Lab Notes PH-315
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Transfer function, Laplace transform, Low pass filter

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1. INTRODUCTION

Transfer functions are used to calculate the response C(t) of a system to a given input signal R(t). Here t stands for the time variable.

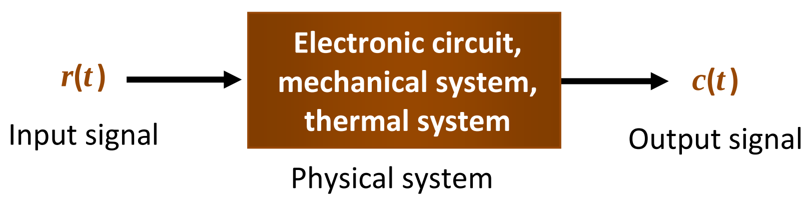


Fig. 1 Given an input signal, we would like to know the system response C(t).

The dynamic behavior of a physical system are typically described by differential (and/or integral) equations:

- For a given input signal R(t), these equations need to be solved in order to find C(t).
- Alternatively, instead of trying to find the solution in the time domain, each time-variable, as well as the differential equations, can be **transformed** to a different variable domain in which the solutions can be obtain in a more straightforward way; then an inverse transform would take place the solution into the time domain

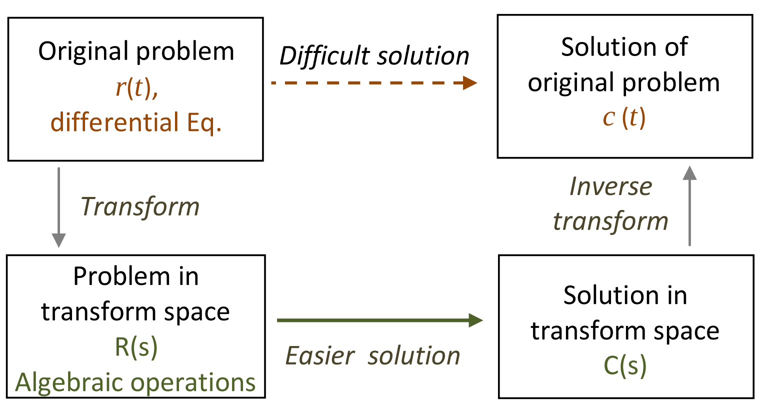


Fig. 2 Solving the equations in a different domain and then applying an inverse transform to obtain the solution in the time domain

One of those transforms is the Laplace transformation

2. THE LAPLACE TRANSFORMATION $\mathcal{L}$

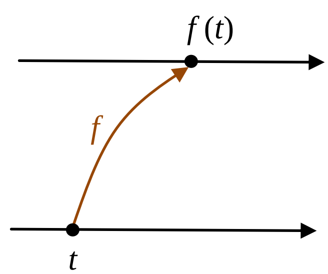
The Laplace transform F=F(s) of a function $f = f(t)$ is defined by,

$$f \xrightarrow{\mathcal{L}} \mathrm{F}$$

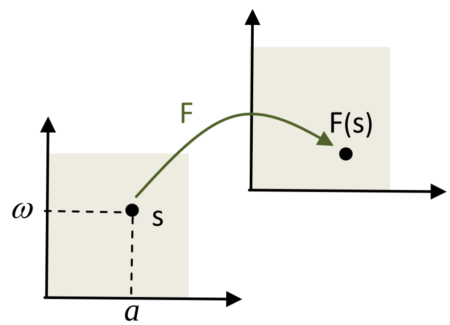
$$\mathcal{L}(f) = \mathrm{F}$$

$$\mathrm{F}(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt \tag{1}$$

The variable s is a complex number, s = a +jω.

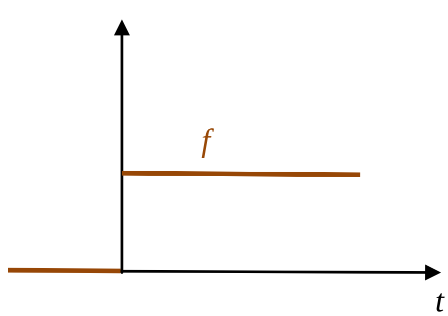


Time domain



Laplace domain

Example f is the unit step function



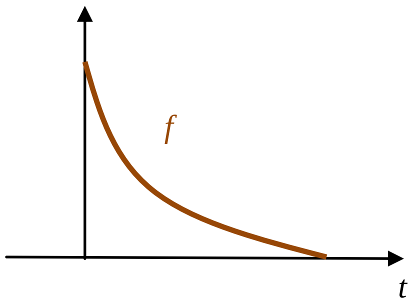
Time domain

$$\begin{aligned} \mathrm{F}(s) &= \int_0^{\infty} f(t)\, e^{-st}\, dt \\ &= \int_0^{\infty} 1 \cdot e^{-st}\, dt \\ &= \frac{1}{s} \end{aligned} \tag{2}$$

Laplace domain

Example A signal response from a system (temperature, for example) respond, after an external; excitation has stopped, by decaying exponentially. Let's find out how such decay is characterized by a Laplace transformation.

f is a decaying exponential $f(t) = \mathrm{A}e^{-\alpha t}$



$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$F(s) = \int_0^\infty \mathrm{A}e^{-\alpha t}\; e^{-st}\, dt$$

$$= \frac{\mathrm{A}}{s + \alpha} \qquad (2)$$

Time domain *Laplace domain*

Example. As we mentioned in the introduction, the system response is governed by differential equations. We would like to know then, how $\frac{df}{dt}$ and $\frac{d^2 f}{dt^2}$ transform by a Laplace transformation. For simplicity, and clarity, let's use the notation: $\frac{df}{dt} = f\,'$ and $\frac{d^2 f}{dt^2} = f\,''$.

- If $F = \mathcal{L}(f)$ evaluate $\mathcal{L}(f\,')$

$$\mathcal{L}(f\,')\big|_s = \int_0^\infty f'(t)\, e^{-st}\, dt$$

$$= f(t)\, e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)\, e^{-st}\, dt$$

$$= -f(0) + s\int_0^\infty f(t)\, e^{-st}\, dt$$

$$= -f(0) + s\,[\mathcal{L}(f)]\big|_s$$

$$= -f(0) + sF(s) \qquad (3)$$

Laplace transformation of the derivative

Typically, one proceeds putting the initial conditions equal to zero. (The situation with initial conditions different than zero are added in a separate simpler procedure). Thus,

$$\mathcal{L}(f\,')\big|_s = s\,F(s) \qquad (3)'$$

Laplace transformation of the derivative with the initial conditions equal to zero

- If $F = \mathcal{L}(f)$ evaluate $\mathcal{L}(f\,'')$

$$\mathcal{L}(f'')\big|_s = \int_0^\infty f''(t)\, e^{-st}\, dt$$

$$= -f'(0) - s\, f(0) + s^2\, F(s) \qquad \textit{Laplace transformation of the second derivative} \qquad (4)$$

Typically, one proceeds putting the initial conditions equal to zero. (The situation with initial conditions different than zero are added in a separate simpler procedure). Thus,

$$\mathcal{L}(f'')\big|_s = s^2\, F(s) \qquad \textit{Laplace transformation of the second derivative with the initial conditions equal to zero} \qquad (4)'$$

Example. Sometimes the response signal of a system (the voltage across a capacitor, for example) must be given in terms of the integral of another quantity (the integral of the corresponding current across the capacitor). It is convenient, then, to obtain the Laplace transformation of an indefinite integral $g(t) = \int_0^t f(u)\, du$

- If $F = \mathcal{L}(f)$ and $g(t) = \int_0^t f(u)\, du$, evaluate $\mathcal{L}(g)$

$$\mathcal{L}(g)\big|_s = \int_0^\infty g(t)\, e^{-st}\, dt = \int_0^\infty \left[\int_0^t f(u)\, du\right] e^{-st}\, dt$$

$$= -\left[\int_0^t f(u)\, du\right] \frac{1}{s} e^{-st}\Big|_0^\infty - \int_0^\infty [\, f(t)\,]\left[-\frac{1}{s}\, e^{-st}\right] dt$$

$$= \frac{1}{s}\int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \frac{1}{s}\, F(s) \qquad \textit{Laplace transformation of the indefinite integral} \qquad (5)$$

3. TRANSFER FUNCTIONS

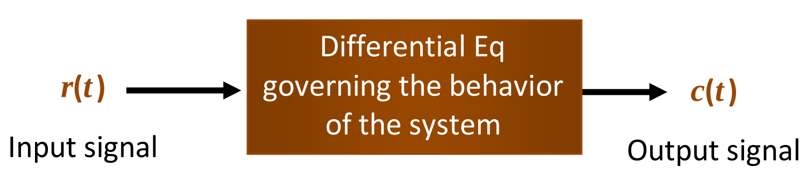


Fig. 3 Schematic of the system response in the time domain

In a simple system, the output c(t) may be governed by a second order differential equation

$$a_2\, c'' + a_1\, c' + a_o\, c = r(t)$$

Applying the Laplace transforms (3)' and (4)', one obtains

$$(a_2 s^2 + a_1 s + a_o) C(s) = R(s)$$

$$C(s) = \frac{1}{a_2 s^2 + a_1 s \;\; a_o} R(s)$$

In a more general case, the differential equation may be of higher order (higher than 2). Also the input may be composed of derivatives of a given function r=r(t). Therefore the factor $\frac{1}{a_2 s^2 + a_1 s \;\; a_o}$ may become a more elaborated function of s.

Thus, for a system in general,

$$C(s) = G(s) \; R(s) \tag{6}$$

Notice, G=G(s) characterizes the physical system. It is called the *transfer function.*

It is more typical to write,

$$G(s) = \frac{C(s)}{R(s)} \tag{7}$$

from which, for a given R=R(s) the function C=C(s) can be obtained.

Fig. 4 Schematic of the system response in the Laplace domain

Example. A system is characterized by the transfer function $G(s) = \frac{2s+3}{(s+1)\;(s+6)}$.

Find out how the system respond to a exponentially decaying input $r(t) = e^{-2t}$.

Answer:

The Laplace transformation of r gives, using expression (2), $R(s) = \frac{1}{s+2}$

The output signal, in the Laplace domain , is then given by, $C(s) = \frac{2s+3}{(s+1)\;(s+6)} \frac{1}{s+2}$

In a typical procedure, when possible, C(s) is re-written in the following form:

$$C(s) = \frac{K_1}{s+1} + \frac{K_2}{s+6} + \frac{K_3}{s+2}$$, with K_1, K_2, K_3, to be determined.

Notice, $K_1 = C(s)(s+1)\big|_{-1} = 0.8$

$$K_2 = C(s)(s+6)\big|_{-6} = -0.3$$

$$K_3 = C(s)(s+2)\big|_{-2} = -0.5$$

Thus,

$$C(s) = \frac{0.8}{s+1} + \frac{-0.3}{s+6} + \frac{-0.5}{s+2}$$

Now, using (2) we identify the time dependent functions these individual Laplace transforms come from,

$$c(t) = 0.8\,e^{-t} - 0.3\,e^{-6t} - 0.5e^{-2t}$$ Answer.

Recapitulating the process,

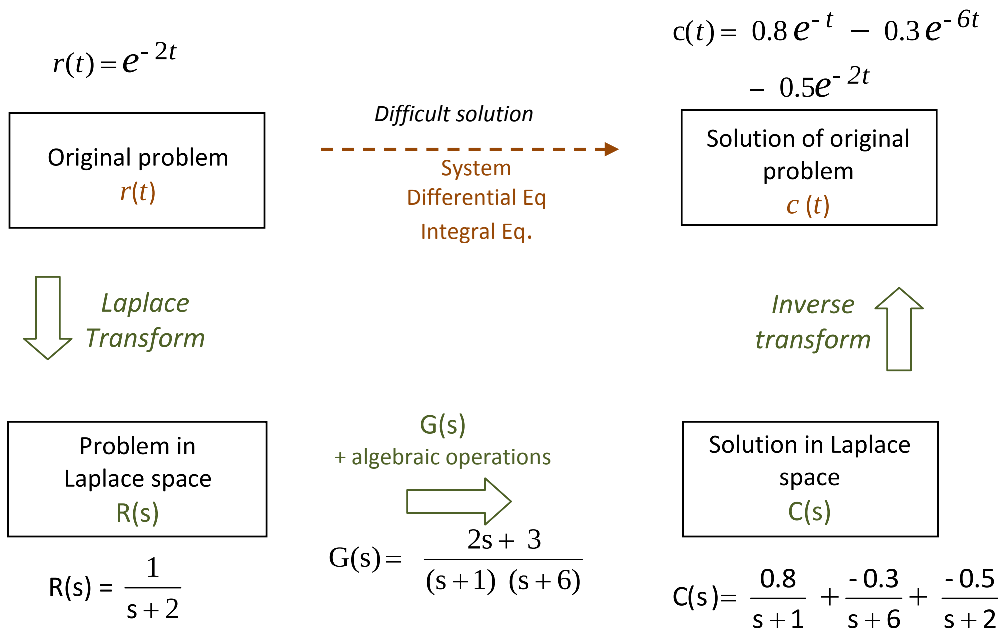


Fig. 5 Schematic representation of the solution procedure in the previous example.

In the previous example, the transfer function was given. In the next section we will figure out the transfer function for the case of electrical systems.

4. ELECTRICAL SYSTEMS

- Let's analyze the three basic elements R, C and L individually

Let $\mathsf{I} = \mathsf{I}(s)$ be the Laplace transform of $i = i\,(t)$.

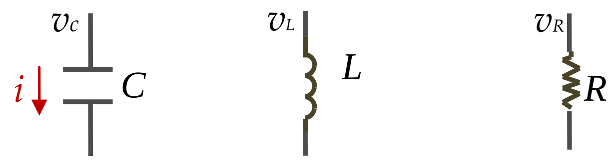


Fig. 6 Elementary passive circuit elements

Capacitor

$$v_c\,(t) = \frac{q(t)}{C}$$

$$= \frac{1}{C}\int_0^t i(u)\;du \xrightarrow{\;using\;\;(5)\;} V_c\,(s) = \frac{1}{C}\int_0^t i(u)\;du = \frac{1}{C}\,\frac{1}{s}\,\mathsf{I}(s)$$

$$V_c\,(s) = \frac{1}{Cs}\;\mathsf{I}(s) \qquad (8)$$

Inductor

$$v_L\,(t) = L\frac{d\,i(t)}{dt} \xrightarrow{\;using\;\;(3)'\;} V_L\,(s) = L\,s\;\mathsf{I}(s) \qquad (9)$$

Resistor

$$v_R\,(t) = R\,i(t) \qquad V_R\,(s) = R\;\;\mathsf{I}(s) \qquad (10)$$

Time domain *Laplace domain*

- Analysis of a simple lag network

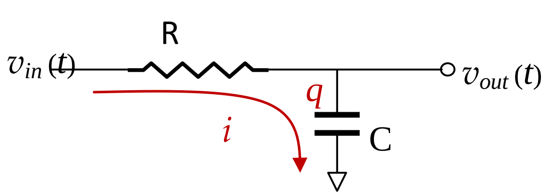


Fig. 7 Low pass filter in the time domain.

$$v_{in}(t) = R\,i(t) + \frac{q(t)}{C}$$

$$= R\,i(t) + \frac{1}{C}\int_0^t i(u)\,du \xrightarrow{\text{using (5)}} V_{in}(s) = R\,I(s) + \frac{1}{Cs}\,I(s)$$

$$= \left(R + \frac{1}{Cs}\right) I(s) \qquad (11)$$

On the other hand

$$v_{out}(t) = \frac{q(t)}{C}$$

$$= \frac{1}{C}\int_0^t i(u)\,du \xrightarrow{\text{using (5)}} V_{out}(s) = \frac{1}{Cs}\,I(s) \qquad (12)$$

From (11) and (12)

$$\frac{V_{out}}{V_{in}} = \frac{1/Cs}{R + 1/Cs}$$

$$G(s) = \frac{V_{out}}{V_{in}} = \frac{1}{RCs + 1} \qquad (13)$$

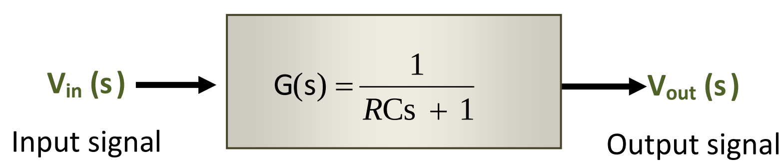


Fig. 8 Low pass filter in the Laplace domain.