

An identity involving the imaginary error function $\operatorname{erfi} x$

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The imaginary error function $\operatorname{erfi}(x)$ is customarily defined as $\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt e^{t^2}$. It is related to the error function by $\operatorname{erfi}(x) = \operatorname{erf}(ix)/i$, and can be also defined as

$$\frac{1}{\pi} \int_{-\infty}^{\infty} dy \frac{e^{ay^2+2by}}{y} = \frac{2}{\pi} \int_0^{\infty} dy e^{ay^2} \frac{\sinh(2by)}{y} = \operatorname{erfi}\left(\frac{b}{\sqrt{-a}}\right) \text{ for } a < 0. \quad (1)$$

In this note we investigate a property of a function related to the imaginary error function.

We can define two functions

$$\phi_0(x) \equiv \frac{\exp(-x^2)}{\sqrt[4]{\pi/2}}, \quad \phi_1(x) \equiv \mathcal{N}(\alpha) \phi_0(x) \operatorname{erfi}(\alpha x), \quad \mathcal{N}(\alpha) = \sqrt{\frac{\pi}{2 \sin^{-1}\left(\frac{\alpha^2}{2-\alpha^2}\right)}}, \quad (2)$$

whose \mathcal{L}^2 -orthogonality is seen from their respective parities.

We here show that ϕ_1 is, like ϕ_0 , \mathcal{L}^2 -normalized; to this end consider the integral

$$I = \int_{-\infty}^{\infty} dx \phi_1^2(x) = \mathcal{N}^2 \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx \exp(-2x^2) \operatorname{erfi}^2(\alpha x). \quad (3)$$

This integral is, to the best of our knowledge, not currently tabulated in standard computer algebra systems (CAS) like [Mathematica](#) or [Maple](#) or the [Wolfram Functions Site](#). We use Eq. (1) to write

$$I = \frac{\mathcal{N}^2}{\pi^2} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx \exp(-2x^2) \int_{-\infty}^{\infty} \frac{du}{u} \exp\left(-\frac{u^2}{\alpha^2} + 2ux\right) \int_{-\infty}^{\infty} \frac{dv}{v} \exp\left(-\frac{v^2}{\alpha^2} + 2vx\right) \quad (4a)$$

$$= \frac{\mathcal{N}^2}{\pi^2} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{du}{u} \int_{-\infty}^{\infty} \frac{dv}{v} \exp\left(-\frac{u^2}{\alpha^2} - \frac{v^2}{\alpha^2}\right) \int_{-\infty}^{\infty} dx \exp(-2x^2 + 2ux + 2vx) \quad (4b)$$

$$= \frac{\mathcal{N}^2}{\pi^2} \int_{-\infty}^{\infty} \frac{du}{u} \exp(-\beta u^2) \int_{-\infty}^{\infty} \frac{dv}{v} \exp(-\beta v^2 + uv), \quad (4c)$$

where we have liberally switched the order of the integrals and written $\beta = 1/\alpha^2 - 1/2$. We now again use Eq. (1) to find

$$I = \frac{\mathcal{N}^2}{\pi^2} \int_{-\infty}^{\infty} \frac{du}{u} \exp(-\beta u^2) \left(\pi \operatorname{erfi}\left(\frac{u}{2\sqrt{\beta}}\right) \right) = \frac{2\mathcal{N}^2}{\pi} \underbrace{\int_0^{\infty} \frac{du}{u} \exp(-\beta u^2) \operatorname{erfi}\left(\frac{u}{2\sqrt{\beta}}\right)}_{\sin^{-1} \frac{1}{2\beta}} = 1. \quad (5)$$

The last integral can be evaluated using any of the aforementioned CAS for $|\beta| \geq 1/2$, $\operatorname{Re}[\beta] > 0$ (i.e., $|\alpha| \leq 1$). Eq. (5) shows that the set $\{\phi_0, \phi_1\}$ forms an (incomplete) orthonormal set in \mathcal{L}^2 .