Nash Equilibria of a Bertrand Price Competition Game

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The Bertrand Dupoly is a market structure where two firms compete over pricing schemes. If one firm prices themselves below the other, they reap the full market demand, and the firm which is undercut gets zero profits. This unique type of competition may lead to both firms pricing at the competitve market level ($p = \text{MC}$), a surprisingly favorable outcome for the consumer given the anticompetitive nature of a duopoly. However, it is also true that if both firms set prices together, they can indeed have positive profits while sharing the market demand. This article will show how to derive the set of prices which are Nash Equilibria for both firms.

Structure

We have two profit-maximizing firms, A and B. Let us suppose we are facing a linear market demand curve, with arbitrary parameters, given by:

$$y = 15 - p \tag{1}$$

Where p is the chosen price level, and y is the corresponding demand for the good at that price. Further suppose both firms, A and B, face an identical cost function with increasing marginal cost:

$$c(y) = \frac{1}{2}y^2 \tag{2}$$

Finally, the profit function for each firm depending on what the prices are:

$$\pi_i = \begin{cases} p(15-p) - \frac{1}{2}(15-p)^2 & \text{if } p_i < p_{-i} \\ p\left(\frac{15-p}{2}\right) - \frac{1}{2}\left(\frac{15-p}{2}\right)^2 & \text{if } p_i = p_{-i} \\ 0 & \text{if } p_i > p_{-1} \end{cases} \tag{3}$$

Solving for Equilibria

To find what prices p constitute a Nash Equilibrium, we must first establish what prices can and cannot satisfy the properties of an equilibrium strategy profile.

Proposition 1. *Every equilibrium price profile involves $p_A = p_B$, where both firms split the market demand.*

Proof. This must be true via contradiction. Let firm A be priced below B, so that $p_A < p_B$. Firm B would have incentive to undercut A, since its profits would go from 0 to some positive number. The two firms continue to do so until they reach the price so low that it cannot cover their costs, giving negative profits. So if there were to exist an optimal price profile where $p_A < p_B$, it must be that $\pi_A < 0$ so that B would not undercut it. However, if $\pi_A < 0$, then A would simply not produce and have 0 profits. A similar argument holds for B having the lower price. Thus, there is no equilibrium in which the firms' prices are not equal. □

Proposition 2. *For either firm, the set of feasible prices are those that give rise to non-negative profits. In this case, $p \in [3, 15]$.*

Proof. Since the firms are profit maximizers, they will always prefer positive profits to zero or negative profits. We have established that equilibrium will always entail splitting the market, so it remains to find what values of p satisfy the inequality:

$$\begin{aligned}
\pi_i &= p\left(\frac{15-p}{2}\right) - \frac{1}{2}\left(\frac{15-p}{2}\right)^2 \geq 0 &\quad (4)\\
&= \frac{1}{2}(15-p)(p - \frac{15-p}{4}) \geq 0 &\quad (5)\\
&\implies 3 \leq p \leq 15 &\quad (6)
\end{aligned}$$

Thus, any equilibrium price must lie between 3 and 15. □

The last thing we will do is find p that satisfies the equilibrium property of having no profitable deviations. If a firm deviated, that would mean getting the entire market, so it must be the case that for all $p_A = p_B = p$:

$$p\left(\frac{15-p}{2}\right) - \frac{1}{2}\left(\frac{15-p}{2}\right)^2 \geq p(15-p) - \frac{1}{2}(15-p)^2 \tag{7}$$

Which yields the condition:

$$p \leq \frac{45}{7} \tag{8}$$

So in conclusion, the equilibrium price profiles of our original game is described by:

$$\left\{(p_A, p_B)\ :\ 3 \leq p_A = p_B \leq \frac{45}{7}\right\} \tag{9}$$