Bollobas - Modern Graph Theory

Chapter 3 - Problem 25

Let G be a connected graph of order at least 4 such that every edge belongs to a 1-factor of G. Show that G is 2-connected.

Proof. Assume that G is not 2-connected.

Then $\exists\ v \in V(G)$ such that

$G - \{v\}$ is disconnected.

Since G has at least one 1-factor, ~~by~~ the Tutte's theorem implies that

$$q(G-S) \leq |S| \quad \forall\ S \subseteq V(G).$$

where $q(H) = \#$ odd components of H.

Applying the Tutte's theorem to $S = \{v\}$ we obtain:

$$q(G-\{v\}) \leq |\{v\}| = 1$$

$$\Rightarrow q(G-\{v\}) = 0 \text{ or } 1.$$

Case 1: Suppose $q(G-\{v\}) = 0$

Then every component of $G-\{v\}$ is of an even order.

$\Rightarrow |V(G)| = |G|$ (for simplicity write like this)

$= \sum$ even numbers (orders of even components)

$+ |\{v\}|$

$=$ an odd number.

But we know that G has a 1-factor only if $|G|$ is an even number.

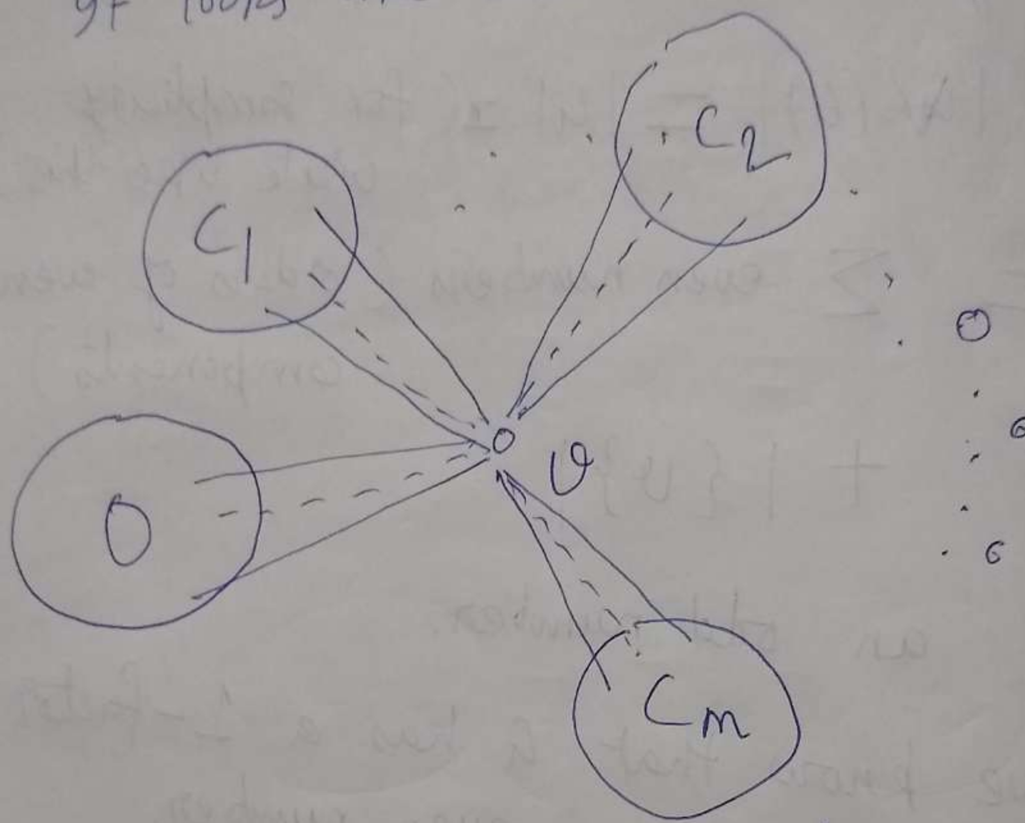
$\Rightarrow$ our graph G does not have a 1-factor.

This is a contradiction.

Case 2: Suppose $q(G-\{v\}) = 1$

Then $G-\{v\}$ has exactly 1 odd order component. Call it O. And the rest of the components of $G-\{v\}$ are of even orders. Call them $C_1, C_2, \ldots, C_n$

So now we have some information about the structure of our graph G. It looks like below.

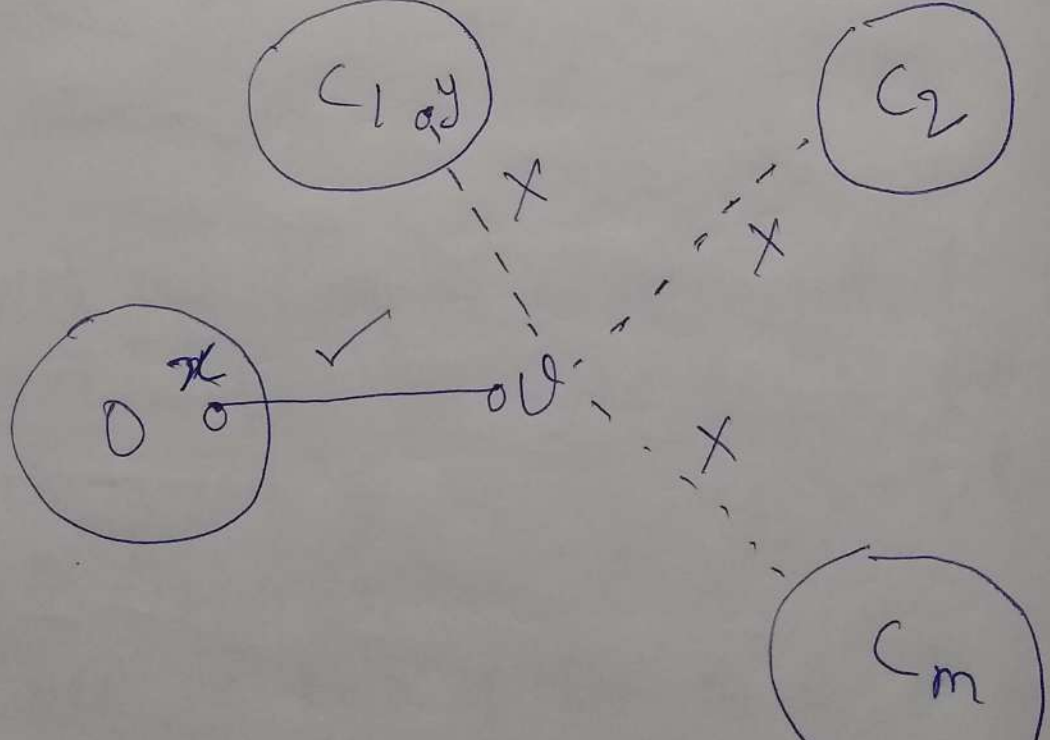


Note that since G is connected so ~~edges exist~~ at least one edge exists from v to each of the components-sets $O, C_1, \dots, C_m$.

Next we think of how a 1-factor of G looks like. So let H be a ~~choice~~ 1-factor of G.

claim: In each ~~a~~ 1-factor H of G, v has exactly one edge connecting it to exactly one element of the odd component O of G and exactly 0 edges connecting it to each of the even components $C_1, C_2, \ldots, C_m$.

In other words the 1-factor H looks like this:



why is this so?

suppose v is connected to an vertex $y \in C_1$ in the 1-factor H.

~~Theor~~ Note that anyway there can be at most one edge from v to C_1. For if there are 2 such edges from v to C_1, then the degree of v in H will be 2. But since H is a 1-factor, degree of v in H is equal to 1.

Coming back to our supposition, it implies that

$|C_1 - \{y\}|$ in H is an odd number. — ①

Since C_1 is a connected component, it is a graph in itself.

① $\Rightarrow$ ~~$C_1 - \{y\}$~~ in H, $C_1 - \{y\}$ cannot have ~~an~~ ~~even~~ covering ~~number~~ independent edges. So there can be no such vertex $y \in C_1$ in H which is connected to the vertex v. Similarly for $C_2, \ldots C_m$.

Next let $|O| = p$ and $|C_i| = 2n_i$;

and let $|G| = 2n$.

Then we have

$$|G| = |O| + |\{v\}| + \sum_{i=1}^{m} |C_i|$$

$$\Leftrightarrow \quad 2n = p + 1 + 2\sum_{i=1}^{m} n_i \quad \text{——} \quad (2)$$

If G is a connected graph in which every edge belongs to a 1-factor, how many distinct 1-factors are possible?

If H is a 1-factor of G, then H has exactly $|G|/2 = n$ independent edges.

From $2n$ vertices in G, we can choose n independent edges in:

$$\binom{2n}{2} \cdot \binom{2n-2}{2} \cdots \binom{2}{2} \text{ ways}$$

$$= \left[\frac{2n(2n-1)}{1\cdot 2}\right]\left[\frac{(2n-2)(2n-3)}{1\cdot 2}\right] \cdots \left[\frac{2\cdot 1}{1\cdot 2}\right]$$

$$= (2n)!/2^n \text{ ways.} \quad \text{——} \quad (3)$$

~~Since in H, each C_i behaves like~~

Next under the claim we have a particular structure for G. How many distinct 1-factors of G are possible under this structure? Since in H, each C_i and $O - \{x\}$ behave like a graph in itself, the 1-factor H of G decomposes into 1-factors H^{C_i} and H^{O} of C_i and O respectively.

(this is because no edges are allowed between C_i & C_j $(i \neq j)$ and also between C_i and O.)

By the reasoning in 3,

$$\# \text{1-factors in } C_i = (2n_i)!/2^{n_i}$$

$$\# \text{1-factors in } O - \{x\} = (p-1)!/2^{\frac{p-1}{2}}$$

and # ways of choosing x in $O = p$.

combining the last three equations we get,

possible distinct 1-factors (under structure)

$$= p \cdot \frac{(p-1)!}{2^{(p-1)/2}} \cdot \prod_{i=1}^{m} \frac{(2n_i)!}{2^{n_i}}$$

$$= \frac{2}{p+1} \cdot \frac{(p+1)!}{2^{(p+1)/2}} \cdot \prod_{i=1}^{m} \frac{(2n_i)!}{2^{n_i}} \quad — ④$$

Ratio of the numbers in ③ & ④

$$= \frac{(2n)!}{\cancel{2^n}} \cdot \frac{p+1}{2} \cdot \frac{\left[\cancel{2^{\frac{p+1}{2} + \sum_{i=1}^{m} n_i}}\right]}{(p+1)! \prod_{i=1}^{m} (2n_i)!}$$

$$= \frac{p+1}{2} \cdot \frac{\left(p+1+\sum_{i=1}^{m} 2n_i\right)!}{(p+1)! \prod_{i=1}^{m} (2n_i)!}$$

$> 1.$

④ results from assuming the G is not 2-connected. ③ $\neq$ ④ is a contradiction

$\Rightarrow$ G is 2-connected.