

Useful Identities in Quantum Optics

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Notation

Latin letters with a hat ($\hat{a}, \hat{x}, \hat{y}, \hat{z}$) denote linear operators.

Greek letters (α, λ) denote \mathbb{C} numbers.

The hermitian adjoint of \hat{a} is denoted \hat{a}^\dagger .

The commutator of two operators \hat{x} and \hat{y} is defined as: $[\hat{x}, \hat{y}] := \hat{x}\hat{y} - \hat{y}\hat{x}$

Leibniz's law

$$[\hat{x}, \hat{y}\hat{z}] = [\hat{x}, \hat{y}]\hat{z} + \hat{y}[\hat{x}, \hat{z}], \quad [\hat{x}\hat{y}, \hat{z}] = [\hat{x}, \hat{z}]\hat{y} + \hat{x}[\hat{y}, \hat{z}]$$

Jacobi Identity

$$[\hat{x}, [\hat{y}, \hat{z}]] + [\hat{y}, [\hat{z}, \hat{x}]] + [\hat{z}, [\hat{x}, \hat{y}]] = 0$$

Baker-Campbell-Hausdorff formula

If $[\hat{x}, [\hat{x}, \hat{y}]] = [\hat{y}, [\hat{x}, \hat{y}]] = 0$ then

$$e^{\hat{x}+\hat{y}} = \exp\left\{\left(-\frac{1}{2}[\hat{x}, \hat{y}]\right)\right\} e^{\hat{x}} e^{\hat{y}} = \exp\left\{\left(\frac{1}{2}[\hat{x}, \hat{y}]\right)\right\} e^{\hat{y}} e^{\hat{x}}$$

Hadamard lemma

$$e^{i\lambda\hat{x}} \hat{y} e^{-i\lambda\hat{x}} = \hat{y} + i\lambda[\hat{x}, \hat{y}] + \frac{(i\lambda)^2}{2!}[\hat{x}, [\hat{x}, \hat{y}]] + \dots$$

Identities for Bosonic Operators

The bosonic operators \hat{a} and \hat{a}^\dagger satisfy $[\hat{a}, \hat{a}^\dagger] = \hat{1}$

$$\begin{aligned} [\hat{a}, f(\hat{a}, \hat{a}^\dagger)] &= \frac{\partial f}{\partial \hat{a}^\dagger}, \quad [\hat{a}^\dagger, f(\hat{a}, \hat{a}^\dagger)] = -\frac{\partial f}{\partial \hat{a}}, \\ e^{-\alpha \hat{a}^\dagger \hat{a}} f(\hat{a}, \hat{a}^\dagger) e^{\alpha \hat{a}^\dagger \hat{a}} &= f(\hat{a} e^\alpha, \hat{a}^\dagger e^{-\alpha}) \end{aligned}$$

References

[1] J.J. Sakurai, *Modern Quantum Mechanics*, Addison Wesley Longman

[2] M.O. Scully, M.S. Zubairy *Quantum Optics*, Cambridge University Press.