Problem:

Suppose we have a 3×3 matrix filled with real numbers in such a way that;

the number in each cell is the SUM of the numbers in its neighbouring cells.

Prove that the only solution to this problem is the trivial matrix filled with 0's in all cells

that is:

0	0	0
0	0	0
0	0	0

Solution

x_{11}	x_{12}	x_{13}
x_{21}	x_{22}	x_{23}
x_{31}	x_{32}	x_{33}

$$x_{11} = x_{12} + x_{21} + x_{22}$$

$$x_{12} = x_{11} + x_{21} + x_{22} + x_{23} + x_{13}$$

$$x_{13} = x_{12} + x_{22} + x_{23}$$

$$x_{21} = x_{11} + x_{12} + x_{22} + x_{32} + x_{31}$$

$$x_{22} = x_{11} + x_{12} + x_{13} + x_{21} + x_{23} + x_{31} + x_{32} + x_{33}$$

$$x_{23} = x_{13} + x_{12} + x_{22} + x_{32} + x_{33}$$

$$x_{31} = x_{21} + x_{22} + x_{32}$$

$$x_{32} = x_{31} + x_{21} + x_{22} + x_{23} + x_{33}$$

$$x_{33} = x_{32} + x_{22} + x_{23}.$$

$$x_{17} = x_{12} + x_{21} + x_{22}$$

$$\cancel{x_{12}} = (\cancel{x_{12}} + \cancel{x_{21}} + x_{22}) + x_{21}$$

$$x_{22} + x_{23} + x_{13}$$

$$2x_{27} + 2x_{22} + x_{23} + x_{13} = 0 \quad — ①$$

$$\cancel{x_{21}} = (x_{12} + \cancel{x_{21}} + x_{22}) + x_{12}$$

$$x_{22} + x_{32} + x_{31}$$

$$2x_{12} + 2x_{22} + x_{32} + x_{31} = 0 \quad — ②$$

$$x_{22} = (x_{11} + x_{33}) + (x_{12} + x_{21})$$

$$+ (x_{13} + x_{23} + x_{32} + x_{31})$$

$$= (x_{12} + x_{27} + x_{22} + x_{32} + x_{23} + x_{22})$$

$$+ (x_{12} + x_{21})$$

$$+ (x_{23} + x_{32})$$

$$+ (x_{13} + x_{31})$$

x_{22}: $0 = x_{22} + 2(x_{12} + x_{21}) + 2(x_{23} + x_{32})$
$+ (x_{13} + x_{31})$ — ③

① $\cancel{x_{32}} = x_{31} + x_{21} + \cancel{x_{22}} + \cancel{x_{23}} + x$
$\cancel{x_{32}} + x_{22} + x_{23}$

$2x_{23} + 2x_{22} + x_{21} + x_{31} = 0$ — ④

② $\cancel{x_{23}} = (x_{32} + \cancel{x_{23}} + x_{22}) +$
$x_{13} + x_{12} + x_{22} + x_{32}$

$2x_{32} + 2x_{22} + x_{13} + x_{12} = 0$ — ⑤

Adding ① & ② we get,

$4x_{22} + 2(x_{12} + x_{21}) +$
$+ (x_{13} + x_{31}) + (x_{23} + x_{32}) = 0$ — ⑥

$4x_{22} + 2(x_{23} + x_{32}) +$
$+ (x_{12} + x_{21}) + (x_{13} + x_{31}) = 0$ — ⑦

$8x_{22} + 3(x_{12} + x_{21}) + 3(x_{23} + x_{32})$
$+ 2(x_{13} + x_{31}) = 0$ — ⑧

$$8x_{22} + \frac{3}{2}\left[-x_{22} - (x_{13}+x_{31})\right]$$

$$+ 2(x_{13}+x_{31}) = 0$$

$$\left(8-\frac{3}{2}\right)x_{22} + \left(2-\frac{3}{2}\right)(x_{13}+x_{31}) = 0$$

$$15x_{22} + (x_{13}+x_{31}) = 0 \quad \text{—} \quad (9)$$

$$x_{13}+x_{31} = 2x_{22} + (x_{12}+x_{21}) + (x_{23}+x_{32})$$

$$\text{—} \quad (10)$$

$$x_{13}+x_{31} = 2x_{22} + \frac{1}{2}\left[-x_{22} - (x_{13}+x_{31})\right]$$

$$\left(1+\frac{1}{2}\right)(x_{13}+x_{31}) = \left(2-\frac{1}{2}\right)x_{22}$$

$$3(x_{13}+x_{31}) = x_{22} \quad \text{—} \quad (11)$$

(9) & (10) together imply that

$$x_{22} = 0, \quad x_{13}+x_{31} = 0$$

$$\Rightarrow (x_{12}+x_{21}) + (x_{23}+x_{32}) = 0$$

$$\text{—} \quad (12)$$

$$x_{22} = x_{11} + (x_{12}+x_{21}) + (x_{23}+x_{32}) + (x_{13}+x_{31}) + x_{33}$$

$$\Rightarrow \cancel{0 = x_{11}+x_{33}}$$

$$x_{11}+x_{33} = 0 \quad \text{—— (13)}$$

But $\cancel{x_{11}+x_{33} = (x_{12}+x_{21}) + (x_{32}+x_{23}) + 2x_{22}}$

$$x_{12} = x_{11}+x_{21}+x_{22}+x_{23}+x_{13}$$

& $x_{21} = x_{11}+x_{12}+x_{22}+x_{32}+x_{31}$

together imply that,

$$(x_{12}+x_{21}) = 2x_{11} + (x_{12}+x_{21}) + 2x_{22} + (x_{23}+x_{32}) + (x_{13}+x_{31})$$

$$\Rightarrow \cancel{2}x_{11} = x_{12}+x_{21} \quad \text{—— (14)}$$

Reduced equation:

$$x_{11} = x_{12}+x_{21} \quad \text{—— (15)}$$

$$x_{13} = x_{12}+x_{23} \quad \text{—— (16)}$$

$$x_{31} = x_{21}+x_{32} \quad \text{—— (17)}$$

$$x_{33} = x_{23}+x_{32} \quad \text{—— (18)}$$

let $x_{11} = m \Rightarrow x_{33} = -m$

let $x_{13} = n \Rightarrow x_{31} = -n$

our matrix now looks like this:

m	x_{12}	n
x_{21}	0	x_{23}
$-n$	x_{32}	$-m$

— (*)

$$x_{12} + x_{21} + 0 = m$$

and $x_{12} + x_{23} + 0 = n$

$$\Rightarrow 2x_{12} + (x_{21} + x_{23}) = m+n.$$

But $x_{12} = (x_{21} + x_{23}) + (m+n)$.

$$\Rightarrow 3x_{12} = 2(m+n)$$

$$\Rightarrow x_{12} = \frac{2}{3}(m+n) \quad \text{— (19)}$$

$$\Rightarrow x_{21} + x_{23} = -\frac{1}{3}(m+n) \quad \text{— (20)}$$

$$\Rightarrow x_{21} = m - x_{12}$$

$$= m - \frac{2}{3}(m+n)$$

$$= \frac{1}{3}m - \frac{2}{3}n \quad \text{—— (21)}$$

& $$x_{23} = n - x_{12}$$

$$= \frac{1}{3}n - \frac{2}{3}m \quad \text{—— (22)}$$

(21) implies that

$$x_{32} = -n - x_{21}$$

$$= -n - \frac{m}{3} + \frac{2}{3}n$$

$$= -\frac{1}{3}(m+n) \quad \text{—— (23)}$$

which is also implied by (22).

~~But, by symmetry~~ But

$$x_{21} + x_{32} = -n,$$

and, $$x_{23} + x_{32} = -n$$

$$\Rightarrow 2x_{32} + (x_{21} + x_{23}) = -(m+n)$$

and,

$$x_{32} - (x_{21} + x_{23}) = -(m+n).$$

$$\Rightarrow \quad 3x_{32} = -2(m+n)$$

$$\Rightarrow \quad x_{32} = -\frac{2}{3}(m+n) \quad \text{—— } 24$$

comparing 23 & 24 we get,

$$-\frac{2}{3}(m+n) = -\frac{1}{3}(m+n)$$

$$\Rightarrow \quad m+n = 0$$

Now our matrix looks like this:

$$\begin{array}{c c} & \begin{array}{|c|c|c|} \hline m & 0 & -m \\ \hline m & 0 & -m \\ \hline m & 0 & -m \\ \hline \end{array} \end{array} \quad \text{—— } (**)$$

x_{21}

$$m = x_{21} = m + 0 + 0 + 0 + m$$

$$\Rightarrow \quad m = 2m \Rightarrow m = 0$$

So the only possible solution is

0	0	0
0	0	0
0	0	0

This completes the proof.