<u>Problem</u>: Prove that $2592 = 2^5 \times 9^2$ is the only number which has the property that

$$N = 1000a + 100b + 10c + d = a^b \cdot c^d$$

<u>Proof</u>: First note that $b = 1 = d$ is an impossible situation. Because then

$$1000a + 100b + 10c + d = ac$$
$$< 10c < 1000a + 100b + 10c + d$$
$$(\because a \leq 9 < 10)$$

So at least one of b or d is > 1.

$\Rightarrow b + d \geq 3$.

~~suppose~~ <u>claim</u> c is odd.

<u>proof of claim</u>: suppose c is even.

$\Rightarrow d$ is even. $\Rightarrow 4 \mid c^d$.

$\Rightarrow 4 \mid (1000a + 100b + 10 + d)$

$\Rightarrow 4 \mid d \quad (\because c \text{ is even})$

$\Rightarrow 2^4 = 16 \mid c^d.$

$\Rightarrow 16 \mid (1000a + 100b + 10c + d)$

$\Rightarrow 16 \mid (8a + 4b + 4c_1 + 4d_1)$

where, $c = 2c_1$ and $d = 4d_1$

(take modulo 16 of all terms).

$\Rightarrow 4 \mid (2a + b + c_1 + d_1)$

where $d_1 \in \{1, 2\}$

& $c_1 \in \{1, 2, 3, 4\}$.

$(\because 0 < c < 9)$

Note that $d_1 = 1$

$\Rightarrow 1000 < a^b \cdot c^{4d_1} < 10000$

$\Rightarrow c = 6$ or 7 or 8 or 9.

But we assumed c is even.

$\Rightarrow c = 6$ or $c = 8$

$\Rightarrow c_1 = c/2 = 3$ or 4.

so $4 \mid (2a+b+3+1)$

$\Rightarrow b$ is even

$\Rightarrow 4 \mid (2a+2b_1) \Rightarrow 2 \mid (a+b_1)$

case i a is even.

$\Rightarrow 2^2 = 4 \mid a^b$

$\Rightarrow 4 \times 16 = 64 \mid N$

i.e. $64 \mid (N = 1000a + 100b + 10c + d)$

or ~~$64 \mid (40a+$~~

$(1000a+100b+10c+d)$

$\Rightarrow 64 \mid (1000a+100b+10c+d)$

$\Rightarrow 64 \mid (16a' + 8b' + 62)$

$\Rightarrow$ ~~$8 \mid (2a' + b' + 4)$~~

$\Rightarrow 8 \mid 16a' + b' + 62$

$64 \mid (40a + 36b + 62)$

$32 \mid = 2(20a + 18b + 31)$

$32 \mid (20a + 18b + 31)$ is impossible.

similar if $c = 8$ then.

$32 \mid (20a + 18b + 41)$ is also impossible.

$\Rightarrow$ a ~~cannot~~ be even.

<u>case ii)</u> a is odd.

then $\cancel{2 \nmid a+b}$ $2 \mid (a + b_1)$

$\Rightarrow b_1$ is even ($\because$ a is even)

$\Rightarrow b_1$ is $\{2, 4\}$

$\Rightarrow b = 2b_1 = \{4, 8\}$

$\Rightarrow a^b \in \{a^4, a^8\}$.

and $a \neq 1$.

But if $a > 1$ then, $a < 3$

$\because$ $\cancel{a^4 \times c^d} =$ if $a \geq 3$ then

$a^4 \cdot c^d = 3^4 \times (688)^4 > 100$

$\Rightarrow$ a has to be equal to 2
which is a contradiction because we assumed a is odd.

Similarly $d_1 = 2$ also leads to contradiction.

Thus c cannot be even

$\Rightarrow$ c is an odd number.

suppose a is also odd.

$\Rightarrow$ d is odd.

Now $1000a + 100b + 10c + d = a^b \cdot c^d$

$\Rightarrow a \mid (100b + 10c + d)$ — (i)

and $c \mid (1000a + 10c + d)$ — (ii)

So let $100b + 10c + d = ma$ — (iii)

& $1000a + 10c + d = nc$ — (iv).

$\Rightarrow (10 + n)c = (1000 + m)a$.

subtracting the equations (iii) & (iv)

Since a & $c \in \{1, 3, 5, 7\}$
are both odd at this point

$a \nmid c$ & $c \nmid a$

$\Rightarrow$ gcd $(a, c) = 1$.

$\Rightarrow$ $c \mid (1000 + m)$

& $a \mid (10 + n)$

$\Rightarrow$ $1000 + m = uc$ — (v) $\times (a)$

$10 + n = va$ — (vi) $\times (c)$

$\Rightarrow$ $(1000a - 10c) + (ma - nc)$ — (v)

$= (u - v) ac.$

But $ma - nc \equiv 0 \pmod{10}$

$\Rightarrow$ (v) $\Rightarrow$ $(u - v) ac \equiv 0 \pmod{10}$

$\Rightarrow$ $2 \mid ac.$ or $5 \mid ac$

or $10 \mid ac.$

$2 \mid ac$ is impossible be a & c are odd

$10 \mid ac \Rightarrow a=2, c=5$
or
$a=5, c=2$

impossible because a & c are odd.

So we are left with

$5 \mid ac \Rightarrow a=1, c=5$
or
$a=5$ & $c=1$.

$a=1 \Rightarrow 1000a+100b+10c = c^d$

or $N = 5^d \ (d>1)$

or $1000+100b+50 = 5^d$

$\Rightarrow 40+4b+2 = 5^{d-2}$

(impossible LHS is even & RHS is odd).

$c=1 \Rightarrow 5000+100b+10 = 5^b$

$\Rightarrow$

$\Rightarrow 1000 + 20b + 2 = 5^{b-2}$

impossible LHS is even, RHS is odd

Thus a is not odd.

$\Rightarrow$ a is even number.

$\Rightarrow$ d is even.

So we get

$\boxed{c \text{ is odd, } a \text{ is even } \& \ d \text{ is even}}$

Let $a = 2a_1$ and $d = 2d_1$.

$$N = 2000a_1 + 100b + 10c + 2d_1$$
$$= 2^b \cdot a_1^b \cdot c^{2d_1}$$

$$\Rightarrow 1000a_1 + 50b + (5c + d_1)$$
$$= 2^{b-1} \cdot a_1^b \cdot c^{2d_1}$$

c is odd $\Rightarrow$ LHS $\equiv 5 + d_1 \pmod{10}$

$\Rightarrow d_1 = 1$ or 3

$\Rightarrow d = 2d_1 = 2$ or 6

~~case 1 $d = 6$. Then,~~

case 1 $d = 6$. Then,

$1000 < N = a^{b} \cdot c^{6} < 10000$

$\Rightarrow 1000 \leq (2a_1)^{b} \cdot c^{6} < 10000$

$\Rightarrow c \leq 4.$

$\Rightarrow c = 1$ or 3 ($\because c$ is odd)

$\Rightarrow a_1 \mid (56b + 8).$

58
checking prime factors
68
we find that
78
a_1 can be 1, 2, 3, 4
88
98
$\Rightarrow a$ can be 2, 4, 6, 8
108
118
Plugging 2, 4, 6, 8 for a
128
138
we get 4 Diophantine
148
equations as follows.
158

$$\left\{\begin{array}{l} 2016 + 100b = 2^b \\ 4016 + 100b = 4^b \\ 6016 + 100b = 6^b \\ 8016 + 100b = 8^b \end{array}\right.$$

Easy ~~exect check~~ checking in MS Excel tool shows that none of them have solution. In fact since LHS is a linear function of b & RHS is a exponential function of b so checking only first few values is sufficient.

$c = 3 \Rightarrow a_1 \mid 50b + 13$

63
73 $\rightarrow$ Most of them are prime
83 $\rightarrow a_1 = 1$
93
103 Others are divisible by 3 or
113
123 & $a_1 = \{1, 3, \text{or } 7\}$
133
143
153 $\rightarrow$ a is 2 or 6. only

$a=6 \Rightarrow$ implies that

$$6000 + 100b + 10c + 6 = 6^b \cdot c^6.$$

$\Rightarrow \; 6 \mid (100b + 10c)$

$\Rightarrow \; 3 \mid (50b + 5c)$

$\Rightarrow \; 3 \mid 5\cdot(10b + c)$

$\Rightarrow \; 3 \mid (10b + c) \quad (\because 3 \nmid 5)$

$\Rightarrow \; 3 \mid (b + c)$

$\Rightarrow \; b + c$ is 6 or 9.

$\Rightarrow \; b = 6$ & $c = 3$

or $b = 3$, $c = 3$.

Thus, we get

$a = 6$, $b = 6$, $c = 3$, $d = 6$

and $a = 6$, $b = 3$, $c = 3$, $d = 6$

Neither is a solution.

$$\Rightarrow a \neq 6.$$

$\Rightarrow a = 2$ & $d = 6$

$\Rightarrow 2000 + 100b + 10c + 6 = 2^b \cdot c^6$

$\Rightarrow c \mid (50b + 1003)$.

listing down for $b = 1, 2, \ldots, 9$
& noting prime factors we get

$c = 1, 3,$ or 7

$c = 7 \Rightarrow$

$2000 + 100b + 10c + 6 = 2^b \cdot 7^6$

LHS < 3000 & RHS > 3000

A contradiction.

$\Rightarrow c = 1$ or 3

If $c = 1$, $2016 + 100b = 2^b$

has no solution (check with MS Excel)

$c = 3 \Rightarrow 2036 + 100b = 2^b \cdot 3^6$

or $9 \mid (b + 2) \Rightarrow b = 7$.

But $a = 2, c = 3, b = 7, d = 6$ is
again not a solution.

we are finally left with the case of

$a = 2$ & $d = 2$

$2000 + 100b + 10c + 2 = 2^b \cdot c^2$

where $c = 1, 3, 7$ or 9.

$c = 1 \Rightarrow 50b + 1006 = 2^{b-1}$

Excel shows no solution.

$c = 3 \Rightarrow 50b + 1016 = 2^{b-1} \cdot 9$

$\Rightarrow 9 | 5 + 5b = 5(b+1)$

$\Rightarrow 9 | b + 1$ impossible

$\Rightarrow b = 8$.

But $a = 2, b = 8, c = 3, d = 2$ is easily checked to be NOT a soln.

$\Rightarrow c \neq 3$.

$c = 7 \Rightarrow 49 | (b + 28)$

impossible $\because b + 28 < 49$

Finally we are left with the case of $c=9$.

$$1010 + 50b = 2^{b-1} \cdot 81.$$

This yields a valid solution of $b=5$.

It is easily checked that $a=2,\ b=5,\ c=9,\ d=2$ is a valid solution and in fact the only solution possible.

Hence the proof is completed.

[QED].