Jack,

I'm confused on how, given contrast and differential phase, we can calculate the 6 ellipse parameters [A,B,C,D,E,F] that satisfy

Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 and B^2 – 4AC < 0.

I follow the work presented in [1] that starts with this background:

2.4 Ellipse Fitting

While the previous section was focused on calculating the theoretical phase shifts in an atom interferometer, the experimental apparatus is in practice used to solve the inverse problem of measuring the phase shifts given an interference pattern. For an initial atom amplitude of A_0, the upper output ports of each Ramsey-Bordé interferometer shown in Fig. 2.3 have theoretical amplitudes of

$$|\Psi_{\ell u}|^2 = \frac{A_0}{8}\cos^2\left(\frac{1}{2}(\phi_c + \phi_d)\right)$$
$$|\Psi_{uu}|^2 = \frac{A_0}{8}\cos^2\left(\frac{1}{2}(\phi_c - \phi_d)\right),$$

where the factor of 1/8 comes from the four $\pi/2$ beam splitters, the last of which interferes two trajectories. The differential phase ϕ_d and common mode phase ϕ_c were previously calculated as

$$\begin{aligned}\phi_d &= 8n(n+N)\omega_r T - n\omega_m T \\ \phi_c &= nk_{\text{eff}}gT(T + T_1' + T_2')\end{aligned} \tag{2.27}$$

In a real world interferometer, the four output ports might have non-ideal interference contrast and can therefore be generalized to

$$|\Psi_{\ell\ell}|^2 = A_\ell \sin\left(\frac{1}{2}(\phi_c + \phi_d)\right)^2 + b_1, \qquad |\Psi_{\ell u}|^2 = A_\ell \cos\left(\frac{1}{2}(\phi_c + \phi_d)\right)^2 + b_2$$

$$|\Psi_{u\ell}|^2 = A_u \sin\left(\frac{1}{2}(\phi_c - \phi_d)\right)^2 + b_3, \qquad |\Psi_{uu}|^2 = A_u \cos\left(\frac{1}{2}(\phi_c - \phi_d)\right)^2 + b_4,$$

where $\{A_\ell, A_u\}$ are the interference amplitudes and $\{b_1, b_2, b_3, b_4\}$ are the amplitude offsets due to backgrounds. Since the total atom number for each pair of outputs $|\Psi_{\ell\ell}|^2 + |\Psi_{\ell u}|^2$ and $|\Psi_{u\ell}|^2 + |\Psi_{uu}|^2$ can fluctuate and drift due to imperfections in the atom source, it is useful to first normalize the signal

$$x = \frac{|\Psi_{\ell u}|^2 - |\Psi_{\ell\ell}|^2}{|\Psi_{\ell u}|^2 + |\Psi_{\ell\ell}|^2}, \qquad y = \frac{|\Psi_{uu}|^2 - |\Psi_{u\ell}|^2}{|\Psi_{uu}|^2 + |\Psi_{u\ell}|^2}$$

such that

$$\begin{aligned} x &= A_x \cos(\phi_c + \phi_d) + b_x \\ y &= A_y \cos(\phi_c - \phi_d) + b_y, \end{aligned} \tag{2.28}$$

0 < phi_d < pi/2 ?

where A_x, A_y are the normalized fringe contrasts and b_x, b_y are offsets. In order to fit data of this form to an ellipse, we first need to rewrite $\{x, y\}$ in the form of a generalized conic section

$$a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6 = 0 \tag{2.29}$$

which describes an ellipse when $a_2^2 - 4a_1 a_3 < 0$. To convert the equations for $\{x, y\}$ into this form, first subtract the offsets and normalize to obtain a set of rescaled points

I'm not exactly sure what these four wave functions refer to, but I assume in the expression for x and y (2.28) that: b_x = b_y = ½. Next, [1] normalizes x and y to form x' and y' below:

$$x' = (x - b_x)/c_x = \cos\phi_c \cos\phi_d + \sin\phi_c \sin\phi_d$$
$$y' = (y - b_y)/c_y = \cos\phi_c \cos\phi_d - \sin\phi_c \sin\phi_d. \tag{2.30}$$

With these rescaled coordinates, it becomes apparent that the relation

$$\left(\frac{x' - y'}{2\sin\phi_c}\right)^2 + \left(\frac{x' + y'}{2\cos\phi_c}\right)^2 - 1 = 0$$

cancels the common mode phase ϕ_c and has the form of (2.29). Expanding this equation using our definitions for x' and y' from (2.30) yields

if phi_d = pi/4, this a_2 term = 0

$$\frac{1}{c_x^2}x^2 - \frac{2\cos 2\phi_d}{c_x c_y}xy + \frac{1}{c_y^2}y^2 + \left(\frac{2b_y\cos 2\phi_d}{c_x c_y} - \frac{2b_x}{c_x^2}\right)x + \left(\frac{2b_x\cos 2\phi_d}{c_x c_y} - \frac{2b_y}{c_y^2}\right)y$$
$$+ \left(\frac{b_x^2}{c_x^2} + \frac{b_y^2}{c_y^2} - \frac{2b_x b_y \cos 2\phi_d}{c_x c_y} - 4\cos^2\phi_d \sin^2\phi_d\right) = 0, \tag{2.31}$$

which can be matched up term for term with the coefficients a_i from (2.29). A simple substitution will then verify that

$$\phi_d = \frac{1}{2}\cos^{-1}\left(\frac{-a_3}{2\sqrt{a_1 a_2}}\right).$$

a1*a2 > 0 must hold, but this is not true for general ellipse, or for phi_d = pi/4

By fitting an ellipse to the data and extracting the parameters a_1, a_2, and a_3, the differential phase and therefore the desired recoil frequency phase can be measured.

Here I don't know what c_x or c_y refer to, and I'm unsure what values I should use for b_x and b_y (2.31). If I construct the model used in the multiplexed clock paper, I believe phi_d = phi/2, where

x = (1 - C cos(theta))/2

y = (1 - C cos(theta + phi))/2

Lastly, there seems to be some issue when phi_d = pi/4 => a_2 = 0 => phi_d is complex. Also, phi_d real => a_1*a_2 > 0 but a_1*a_2 > 0 is not true in general for ellipses.

[1] B. V. Estey, "Precision Measurement in Atom Interferometry Using Bragg Diffraction," p. 203.