CENG 460 - Introduction to Robotics
Fall 2018

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Homework 3 - Serial-Link Manipulators Kinematics and Motion

1. As found in the last question of the previous homework,

$$^{O}T_E = R(\theta_1)t_x(a_1)R(\theta_2)t_x(a_2)R(\theta_3)t_x(a_3)$$
$$= \begin{bmatrix} cos(\theta_1+\theta_2+\theta_3) & -sin(\theta_1+\theta_2+\theta_3) & a_1cos(\theta_1)+a_2cos(\theta_1+\theta_2)+a_3cos(\theta_1+\theta_2+\theta_3) \\ sin(\theta_1+\theta_2+\theta_3) & cos(\theta_1+\theta_2+\theta_3) & a_1sin(\theta_1)+a_2sin(\theta_1+\theta_2)+a_3sin(\theta_1+\theta_2+\theta_3) \\ 0 & 0 & 1 \end{bmatrix}$$

We are asked to derive expressions for $\theta_1, \theta_2, \theta_3$ values that would result in a specified end effector pose

$$^{O}T_E = \, ^{O}T_E^* = \begin{bmatrix} cos\phi & -sin\phi & x \\ sin\phi & cos\phi & y \\ 0 & 0 & 1 \end{bmatrix}$$

$$x = a_1cos(\theta_1)+a_2cos(\theta_1+\theta_2)+a_3cos(\theta_1+\theta_2+\theta_3) \quad (1)$$
$$y = a_1sin(\theta_1)+a_2sin(\theta_1+\theta_2)+a_3sin(\theta_1+\theta_2+\theta_3) \quad (2)$$
$$\phi = \theta_1+\theta_2+\theta_3 \quad (3)$$

Replacing the terms $(\theta_1+\theta_2+\theta_3)$ by ϕ in eq(1) and eq(2) , we get:

$x - a_3cos\phi = a_1cos(\theta_1)+a_2cos(\theta_1+\theta_2)$
$y - a_3sin\phi = a_1sin(\theta_1)+a_2sin(\theta_1+\theta_2)$

(square and add these equations)
$x^2+y^2+a_3^2-2a_3(xcos\phi+ysin\phi) = a_1^2+a_2^2+2a_1a_2cos\theta_2$

From this we can solve for θ_2:

$$\theta_2 = \pm acos\frac{x^2+y^2+a_3^2-2a_3(xcos\phi+ysin\phi)-a_1^2-a_2^2}{2a_1a_2}$$

Applying the trigonometric equalities in eq(2), we get:

$$\begin{aligned} y - a_3sin\phi &= a_1sin(\theta_1)+a_2sin(\theta_1+\theta_2) \\ &= a_1sin\theta_1+a_2cos\theta_1sin\theta_2+a_2sin\theta_1cos\theta_2 \\ &= (a_1+a_2cos\theta_2)sin\theta_1+a_2sin\theta_2cos\theta_1 \end{aligned}$$

From this we can solve for θ_1:

$$\begin{aligned}\theta_1 =& atan2(y - a_3 sin\phi, \sqrt{a_2^2 sin^2\theta_2 + (a_1 + a_2 cos\theta_2)^2 - (y - a_3 sin\phi)^2})\\ &- atan2(a_2 sin\theta_2, a_1 + a_2 cos\theta_2)\\ =& atan2(y - a_3 sin\phi, x - a_3 cos\phi) - atan2(a_2 sin\theta_2, a_1 + a_2 cos\theta_2)\end{aligned}$$

It is clear from the above equation that for positive and negative θ_2, the values of θ_1 will be different because of the sign change in the term $sin\theta_2$.

Finally, θ_3 can be found from eqn(3):
$\theta_3 = \phi - \theta_1 - \theta_2$

2. Our goal is to have the end effector pose

$${}^{O}T_E(\gamma) = \begin{bmatrix} cos(\theta(\gamma)) & -sin(\theta(\gamma)) & x(\gamma) \\ sin(\theta(\gamma)) & cos(\theta(\gamma)) & y(\gamma) \\ 0 & 0 & 1 \end{bmatrix}$$

we readily have the expressions for $x(\gamma)$ and $y(\gamma)$:

$$\begin{aligned} x(\gamma) &= (0.9 + 0.4 sin(8\pi\gamma)^2) cos(2\pi\gamma) \\ y(\gamma) &= (0.9 + 0.4 sin(8\pi\gamma)^2) sin(2\pi\gamma) \end{aligned}$$

We can find the orientation by finding the angle that the normal to the curve makes with the x axis of the world frame. To do this, we can simply find the gradient of the curve (m) (namely the tangent of the angle that the y axis of the end effector frame makes with the x axis of the world frame), and the desired angle can be expressed as: $\theta(\gamma) = atan2(dy, dx) - \frac{\pi}{2}$

$$\begin{aligned} m(\gamma) &= \frac{dy}{dx} = \frac{\frac{dy}{d\gamma}}{\frac{dx}{d\gamma}} \\ &= \frac{2\pi cos(2\pi\gamma)(0.9 + 0.4 sin(8\pi\gamma)^2) + 0.4 * 2 * 8\pi sin(8\pi\gamma) cos(8\pi\gamma) sin(2\pi\gamma)}{-2\pi sin(2\pi\gamma)(0.9 + 0.4 sin(8\pi\gamma)^2) + 0.4 * 2 * 8\pi sin(8\pi\gamma) cos(8\pi\gamma) cos(2\pi\gamma)} \end{aligned}$$

So, the end effector pose ${}^{O}T_E(\gamma)$ can be written as:

$${}^{O}T_E(\gamma) = \begin{bmatrix} cos(atan2(dy, dx) - \frac{\pi}{2}) & -sin(atan2(dy, dx) - \frac{\pi}{2}) & x(\gamma) \\ sin(atan2(dy, dx) - \frac{\pi}{2}) & cos(atan2(dy, dx) - \frac{\pi}{2}) & y(\gamma) \\ 0 & 0 & 1 \end{bmatrix}$$

3. The end effector pose as γ goes from 0 to 1 in 100 steps (run hw3_e2177269_scr1.m):

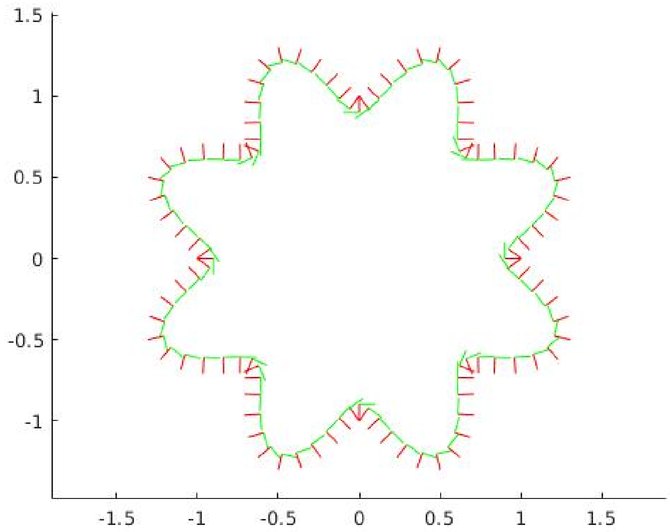

Figure 1: Desired end effector frame poses

4. As discussed in the second question, 2 different configurations (choosing the sign of θ_2 to be positive or negative) will result in the same end effector pose (run hw3_e2177269_scr2.m):

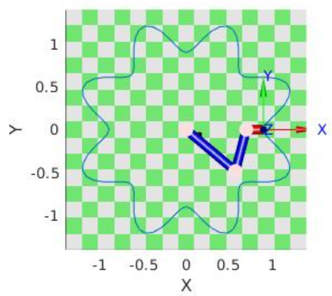

Figure 2: positive θ_2

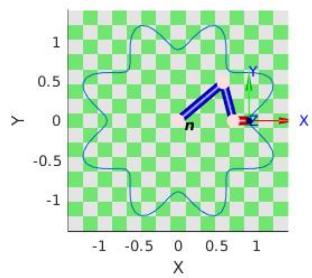

Figure 3: negative θ_2

5.

6. For the points on the curve having the largest distance from the origin, $\sqrt{x^2+y^2} = 0.9+0.4sin(8\pi\frac{2k+1}{16})^2 = 1.3$. If all three links of this robot are aligned, its end effector can reach the points which are farthest from its base, i.e. the points that have a

distance of $d = a_1 + a_2 + a_3$ from the origin. So, to be able to follow the path in the left plot of Figure 2 (given in the homework pdf), minimum a_1 value would have to be $a_1 = 1.3 - a_2 - a_3 = 1.3 - 0.5 - 0.2 = 0.6$.
However, with $a_1 = 0.6$, the three links of the robot arm would have to be aligned to be able to reach the farthest points, and the arm would go through singularities at those points (with $\theta_2 = 0$, $\theta_3 = 0$). Because of loss of DOFs, the robot can not go through those points with arbitrary orientation (it would have a single configuration that would put its end effector at the desired position, just like a single DOF robot). So, if the robot were to follow the path with its end effector orientation as shown in the right plot of Figure 2, with $a_1 = 0.6$, the inverse kinematics equations derived in q1 cannot be solved for the points near singularities.