High-gain twin-beam generation in waveguides: from Maxwell's equations to efficient simulation

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Abstract: We provide an efficient method for the calculation of the quantum state describing high-gain, twin-beam generation in waveguides that is derived from a canonical treatment of Maxwell's equations and accommodates self- and cross-phase modulation.

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1. Introduction

The generation of twin beams is an important technique for the production of nonclassical light. In early experiments, the twin beams were generated with low intensities over a manifold of modes. Recent developments in photonics have allowed for the tight confinement of the travelling waves participating in the three- or four-wave mixing process necessary for the generation of twin beams [1]. These developments have moved the focus of theoretical descriptions of twin beam generation from the perturbative regime to the nonperturbative regime. In this work, we construct a theory for the description of high-gain squeezing satisfying the following criteria:

- The theory is directly based on a canonical Hamiltonian formulation of the nonlinear macroscopic Maxwell equations.
- The creation and annihilation operators describe photons in the modes of integrated optical structures, and their evolution fully respects the effects of self- and cross-phase modulation as well as twin beam generation beyond the perturbative regime..
- The theory can be used to calculate the full quantum state $|\Psi\rangle$ characterizing the twin beams.
- The equations of motion that result can be solved numerically with unprecedented ease.

Using these results we investigate and benchmark the simulation of the simplest waveguide SPDC geometry: a nonlinear region that is uniform over a finite region of 1-D space.

2. Deriving canonical equations of motion and solving them

We take $\mathbf{B}$ and $\mathbf{D}$ as the fundamental fields, which allows for a correct canonical quantization of the electromagnetic field in a nonlinear material [2]. The linear Hamiltonian of the field is $H_{\mathrm{L}} = \int d\mathbf{r} \left(\frac{\mathbf{B}^2}{2\mu_0} + \frac{(1+\chi^{(1)})^{-1}}{2\varepsilon_0}\mathbf{D}^2 \right) = \int dk \sum_\mu \hbar\omega_\mu(k) b^\dagger_{\mu k} b_{\mu k}$, and we write the displacement field as $\mathbf{D}(\mathbf{r}) = \sum_\mu \int dk \sqrt{\frac{\hbar\omega_{\mu k}}{4\pi}} b_{\mu k} \mathbf{d}_{\mu k(x,y)} e^{ikz} + \text{H.c.},)$ where $\mathbf{d}_{\mu k}(x,y)$ is the (properly normalized [2]) transverse profile of the field, μ is a waveguide mode index, and $\omega_{\mu k}$ is the frequency associated with wavevector k in mode μ; a similar decomposition holds for $\mathbf{B}$ The boson operators $b_{\mu,k}$ satisfy the *equal-time* commutation relations $[b_{\mu k}(t), b_{\mu' k'}(t)] = 0$ and $[b_{\mu k}(t), b^\dagger_{\mu' k'}(t)] = \delta_{\mu,\mu'}\delta(k-k')$.

In what follows we assume that the signal ($\mu = s$) and idler ($\mu = i$) beams generated in the nonlinear region are narrow enough in frequency that for each field we can approximate locally the dispersion relation as linear, $k - \bar{k}_\mu \approx (\omega - \bar{\omega}_\mu)/v_\mu$ where v_μ is the group velocity of mode μ and $\bar{k}_\mu$ and $\bar{\omega}_\mu$ are suitably defined central wavevectors and frequencies for each mode. We make the same assumption for the pump field $\mu = p$ sent into the nonlinear region. We now consider the nonlinear part of the Hamiltonian, $H_{NL} = \int d\mathbf{r} \left(-\frac{\chi^{(2)}\mathbf{D}^3}{3\varepsilon_0^2 n_0^6} - \frac{\chi^{(3)}\mathbf{D}^4}{4\varepsilon_0^2 n_0^8} \right)$ (n_0 is a typical refractive index and $\chi^{(i)}$ are the standard nonlinear susceptibilities) in which we use our expansion for the $\mathbf{D}$ field and only keep energy-

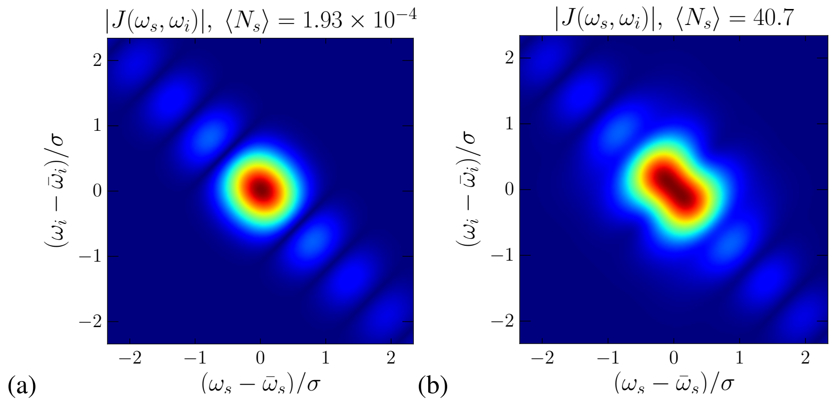

Fig. 1. Absolute value of the (a) low- and (b) high-gain joint spectral amplitude for a top-hat nonlinearity profile.

and phase-matched terms. Thus for the three-wave mixing terms we only keep terms of the form $b_{pk}b^{\dagger}_{sk'}b^{\dagger}_{ik''}+\text{H.c.}$ giving rise to spontaneous parametric down-conversion, whereas for the four-wave wave mixing we keep only terms of the form $b^{\dagger}_{pk}b_{pk'}b^{\dagger}_{pk''}b_{pk'''}$ and $b^{\dagger}_{pk}b_{pk'}b^{\dagger}_{xk''}b_{xk'''}$ $(x\in\{s,i\})$ giving rise to self-phase modulation of the pump and cross-phase modulation of the pump on the signal and idler. With the Hamiltonian in hand we can now calculate the equations of motion for the operators $b_{\mu k}$ by simply taking $\frac{\partial}{\partial t}b_{\mu k}(t)=\frac{i}{\hbar}[H_L+H_{NL},b_{\mu k}(t)]$. In the classical-undepleted pump limit one can introduce suitably defined Fourier transformed operators $a_x(z,\omega)$, with equations [3]

$$\frac{\partial}{\partial z}a_s(z,\omega)=i\Delta k_s(\omega)a_s(z,\omega)+i\frac{\gamma_{\text{XPM},s}h_s(z)}{2\pi}\int d\omega'\mathscr{E}_p(\omega-\omega')a_s(z,\omega')+i\frac{\gamma_{\text{PDC}}g(z)}{\sqrt{2\pi}}\int d\omega'\beta_p(z,\omega+\omega')a^{\dagger}_i(z,\omega'),$$

and likewise for the idler by switching $s\leftrightarrow i$. In the last equation $\Delta k_x=\left(\frac{1}{v_x}-\frac{1}{v_p}\right)(\omega-\bar{\omega}_x)$, $\gamma_{\text{PDC}}\propto\chi^{(2)}$ and $\gamma_{\text{XPM}}\propto\chi^{(3)}$ characterize the strength of the three- and four-wave mixing interactions and $h_x(z)$ and $g(z)$ their spatial extent. Finally, $\beta_p(z,\omega)$ is the classical spectral distribution of the pump, which evolves due to SPM according to $\partial_z\beta_p(z,\omega)=i\gamma_{\text{SPM}}\int d\omega'\mathscr{E}_p(\omega-\omega')\beta_p(z,\omega')$ where $\mathscr{E}_p(\omega)=\int d\omega\beta^*_p(z_0,\omega''-\omega)\beta_p(z_0,\omega'')$. These equations can be solved numerically using matrix exponentiation, and lead to the following linear Bogoliubov transformation

$$a^{(\text{out})}_s(\omega)=\int d\omega' U^{s,s}(\omega,\omega')\,a^{(\text{in})}_s(\omega')+\int d\omega' U^{s,i}(\omega,\omega')\,a^{\dagger(\text{in})}_i(\omega'),\tag{1}$$

The input/output $a^{\text{in/out}}_x(\omega)$ operators are related to the proper-evolving-in-time Heisenberg operators via

$$b^{\text{in/out}}_{xk}\equiv b_{xk}(t_{0/1})e^{i\omega(k)t_{0/1}}=\sqrt{v_x}e^{-\Delta k_x z_{0/1}}a(z_{0/1},\omega(k))=\sqrt{v_x}a^{\text{in/out}}(\omega(k)),\tag{2}$$

where recall $x\in\{s,i\}$, we assumed a linear dispersion relation, and$z_{0/1}$ and $t_{0/1}$ are positions and times long before/after the pump as entered the nonlinear region. Due to the $SU(1,1)$ symmetry of the equations of motion one can show that the transfer functions $U^{x,x'}(\omega,\omega')$ can be jointly decomposed

$$U^{s,i}(\omega,\omega')=\sum_l\sinh(r_l)[\rho^{(l)}_s(\omega)][\tau^{(l)}_i(\omega')],\quad (U^{i,i}(\omega,\omega'))^*=\sum_l\cosh(r_l)[\rho^{(l)}_i(\omega)]^*[\tau^{(l)}_i(\omega')],\tag{3}$$

and a similar equation for the signal by letting $s\leftrightarrow i$. Based on this decomposition, and noting that the initial state in the signal and idler beams is vacuum, we can finally write the output state $|\Psi\rangle=\exp\left(\int d\omega d\omega' J(\omega,\omega')a^{(\text{in})\dagger}_s(\omega)a^{(\text{in})\dagger}_i(\omega')-\text{H.c.}\right)|\text{vac}\rangle$ where the joint spectral amplitude (JSA) is given by $J(\omega,\omega')=\sum_l r_l\rho^{(l)}_s(\omega)\rho^{(l)}_i(\omega')$. This completes the program sketched in the Introduction. In Fig. 1 we use the method just described to obtain the low- and high- gain regimes for a waveguide with a top-hat nonlinearity profile. In the low gain regime where the mean photon number in the signal or idler modes is $\langle N_s\rangle=\langle N_i\rangle\sim 2\times10^{-4}\ll 1$ the JSA is simply the product of the pump function (which we assumed Gaussian) and the Fourier transform of the nonlinearity profile, which for a top-hat functions results in a sinc function. This can be contrasted with the high -gain regime $\langle N_s \rangle = \langle N_i \rangle \sim 41$ where due to time-ordering effects the JSA has a much more complicated frequency dependence that can be understood as interference between many different Magnus-expansion terms [4]. Note that for a grid size of 600×600 points the JSAs presented in the figure require just a few seconds to be generated in a desktop computer, which should be compared with the hours it takes using other methods [5].

The theory we have developed has also been used to explain a recent high-gain experiment [6] where the JSA is not only modified by the effects of time-ordering but also by self- and cross-phase modulation.

References

1. G. Harder, *et al.*, "Single-Mode Parametric-Down-Conversion States with 50 Photons as a Source for Mesoscopic Quantum Optics", Phys. Rev. Lett. **116**, 143601 (2016).
2. N. Quesada and J.E. Sipe, "Why you should not quantize using the electric field in nonlinear optics", Opt. Lett. **42**, 3443-3446 (2017).
3. N. Quesada, *et al.*"Efficient simulation of high-gain twin-beam generation in waveguides", arXiv:1907.01958 (2019)
4. N. Quesada and J.E. Sipe, "Time-ordering effects in the generation of entangled photons using nonlinear optical processes", Phys. Rev. Lett. **114**, 093903 (2015).
5. A. Christ *et al.*, "Theory of quantum frequency conversion and type-II parametric down-conversion in the high-gain regime", New J. Phys. **15**, 053038 (2013).
6. G. Triginer *et al.*, "Understanding high gain twin beam sources using cascaded stimulated emission", arXiv:1911.05708 (2019).