the order of the integrals and written $\beta = 1/\alpha^2 - 1/2$. We now again use Eq. (1) to find

$$I = \frac{\mathcal{N}^2}{\pi^2} \fint_{-\infty}^{\infty} \frac{du}{u} \exp\left(-\beta u^2\right) \left(\pi\ \mathrm{erfi}\left(\frac{u}{2\sqrt{\beta}}\right)\right) = \frac{2\mathcal{N}^2}{\pi} \underbrace{\int_0^{\infty} \frac{du}{u} \exp\left(-\beta u^2\right) \mathrm{erfi}\left(\frac{u}{2\sqrt{\beta}}\right)}_{\sin^{-1}\frac{1}{2\beta}} = 1. \tag{5}$$

The last integral can be evaluated using any of the aforementioned CAS for $|\beta| \geq 1/2, \Re\left[\beta\right] > 0$ (i.e., $|\alpha| \leq 1$). Eq. (5) shows that the set $\{\phi_0, \phi_1\}$ forms an (incomplete) ortho*normal* set in $\mathcal{L}^2$.