CENG 460 - Introduction to Robotics Narmin Aliyeva
Fall 2018 2177269

Homework 1 - Pose Representations

1. *Proof.* For any point $(x,\ y,\ z)$ on the sphere,

$$\left(\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} - \begin{bmatrix} c_x \\ c_y \\ c_z \\ 1 \end{bmatrix}\right)^T \left(\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} - \begin{bmatrix} c_x \\ c_y \\ c_z \\ 1 \end{bmatrix}\right) = r^2$$

The center of the sphere after transformation T is located at $T[c_x, c_y, c_z]^T$.

$$\begin{aligned}
&\left(T\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} - T\begin{bmatrix} c_x \\ c_y \\ c_z \\ 1 \end{bmatrix}\right)^T \left(T\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} - T\begin{bmatrix} c_x \\ c_y \\ c_z \\ 1 \end{bmatrix}\right) \\
&\qquad = \left(R\begin{bmatrix} x \\ y \\ z \end{bmatrix} + t - R\begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} - t\right)^T \left(R\begin{bmatrix} x \\ y \\ z \end{bmatrix} + t - R\begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} - t\right) \\
&\qquad = \left(R\begin{bmatrix} x - c_x \\ y - c_y \\ z - c_z \end{bmatrix}\right)^T \left(R\begin{bmatrix} x - c_x \\ y - c_y \\ z - c_z \end{bmatrix}\right) \\
&\qquad = \begin{bmatrix} x - c_x \\ y - c_y \\ z - c_z \end{bmatrix}^T R^T R \begin{bmatrix} x - c_x \\ y - c_y \\ z - c_z \end{bmatrix} \\
&\qquad = \begin{bmatrix} x - c_x \\ y - c_y \\ z - c_z \end{bmatrix}^T R^{-1} R \begin{bmatrix} x - c_x \\ y - c_y \\ z - c_z \end{bmatrix} \\
&\qquad = r^2
\end{aligned}$$

After being transformed through homogeneous transformation T, any point $(x,\ y,\ z)$ on the sphere is located at a distance of r from the new center. $\square$

2. (a) For an eigenvalue λ of the matrix $[\hat{w}]_\times$, $\det([\hat{w}]_\times - \lambda I) = 0$.

$$
\begin{aligned}
\begin{vmatrix} -\lambda & -w_z & w_y \\ w_z & -\lambda & -w_x \\ -w_y & w_x & -\lambda \end{vmatrix} &= -\lambda \begin{vmatrix} -\lambda & -w_x \\ w_x & -\lambda \end{vmatrix} + w_z \begin{vmatrix} w_z & -w_x \\ -w_y & -\lambda \end{vmatrix} + w_y \begin{vmatrix} w_z & -\lambda \\ -w_y & w_x \end{vmatrix} \\
&= -\lambda(\lambda^2 + w_x^2) + w_z(-\lambda w_z - w_x w_y) + w_y(w_z w_x - \lambda w_y) \\
&= \lambda^3 + \lambda w_x^2 + \lambda w_z^2 + \lambda w_y^2 \\
&= \lambda^3 + \lambda = \lambda(\lambda^2 + 1) = 0 \qquad \to \lambda = 0, \pm i
\end{aligned}
$$

For an eigenvector v corresponding to eigenvalue λ, $([\hat{w}]_\times - \lambda I)v = 0$.
The unit eigenvector corresponding to $\lambda = 0$ is $\hat{w} = [w_x,\ w_y,\ w_z]^T$.

(b) The matrix R has the same eigenvectors as the matrix $[\hat{w}]_\times$, corresponding to eigenvalues $e^{\lambda\theta}$:

$$
\begin{aligned}
e^{[\hat{w}]_\times \theta} v &= \sum_{k=0}^{\infty} \frac{1}{k!}([\hat{w}]_\times \theta)^k v \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\theta^k [\hat{w}]_\times^k v \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\theta^k \lambda^k v \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}(\lambda\theta)^k v \\
&= e^{\lambda\theta} v
\end{aligned}
$$

So, the eigenvalues of R are $1, e^{i\theta}, e^{-i\theta}$.
The eigenvector with eigenvalue 1 is $\hat{w}$.

(c)

$$
\begin{aligned}
r_1^T(r_2 \times r_3) &= r_1^T([r_2]_\times r_3) \\
&= \begin{bmatrix} r_{11} & r_{21} & r_{31} \end{bmatrix} \left(\begin{bmatrix} 0 & -r_{32} & r_{22} \\ r_{32} & 0 & -r_{12} \\ -r_{22} & r_{12} & 0 \end{bmatrix} \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} \right) \\
&= \begin{bmatrix} r_{11} & r_{21} & r_{31} \end{bmatrix} \begin{bmatrix} r_{22}r_{33} - r_{23}r_{32} \\ r_{32}r_{13} - r_{12}r_{33} \\ r_{12}r_{23} - r_{22}r_{13} \end{bmatrix} \\
&= r_{11}(r_{22}r_{33} - r_{23}r_{32}) + r_{21}(r_{32}r_{13} - r_{12}r_{33}) + r_{31}(r_{12}r_{23} - r_{22}r_{13}) \\
&= \det(R)
\end{aligned}
$$

3. (a) Homogeneous transformation matrices:

$$
{}^{0}T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{1}T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{3}T_4 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

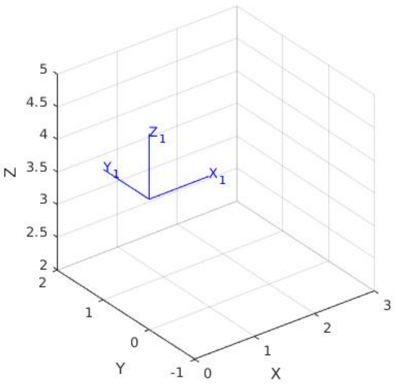

Figure 1: frame 1

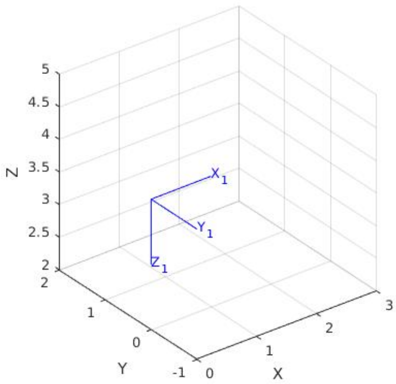

Figure 2: frame 2

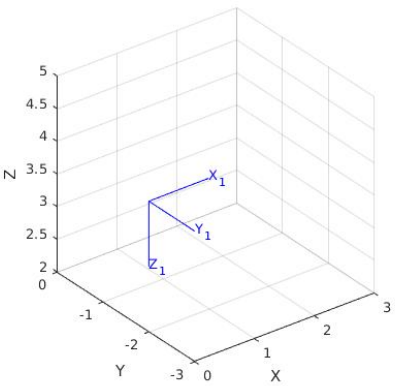

Figure 3: frame 3

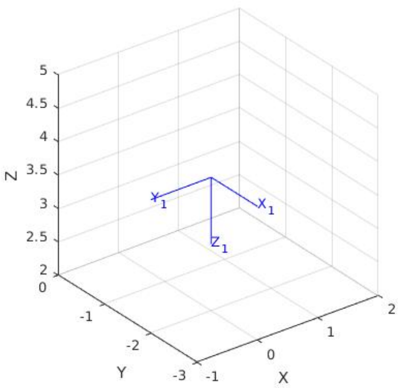

Figure 4: frame 4

Coordinates of the frame origins:

$\widetilde{{}^{i}o_i} = [0\ 0\ 0\ 1]^T$

$$\begin{aligned}
\widetilde{{}^{0}o_1} &={}^0 T_1\ \widetilde{{}^{1}o_1} = [0\ 0\ 4\ 1]^T && \longrightarrow \quad O_1 = (0,0,4)\\
\widetilde{{}^{0}o_2} &={}^0 T_1\ {}^1T_2\ \widetilde{{}^{2}o_2} = [0\ 0\ 4\ 1]^T && \longrightarrow \quad O_2 = (0,0,4)\\
\widetilde{{}^{0}o_3} &={}^0 T_1\ {}^1T_2\ {}^2T_3\ \widetilde{{}^{3}o_3} = [0\ -2\ 4\ 1]^T && \longrightarrow \quad O_3 = (0,-2,4)\\
\widetilde{{}^{0}o_4} &={}^0 T_1\ {}^1T_2\ {}^2T_3\ {}^3T_4\ \widetilde{{}^{4}o_4} = [0\ -2\ 4\ 1]^T && \longrightarrow \quad O_4 = (0,-2,4)
\end{aligned}$$

(b) $\mathring{q} = \cos\frac{\theta}{2} < \hat{v}\sin\frac{\theta}{2} >$

Step 1: $\theta = 0 \quad \longrightarrow \quad \mathring{q}_1 = 1 < 0,\ 0,\ 0 >, \quad t_1 = [0,\ 0,\ 4]^T$

Step 2: $\theta = -\pi, \quad \hat{v} = [1,\ 0,\ 0]^T \longrightarrow \quad \mathring{q}_2 = 0 < -1,\ 0,\ 0 >, \quad t_2 = [0,\ 0,\ 0]^T$

Step 3: $\theta = 0 \quad \longrightarrow \quad \mathring{q}_3 = 1 < 0,\ 0,\ 0 >, \quad t_3 = [0,\ 2,\ 0]^T$

Step 4: $\theta = \frac{\pi}{2}, \quad \hat{v} = [0,\ 0,\ 1]^T \longrightarrow \quad \mathring{q}_4 = \frac{\sqrt{2}}{2} < 0,\ 0,\ \frac{\sqrt{2}}{2} >, \quad t_4 = [0,\ 0,\ 0]^T$

4.

$$ {}^0T_1 = \begin{bmatrix} 0 & 1 & 0 & 0\\ \cos45^\circ & 0 & -\cos45^\circ & 0\\ -\cos45^\circ & 0 & -\cos45^\circ & 1\\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$ {}^0T_2 = \begin{bmatrix} 0 & 0 & -1 & 1\\ -1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$ {}^1T_2 = \begin{bmatrix} -\cos45^\circ & -\cos45^\circ & 0 & \sqrt{2}\\ 0 & 0 & -1 & 1\\ \cos45^\circ & -\cos45^\circ & 0 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_2 = {}^0T_1{}^1T_2$$

5. (a)

$$\text{Let } {}^WR_B = \begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1 \end{bmatrix} \quad \text{and } t = \begin{bmatrix} x\\ y\\ 0 \end{bmatrix}.$$

$${}^WT_R ={}^W T_B\ {}^BT_R = \begin{bmatrix} {}^WR_B & t\\ 0_{1\times3} & 1 \end{bmatrix} \begin{bmatrix} I_{3\times3} & {}^Br\\ 0_{1\times3} & 1 \end{bmatrix} = \begin{bmatrix} {}^WR_B & t +{}^W R_B\ {}^Br\\ 0_{1\times3} & 1 \end{bmatrix}$$

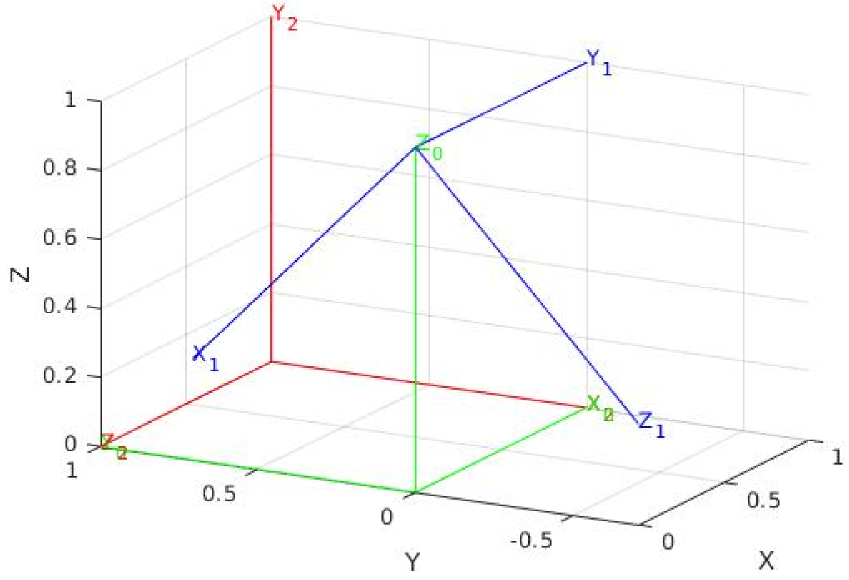

Figure 5: 3 frames asked in Q4.

(b)

$$
{}^{W}T_{desired} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & a \\ \sin\phi & \cos\phi & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ where } \phi, a, b, c \text{ are constants.}
$$

$$
{}^{W}e = {}^{W}T_{R}\, {}^{R}e = {}^{W}T_{R}\, {}^{R}T_{E}\, {}^{E}e = {}^{W}T_{desired}\, {}^{E}e
$$

$$
\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} + \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}
$$

$$
\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = \begin{bmatrix} a - x \\ b - y \\ c \end{bmatrix}
$$

$$
\begin{bmatrix} e_x(t) \\ e_y(t) \\ e_z(t) \end{bmatrix} = \begin{bmatrix} \cos(\theta(t)) & \sin(\theta(t)) & 0 \\ -\sin(\theta(t)) & \cos(\theta(t)) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a - x(t) \\ b - y(t) \\ c \end{bmatrix}
$$

$$
\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos e_\theta & -\sin e_\theta & 0 \\ \sin e_\theta & \cos e_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} \cos e_\theta & -\sin e_\theta & 0 \\ \sin e_\theta & \cos e_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos(\phi-\theta) & -\sin(\phi-\theta) & 0 \\ \sin(\phi-\theta) & \cos(\phi-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
e_\theta(t) = \phi - \theta(t)
$$

6. The cube is moving around the z axis with $w = 1rad/s$, while traveling upwards with $v_z = 1m/s$. $p(t) = [p_x(t), p_y(t), p_z(t)]^T$ describes the motion of the center of the cube:

$$
\begin{aligned}
p_x(t) &= rsin(wt) = sin(t) \\
p_y(t) &= rcos(wt) = sin(t) \\
p_z(t) &= v_z t = t
\end{aligned}
$$

See $q6.m$ for animation of the motion.

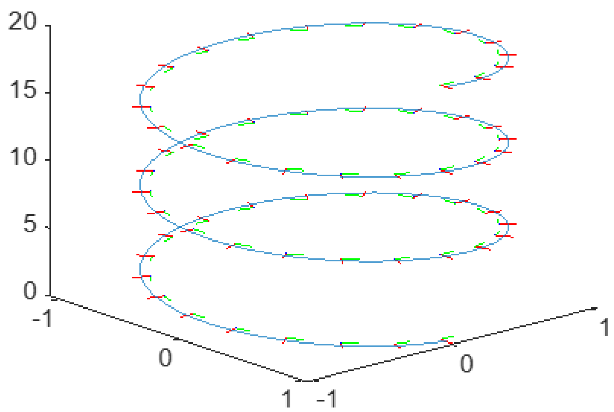

Figure 6: Trajectory of the cube