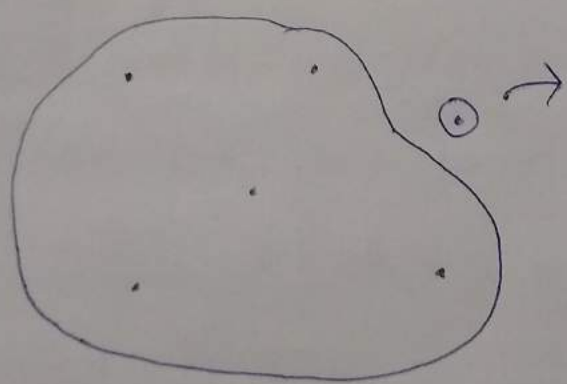
Bollobas - Modern Graph Theory

Chapter 1 - Problem 69

Let G be a graph of order $n \geq 4$. Such that every graph obtained from G by deleting a vertex is regular (i.e., all vertices have the same degree). Show that G is either the complete graph K_n or the empty graph E_n.

<u>Proof</u>: Let $d_1, d_2, \ldots, d_n$ be the degree sequence of the graph.



i-th vertex removed

$G = (V, E)$

$G - \{i\}$

Now $G - \{i\}$ is regular.

Let k_i = regularity of $G - \{i\}$.

Then the number of edges of $G - \{i\}$

$$= \frac{1}{2}(n-1)k_i$$

Thus we have the following relation:

$$|E| = d_i + \frac{1}{2}(n-1)k_i$$

consider a pair of vertices $i \neq j$. Then:

$$|E| = d_i + \frac{1}{2}(n-1)k_i$$

and $|E| = d_j + \frac{1}{2}(n-1)k_j$

$$\Rightarrow (n-1)|k_i - k_j| = 2|d_i - d_j| \quad \text{— (1)}$$

But since $0 \leq d_i \leq n-1$, so,

$0 \leq |d_i - d_j| \leq n-1$.. Also note that

(1) implies that $n-1$ divides $|d_i - d_j|$.

$\Rightarrow$ $|d_i - d_j|$ is 0 or $n-1$

And this is true for all pairs of

vertices $i \neq j$. 1

This tells us something about the

form of the graph G.

$\Rightarrow$ G has the following form:

$E_G = S_1 \cup S_2$ and $S_1 \cap S_2 = \phi$

where

$S_1 = \{(i,j) : d_i = d_j\}$

$S_2 = \{(i,j) : d_i = 0$ and $d_j = n-1$
or $d_i = n-1$ and $d_j = 0\}$

Consider the ~~size~~ case of $|S_2| \geq 1$ and $|S_1| \geq 1$

If $|S_2| > 1$ then S_2 has the following form:

S_2 has p vertices of degree 0
and S_2 has q vertices of degree $n-1$
and $p + q \leq n$.

But to have a vertex of degree $n-1$, we need at least n vertices.

So only solution is:

$p = 0$ and $q = n$

or $p = n$ and $q = 0$

Now $|S_2| > 1$ implies there exists at least one pair (i_0, j_0) such that

$$d_{i_0} = 0 \text{ and } d_{j_0} = n-1$$

$\Rightarrow q \geq 1 \Rightarrow q$ has to be n.

This is a contradiction to $p + q < n$.

$$\Rightarrow |S_2| = 0 \Rightarrow S_2 = \phi.$$

$\Rightarrow$ | ~~Spqecah~~ $\Rightarrow S_1 = E$.

$\Rightarrow$ All vertices have same degree. which can either be 0 or $n-1$.

Why? Because:

Let $d_i = d$ for all $i = 1, 2, \dots, n$

Then equation ⓪ gives:

$$|E| = d + \frac{1}{2}(n-1)k$$

But $d_i = d \ \forall i \Rightarrow G$ is regular

$$\Rightarrow |E| = \frac{nd}{2}.$$

So $\frac{nd}{2} = d + \frac{1}{2}(n-1)k$

$\Rightarrow (n-2)d = (n-1)k$

where we have the constraints:

$$0 \le d \le n-1$$

and $0 \le k \le n-2$

with these constraints the only possible solution is:

$$d = n-1 \text{ and } k = n-2. \quad — (2)$$

or $d = 0$ and $k = 0$ — (3)

(2) corresponds to the complete graph K_n

(3) corresponds to the empty graph E_n.

This completes the proof.

Book — Bipartite Graphs and their Applications

chapter 2 — Problem 2·2·3

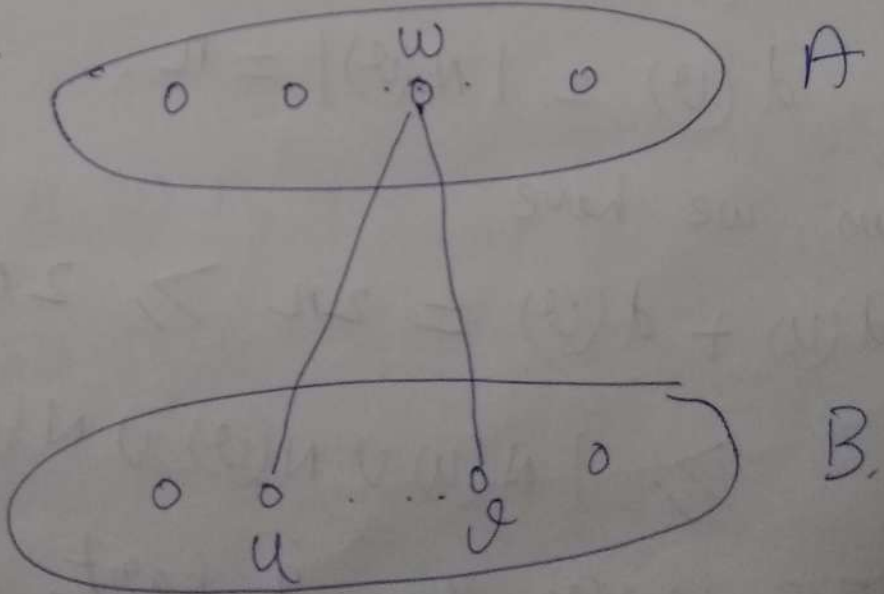
Let G be a simple connected bipartite graph with $|V(G)| \geq 4$. Show that G is a complete balanced bipartite graph if and only if

$$d(u) + d(v) \geq |N(u) \cup N(w) \cup N(v)|$$

for any path uwv of G.

(Asratian, Sarkisian (1991))

<u>Proof</u>:



G is balanced

$\Rightarrow |A| = |B| = n$ (say).

let uwv be a path as shown in the diagram.

G is complete

$\Rightarrow N(u) = A = N(v)$

and $N(w) = B$.

$\Rightarrow |N(u) \cup N(w) \cup N(v)|$

$= |A \cup B| = 2n$

and $d(u) = |N(u)| = n$

$d(v) = |N(v)| = n$.

Thus we have

$d(u) + d(v) = 2n \geq 2n$

$\geq |N(u) \cup N(v) \cup N(w)|$

This proves the $\dot{y}$ part.

Next let's prove the only if part.

Let uwv be a path in ~~the~~ G.

Then we have,

$$d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)|.$$

$$\Rightarrow |N(u)| + |N(v)|$$

$$\geq |(N(u) \cup N(v)) \cup N(w)|$$

$$= |N(u) \cup N(v)| + |N(w)|$$

(because $N(u) \cup N(v)$ and $N(w)$ are disjoint sets).

$$= |N(u)| + |N(v)| - |N(u) \cap N(v)| + |N(w)|$$

$$\Rightarrow |N(u) \cap N(v)| \geq |N(w)|.$$

for all such uwv paths in G.

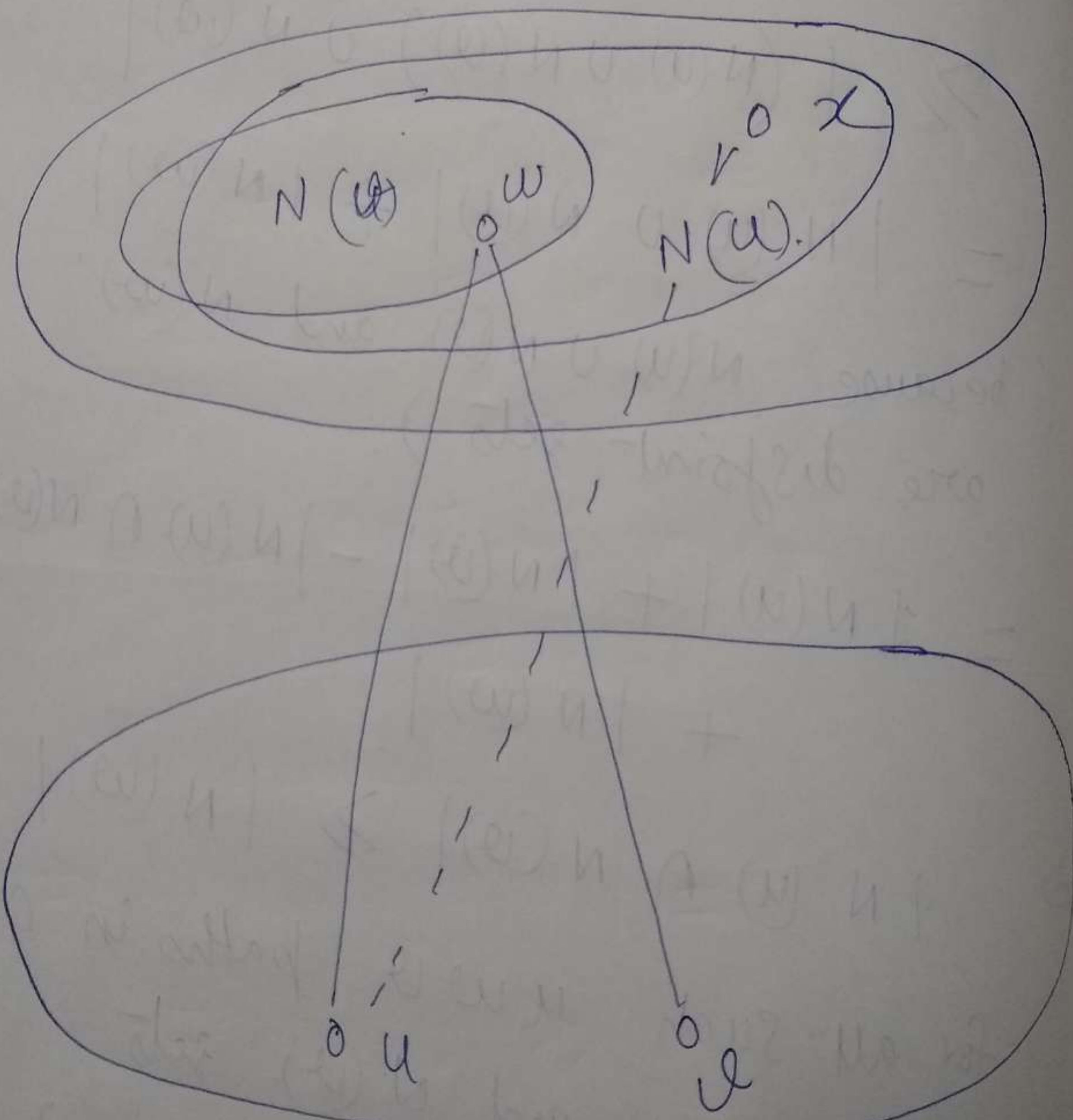
Consider $N(u)$ and $N(v)$ sets.

What can we say about them?

claim $N(u) = N(v)$.

proof of claim:

Suppose $\exists\, x \in N(u)$ such that $x \notin N(v)$.



consider the path wux in G as shown in the diagram.

Then, we have,

$$|N(w) \cap N(x)| \geq |N(u)|$$

$$\geq |N(u) \cap N(v)| \geq |N(w)|.$$

So $|N(w) \cap N(x)| \geq |N(w)|$

$$\Rightarrow \quad N(x) \subseteq N(w) \quad — ①$$

But then, we also have,

$$|N(x)| \geq |N(w) \cap N(x)| \geq |N(w)|$$

$$\Rightarrow \quad |N(x)| \geq |N(w)|. \quad — ②$$

① & ② imply that

$$N(x) = N(w).$$

Now $v \in N(w)$

$$\Rightarrow v \in N(x).$$

which contradicts our starting assumption that $v \notin N$

$\Rightarrow x \in N(v)$.

which contradicts our assumption that $x \notin N(v)$.

$\Rightarrow$ there is no such x which $\in N(u)$ and $\notin N(v)$.

By symmetry there is no such x which $\in N(v)$ and $\notin N(u)$.

These 2 statements imply that

$$N(u) = N(v).$$

as claimed.

$$\Rightarrow |N(u) \cap N(v)| = |N(u)|$$

$$= |N(v)| \geq |N(w)|.$$

$$\Rightarrow |N(u)| \geq |N(w)| \quad — \quad (3)$$

for all such paths.

But now considering the path wux we also have,

$$|N(w)| \geq |N(u)|. \quad — (4)$$

(3) & (4) imply together that

$$|N(w)| = |N(u)| = |N(v)|.$$

for all paths uwv in G.

$\Rightarrow$ $N(a)$ is the same set $\forall\, a \in A$
$N(b)$ is the same set $\forall\, b \in B$.

$\Rightarrow$ $\begin{pmatrix} N(a) = B \\ \forall\, a \in A \end{pmatrix}$ and $\begin{pmatrix} N(b) = A \\ \forall\, b \in B \end{pmatrix}$

Also, $|N(a)| = |N(b)| \;\forall\, a \in A, b \in B$

$\Rightarrow$ $|A| = |B|$, G is balanced.

clearly the above also implies that G is complete.

Thus G is balanced and complete bipartite graph. [QED]

Bollobas — Modern Graph Theory

Chapter 3 — Problem 40 (Hakimi-Havel)

Call a sequence $d_1, \ldots, d_n$ of integers graphic if there is a graph such that $d(x_i) = d_i$, $1 \leq d_i \leq n$. Show that $d_1 \geq d_2 \geq \cdots \geq d_n$ is graphic iff so is the sequence

$$d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n.$$

Proof: First we prove the if part.

let $\{\Delta = d_1 \geq \cdots \geq d_n\} = D$

and let $\{d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n\} = D'$.

Let G' be a graph with degree sequence D'. And let $\{x_1', \ldots, x_{n-1}'\} = V(G')$ be the vertices of G'. where

$deg(x_1') = d_2 - 1$, $deg(x_2') = d_3 - 1$, ... and so on.

Now if we add a vertex x_n' to G' such that x_n' is connected to each of the vertices $(x_1', x_2', \dots, x_{d_1}') \subset V(G')$. then the new graph will have the degree sequence

$$\begin{array}{ccccccc} (d_1, & d_2, & d_3, & \cdots & d_{d_1}, & d_{d_1+1}, \cdots & d_n) \\ | & | & | & & | & | & | \\ (x_1' & x_2' & x_3' & & x_{d_1}' & x_{d_1+1}' & x_n') \end{array}$$

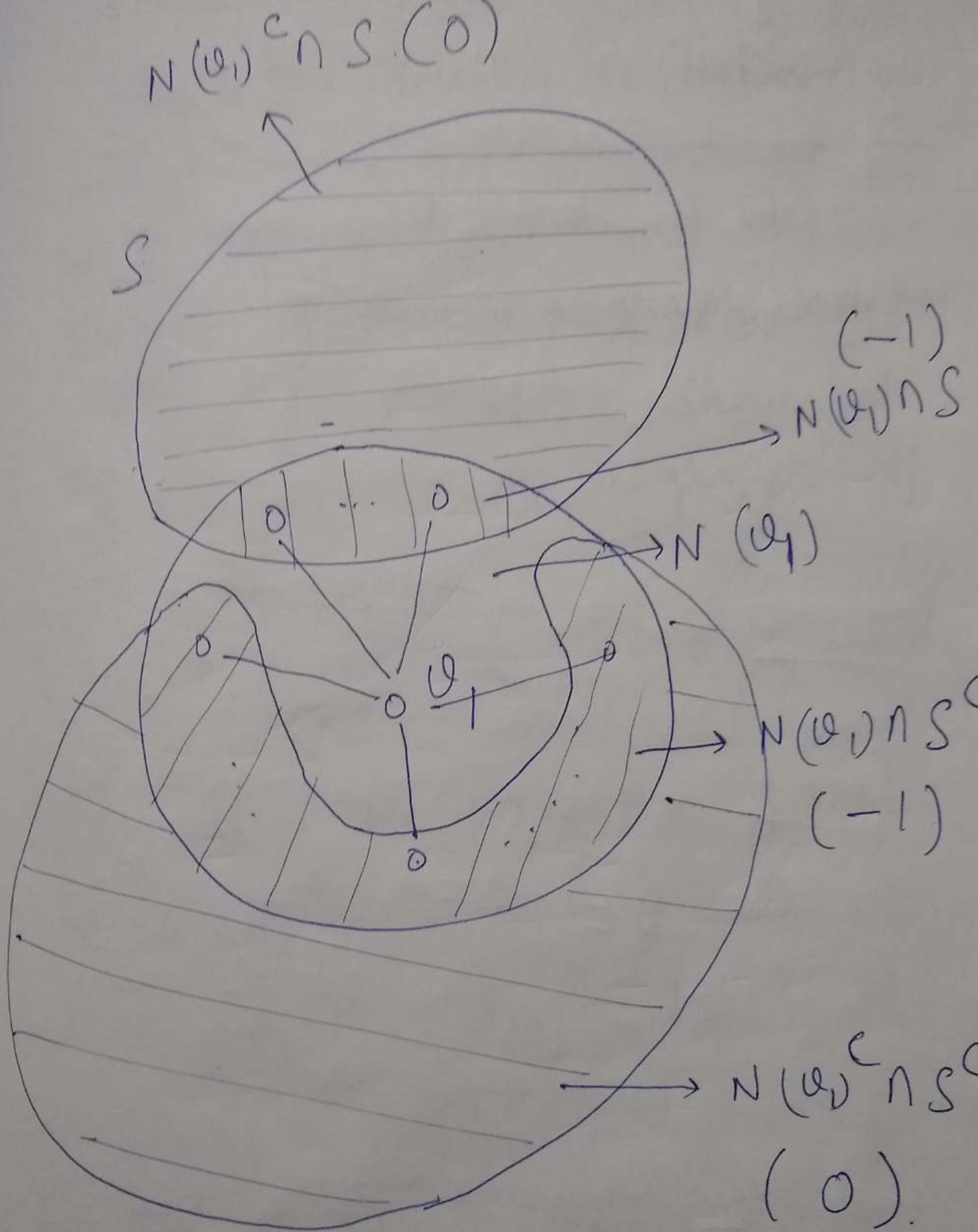
This proves the if part.

Next let's prove the only if part.

So, let G be a graph with degree sequence $D = \{\Delta = d_1, d_2, \dots, d_n\}$ and let S be the set of vertices of G whose degrees are

$$d_2, d_3, \dots, d_{d_1+1}.$$

Consider the decomposition of the graph as shown in the next page. where v_1 is a vertex with degree $= d_1$.



$N(v_1)^c \cap S \, (0)$
S
(-1)
$N(v_1) \cap S$
$N(v_1)$
v_1
$N(v_1) \cap S^c$
(-1)
$N(v_1)^c \cap S^c$
(0).

First note that:

$$|N(v_1) \cap S^c| = |N(v_1)^c \cap S| \quad \text{— (D).}$$

~~This~~ Why is this so?

Let $|N(v) \cap S| = p$.

Then, $|N(v_1)| = d_1 \Rightarrow |N(v_1) \cap S^c| = d_1 - p$.

Also, $|S| = d_1 \Rightarrow |N(v_1)^c \cap S| = d_1 - p$.

which proves the statement (D)

Next note that if the delete the vertex v_1, then the following changes happen to the degrees of the vertices of different sets:

1. degrees of vertices in $N(v_1) \cap S$ get reduced by 1.
2. degrees of vertices in $N(v_1) \cap S^c$ get reduced by 1.

③ degrees of vertices in $N(v_1)^c \cap S$ remain unchanged.

④ degrees of vertices in $N(v_1)^c \cap S^c$ remain unchanged.

① implies that our objective is only partially reached.

② and ③ are unwanted changes.

<u>Question</u>: can we do something to reverse the changes in ② & ③?

<u>Answer</u>: Yes.
How?

Since $|N(v_1) \cap S^c| = |N(v_1)^c \cap S|$

let $N(v_1) \cap S^c = \{x_1, \dots, x_{d_1-p}\}$

and let $N(v_1)^c \cap S = \{y_1, \dots, y_{d_1-p}\}$

where $\deg(x_1) \geq \dots \geq \deg(x_{d_1-p})$

and $\deg(y_1) \geq \dots \geq \deg(y_{d_1-p})$.

Now note that, after deleting v_1,

$\deg(y) \geq d_{d_1+1} \quad \forall\, y \in N(v_1)^c \cap S$

and $\deg(x) \leq d_{d_1+1} - 1 \quad \forall\, x \in N(v_1) \cap S^c$

The second statement is true because, $N(v_1) \cap S^c$ has vertices with degrees which belong to the set $\{d_{d_1+1}-1, \ldots, d_n - 1\}$.

Thus,

$$\deg(y_i) > \deg(x_i).$$

for all pairs $x_i, y_i \quad i = 1, \ldots, d_1 - p$.

Also every vertex in both $N(v_1) \cap S^c$ and $N(v_1)^c \cap S$ are only connected to vertices in T ~~$= V(G) - N(v_1) \cap$~~

$$T = V(G) - N(v_1) \cap S^c - N(v_1)^c \cap S.$$

Start with the pair (x_1, y_1).

$\deg(y_1) > \deg(x_1)$.

Start with the pair (x_1, y_1).

$\deg(y_1) > \deg(x_1)$

and both y_1 and x_1 are only connected to vertices in T together imply that

$\exists\ w_1 \in T$ such that

① y_1 is connected to w_1

② x_1 is not connected to w_1.

~~Remo~~ Delete the edge $y_1 w_1$
and add the edge $x_1 w_1$.

Then $\deg(y_1)$ gets reduced by 1
and $\deg(x_1)$ gets increased by 1.

This reverses the change in ② for x_1
and also reverses the change in ③ for y_1.

Next consider the pair (x_2, y_2) and repeat the procedure. We get a vertex $w_2 \neq w_1$. Delete $y_2 w_2$ and add $x_2 w_2$. This reverses changes for x_2 and y_2.

Repeating this procedure for all pairs

$(x_1, y_1), (x_2, y_2), \ldots, (x_{d_1-p}, y_{d_1-p})$.

we end up reversing all the changes in 2 and 3.

~~we are now left with a graph G'.~~

Note that in this entire process the degrees of vertices remain unchanged because for each w_i, we simultaneously ~~inser~~ delete an edge and also add a new edge.

After all this we are now left with a new graph G' all vertices in the set S have their ~~vertices~~ degrees reduced by 1. and the new values are $d_2-1, d_3-1, \ldots, d_{d_1+1}-1$.

and the rest of the vertices have their degrees same as in G that is their values are $d_{d_1+2}, \ldots, d_n$.

$\Rightarrow$ degree sequence of G' is

$d_2-1, \ldots, d_{d_1+1}-1, d_{d_1+2}, \ldots, d_n$. [QED]

Nathanson [88]: Prove that there does not exist a polynomial identity of the form:

$$(x_1^2+x_2^2+x_3^2)(y_1^2+y_2^2+y_3^2) = z_1^2+z_2^2+z_3^2$$

where z_1, z_2, z_3 are polynomials in $x_1, x_2, y_1, y_2, x_3, y_3$ with integral coefficients:

Proof: Setting $x_3 = y_3 = 0$ we obtain:

$$(x_1^2+x_2^2)(y_1^2+y_2^2) = z_1'^2+z_2'^2+z_3'^2$$

(with $x_3 = y_3 = 0$)

$$\Leftrightarrow \quad x_1^2y_1^2 + x_2^2y_1^2 + x_1^2y_2^2 + x_2^2y_2^2$$

$$= z_1'^2 + z_2'^2 + z_3'^2$$

where z_1', z_2', z_3' are polynomials in x_1, x_2, y_1, y_2 only:

Any general polynomial in x_1, x_2, y_1, y_2 is of the form:

$$ax_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2)$$

$$a, b, c, d \in \mathbb{Z}.$$

so let us assume that:

$$x_1^2y_1^2 + x_2^2y_1^2 + x_1^2y_2^2 + x_2^2y_2^2$$

$$= (ax_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2)^2$$

$$+ (ex_1y_1 + fx_1y_2 + gx_2y_1 + hx_2y_2)^2$$

$$+ (ix_1y_1 + jx_1y_2 + kx_2y_1 + lx_2y_2)^2$$

comparing coefficients of x_1y_1, x_1y_2, x_2y_1 and x_2y_2 on both sides give us:

$$\begin{cases} a^2+e^2+i^2=1 & (1) \\ b^2+f^2+j^2=1 & (2) \\ c^2+g^2+k^2=1 & (3) \\ d^2+h^2+l^2=1 & (4) \end{cases}$$

Next coefficients of cross terms like $x_1^2y_1y_2$, $x_2^2y_1y_2$ etc should be zero on the RHS because no such terms appear on the LHS. ~~This~~

~~zero~~

This gives us the equations:

$$\begin{cases} ab + ef + ij = 0 & (1)' \\ ac + eg + ik = 0 & (2)' \\ ad + eh + il = 0 & (3)' \\ bc + fg + jk = 0 & (4)' \\ bd + fh + jl = 0 & (5)' \\ cd + gh + kl = 0 & (6)' \end{cases}$$

(1) $\Rightarrow a^2 + e^2 + i^2 = 1$

where $a, e, i \in \mathbb{Z}$.

w.l.o.g assume $a = \pm 1$, $e = i = 0$.

With $e = i = 0$ and $a = \pm 1$, (1)' implies

$b = 0$.

Now (2) $\Rightarrow f^2 + j^2 = 1$.

w.l.o.g assume $f = \pm 1$ ~~and~~ $j = 0$.

(4)' $\Rightarrow g = 0$

(5)' $\Rightarrow h = 0$.

③ $\iff c^2+g^2+k^2=1$

$\implies c^2+k^2=1 \quad (\because g=0)$

~~w.l.o.g. assume~~ But,

$ac+eg+ik=0$

$\iff (\pm 1)c + 0\cdot 0 + 0\cdot k = 0$

$\implies c=0.$

~~So~~ ③ $\implies c^2+k^2=1$

$\iff k=\pm 1.$

④$'$ $\implies j=0$

⑥$'$ $\implies l=0$

③$'$ $\implies ad+0+0=0$

$\implies d=0$

So $d^2+h^2+l^2 = 0^2+0^2+0^2$

$= 0$

But ④ $\implies d^2+h^2+l^2=1$

A contradiction.