Geometric Brownian Motion in modelling Options and Stock Prices

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Abstract

This paper provides an expository review of geometric Brownian motion (GBM) as a stochastic model for asset prices and its role in the Black–Scholes framework for pricing options. Beginning with foundational concepts from probability theory and stochastic processes, the paper develops the mathematical structure of Brownian motion, introduces GBM through stochastic differential equations, and explains how the lognormal price model leads to the Black–Scholes formula under ideal market assumptions. A simulation study then illustrates how the approximated price of a hypothetical European call option under GBM, computed by averaging discounted payoffs, is consistent with the Black–Scholes closed-form solution. Finally, the model is applied to historical S&P 500 data to estimate volatility, simulate future index paths, and price a hypothetical European call option. While the empirical application highlights certain limitations of the GBM assumptions in real markets, it also illustrates the practical usefulness of GBM as an approximation method for asset prices due to its relative success in capturing market behavior over other mathematical models.

Keywords: stochastic processes, Markov property, financial modelling, estimation, Black-Scholes framework

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1 Introduction

In the financial realm, options are defined as a financial instrument or legal contract that gives its owner the right to either buy or sell a quantity of assets, typically stocks, at a predetermined price, known as a strike price, at or within a certain date, called an expiration date Fidelity Investments (2024). Options that can be exercised at any time within the expiration date are termed as 'American' while those that can only be exercised on the expiration date are referred to as 'European.' Options that give its owner the right to buy the underlying stocks within the contract are 'call' options, and those that come with a legal right to sell the contract's underlying assets are called 'put' options Fidelity Investments (2024).

In general, it is safe to assume that "the higher the price of the stock, the greater the value of the option," Black & Scholes (1973) and therefore, accurately forecasting the behavior of the underlying assets or stocks of an option is critical for a trader to understand what kind of options-based trading strategy might be most profitable for them to pursue. For instance, if a stock price can be forecasted to be much greater than its strike price negotiated in a call option on or within its expiration date, then it is profitable for the trader to buy that call option and then, if the forecast holds, the option will most likely be exercised by the trader Black & Scholes (1973). Therefore, traders, risk managers, and financial analysts rely on statistical models to forecast asset price movements, assess risk, and determine whether an option is "fairly priced."

However, modeling stock prices deterministically is challenging since in real markets, prices fluctuate continuously, respond to unpredictable information, and exhibit randomness. Financial analysts assume that the stock market operates under the efficient market hypothesis (EMH), which states that given a particular information set, the market already captures all of that information and the prices would be unaffected if the information was fully revealed to market participants Sewell (2012). So, early work in options pricing posit that asset prices evolve in a time-dependent and uncertain manner and require stochastic modelling assumptions Samuelson (1965). The most prominent model to forecast options price is Geometric Brownian Motion (GBM), which is a "continuous-time stochastic process," *A Survey of Geometric Brownian Motion and Its Applications* (2020) where the logarithm of the stock prices follows a Brownian motion with drift. This theory was central to the development of the 'Black-Scholes' formula, which prices European options under the assumption that the underlying asset follows a lognormal distribution and therefore, can be modeled using a GBM framework Black & Scholes (1973). In this expository review, I will introduce the necessary background for stochastic processes, develop the mathematical intuition behind GBM, and explain how the Black–Scholes framework links GBM to option valuation. I will then briefly explore prominent examples of GBM applied to financial problems in literature. Finally, I will build a demonstration model that applies GBM and the Black–Scholes formula to real stock-market data.

2 An Expository Review of Geometric Brownian Motion (GBM)

2.1 Introduction to Stochastic Processes

We recall the definition of a probability space as the triple $(\Omega, \mathcal{F}, \mathbb{P})$ where,

1. Ω is the sample space, representing the set of all possible outcomes of the random

experiment under consideration.

2. $\mathcal{F}$ is a collection of subsets of Ω, called *events*, to which probabilities can be assigned. This collection is chosen so that it contains all events of interest and is closed under complements and countable unions. Intuitively, $\mathcal{F}$ specifies those events that are observable and meaningful.

3. $\mathbb{P}$ is a probability measure defined on the domain $\mathcal{F}$, satisfying

$$\mathbb{P} : \mathcal{F} \to [0, 1], \qquad \mathbb{P}(\Omega) = 1 \tag{1}$$

and countable additivity:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i), \qquad \text{for all disjoint } A_i \in \mathcal{F}. \tag{2}$$

Petters & Dong (2016)

A stochastic (or random) process is a collection of random variables defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Each of these random variables represent the same underlying quantity, but capture observations of the quantity at different points in time. Taken together, the stochastic process models the evolution of that uncertain quantity over time Petters & Dong (2016). Formally, following Petters and Dong, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a stochastic process is defined as

$$\{X(t) : t \in J\}, \tag{3}$$

where each $X(t)$ is a real-valued random variable defined on the common probability space and J, a non-empty subset of the reals, is an index set of the process X representing time

Petters & Dong (2016). If $J = \mathbb{N}$, then the random process evolves over discrete time and if $J = [0, \infty)$, it evolves over continuous time Petters & Dong (2016).

As discussed by Samuelson in his foundational 1965 paper, asset prices evolve over time in a way that incorporates both deterministic trends and random fluctuations, motivating the use of continuous-time stochastic models when forecasting stock prices Samuelson (1965).

2.2 Markov Processes and the Markov Property

In this section, we review necessary definitions for a widely used stochastic process fundamental to understanding the theory for GBM. First, we recall that a set S is called the *state space* of a system or process if it consists of all possible values that the random variables associated with that system may take Hoel et al. (1972). This state space is called discrete if S is a finite or countable set, and *continuous* if it is an uncountable subset of $\mathbb{R}^d$, an interval of real numbers.. For the purposes of illustration, we first restrict attention to stochastic processes with discrete state spaces. This restriction will be relaxed later when we introduce continuous-state Markov processes such as Brownian motion and GBM.

Now, let, $\{X_n : n \in J\}$ where $J = \{0, 1, 2, 3, ...\}$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a discrete state space S, as defined above. Following Hoel, Port, and Stone, this stochastic process is called a *Markov process* if, for all states $x_0, x_1, \ldots, x_{n-1}, x_n, x_{n+1} \in S$, the transition probabilities satisfy the following *Markov rule*:

$$\mathbb{P}(X_{n+1} = x_{n+1} \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x_n) = \mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n), \tag{4}$$

whenever these conditional probabilities are well-defined Hoel et al. (1972). Intuitively, the Markov rule formalizes the idea that the future evolution of a Markov process depends only on its present state and not on past states. Once the current state $X_n = x_n$ is known, any additional information about earlier states $X_0, \dots, X_{n-1}$ provides no further predictive power for determining the distribution of X_{n+1} Hoel et al. (1972). For our purposes, Markov processes with discrete state spaces will be referred to as *Markov chains* as per Hoel, Port, and Stone Hoel et al. (1972), while the focus of this paper is on Markov processes with continuous state spaces, which we will simply refer to as *Markov processes*.

For a Markov process, the conditional rules

$$\mathbb{P}(X_{n+1} = y \mid X_n = x), \tag{5}$$

are called the *transition probabilities* to 'go' from state x at time n to state y at time $n+1$. These transition probabilities are called 'stationary' if they are independent of n Hoel et al. (1972). Generalizing this definition, if $\{X_n : n \geq 0\}$ is a Markov process with state space S, then for $x, y \in S$, the function

$$P(x, y) = \mathbb{P}(X_1 = y,\, X_0 = x), \qquad x, y \in S, \tag{6}$$

is called the 'transition function' of the chain which satisfies the following:

$$P(x, y) \geq 0 \quad \text{for all } x, y \in S, \qquad \text{and} \qquad \sum_{y \in S} P(x, y) = 1 \quad \text{for all } x \in S. \tag{7}$$

These are also known as 'one-step' transition probabilities of the Markov chain Hoel et al. (1972). Finally, the function

$$\pi_0(x) = \mathbb{P}(X_0 = x), \qquad x \in S, \tag{8}$$

is called the initial distribution of the Markov chain such that the following hold:

$$\pi_0(x) \geq 0 \quad \text{for all } x \in S, \qquad \text{and} \qquad \sum_{x \in S} \pi_0(x) = 1, \tag{9}$$

Hoel et al. (1972).

In continuous-state Markov processes, transition probabilities are described by probability density functions rather than discrete transition probabilities, and summations over states are replaced by integrals Hoel et al. (1972).

Both Brownian motion, which underlies geometric Brownian motion (GBM), and GBM itself are continuous-time Markov processes and therefore, satisfy the Markov property Petters & Dong (2016). The relevance of the Markov property to financial modeling lies in its role as a foundational assumption for continuous-time asset price dynamics. In particular, if the price of a common stock is modeled using GBM, the Markov property implies that the conditional distribution of the stock's future prices depends only on the current price and not on the history of past prices Samuelson (1965).

2.3 Illustrative Example: The Weather Chain

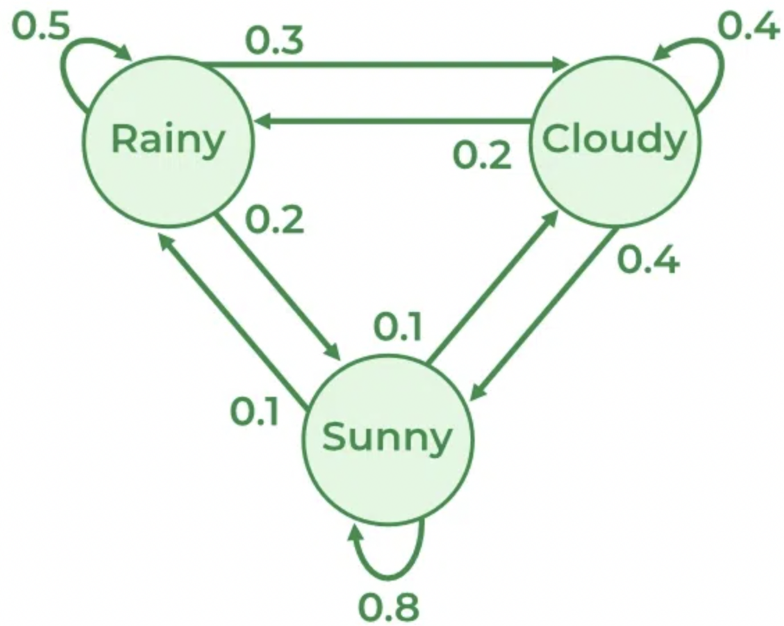

Figure 1: An Illustrative Markov Chain: The Weather Chain.

Here, we introduce a simple illustrative example of a *Markov chain* to solidify understanding of the Markov property before transitioning to continuous-time Markov processes. Figure 1 illustrates a simple discrete-time Markov chain with state space

$$S = \{\text{Rainy}, \text{Cloudy}, \text{Sunny}\}, \tag{10}$$

which models the evolution of daily weather conditions. At each time step, exactly one state is occupied, and transitions to a new state at the next time step occur according to the labeled transition probabilities Hoel et al. (1972), GeeksforGeeks (2025).

The arrows indicate possible one-step transitions between weather states, while the numerical labels specify the corresponding transition probabilities. For example, if the current state is Rainy, the chain remains Rainy with probability 0.5, transitions to Cloudy with probability 0.3, and transitions to Sunny with probability 0.2. Similarly, if the current state is Sunny, the chain remains Sunny with probability 0.8.

This diagram provides a concrete illustration of the Markov property. The probability distribution of tomorrow's weather depends only on today's weather and not on the sequence of weather outcomes observed on previous days. Once the current state is known, all relevant information needed to determine the distribution of the next state is fully captured Hoel et al. (1972).

2.4 Brownian Motion

We now introduce the continuous-time stochastic process 'Brownian Motion', which serves as the foundational definition for GBM. Brownian motion, also known as a *Wiener process*, is the classical model for the continuous random fluctuations and the highly irregular motion observed in microscopic particles suspended in liquid. Following the standard definition, a stochastic process $\{B(t) : t \geq 0\}$ is called a *standard Brownian motion* if it satisfies the following properties:

1. $B(0) = 0$
2. For all $0 \leq s < t$,

$$B(t) - B(s) \sim \mathcal{N}(0, t - s).$$

3. For any $0 \leq t_1 < t_2 < \cdots < t_n$, the random variables

$$B(t_2) - B(t_1),\ B(t_3) - B(t_2),\ \ldots,\ B(t_n) - B(t_{n-1})$$

are independent Hoel et al. (1972), Petters & Dong (2016).

Using these defining properties of Brownian motion, we make comments on some important results related to the process below:

2.4.1 Mean and Variance of Brownian Motion

From the above definition, particularly from property 2, it immediately follows that Brownian motion has mean zero at all times,

$$\mathbb{E}[B(t)] = 0, \qquad t \geq 0, \tag{11}$$

and variance grows linearly in time,

$$\mathrm{Var}(B(t)) = t, \qquad t \geq 0. \tag{12}$$

2.4.2 Brownian Motion as a Gaussian Process and its Covariance Function

Brownian Motion $\{B(t) : t \geq 0\}$ is also a Gaussian process, which means that every finite linear combination of the random variables $B(t)$, for all $t \in T$, is normally distributed Hoel et al. (1972). The defining properties of Brownian motion also allow us to mathematically derive the covariance function for two random variables $B(t)$ and $B(s)$, which is given by

$$\mathrm{Cov}(B(s), B(t)) = \min(s, t), \qquad s, t \geq 0. \tag{13}$$

A proof of this result is provided in Section 6.

For Gaussian processes, if two random variables have the same mean and covariance functions, as above, "they also have the same joint distribution functions," Hoel et al. (1972). So, the mean and covariance functions completely determine all finite-dimensional distributions and fully characterize a Brownian motion process Hoel et al. (1972).

2.4.3 Brownian Motion as a Markov Process

An important consequence of the defining properties of Brownian motion, especially property 3, is that it satisfies the Markov property Petters & Dong (2016). Intuitively, given the current value $B(t)$, the future increment $B(t+h) - B(t)$ is independent of the past history of the process prior to time t, and depends only on the length of the time interval h Petters & Dong (2016). As a result, once the present state is known, past values provide no additional information for predicting future behavior for a system modeled by this process Hoel et al. (1972), Petters & Dong (2016).

While L.~Bachelier's seminal work established Brownian motion as a mathematical foundation for modeling the continuous random fluctuations of stock prices and is widely regarded as the birth of mathematical finance Bachelier (2018), it leads to the result that in the long run, an option "will increase in price indefinitely, coming even to exceed the price of the common stock itself," Samuelson (1965). This outcome is counterintuitive, since ownership of the stock is economically equivalent to holding a perpetual option exercisable at zero cost Samuelson (1965).

As Samuelson later emphasized, this anomaly arises because modeling stock prices directly using Brownian motion allows prices to take on negative values with positive probability Samuelson (1965). Such behavior is incompatible with financial reality, where stock prices are strictly positive Samuelson (1965). These considerations motivate modeling the logarithm of the asset price as a Brownian motion, leading to the geometric Brownian motion (GBM) model. By exponentiating Brownian motion, GBM preserves the Markov property and yields lognormally distributed stock prices that takes on strictly positice values, a feature consistent with observed market dynamics and foundational to the Black–Scholes framework Samuelson (1965), Black & Scholes (1973), Petters & Dong (2016). We formally develop the GBM model as well as the Black–Scholes framework in the following sections.

2.5 Stochastic Differential Equations (SDE) and It^o's Lemma

In modeling continuously evolving random systems, it is useful to separate systematic trends from random fluctuations. On that note, we present a parameter μ, representing the *drift* of Brownian motion, capturing its deterministic rate of change over time, and a second parameter σ, representing the *volatility*, which controls the magnitude of random fluctuations in the process. In financial applications, μ reflects the expected rate of return of an asset such as a stock, while σ measures the uncertainty or risk associated with its price movements Petters & Dong (2016).

Now, a stochastic process $\{X(t)\}$ is called a Brownian motion with drift μ and scaling σ if it satisfies the stochastic differential equation

$$dX(t) = \mu\, dt + \sigma\, dB(t), \tag{14}$$

where $\{B(t)\}$ is a standard Brownian motion Petters & Dong (2016). It is possible to show that integrating this equation yields the explicit representation

$$X(t) = x_0 + \mu t + \sigma B(t), \tag{15}$$

for some initial value $x_0 \in \mathbb{R}$ Petters & Dong (2016). Thus, {X(t)} is a Brownian motion with drift μ and scaling σ if and only if it satisfies equation 15 for some initial condition $x_0 \in \mathbb{R}$.

Let us suppose that the increment of a stochastic process $X(t)$ over a small time interval $[t, t + \Delta t]$ can be approximated by

$$\Delta X(t) \approx \mu(X(t), t)\, \Delta t + \sigma(t)\, \Delta B(t), \tag{16}$$

where $\Delta B(t)$ denotes the increment of a Brownian motion, and μ and σ are as defined above Petters & Dong (2016). In differential form, this relationship is written as the stochastic differential equation

$$dX(t) = \mu(X(t), t)\, dt + \sigma(t)\, dB(t). \tag{17}$$

This expression separates the evolution of $X(t)$ into a deterministic component, governed by the drift term μ, and a random noise component σ driven by Brownian motion Petters & Dong (2016). The differential equation itself provides a mathematical framework for modeling the rate of change in Brownian motion, which is continuously evolving process.

To analyze functions of stochastic processes that satisfy stochastic differential equations, we use It^o's formula, which plays the role of the chain rule in stochastic calculus and provides us with a framemwork to obtain solutions of s.d.e's. Let us suppose that $X(t)$ is a stochastic process that satisfies the stochastic differential equation above, and let $Y(t) = f(X(t), t)$, where f has continuous first and second partial derivatives. Then It^o's formula states that

$$dY = \left(f_t + \mu f_x + \tfrac{1}{2}\sigma^2 f_{xx}\right) dt + \sigma f_x \, dB(t), \tag{18}$$

where subscripts denote partial derivatives Petters & Dong (2016).

In the next section, we define geometric Brownian motion (GBM) as the solution to a stochastic differential equation and examine the application of It$\hat{}$o's formula to analyze functions of the GBM process and derive the s.d.e.'s explicit solution Petters & Dong (2016).

2.6 Geometric Brownian Motion

We now introduce Geometric Brownian Motion (GBM), the stochastic process used to model asset prices in modern-day financial mathematics. A stochastic process $\{X(t) : t \geq 0\}$ is called a *Geometric Brownian Motion (GBM)* if it satisfies the It$\hat{}$o stochastic differential equation 17 Petters & Dong (2016). It can be shown that solving this equation with initial condition $X(0) = x_0 > 0$, where $x_0 > 0$ by applying It$\hat{}$o's formula with $f(x) = ln(x)$ yields the explicit solution

$$X(t) = x_0 \exp\left[\left(\mu - \tfrac{1}{2}\sigma^2\right) t + \sigma B(t)\right], \tag{19}$$

Petters & Dong (2016).

Thus, $\{X(t)\}$ with $X(t) = exp[Y(t)]$ is a geometric Brownian motion if and only if the corresponding log-process $\{Y(t)\}$ is a Brownian motion with drift and scaling Petters & Dong (2016).

In financial applications, $X(t)$ represents the price of an asset, most commonly a stock, and we therefore write $S(t)$ in place of $X(t)$ for cleaner notation. We define the log-return process by

$$X(t) := \ln\left(\frac{S(t)}{S(0)}\right). \tag{20}$$

From the explicit solution of the GBM stochastic differential equation, we see that since $B(t)$ is normally distributed with mean 0 and variance t Petters & Dong (2016), $X(t)$ as defined as the log-return process is then normally distributed with mean μt and variance $\sigma^2 t$ Petters & Dong (2016). Consequently, the stock price process $\{S(t) : t \geq 0\}$ defined by

$$S(t) = S(0)e^{X(t)} = S(0)e^{\mu t+\sigma B(t)}, \qquad t \geq 0, \tag{21}$$

is a geometric Brownian motion with parameters μ and σ Petters & Dong (2016). Finally, because $X(t)$ is normally distributed, the random variable $S(t)$ follows a *lognormal distribution*, written

$$S(t) \sim \text{lognormal}(\mu t, \sigma^2 t), \tag{22}$$

as shown in Petters & Dong (2016).

Geometric Brownian motion, as defined above, is a more appropriate model for stock prices than standard Brownian motion, since stock prices are strictly nonnegative and exhibit fluctuations that scale proportionally with their current level Petters & Dong (2016). Clearly, when modeling stock prices under GBM, the prices themselves are lognormally distributed and take on strictly non-negative values. This property make GBM the natural foundation for the Black–Scholes option pricing framework, which we develop in the next section.

2.7 The Black–Scholes Options Pricing Framework

Before presenting the Black–Scholes formula, we briefly review the structure of a *European call option*. A European call option gives its holder the right, but not the obligation, to purchase an underlying asset at a fixed price K, called the *strike price*, at a fixed future time T, called the *maturity* Black & Scholes (1973). The payoff of the option at time T is

$$\max(S(T) - K,\, 0), \tag{23}$$

where $S(T)$ denotes the price of the asset at maturity. This payoff depends only on the value of the asset at time T and not on how the price evolved over time Petters & Dong (2016).

In Section 2.6, we modeled the asset price, or the price of a stock $\{S(t)\}$ using geometric Brownian motion (GBM). Under this model, $S(T)$ is lognormally distributed, and therefore the uncertainty in the option's payoff arises entirely from the randomness in the terminal stock price $\{S(t)\}$. Thus, once the parameters $S(0)$, μ, σ, and T are specified, the distribution of $S(T)$ is fully determined Petters & Dong (2016).

Under certain 'ideal conditions' within the market, Black and Scholes posit that "the value of the option will depend only on the value of the stock and time and on certain variables that are taken to be known constants," Black & Scholes (1973). Essentially, Black and Scholes argue that the option payoff depends on the same underlying source of randomness as the stock price itself, and thus, it is possible to offset the randomness in the option's value by appropriately combining it with the stock, or by modeling the option's price as a function of the stock's price Black & Scholes (1973). When this randomness is eliminated, the remaining evolution of value would be deterministic.

Black and Scholes obtain a mathematical formulation of this argument, which leads to a partial differential equation (p.d.e.), known as the Black—Scholes equation, which the option price must satisfy Black & Scholes (1973). Solving this p.d.e. using It^o's Lemma yields the following Black–Scholes formula for the price of a European call option with strike price K and maturity T at time 0:

$$C(S_0, K, T, r, \sigma) = S_0\, \Phi(d_1) - Ke^{-rT}\, \Phi(d_2), \tag{24}$$

where,

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}. \tag{25}$$

Additionally, r is the risk-free interest rate representing the constant rate of return on an asset that is assumed to have no uncertainty, and $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution Black & Scholes (1973). This Black–Scholes formula provides a closed-form expression for the value of a European call option under the assumptions that asset prices follow geometric Brownian motion and that markets exist under certain ideal conditions. Despite its simplifying assumptions, the model has since proven to hold a variety of assets other than European call options, cementing it as one of the foundational results of modern financial mathematics Black & Scholes (1973), Petters & Dong (2016).

2.8 Concluding Remark on the Black-Scholes Framework and its Connection to the Lognormal Model

The closed-form solution to the Black–Scholes equation presented in Section 2.7 assumes the lognormal distribution of the terminal stock price $S(T)$, as established in Section 2.6. Under the GBM model, this lognormal structure implies that the expected payoff of an option has an explicit analytical expression, which is the Black–Scholes price.

In Section 3, we exploit the same GBM-implied distribution of $S(T)$ to compute a hypothetical option's price numerically. Specifically, we simulate independent realizations of the terminal stock price under the GBM model, evaluate the corresponding option payoffs, and approximate their expectation via Monte Carlo averaging. This allows us to compare the closed-form Black–Scholes price with a simulation-based estimate of the same theoretical quantity.

3 Simulation Study: GBM and the Black–Scholes Formula

In this section, we illustrate how the GBM modelling assumptions for stock prices connect to the Black–Scholes option-pricing formula through a simple simulation experiment. Our goal is to start from the GBM dynamics introduced earlier and numerically approximate the price of a hypothetical European call option by averaging simulated payoffs. We then compare this estimate to the closed-form Black–Scholes price of this hypothetical option by following the framework presented in Section 2.7, and we expect them to be approximately the same.

Before we describe the study further, we note that the Black–Scholes formula and the simulation-based price of an option are both derived from the same GBM model, but they compute the payoff value in different ways. The Black–Scholes formula evaluates the expected payoff exactly using an analytical expression, while the simulation-based price approximates the same expectation by averaging payoffs across many simulated GBM outcomes. Comparing the two therefore checks whether the numerical simulation reproduces the theoretical Black–Scholes price, with any differences arising from Monte Carlo approximation error.

3.1 Design of the Toy Option

We consider a single stock whose price at time t is modeled by the GBM process

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dB(t), \tag{26}$$

with the initial price $S(0) = S_0$. For the simulation, we fix the following parameter values, with the parameters being defined the same as in Section 2.5, Section 2.6, and Section 2.7.

$$S_0 = 100, \quad K = 100, \quad T = 1, \quad \mu = 0.02, \quad \sigma = 0.20. \tag{27}$$

Here, we recall that K is the strike price, in dollars and T is the maturity of the option, in years. The quantity μ plays is essenitally a constant interest rate, and σ is the volatility parameter from the GBM model that exhibits randomness. We recall from Section 2.7 that the payoff of a European call option with strike K and maturity T is,

$$\max(S(T) - K, 0). \tag{28}$$

Petters & Dong (2016).

Since $S(T)$ is lognormally distributed under the GBM model, this payoff quantity will have a well-defined distribution determined by the parameters above.

3.2 Simulating Terminal Prices Under GBM

From equation 19 and by using the notation in Section 3.1, we have,

$$S(T) = S(0) \exp\left((\mu - \tfrac{1}{2}\sigma^2)T + \sigma B(T)\right). \tag{29}$$

In the simulation, we will use the same volatility parameter σ as in the Black–Scholes formula. To generate a single simulated future price of the option at maturity time T, termed $S(T)$, we require a realization of the Brownian motion $B(T)$. By definition of Brownian motion as presented in Section 2.4, $B(T)$ is normally distributed with mean 0 and variance T:

$$B(T) \sim N(0, T). \tag{30}$$

We know, a well-known property of the normal distribution is that if $Z \sim N(0, 1)$, then for any constant $a > 0$,

$$aZ \sim N(0, a^2). \tag{31}$$

Blitzstein & Hwang (2019). Choosing $a = \sqrt{T}$ gives,

$$\sqrt{T} Z \sim N(0, T), \tag{32}$$

and therefore,

$$B(T) \overset{d}{=} \sqrt{T}Z, \qquad Z \sim N(0,1). \tag{33}$$

Thus, to generate a realization of $B(T)$ in our simulation, it suffices to draw a standard normal random variable Z and set $B(T) = \sqrt{T}Z$. Substituting this into the explicit GBM solution yields the simulation formula,

$$S(T) = S_0 \exp\Big((\mu - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z\Big), \qquad Z \sim N(0,1). \tag{34}$$

Repeating this procedure N times produces N independent realizations,

$$S^{(1)}(T),\ S^{(2)}(T),\ \dots,\ S^{(N)}(T). \tag{35}$$

of simulated terminal or strike prices of the stock at time T under the GBM model.

3.3 Approximation of the Call Price under GBM

For each simulated terminal stock price $S^{(i)}(T)$ from Section 3.2, the corresponding payoff or final price of the option at maturity is,

$$\text{payoff}^{(i)} = \max\big(S^{(i)}(T) - K, 0\big). \tag{36}$$

Mathematically, it is possible to show that for any option payoff depending only on the terminal price of a stock, $S(T)$, the time–0 price of the option is the discounted expectation of that payoff under the lognormal distribution of $S(T)$ generated by the GBM model

Petters & Dong (2016). If we denote the theoretical price of the call option at $T = 0$ as C_0, then intuitively, the Black–Scholes framework expresses C_0 as a discounted expectation of the option's payoff at time T. In other words, since the expected payoff of the option is computed at time T, we compute its payoff at time $T = 0$ through 'discounting,' Petters & Dong (2016).

Now, since the payoff of a European call option is $\max(S(T)-K, 0)$ as presented Section 2.7, the mathematical representation of the above intuition yields,

$$C_0 = e^{-\mu T}\,\mathbb{E}[\max(S(T) - K, 0)]\,. \tag{37}$$

In practice, the expectation in 37 cannot be evaluated directly, and is instead approximated numerically. We do so by simulating the terminal stock prices at time T, and then by replacing the expectation in 37 with a sample average of N simulated and independent discounted payoffs, as demonstrated in Section 3.2. This yields the GBM estimator of the time–0 call price, $\widehat{C}_N$. An estimate of this approximation's standard error is given by,

$$\mathrm{SE}(\widehat{C}_N) = \frac{s}{\sqrt{N}}, \tag{38}$$

where s is the sample standard deviation of the discounted payoffs.

3.4 Summary Of Process: Comparing the GBM Simulated Price with the Analytical Black-Scholes Price

In this section, we compute the time–0 price of a European call option under the GBM model in two ways: analytically, using the Black–Scholes explicit formula, and numerically using a simulated GBM estimate. The analytical Black–Scholes price, C_{BS}, is computed using the closed-form formula in 24, where the terms d_1 and d_2 are defined in 25. This formula gives the option price directly as a function of ((S_0, K, T, r,)).

The GBM simulation-based price approximates the same time–0 value by simulating many possible terminal stock prices under GBM. Specifically, we generate independent draws (S^{(i)}(T)) using the terminal simulation formula in 34, compute the corresponding option payoffs using 39, and then average the *discounted* payoffs to form the GBM estimator $\widehat{C}_N$. The standard error of this Monte Carlo estimate is summarized by 38.

Because both methods are derived from the same GBM assumptions, they are estimating the same theoretical price. So, we expect that with increasing N, the value of $\widehat{C}_N$ converges to the analytical value of C_{BS}. Any differences between C_{BS} and $\widehat{C}_N$ are expected to arise due to sampling error.

3.5 Implementation

In this sub-section, we briefly describe the reproducible Python code used to carry out the simulation study introduced in Section 3.2, Section 3.3, and Section 3.4. The full implementation of the code is provided in a reproducible Google Colab notebook Rashha (2025*b*), which will be referring to throughout this sub-section.

The goal of the implementation is to consider a hypothetical European call option with pre-specified parameters as defined in Section 3.1, generate simulated terminal stock prices under the GBM model, compute the corresponding payoffs for the hypotehtical option, and compare the resulting GBM estimates with the analytical Black–Scholes value of the option's payoff. The code is organized into three key components: (1) a function that evaluates the Black–Scholes closed-form formula to compute the payoff of the hypothetical option, (2) a function that simulates terminal stock prices using the explicit GBM solution, and (3) an estimator function that averages the simulated payoffs, discounts it to time–0, and thus computes the GBM simulated payoff of the option.

3.5.1 Black–Scholes Pricing Function

The first component of the code is a function that evaluates the theoretical price of a European call option using the Black–Scholes formula presented in in Section 2.7. Given the parameter tuple (S_0, K, T, r, σ), the pricing function computes the quantities d_1 and d_2, and returns the corresponding closed-form option value at time–0.

3.5.2 GBM Simulator Function

To simulate realizations of the terminal stock price $S(T)$ under the GBM model, we use the explicit solution to the stochastic differential equation presented in Section 2.6. Under this model, the terminal price is given by equation 34. Independent draws of the normally distributed random variable Z yield independent realizations of $S(T)$, as explained in Section 3.2, allowing for an approximation of expectations involving the terminal stock price.

3.5.3 GBM Estimator Function

Using simulated terminal prices $S^{(i)}(T)$, the GBM estimator of the European call option price is constructed by averaging discounted payoffs across N independent simulation paths. The estimator is given by

$$\widehat{C}_N = e^{-rT}\frac{1}{N}\sum_{i=1}^{N}\max\bigl(S^{(i)}(T) - K,\, 0\bigr)\,. \tag{39}$$

Petters & Dong (2016), Black & Scholes (1973).

In order to ensure consistency with the Black–Scholes framework, the drift parameter μ used in the simulation is set equal to the risk-free rate r since they both capture the deterministic rate of change over time. The variability of this estimator is quantified by its standard error, which this function computes, along with the following 95% confidence interval,

$$\widehat{C}_N \pm 1.96 \times \mathrm{SE}\Bigl(\widehat{C}_N\Bigr)\,. \tag{40}$$

3.6 Simulation Results and Analysis

Finally, we compare the Black-Scholes analytical price and the GBM simulated price of the payoff of our hypothtical European call option based on the results obtained from the implementation of this study, as described in previous sections.

Using the code in Rashha (2025*b*), the Black-Scholes price of the option is computed as,

$$C_{\mathrm{BS}} = 8.9160. \tag{41}$$

We treat this value serves as a reference point for assessing the accuracy of the simulation-based estimator using GBM price paths of the stocks. Table 1 summarizes the estimates of the option prices for a varying number of simulated GBM paths that were tested:

N_{paths}	$\widehat{C}_N$	Std. Error	95% CI Lower	95% CI Upper	$\lvert\widehat{C}_N - C_{\text{BS}}\rvert$
1,000	8.4335	0.4229	7.6046	9.2623	0.4826
5,000	9.0358	0.1926	8.6584	9.4132	0.1197
20,000	9.0621	0.0984	8.8692	9.2549	0.1460
100,000	8.9361	0.0437	8.8505	9.0216	0.0200

Table 1: GBM Estimates of the European Call Price using Different Numbers of Simulated Paths.

From the results above, we observe that as the number of simulated paths N increases, the GBM price estimate converges toward the Black–Scholes benchmark price. For instance, the estimate based on 1,000 paths differs from C_{BS} by approximately 0.48 units, whereas the estimate based on 100,000 paths differs by only 0.02 units. This convergent behavior is what we expect from this simulation, since the GBM estimator is essentially a sample average of payoffs computed from many GBM-simulated terminal stock prices, and therefore converges to the true expectation C_0 as N increases. This is also consistent with the theory of the Black-Scholes Pricing Framework, which assumes GBM as the underlying distribution of the assets when computing the analytical price of the options Petters & Dong (2016), Black & Scholes (1973).

4 Application of Geometric Brownian Motion and The Black-Scholes Framework: Real Markets

4.1 Motivations

In this section, we will apply the Geometric Brownian Motion (GBM) framework and the Black–Scholes option pricing model to real financial data in order to assess how well the theoretical assumptions examined earlier align with observed market behavior. Because publicly available data on option payoffs are limited, this application focuses on modeling stock prices, for which high-frequency historical data are readily available. This approach is standard in empirical finance, since GBM and the Black-Scholes formula have been shown to hold for a variety of assets other than European call options Petters & Dong (2016).

Using daily closing values of the S&P~500 index, we estimate the parameters of a GBM model from historical data, examine whether empirical log returns are consistent with the normality assumption implied by the model, and use the fitted parameters to price a hypothetical European call option. By comparing GBM–based price estimates computed following the framework in Section 3.5 with the corresponding Black–Scholes closed-form value, this application illustrates how the GBM model performs when calibrated to real market data and highlights the extent to which theoretical option pricing results are supported in practice.

All code for this section has been written in Python, is fully reproducible, and can be found in Rashha (2025*a*).

4.2 The Dataset

For the application, we use a dataset of the daily S&P~500 index values obtained from the publicly available `xLSTM-TS` repository maintained by Lopez Gil Lopez Gil (2024).

```
sp500_df <- read_csv("/Users/nahianrashha/Desktop/sp500_daily.csv")
```

```
Rows: 6037 Columns: 8
-- Column specification --------------------------------------------------------
Delimiter: ","
dbl  (7): Open, High, Low, Close, Volume, Dividends, Stock Splits
dttm (1): Date

i Use `spec()` to retrieve the full column specification for this data.
i Specify the column types or set `show_col_types = FALSE` to quiet this message.
```

```
glimpse(sp500_df)
```

```
Rows: 6,037
Columns: 8
$ Date            <dttm> 2000-01-03 05:00:00, 2000-01-04 05:00:00, 2000-01-
05 0~
$ Open            <dbl> 1469.25, 1455.22, 1399.42, 1402.11, 1403.45, 1441.47, 1~
$ High            <dbl> 1478.00, 1455.22, 1413.27, 1411.90, 1441.47, 1464.36, 1~
$ Low             <dbl> 1438.36, 1397.43, 1377.68, 1392.10, 1400.73, 1441.47, 1~
$ Close           <dbl> 1455.22, 1399.42, 1402.11, 1403.45, 1441.47, 1457.60, 1~
```

```
$ Volume         <dbl> 931800000, 1009000000, 1085500000, 1092300000, 12252000~
$ Dividends      <dbl> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0~
$ `Stock Splits` <dbl> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0~
```

The dataset spans the period from March 1,~2000 to December 29,~2023 and contains daily observations of the S&P~500 index. The dataset is stored locally as `sp500_daily.csv`, has 6037 observations, each representing a single trading day within the described period, and contains eight columns: `Date`, `Open`, `High`, `Low`, `Close`, `Volume`, `Dividends`, and `Stock Splits`.

The `Close` column records the closing level or prices of the S&P~500 index on each trading day and serves as the price process $\{S(t)\}$ used throughout the analysis. Because the data are recorded at a daily frequency, we will set the time increment used for computing log returns to $\Delta t = 1/252$, reflecting the conventional assumption of approximately 252 trading days per year in U.S. equity markets.

In the following section, we use the daily closing prices from this dataset to estimate the parameters of a Geometric Brownian Motion model and to assess whether the empirical distribution of log returns is consistent with the normality assumption implied by the GBM framework.

4.3 Estimation of the GBM Model from Market Data

Using the daily closing prices of the S&P500 index described above, we model the price process $\{S_t\}$ as a Geometric Brownian Motion, following the theoretical framework introduced in Section 2.6. The estimation procedure is based on continuously compounded log returns,

$$r_t = \log\left(\frac{S_t}{S_{t-1}}\right), \tag{42}$$

which, under the GBM assumption, are independent and normally distributed with parameters determined by the drift μ and volatility σ Petters & Dong (2016).

In the application code, the dataset is first ordered chronologically, rows with missing values are omitted, and then, the `Close` column is extracted as the observed price series. Daily log returns are then computed using 42. Because the data are observed at a daily frequency, the time increment is fixed at

$$\Delta t = \frac{1}{252}, \tag{43}$$

corresponding to the standard convention of 252 trading days per year.

We then estimate the GBM parameters using sample moments of the daily log returns. Specifically, the sample variance of log returns is used to estimate the volatility parameter σ, while the drift parameter μ is obtained by rescaling the sample mean of log returns and accounting for the Itô correction implied by the GBM solution, as discussed Section 2.6. This moment-based estimation approach follows from the closed-form distribution of GBM log returns.

4.4 Theoretical Basis for Parameter Estimation

The estimation strategy used in this application follows from the analytical properties of the Geometric Brownian Motion (GBM) model introduced earlier in the report. Under GBM, the asset price process satisfies

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t, \tag{44}$$

where μ is the drift parameter and σ is the volatility parameter.

As shown in Section Section 2.6, from 19, this stochastic differential equation has the exact closed-form solution

$$S_t = S_0 \exp\bigl((\mu - \tfrac{1}{2}\sigma^2)t + \sigma B_t\bigr)\,, \tag{45}$$

Petters & Dong (2016).

Now, taking logarithms of price ratios over a fixed time increment Δt yields the log return

$$\log\left(\frac{S_t}{S_{t-\Delta t}}\right) = (\mu - \tfrac{1}{2}\sigma^2)\Delta t + \sigma(B_t - B_{t-\Delta t}). \tag{46}$$

From Section 2.4 we know that Brownian motion has independent increments satisfying

$$B_t - B_{t-\Delta t} \sim \mathcal{N}(0, \Delta t), \tag{47}$$

Petters & Dong (2016). From this, it follows that the log returns are normally distributed,

$$r_t := \log\left(\frac{S_t}{S_{t-1}}\right) \sim \mathcal{N}\bigl((\mu - \tfrac{1}{2}\sigma^2)\Delta t,\ \sigma^2\Delta t\bigr)\,. \tag{48}$$

Thus, the population mean and variance of log returns would staisfy the following,

$$\mathbb{E}[r_t] = (\mu - \tfrac{1}{2}\sigma^2)\Delta t, \tag{49}$$

and

$$\mathrm{Var}(r_t) = \sigma^2\Delta t. \tag{50}$$

These relationships provide the necessary theoretical justification for the estimation approach used in our application. Under the GBM assumptions, we expect that the sample mean and sample variance of observed log returns would converge to 49 and 50, respectively. The volatility parameter σ can therefore be estimated from the sample variance, while the drift parameter μ is recovered by rearranging the mean relationship. Using the estimated parameters, the implied daily mean and variance of log returns would then be,

$$\mathbb{E}[r_t] = (\hat{\mu} - \tfrac{1}{2}\hat{\sigma}^2)\Delta t, \tag{51}$$

and

$$\mathrm{Var}(r_t) = \hat{\sigma}^2 \Delta t. \tag{52}$$

In the application, these quantities are used to compare the empirical distribution of observed log returns to the normal distribution implied by the fitted GBM model to the data Rashha (2025*a*). We coduct this comparison by visually inspecting the distribution of the simulated of the log-returns and comparing it to the normal distribution to evaluate whether the normality assumption underlying GBM is reasonable for the S&P~500 data.

4.5 Hypothetical Option Pricing under Black–Scholes

With the volatility parameter estimated from historical data, we apply the Black–Scholes framework introduced in 2.7 to price a hypothetical European call option specified by a maturity T, strike price K, and a constant risk-free interest rate r.

The drift used in simulation is set equal to the risk-free rate r, while volatility is fixed at the estimated value $\hat{\sigma}$. Two pricing approaches are implemented. First, the Black–Scholes closed-form solution is evaluated using the fitted parameters. Second, a GBM simulated estimate is obtained by simulating terminal prices from the GBM solution and obtaining the discounted average payoff. This comparison mirrors the simulation-based pricing framework developed in 3.5 and could aid in illustrating how the analytical and numerical approaches relate when calibrated to real market data.

4.6 Implementation Overview

The application code performs the following steps:

1. Loads daily S&P~500 index data and extracts the closing price series.
2. Computes daily log returns and sets the time increment according to equation 43.
3. Estimates the GBM drift and volatility using sample moments of log returns.
4. Evaluates the normality assumption implied by the GBM model using graphical diagnostics.
5. Prices a hypothetical European call option using both the Black–Scholes formula and the GBM estimated volatility parameter.

Rashha (2025*a*)

In the next section, we examine the results obtained from this application example.

4.7 Application Parameters and Results

4.7.1 Empirical Log Returns and GBM Fit

Figure 2 compares the empirical distribution of daily log returns for the S&P~500 to the normal distribution implied by the fitted GBM model. We observe that while the fitted distribution of log returns captures the center of the normal distribution reasonably well, the empirical log returns exhibit substantial departures from normality.

The estimated sample moments of daily log returns are,

$$\hat{m} = 0.00053742, \qquad \hat{v} = 3.58197760 \times 10^{-4}, \tag{53}$$

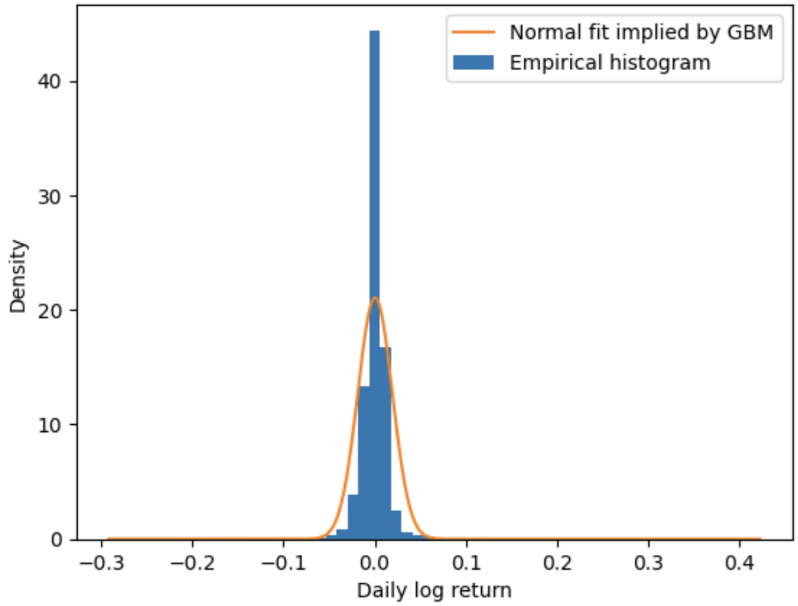

Figure 2: Histogram of daily S&P~500 log returns overlaid on GBM-implied normal distribution.

Using equations 51 and 52 and inputting the above values of the sample moments, we obtain the following estimated GBM parameters for drift and volatility respectively:

$$\hat{\mu} = 0.1806, \qquad \hat{\sigma} = 0.3004. \tag{54}$$

The computed higher-order moments indicate pronounced departures from normality. The empirical skewness and excess kurtosis are,

$$\text{Skewness} = 3.4924, \qquad \text{Excess kurtosis} = 156.1138. \tag{55}$$

These results likely capture the well-known heavy-tailed nature real of equity markets, and a key limitation of the GBM assumption Koronkiewicz & Jamroz (2014), Petters & Dong (2016).

4.7.2 GBM Forecast Distribution

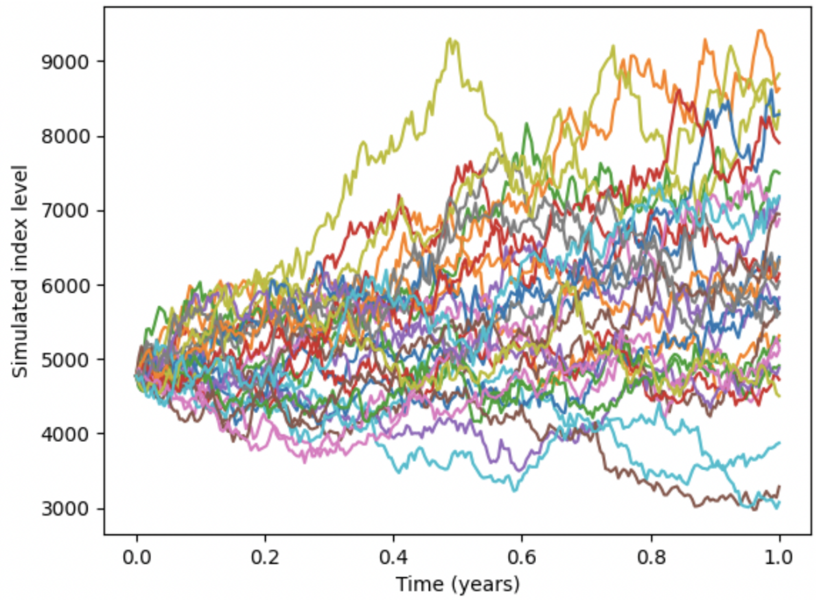

Figure 3: Sample of Simulated GBM Price Paths fitted on the S&P500.

Using the above fitted GBM parameters in 54, we generate a one-year-ahead distribution for the S&P~500 index. Starting from the current index level or the last observed closing value of the index,

$$S_0 = 4769.83, \tag{56}$$

the simulated GBM forecast distribution yields the following quantiles:

$$S_{0.05}(T) \approx 3333.80, \quad S_{0.50}(T) \approx 5440.97, \quad S_{0.95}(T) \approx 8949.06. \tag{57}$$

Figure 3 illustrates a subset of simulated GBM index price paths over the one-year horizon. The wide dispersion of trajectories reflects the relatively large estimated volatility and likely indicates the uncertainty inherent in long-term equity forecasts under the GBM model Petters & Dong (2016).

4.8 Option Pricing Results

Using the historical volatility estimate $\hat{\sigma}$, a hypothetical European call option is priced under the Black–Scholes framework. The option parameters are

$$T = 1.0, \qquad r = 0.04, \qquad K = 1.05 S_0 = 5008.32. \tag{58}$$

The Black–Scholes closed-form price is

$$C_{\mathrm{BS}} = 551.2667. \tag{59}$$

A GBM estimate based on $N = 200{,}000$ simulated GBM paths yields

$$\hat{C}_{\mathrm{MC}} = 553.1366, \tag{60}$$

with standard error

$$\mathrm{SE} = 2.2193, \tag{61}$$

and corresponding 95% confidence interval

$$[548.7868,\ 557.4863]. \tag{62}$$

The absolute difference between the Monte Carlo estimate and the Black–Scholes price is

$$|\hat{C}_{\mathrm{MC}} - C_{\mathrm{BS}}| = 1.8699. \tag{63}$$

The relatively narrow absolute difference between the GBM simulated price of the hypothetical option and the analytical Black-Scholes framework reaffirms our discussion in Section 3.6, where we saw that for large N, we expect the GBM simulated price of an asset to converge to the Black-Scholes one. This agreement shows that, conditional on the fitted volatility parameter, the simulation-based pricing procedure is numerically consistent with the Black–Scholes formula.

Overall, this section illustrates how the GBM framework operates when calibrated to real market data. Historical S&P~500 returns provide a natural basis for estimating the model's first two moments, which in turn determine the volatility parameter used for forecasting and option pricing. While the fitted GBM model produces coherent simulations and yields fairly consistent Black–Scholes and GBM-simulated option prices, diagnostic analysis of log returns reveals substantial departures of its distribution from the normality assumption underlying GBM. As a result, the application highlights both the practical usefulness of GBM as an approximation of the behavior of asset prices and the importance of interpreting its outputs with caution when modeling real markets.

5 Conclusion

In this paper, we examined geometric Brownian motion (GBM) as a foundational model for asset prices in equity markets, and its role in developing the Black–Scholes framework for pricing options. In Section 2, we explored how GBM arises from Brownian motion, its stochastic properties, including how it yields lognormally distributed prices for assets, and its closed-form solution for terminal prices of assets. Finally, we observed GBM's connection to the Black–Scholes formula for options under ideal market conditions.

In Section 3, the simulation study served as a bridge between theory and computation. By simulating terminal prices of stocks directly using the closed-form solution of the GBM stochastic differential equation, the study demonstrated how it is possible to approximate option values as discounted expected payoffs under GBM. The close agreement between the GBM estimates and the Black–Scholes closed-form price in the toy example illustrated the consistency of the models and solidified the theoretical connection between two foundational contributions to modern-day financial mathematics by Paul Samuelson, Fischer Black, and Merton Scholes Samuelson (1965), Black & Scholes (1973).

The final application to real S&P 500 data highlighted both the practical utility and the limitations of the GBM model. Calibrating GBM parameters using historical daily log returns allowed us to construct many possible future index paths, and the price a hypothetical European call option using the Black–Scholes framework. While the fitted model matched the empirical mean and variance of log returns by construction, diagnostic plots revealed substantial deviations from normality in higher-order moments, reflecting skewness and heavy tails commonly commonly observed in equity markets Koronkiewicz & Jamroz (2014). However, conditional on the estimated volatility parameter, the Black–Scholes price and its GBM approximation of the option price remained numerically consistent, reinforcing the interpretation of both GBM and Black–Scholes as model-based pricing tools and the notion of exercising caution when using them to capture real market dynamics .

6 Appendix: Covariance of Brownian Motion

Let $\{B(t) : t \geq 0\}$ be a standard Brownian motion. We derive the covariance function $\mathrm{Cov}(B(s), B(t))$ using the defining properties of Brownian motion—independent increments and Gaussian increments with variance equal to the time increment—as in Petters and Dong Petters & Dong (2016).

Fix $0 \leq s \leq t$. Then we can write

$$B(t) = B(s) + (B(t) - B(s)). \tag{64}$$

By the independent-increments property of Brownian motion, $B(s)$ and $B(t) - B(s)$ are independent random variables. Therefore,

$$\mathrm{Cov}\big(B(s), B(t)\big) = \mathrm{Cov}\big(B(s),\ B(s) + (B(t) - B(s))\big). \tag{65}$$

Using the bilinearity of covariance,

$$\mathrm{Cov}\big(B(s),\ B(s) + (B(t) - B(s))\big) = \mathrm{Cov}\big(B(s), B(s)\big) + \mathrm{Cov}\big(B(s), B(t) - B(s)\big). \tag{66}$$

Since $B(s)$ and $B(t) - B(s)$ are independent, their covariance is zero:

$$\mathrm{Cov}\big(B(s), B(t) - B(s)\big) = 0. \tag{67}$$

Hence,

$$\mathrm{Cov}\big(B(s), B(t)\big) = \mathrm{Cov}\big(B(s), B(s)\big) = \mathrm{Var}\big(B(s)\big). \tag{68}$$

From the definition of Brownian motion, we know that

$$B(s) - B(0) \sim \mathcal{N}(0, s), \tag{69}$$

which implies

$$\mathrm{Var}\big(B(s)\big) = s. \tag{70}$$

Therefore, for $0 \leq s \leq t$,

$$\mathrm{Cov}(B(s), B(t)) = s. \tag{71}$$

By symmetry of covariance,

$$\mathrm{Cov}(B(s), B(t)) = \mathrm{Cov}(B(t), B(s)), \tag{72}$$

and thus, for arbitrary $s, t \geq 0$,

$$\mathrm{Cov}(B(s), B(t)) = \min(s, t). \tag{73}$$

This proves the result.

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