Commonly used matrix decompositions in Quantum Optics

Nicolás Quesada

This documents aims to provide recipes to compute certain decomposition that are commonly used when doing (Gaussian) quantum optics.

TODO: Add acknowledgement and bibliography and historical background.

1 Takagi (Autonne)

Given a symmetric (and in general complex) matrix $\boldsymbol{M} = \boldsymbol{M}^T \in \mathbb{C}^{\ell\times\ell}$ its Takagi (Autonne) decomposition is given by

$$\boldsymbol{M} = \boldsymbol{W}\boldsymbol{\Lambda}\boldsymbol{W}^T, \quad \boldsymbol{\Lambda} = \left[\oplus_{i=1}^{\ell}\lambda_i\right] \tag{1}$$

where $\boldsymbol{W}$ is unitary, $\boldsymbol{W}\boldsymbol{W}^\dagger = \mathbb{1}_\ell$ and $\lambda_i \geq 0$. This decomposition is a singular value decomposition (SVD), albeit one with a special symmetry that makes explicit that the matrix being decomposed is symmetric, indeed the decomposition in the right-hand side of the Eq. (1) makes explicit that the object in the left hand side is symmetric.

To obtain the decomposition we first obtain an SVD of $\boldsymbol{M}$

$$\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}^\dagger \tag{2}$$

For arbitrary matrices there is nothing we can about the relation between the unitary matrices $\boldsymbol{U}$ and $\boldsymbol{V}$. However, $\boldsymbol{M}$ is no arbitrary matrix and indeed we will show that for *symmetric* $\boldsymbol{M}$ is holds that $\boldsymbol{V}^T\boldsymbol{U}$ is a *diagonal* unitary matrix, i.e.,

$$e^{i\boldsymbol{\Phi}} = \boldsymbol{V}^T\boldsymbol{U} = \boldsymbol{U}^T\boldsymbol{V} = \oplus_{j=1}^{\ell} e^{i\phi_j}. \tag{3}$$

Note that the equation above can be easily manipulated to give

$$\boldsymbol{U}^T = e^{i\boldsymbol{\Phi}}\boldsymbol{V}^\dagger \longleftrightarrow e^{-i\boldsymbol{\Phi}/2}\boldsymbol{U}^T = e^{i\boldsymbol{\Phi}/2}\boldsymbol{V}^\dagger \longleftrightarrow \left(\boldsymbol{U}e^{-i\boldsymbol{\Phi}/2}\right)^T = e^{i\boldsymbol{\Phi}/2}\boldsymbol{V}^\dagger \tag{4}$$

Once we establish the equation above the Takagi-Autonne decomposition simply follows from manipulating the SVD

$$\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}^\dagger = \boldsymbol{U}e^{-i\boldsymbol{\Phi}/2}e^{i\boldsymbol{\Phi}/2}\boldsymbol{\Lambda}\boldsymbol{V}^\dagger = \boldsymbol{U}e^{-i\boldsymbol{\Phi}/2}\boldsymbol{\Lambda}e^{i\boldsymbol{\Phi}/2}\boldsymbol{V}^\dagger \tag{5}$$

$$= \boldsymbol{U}e^{-i\boldsymbol{\Phi}/2}\boldsymbol{\Lambda}e^{i\boldsymbol{\Phi}/2}\boldsymbol{V}^\dagger = \boldsymbol{U}e^{-i\boldsymbol{\Phi}/2}\boldsymbol{\Lambda}(\boldsymbol{U}e^{-i\boldsymbol{\Phi}/2})^T \tag{6}$$

From which we identity relative Eq. (1) that

$$\boldsymbol{W} = \boldsymbol{U}e^{-i\boldsymbol{\Phi}/2} = \boldsymbol{U}\sqrt{\boldsymbol{U}^T\boldsymbol{V}} \tag{7}$$

To see why the $\boldsymbol{W}$ above corresponds to the matrix giving the correct decomposition in Eq. (1) note that the following matrix is symmetric

$$\boldsymbol{L} = \boldsymbol{L}^T = \boldsymbol{V}^T \boldsymbol{M} \boldsymbol{V} = \boldsymbol{V}^T \boldsymbol{U} \boldsymbol{\Lambda} \tag{8}$$

and moreover we easily find

$$\boldsymbol{L}^\dagger \boldsymbol{L} = \boldsymbol{V}^\dagger \boldsymbol{M}^\dagger \boldsymbol{V}^* = \boldsymbol{V}^\dagger \boldsymbol{M}^\dagger \boldsymbol{M} \boldsymbol{V} = \boldsymbol{\Lambda} \tag{9}$$

$$\boldsymbol{L} \boldsymbol{L}^\dagger = \boldsymbol{V}^T \boldsymbol{M} \boldsymbol{V} \boldsymbol{V}^\dagger \boldsymbol{M}^\dagger \boldsymbol{V}^* = \boldsymbol{V}^T \boldsymbol{M} \boldsymbol{M}^\dagger \boldsymbol{V}^* = (\boldsymbol{V}^\dagger \boldsymbol{M}^\dagger \boldsymbol{M} \boldsymbol{V})^* = \boldsymbol{\Lambda}^* = \boldsymbol{\Lambda} \tag{10}$$

The two equations above imply that $\boldsymbol{L}$ must be a complex diagonal matrix of the form

$$\boldsymbol{L} = \left[\oplus_{j=1}^{\ell} e^{i\phi_j} \right] \boldsymbol{\Lambda} = e^{i\boldsymbol{\Phi}} \boldsymbol{\Lambda} \tag{11}$$

From which it follows that $\boldsymbol{V}^T \boldsymbol{U}$ is diagonal, as claimed. This derivation follows the presentation of Caves [**cavesnote**].

2 Bloch-Messiah (Euler)

3 Williamson