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The Classical Complexity of Gaussian Boson Sampling

N. Quesada
Xanadu, 777 Bay Street, Toronto, ON, Canada

Abstract

Gaussian Boson Sampling (GBS) is a recently introduced sampling problem where squeezed states are sent into an $M \times M$ interferometer whose outputs are measured using photon-number resolving detectors. A GBS sample is specified by a set of M non-negative integers $S = \{n_1, n_2, \dots, n_M\}$ and has a probability that is proportional to the Hafnian of a certain matrix $\boldsymbol{A}_S$ of size $2\ell = 2\sum_{k=1}^{M} n_k$. We show that one can generate exactly a sample S with computational effort that is comparable to the time required for calculating the Hafnian of the matrix $\boldsymbol{A}_S$ that determines the probability of the sample.

Let $\boldsymbol{Q}$ be the covariance matrix of the Q function of the M-mode Gaussian state ϱ in the $\boldsymbol{\alpha}, \boldsymbol{\alpha}^*$ basis at the output of the interferometer. Let $\bar{I} = \{i_1, i_2, \dots, i_m\}$ be a set of indices (integers) specifying a subset of the M modes and in particular we write $\bar{I} = [k] = \{1, 2, \dots, k\}$ for the first k modes. We write $\boldsymbol{Q}^{\bar{I}}$ to indicate the covariance matrix of the modes specified by the index set $\bar{I}$. Define $\boldsymbol{O}^{\bar{I}} = \mathbb{1} - \left(\boldsymbol{Q}^{\bar{I}}\right)^{-1}$, $\boldsymbol{A}^{\bar{I}} = \boldsymbol{X}\boldsymbol{O}^{\bar{I}}$, with $\boldsymbol{X} = \left[\begin{smallmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right]$. These matrices can be used to calculate photon-number and threshold detection probabilities as

$$p(\bar{N} = \bar{n}) = \frac{\mathrm{Haf}(\boldsymbol{A}^{\bar{I}}_{\bar{n}})}{\bar{n}!\sqrt{\det(\boldsymbol{Q}^{\bar{I}})}} \quad \text{(PNR)}, \qquad p(\bar{N} = \bar{n}) = \frac{\mathrm{Tor}(\boldsymbol{O}^{\bar{I}}_{\bar{n}})}{\sqrt{\det(\boldsymbol{Q}^{\bar{I}})}} \quad \text{(Threshold)}, \tag{1}$$

where $\bar{N} = \{N_{i_1}, N_{i_2}, \dots, N_{i_m}\}$ is the collection of random variables that represent photon number measurements in mode subset $\bar{I}$, $\bar{n} = \{n_{i_1}, n_{i_2}, \dots, n_{i_m}\}$ is a set of integers in the support of the random variables $\bar{N}$ and Haf is the Hafnian matrix function [1]. Similarly, for threshold detection, $\bar{N}$ is a collection of binary random variables, $\bar{n}$ is a bitstring specifying which detectors clicked and Tor is the Torontonian [2].

Now, we want to introduce an algorithm to generate samples of the random variable $\{N_1, \dots, N_M\}$ that distribute according to the probabilities in Eq. (1). To generate samples, we proceed as follows: First, we can always calculate the following probabilities $p(N_1 = n_1) = \mathrm{Haf}\left(\boldsymbol{A}^{[1]}_{\{n_1\}}\right) / \left(n_1!\sqrt{\det(\boldsymbol{Q}^{[1]})}\right)$. Having constructed said probabilities, we can always generate a sample for the first mode. This will fix $N_1 = n_1$. Now we want to sample from $N_2|N_1 = n_1$. To this end, we use the definition of conditional probability

$$p(N_2 = n_2|N_1 = n_1) = \frac{p(N_2 = n_2, N_1 = n_1)}{p(N_1 = n_1)} = \frac{\mathrm{Haf}\left(\boldsymbol{A}^{[2]}_{\{n_1,n_2\}}\right)}{n_1!n_2!\sqrt{\det(\boldsymbol{Q}^{[2]})}} \frac{1}{p(N_1 = n_1)}. \tag{2}$$

We can, as before, calculate this quantity for a set of values of n_2 and then generate a sample of N_2 with value n_2. Note that the factor $p(N_1 = n_1)$ is already known from the previous step. We can now do induction:

$$p(N_k = n_k|N_{k-1} = n_{k-1}, \dots, N_1 = n_1) = \frac{p(N_k = n_k, N_{k-1} = n_{k-1}, \dots, N_1 = n_1)}{p(N_{k-1} = n_{k-1}, \dots, N_1 = n_1)} \tag{3}$$

$$= \frac{\mathrm{Haf}\left(\boldsymbol{A}^{[k]}_{\{n_1,n_2,\dots,n_k\}}\right)}{n_1!n_2!\dots n_k!\sqrt{\det(\boldsymbol{Q}^{[k]})}} \frac{1}{p(N_{k-1} = n_{k-1}, \dots, N_1 = n_1)}. \tag{4}$$

We iterate for a finite set of values n_k and sample the random variable N_k conditioned on the previous values. We can do this for all the M modes to obtain an M-mode sample S. The argument just presented also works for Torontonians with the replacements $\boldsymbol{A} \to \boldsymbol{O}$, Haf $\to$ Tor. Now, let us consider the collision-free regime, relevant to quantum supremacy. In that case, the two probabilities in Eq. (1) are identical and we can estimate the complexity of sampling. The calculation of the numerator in Eq. (4) where $\ell = \sum_{i=1}^{k} n_i$ clicks have been recorded requires the Torontonian of matrix of size $2\ell \times 2\ell$ which scales like $O(2^\ell)$. So, for a sample in which a total of ℓ clicks are recorded, the complexity scales like $\sum_{r=0}^{\ell} O(2^r) \sim 2O(2^\ell)$ which is the same complexity as calculating a Hafnian or Torontonian of size $2\ell \times 2\ell$. This shows that generating a GBS sample using a classical computer has (except for prefactors) the same complexity as calculating a GBS probability.

[1] C. Hamilton et al., *Gaussian Boson Sampling*, Phys. Rev. Lett. **119**, 170501 (2017)

[2] N. Quesada et al., *Gaussian boson sampling with threshold detectors*, Phys. Rev. A **98**, 062322 (2018)