Book - Quantum Computation and Quantum Information.

Authors - Michael A. Nielsen & Isaac L. Chung

Chapter - The Quantum Search Algorithm

Exercise 6.1: Show that the unitary operator corresponding to the phase shift in the Grover iteration is $2|0\rangle\langle 0| - I$.

Proof: Indeed we have:

$$2|0\rangle\langle 0| - I$$

$$= 2\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ & & \ddots & -1 \\ 0 & 0 & \cdots & \end{pmatrix}$$

How does this unitary operator act on qubits in the computational basis state?

$$(2|0\rangle\langle 0| - I)|0\rangle$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(2|0\rangle\langle 0| - I)|1\rangle$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

(Note the minus sign).

In general for 1 in i-th position with $i \geq 2$ we get

$$(2|0\rangle\langle 0| - I)|i\rangle = -|i\rangle$$

Thus we have:

$$(2|0\rangle\langle 0| - I)|x\rangle = \begin{cases} |x\rangle & \text{if } x=0 \\ -|x\rangle & \text{if } x>0. \end{cases}$$

The right hand side is exactly the phase shift happening in the Grover iteration. Hence we are done.

Exercise 6.14. $\boxed{\text{Verification of Equation 6.6}}$

$$H^{\otimes n}(2|0\rangle\langle 0| - I)H^{\otimes n} = 2|\psi\rangle\langle\psi| - I.$$

where $|\psi\rangle$ is the equally weighted superposition which is the starting state

$$|\psi\rangle = \frac{1}{N}\sum_{x=0}^{N-1}|x\rangle \quad (N=2^n).$$

Indeed we have:

$H^{\otimes n}(2|0\rangle\langle 0| - I)H^{\otimes n}$

$$= 2H^{\otimes n}|0\rangle\langle 0|H^{\otimes n} - H^{\otimes n}H^{\otimes n}$$

Now note that:

$$H^{\otimes n}|0\rangle = |\psi\rangle$$

$$\Rightarrow \langle 0|H^{\otimes n} = \langle\psi|$$

$$\Rightarrow 2H^{\otimes n}|0\rangle\langle 0|H^{\otimes n} - H^{\otimes n}H^{\otimes n}$$

$$= 2\,\cancel{H^{\otimes n}}\,|\psi\rangle\langle\psi| - H^{\otimes n}.H^{\otimes n}$$

Now we only have to prove that:

$$H^{\otimes n}H^{\otimes n} = I.$$

We will prove this by induction n.

For $n=1$ we have:

$$H^{\otimes 1}H^{\otimes 1} = H^2$$

$$= \left\{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right\}^2$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Assume true for some n.

Now for n+1, we have,

$$H^{\otimes n+1} \cdot H^{\otimes n+1}$$

$$= \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} H^{\otimes n} & H^{\otimes n} \\ H^{\otimes n} & -H^{\otimes n} \end{pmatrix} \right\}^2$$

$$= \frac{1}{2} \begin{pmatrix} H^{\otimes n} & H^{\otimes n} \\ H^{\otimes n} & -H^{\otimes n} \end{pmatrix} \begin{pmatrix} H^{\otimes n} & H^{\otimes n} \\ H^{\otimes n} & -H^{\otimes n} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2(H^{\otimes n})^2 & 0 \\ 0 & 2(H^{\otimes n})^2 \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$ [by the induction hypothesis is true for n]

$$= I$$

Thus result is true for n+1 also

$\Rightarrow$ result is true for all $n \geq 1$

Therefore $H^{\otimes n} \cdot H^{\otimes n} = I$.

Here we have,

$$H^{\otimes n}(2|0\rangle\langle 0| - I) H^{\otimes n}$$

$$= 2|\psi\rangle\langle\psi| - I.$$

as desired.

Exercise 6.2: Show that the operation $(2|\psi\rangle\langle\psi| - I)$ applied to a general state $\sum_k \alpha_k |k\rangle$ produces

$$\sum_k \left[-\alpha_k + 2\langle\alpha\rangle\right]|k\rangle$$

where $\langle\alpha\rangle \equiv \sum_k \alpha_k / N$ is the mean value of the α_k.

Proof: We have,

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Therefore,

$$|\psi\rangle\langle\psi| = \left(\frac{1}{\sqrt{N}}\right)^2 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$= \frac{1}{N} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \quad \left(\begin{matrix}\text{where} \\ N = 2^n\end{matrix}\right)$$

$$\Rightarrow 2|\psi\rangle\langle\psi| - I$$

$$= \frac{2}{N} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{N}-1 & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & \frac{2}{N}-1 & \cdots & \frac{2}{N} \\ \vdots & & \ddots & \vdots \\ \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N}-1 \end{pmatrix}$$

Now

$$\sum_k \alpha_k |k\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}$$

$$\Rightarrow \left(2|\psi\rangle\langle\psi| - I\right)\left(\sum_k \alpha_k |k\rangle\right)$$

$$= \begin{pmatrix} \frac{2}{N}-1 & \frac{2}{N} & \cdots & \frac{2}{N} \\ \frac{2}{N} & \frac{2}{N}-1 & \cdots & \frac{2}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N}-1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{N}\sum_k \alpha_k - \alpha_1 \\ \frac{2}{N}\sum_k \alpha_k - \alpha_2 \\ \vdots \\ \frac{2}{N}\sum_k \alpha_k - \alpha_N \end{pmatrix}$$

$$= \frac{2}{N}\sum \alpha_k \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} - \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}$$

$$= 2\langle\alpha\rangle \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} - \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}$$

$$= 2\langle\alpha\rangle \sum_k |k\rangle - \sum_k \alpha_k |k\rangle$$

$$= \sum_k \left(2\langle\alpha\rangle - \alpha_k\right) |k\rangle$$

$$= \sum_k \left[-\alpha_k + 2\langle\alpha\rangle\right] |k\rangle$$

Thus we have prove that

$$2|\psi\rangle\langle\psi| - I$$

$$= \sum_k \left[-\alpha_k + 2\langle\alpha\rangle\right] |k\rangle$$

which is what was required to be proved.

Book - Explorations in Quantum Computing
(Texts in Computer Science)

Author - Colin P. Williams

Exerise - 5.2 (Performing search with a Quantum Computer)

Verification of a step in the analysis of the Grover's Algorithm

---

The analysis of Grover's algorithm required us to compute the k-th power of the matrix Q where:

$$Q = \begin{pmatrix} 1-4|u|^2 & 2u \\ -2u^* & 1 \end{pmatrix} \quad \text{--- (1)}$$

where u is a an arbitrary complex number. However Q^k can be exactly computed in terms of the Chebyshev's polynomials. We find that:

$$Q^k = (-1)^k \begin{pmatrix} U_{2k}(|u|) & -\frac{u}{|u|} U_{2k-1}(|u|) \\ \frac{u^*}{|u|} \cdot U_{2k-1}(|u|) & -U_{2k-2}(|u|) \end{pmatrix} \quad \text{— (2)}$$

where $U_k(\cos\theta) = \dfrac{\sin((k+1)\theta)}{\sin(\theta)}$

is the Chebyshev polynomial of the second kind. Use a proof by induction to show that this form is correct. You will find the following facts to be useful:

$$U_0(x) = 1$$

$$U_1(x) = 2x$$

$$U_2(x) = 4x^2 - 1$$

and the Chebyshev polynomials are related to one another via the recursion formula:

$$U_{k+1}(x) - 2x\, U_k(x) + U_{k-1}(x) = 0. \quad \text{— (3)}$$

Proof: we see that for $k=1$,

$$\Phi^1 = (-1)^1 \begin{pmatrix} U_2(|u|) & -\frac{u}{|u|} U_1(|u|) \\ \frac{u^*}{|u|} \cdot U_1(|u|) & -U_0(|u|) \end{pmatrix}$$

$$= -\begin{pmatrix} 4|u|^2 - 1 & -\frac{u}{|u|} \cdot 2|u| \\ \frac{u^*}{|u|} \cdot 2|u| & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 4|u|^2 & 2u \\ -2u^* & 1 \end{pmatrix}$$

$$= \Phi$$

Hence the induction hypothesis is true for $k=1$.

Next assume the induction hypothesis is true for k. That is (2) holds. We will show that it is true for $k+1$.

Before we go ahead that step first we note that:

$$u_{k+1}(x) - 2x\,u_k(x) + u_{k-1}(x) = 0$$

$$\Rightarrow u_{2k+2}(x) - 2x\,u_{2k+1}(x) + u_{2k}(x) = 0$$

and $u_{2k+1}(x) - 2x\,u_{2k}(x) + u_{2k-1}(x) = 0$

together imply that:

$$u_{2k+2}(x) - 2x\left[2x\cdot u_{2k}(x) - u_{2k-1}(x)\right] + u_{2k} = 0$$

$$\Rightarrow u_{2k+2}(x) = (4x^2-1)\,u_{2k}(x) - 2x\cdot u_{2k-1} \qquad (4)$$

similarly we can also get:

$$u_{2k+1}(x) = (4x^2-1)\,u_{2k-1}(x) - 2x\,u_{2k-2}(x) \qquad (5)$$

we will use these in what follows;

$$Q^{k+1} = Q^{k} \cdot Q$$

$$= (-1)^{k} \begin{pmatrix} u_{2k}(|u|) & -\frac{u}{|u|} \cdot u_{2k-1}(|u|) \\ \frac{u^*}{|u|} \cdot u_{2k-1}(|u|) & -u_{2k-2}(|u|) \end{pmatrix}$$

$$\times (-1) \begin{pmatrix} 4u^2-1 & -2u \\ 2u^* & -1 \end{pmatrix}$$

$$= (-1)^{k+1} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

$$= (-1)^{k+1} \left(\begin{array}{l} u_{2k}(|u|)(4|u|^2-1) - \frac{u}{|u|} \cdot u_{2k-1}(|u|) \cdot 2u^* \\ \qquad - u_{2k}(|u|) \cdot 2u + \frac{u}{|u|} \cdot u_{2k-1}(|u|) \\ \frac{u^*}{|u|} \cdot u_{2k-1}(|u|) \cdot (4|u|^2-1) + u_{2k-2}(|u|) \cdot 2u^* \\ \qquad - \frac{u^*}{|u|} \cdot u_{2k-1}(|u|) \cdot 2u + u_{2k-2}(|u|) \end{array} \right)$$

Now, we look at each term:

$$Q_{11} = u_{2k}(|u|)(4|u|^2-1) + 2|u| \cdot u_{2k-1}(|u|)$$

$$= u_{2k+2}(|u|) \quad (\text{by } 4).$$

$$Q_{12} = -u_{2k}(|u|) \cdot 2u + \frac{u}{|u|} \cdot u_{2k-1}(|u|)$$

$$= -\frac{u}{|u|} \cdot \left[2|u| \, u_{2k}(|u|) - u_{2k-1}(|u|)\right]$$

$$= -\frac{u}{|u|} \cdot u_{2k+1}(|u|) \quad (\text{by } 3).$$

$$Q_{21} = \frac{u^*}{|u|} u_{2k-1}(|u|)(4|u|^2-1) + u_{2k-2}(|u|) \cdot 2u^*$$

$$= \frac{u^*}{|u|}\left[u_{2k-1}(|u|)(4|u|^2-1) + 2|u| \cdot u_{2k-2}(|u|)\right]$$

$$= \frac{u^*}{|u|} \cdot u_{2k+1}(|u|) \cdot \quad (\text{by } 5).$$

$$Q_{22} = -\frac{u^*}{|u|} \cdot u_{2k-1}(|u|) \cdot 2u + u_{2k-2}(|u|)$$

$$= -2|u|\, u_{2k-1}(|u|) + u_{2k-2}(|u|)$$

$$= -\left[u_{2k-1}(|u|) \cdot 2|u| - u_{2k-2}(|u|) \right]$$

$$= -\, u_{2k}(|u|) \quad (\text{by } 3).$$

Hence:

$$Q^{k+1} = (-1)^{k+1} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

$$= \begin{pmatrix} u_{2k+2}(|u|) & -\frac{u}{|u|} \cdot u_{2k+1}(|u|) \\ \frac{u^*}{|u|} \cdot u_{2k+1}(|u|) & -u_{2k}(|u|) \end{pmatrix}$$

which has the same form as Q^k except that k is replaced by $k+1$. Thus the result is true for $k+1$ also.

Hence by the principle of Mathematical Induction, the result is true for all $k \geq 1$

[QED]

Book – Quantum Walks & Search Algorithms

Author – Renato Portugal

Exercise – 6.1 & 6.3

Chapter – Coined Walks with Cyclic Boundary Conditions

<u>Problem 1</u>: The Fourier Transform of the Spatial part of the computational basis is:

$$|\tilde{k}\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \omega_N^{jk} |j\rangle$$

where $\omega_N = e^{2\pi i/N}$ and the range of k is the same as j.

Show the following properties of the Fourier Transform:

(i) $\{|\tilde{k}\rangle : 0 \leq k \leq N-1\}$ is an orthonormal basis of the Hilbert space $\mathcal{H}^N$.

(ii) $$|0\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |\tilde{k}\rangle$$

(iii) $$|j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{-jk} |\tilde{k}\rangle.$$

Proof.

(i) let $0 \leq k_1, k_2 \leq N-1$.

$$|\tilde{k}_2\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \omega_N^{jk_2} |j\rangle$$

$$\langle \tilde{k}_1| = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \omega_N^{*\,jk_1} \langle j_1|.$$

where $\omega_N^* = e^{-2\pi i/N} = \frac{1}{\omega_N}$.

(the conjugate of ω_N).

$\Rightarrow \langle \tilde{k}_1|\tilde{k}_2\rangle$

$$= \frac{1}{\sqrt{N}} \cdot \sum_{j_1=0}^{N-1} \sum_{j_2=0}^{N-1} \omega_N^{*\,jk_1} \omega_N^{jk_2} \langle j_1|j_2\rangle$$

Note that $\langle j_1| = (0, \ldots 0, 1, 0, \ldots 0)$
$\downarrow$
j_1-th position.

and $|j_2\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \rightarrow j_2$ th position.

$$\Rightarrow \langle j_1|j_2\rangle = \begin{cases} 1 & \text{if } j_1 = j_2 \\ 0 & \text{if } j_1 \neq j_2. \end{cases}$$

So $\langle \tilde{k}_1 | \tilde{k}_2 \rangle = \frac{1}{N} \sum_{j=0}^{N-1} \omega_N^{*jk_1} \cdot \omega_N^{jk_2}$

$= \frac{1}{N} \sum_{j=0}^{N-1} \omega_N^{j(k_2-k_1)}$.

Now note that $0 \leq k_2 - k_1 \leq N-1$.

If $k_2 = k_1$ then we get,

$\langle \tilde{k}_1 | \tilde{k}_2 \rangle = \frac{1}{N} \sum_{j=0}^{N-1} \omega_N^{0} = \frac{1}{N} \sum_{j=0}^{N-1} 1$

$= \frac{1}{N} \cdot N = 1.$

If $k_2 \neq k_1$ then ~~since~~ $0 \leq k_2 - k_1 \leq N-1$.

<u>Claim</u>: After division by N, the remainders of the numbers in $\{0, (k_2-k_1), \dots, (N-1)(k_2-k_1)\}$ is the same as $\{0, 1, \dots, N-1\}$ in some order.

Let $j_1 k \equiv j_2 k \pmod{N}$ $(k = k_2 - k_1)$

$\Rightarrow (j_1 - j_2) k \equiv 0 \pmod{N}$

$\Rightarrow j_1 - j_2 \equiv 0 \pmod{N}$ $(\because k \neq 0$, $k \not\equiv 0 \bmod N)$

$\Rightarrow j_1 = j_2$ $(\because j_1 - j_2 \leq N-1)$.

So remainders of the N numbers

$0,\ 1\cdot(k_2-k_1),\ \dots,\ (N-1)(k_2-k_1)$

upon division by N are distinct.

$\Rightarrow$ they have to be $0, 1, \dots, N-1$ in some order.

$\Rightarrow$ If $k_2 \neq k_1$, then,

$$\langle \tilde{k}_1 | \tilde{k}_2 \rangle = \frac{1}{N} \sum_{j=0}^{N-1} \omega_N^{j(k_2-k_1)}$$

$$= \frac{1}{N} \sum_{j=0}^{N-1} \omega_N^{j} = \frac{1}{N}\left(1+\omega_N+\dots+\omega_N^{N-1}\right).$$

Now note that $\omega_N = e^{2\pi i/N}$

$$\Rightarrow \omega_N^N = \cos(2\pi) + i\sin(2\pi) = 1$$

$$\Rightarrow \omega_N^N - 1 = 0$$

$$\Rightarrow (\omega_N - 1)\left(1+\omega_N+\dots+\omega_N^{N-1}\right) = 0$$

$$\Rightarrow 1+\omega_N+\dots+\omega_N^{N-1} = 0 \quad (\because \omega_N \neq 1).$$

$$\Rightarrow \langle \tilde{k}_1 | \tilde{k}_2 \rangle = \frac{1}{N}\cdot 0 = 0$$

$$\Rightarrow \langle \tilde{k}_1 | \tilde{k}_2 \rangle = \begin{cases} 1 & \text{if } k_1 = k_2 \\ 0 & \text{if } k_1 \neq k_2 \end{cases}$$

$\Rightarrow \{|\tilde{k}\rangle : 0 \le k \le N-1\}$ is an orthonormal basis of the Hilbert Space $\mathcal{H}^N$.

This completes the proof of (i).

(ii) $|\tilde{k}\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \omega_N^{jk} |j\rangle$

$$\Rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |\tilde{k}\rangle = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \omega_N^{jk} |j\rangle$$

$$= \frac{1}{N} \sum_{j=0}^{N-1} \left[\sum_{k=0}^{N-1} \omega_N^{jk} \right] |j\rangle$$

(interchanging the summands of j & k)

Now note that,

$$\sum_{k=0}^{N-1} \omega_N^{jk} = \begin{cases} N & \text{if } j = 0 \\ 1 + \omega_N + \dots + \omega_N^{N-1} & \text{if } k \ne 0 \end{cases}$$

$$= \begin{cases} N & \text{if } j = 0 \\ 0 & \text{if } j \ne 0. \end{cases}$$

So $\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |\tilde{k}\rangle = \frac{1}{N} \cdot N|0\rangle + (\underbrace{0 + \dots + 0}_{N-1 \text{ times}})$

$= |0\rangle$. (as required)

(iii) $\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{-jk} |\tilde{k}\rangle$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \omega_N^{-jk} \sum_{j'=0}^{N-1} \omega_N^{j'k} |j'\rangle$$

$$= \frac{1}{N} \sum_{j'=0}^{N-1} \left(\sum_{k=0}^{N-1} \omega_N^{(j'-j)k} \right) |j'\rangle$$

$\cancel{= \frac{1}{N} \sum_{j'=0}^{N-1}}$ inner summand, is

$$\left(\sum_{k=0}^{N-1} \omega_N^{(j'-j)k} \right) = \begin{cases} N & \text{if } j'=j \\ 0 & \text{if } j' \neq j \end{cases}$$

So, $\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{-jk} |\tilde{k}\rangle$

$$= \frac{1}{N} \sum_{j'=0}^{N-1} \begin{Bmatrix} N & \text{if } j'=j \\ 0 & \text{if } j' \neq j \end{Bmatrix} |j\rangle$$

$$= \frac{1}{\cancel{N}} \cdot \cancel{N} \cdot |j\rangle + \frac{1}{N} \sum_{j' \neq j} 0 \cdot |j'\rangle$$

$$= |j\rangle \text{ (as required)}$$

Next we look at $|\psi_2\rangle$.

$$|\psi_2\rangle = \frac{1}{2}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right.$$

$$= \begin{pmatrix}1/2\\1/2\\1/2\\-1/2\end{pmatrix} = \begin{pmatrix}ac\\ad\\bc\\bd\end{pmatrix}$$

$$\Rightarrow \begin{cases} ac = \frac{1}{2} \text{ and } ad = \frac{1}{2} & \text{(i)} \\ bc = \frac{1}{2} \text{ and } bd = -\frac{1}{2} & \text{(ii)} \end{cases}$$

From (i) we obtain by multiplication

$$(ac)(ad) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$\Rightarrow a^2(cd) = \frac{1}{4} \Rightarrow cd = \frac{1}{4a^2} > 0 \quad \text{(iii)}$$

From (ii) we obtain

$$(bc)(bd) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$\Rightarrow b^2(cd) = -\frac{1}{4} \Rightarrow cd = \frac{-1}{4b^2} < 0 \quad \text{(iv)}$$

(iii) $\Rightarrow$ $cd > 0$

and (iv) $\Rightarrow$ $cd < 0$

clearly cd cannot be both > 0 and < 0 at the same time.

So this contradiction suggests that no solution for a, b, c, d exists

$\Rightarrow$ $|\psi_2\rangle$ ~~is~~ cannot be factored

$\Rightarrow$ $|\psi_2\rangle$ is entangled.

A correction:

a, b, c, d are in general complex numbers. But in the above argument we assumed that a, b, c, d are real numbers.. The the argument is not valid. I now present the correct proof.

$$|\psi_2\rangle = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}.$$

Since $a, b, c, d \in \mathbb{C}$ (set of complex numbers)

So let
$$a = a_1 + i a_2$$
$$b = b_1 + i b_2$$
$$c = c_1 + i c_2$$
$$d = d_1 + i d_2$$

where $a_1, a_2 \in \mathbb{R}$
$b_1, b_2 \in \mathbb{R}$
$c_1, c_2 \in \mathbb{R}$
$d_1, d_2 \in \mathbb{R}$

Since $ac = \frac{1}{2}$, $ad = \frac{1}{2}$, $bc = \frac{1}{2}$, $bd = -\frac{1}{2}$

we get after multiplications :

$$\underbrace{\begin{aligned} a_1 c_2 + a_2 c_1 &= 0 \\ a_1 d_2 + a_2 d_1 &= 0 \\ b_1 c_2 + b_2 c_1 &= 0 \\ b_1 d_2 + b_2 d_1 &= 0 \end{aligned}}_{\text{imaginary parts}} \quad \Bigg| \quad \underbrace{\begin{aligned} a_1 c_1 - a_2 c_2 &= \tfrac{1}{2} \\ a_1 d_1 - a_2 d_2 &= \tfrac{1}{2} \\ b_1 c_1 - b_2 c_2 &= \tfrac{1}{2} \\ b_1 d_1 - b_2 d_2 &= -\tfrac{1}{2} \end{aligned}}_{\text{real parts}}$$

$a_1 c_2 + a_2 c_1 = 0$ & $a_1 c_1 - a_2 c_2 = \frac{1}{2}$

together imply that

$$c_1 = \frac{\cancel{2}\, a_1}{2(a_1^2 + a_2^2)} \quad \text{—— (i)}$$

$a_1 d_2 + a_2 d_1 = 0$ & $a_1 d_1 - a_2 d_2 = \frac{1}{2}$

together imply that

$$d_1 = \frac{a_1}{2(a_1^2 + a_2^2)} \quad \text{—— (ii)}$$

Similarly,

$b_1 c_2 + b_2 c_1 = 0$ & $b_1 c_1 - b_2 d_2 = \frac{1}{2}$

together imply that

$$c_1 = \frac{b_1}{2(b_1^2 + b_2^2)} \quad \text{--- (iii)}$$

and $b_1 d_2 + b_2 d_1 = 0$ & $b_1 d_1 - b_2 d_2 = -\frac{1}{2}$

together imply that

$$d_1 = -\frac{b_1}{2(b_1^2 + b_2^2)} \quad \text{--- (iv)}.$$

From (i) & (iii), we have,

$$c_1 = \frac{a_1}{2(a_1^2 + a_2^2)} = \frac{b_1}{2(b_1^2 + b_2^2)} \quad \text{--- (v)}$$

$a_1, a_2, b_1, b_2 \in \mathbb{R} \Rightarrow a_1^2 + a_2^2 > 0$ and $b_1^2 + b_2^2 > 0$

$\Rightarrow$ either $a_1 \geq 0$ & $b_1 \geq 0$

or $a_1 \leq 0$ & $b_1 \leq 0$.

From (ii) & (iv) we have

$$d_1 = \frac{a_1}{2(a_1^2 + a_2^2)} = -\frac{b_1}{2(b_1^2 + b_2^2)} \quad \text{--- (vi)}$$

$\Rightarrow$ either $a_1 \geq 0$ & $b_1 \leq 0$

or $a_1 \leq 0$ & $b_1 \geq 0$

(v) & (vi) are compatible if and only if

$$a_1 = b_1 = 0 \Rightarrow c_1 = d_1 = 0.$$

Now our "real parts" equations become reduced to:

$$a_2 c_2 = -\frac{1}{2} \text{ \& } a_2 d_2 = -\frac{1}{2} \quad \text{— (vii)}$$

$$\text{\&} \quad b_2 c_2 = -\frac{1}{2} \text{ \& } b_2 d_2 = \frac{1}{2} \quad \text{— (viii)}$$

Multiplying equation in (vii) we get

$$a_2^2 (c_2 d_2) = \frac{1}{4}$$

$$\Rightarrow c_2 d_2 = \frac{1}{4a_2^2} > 0 \left(\because \begin{matrix} c_2, d_2 \\ a_2 \in \mathbb{R} \end{matrix}\right)$$

From (viii) we get

$$b_2^2 (c_2 d_2) = -\frac{1}{4}$$

$$\Rightarrow c_2 d_2 = -\frac{1}{4b_2^2} < 0 \left(\because \begin{matrix} \cancel{b_2 \in \mathbb{R}} \\ b_2 \in \mathbb{R} \end{matrix}\right)$$

Since $c_2 d_2$ cannot be both > 0 & < 0, we get a contradiction.

$\Rightarrow a_2, b_2, c_2, d_2 \in \mathbb{R}$ do not exist.

So $|\psi_2\rangle$ is entangled.

Book - Quantum walks and Search Algorithms

Author - Renato Portugal

Chapter - Element Distinctness.

Exercise - 10.10

Problem. The goal of this exercise is to find a 5-dimensional matrix associated with U_α.

Use Eqn. (10.2) to show that

$$\langle \alpha_S | \eta_\ell \rangle = \frac{c_\ell}{\sqrt{|\eta_\ell|} \cdot \sqrt{N-r}}$$

where, $c_0 = (N-r)\, \delta_{|S \cap \{i, i_2\}| = 2}$

$$c_1 = (N-r-2)\, \delta_{|S \cap \{i_1, i_2\}| = 0}$$

$$c_2 = 2\, \delta_{|S \cap \{i_1, i_2\}| = 0}$$

$$c_3 = (N-r-1) \cdot \delta_{|S \cap \{i, i_2\}| = 1}$$

$$c_4 = \delta_{|S \cap \{i, i_2\}| = 1}$$

Show that

$$\sum_{\substack{S \in S_r \\ |S \cap \{i, i_2\}| = 0}} |\alpha_S\rangle = \frac{1}{\sqrt{N-r}} \left(\sqrt{|\eta_1|} \cdot |\eta_1\rangle + \sqrt{|\eta_2|} \cdot |\eta_2\rangle \right)$$

and find similar equations for when $|S \cap \{i_1, i_2\}| = 1$ and $|S \cap \{i_1, i_2\}| = 2$.

Use the above equations and Eqn. (10.3) to find the entries $(\mathcal{U}_\alpha)_{kj}$ of

$$U_\alpha |\eta_\ell\rangle = \sum_{k=0}^{4} (\mathcal{U}_\alpha)_{kj} |\eta_k\rangle$$

<u>Solution</u>:

$$|\eta_0\rangle = \frac{1}{\sqrt{|\eta_0|}} \cdot \sum_{(S', y') \in \eta_0} |S', y'\rangle$$

$$= \frac{1}{\sqrt{|\eta_0|}} \cdot \sum_{(S, y') \in \eta_0} |S, y'\rangle + \frac{1}{\sqrt{|\eta_0|}} \cdot \sum_{(S'', y') \in \eta_0} |S'', y'\rangle$$

when we take the inner product of $\langle \alpha_S|$ with $|\eta_\ell\rangle$ only the first term in the above is relevant, because it is related to the set S.

By definition, $|\alpha_S\rangle$ is:

$$|\alpha_S\rangle = \frac{1}{\sqrt{N-r}} \sum_{y \in [N] \setminus S} |S, y\rangle$$

Therefore on taking the inner products we have,
(note that S is a fixed set in S_r)

$\cancel{\langle \alpha_S | \eta_\ell \rangle}$

$$\langle \alpha_S | \eta_0 \rangle = \frac{1}{\sqrt{|\eta_0|}} \cdot \frac{1}{\sqrt{N-r}} \cdot \times$$

$$\sum_{(S,y') \in \eta_0} \sum_{y \in [N] \setminus S} \langle y, S | S, y' \rangle$$

$$= \frac{1}{\sqrt{|\eta_0|}\sqrt{N-r}} \cdot \sum_{(S,y') \in \eta_0} \sum_{y \in [N] \setminus S} \langle y | y' \rangle \langle S | S \rangle$$

$$= \frac{1}{\sqrt{|\eta_0|}} \cdot \frac{1}{\sqrt{N-r}} \sum_{(S,y') \in \eta_0} \left[\sum_{y \in [N] \setminus S} \langle y | y' \rangle \right]$$

$$\left(\because \langle S | S \rangle = 1 \right).$$

$$= \frac{1}{\sqrt{|\eta_0|}\sqrt{N-r}} \sum_{y' \in [N] \setminus S} \sum_{y \in [N] \setminus S} \langle y | y' \rangle$$

($\because$ $(S, y') \in \eta_0 \Rightarrow S$ has exactly 2 marked elements and y' is free to vary over that set $[N] \setminus S$).

$$\Rightarrow C_0 = \sum_{y' \in [N]\setminus S} \sum_{y \in [N]\setminus S} \langle y | y' \rangle$$

$\left(\because \langle y | y' \rangle = \begin{cases} 1 & y = y' \\ 0 & y \neq y' \end{cases} \right.$

$$= (N - r).$$

under the assumption that $|S \cap \{i, i_2\}| = 2$

If on the other hand, $|S \cap \{i, i_2\}| \neq 2$

then $C_0 = 0$.

Thus we get $C_0 = \delta_{|S \cap \{i, i_2\}| = 2}$.

Hence,

$$\langle \alpha_S | \eta_0 \rangle = \frac{C_0}{\sqrt{|\eta_0|} \cdot \sqrt{N - r}} \quad \text{as desired.}$$

Next we consider $\langle \alpha_S | \eta_1 \rangle$.

$$\langle \alpha_S | \eta_1 \rangle = \frac{1}{\sqrt{|\eta_1|}} \cdot \frac{1}{\sqrt{N - r}} \times$$

$$\sum_{(S, y') \in \eta_1} \sum_{y \in [N]\setminus S} \langle y | y' \rangle$$

$$= \frac{1}{\sqrt{|\eta_1|}\sqrt{N - r}} \sum_{\substack{y' \in [N]\setminus S \cup \{i_1\} \\ \cup \{i_2\}}} \sum_{y \in [N]\setminus S} \langle y | y' \rangle$$

Note that the outer sum is over the range,

$$y' \in [N] \setminus (S \cup \{i_1, i_2\}).$$

because by definition,

η_1 = set of vertices (S, y) such that S has no marked element and y' ~~has~~ is not a marked index.

$$\Rightarrow \; y' \notin S \;\&\; y' \notin \{i_1, i_2\}$$

$$\Rightarrow \; y' \notin S \cup \{i_1, i_2\}.$$

For illustration suppose we arrange the sum in the form of rows & columns letting $\{y_1, \dots, y_{N-r}\}$ be the set $[N] \setminus S$:

$$\langle y_1 | y_1 \rangle + \cdots + \langle y_{N-r} | y_1 \rangle \longrightarrow 1$$

$$\langle y_1 | y_2 \rangle + \cdots + \langle y_{N-r} | y_2 \rangle \longrightarrow 1$$

$$\vdots$$

$$\langle y_1 | y_{N-r} \rangle + \cdots + \langle y_{N-r} | y_{N-r} \rangle \longrightarrow 1$$

There will be only $(N-r-2)$ rows because rows corresponding to $y' \in \{i_1, i_2\}$ will be missing and will not appear.

Each row contributes 1 to the sum.
And since there are $(N-r-2)$ rows.
So we get

$$c_1 = (N-r-2)$$

under the assumption that $|S \cap \{i, i_2\}| = 0$
otherwise $c_1 = 0$

$$\Rightarrow c_1 = \delta_{|S \cap \{i, i_2\}| = 0}$$

$$\Rightarrow \langle \alpha_S | \eta_1 \rangle = \frac{c_1}{\sqrt{|\eta_1|} \cdot \sqrt{N-r}}$$

as desired.

By similar reasoning we can also obtain the answers for,

$\langle \alpha_S | \eta_2 \rangle$, $\langle \alpha_S | \eta_3 \rangle$ and $\langle \alpha_S | \eta_4 \rangle$.

$\langle \alpha_S | \eta_2 \rangle$ ↓ only 2 rows contribution to sum is 2

$\langle \alpha_S | \eta_3 \rangle$ ↓ only $(N-r-1)$ rows contribution to sum is $(N-r-1)$.

$\langle \alpha_S | \eta_4 \rangle$ ↓ only 1 row contribution to sum is 1

We move on to the second part of the problem.

$$\sum_{\substack{S\in S_r \\ |S\cap\{i_1,i_2\}|=0}} |\alpha_S\rangle =$$

$$= \frac{1}{\sqrt{N-r}} \sum_{\substack{S\in S_r \\ |S\cap\{i_1,i_2\}|=0}} \; \sum_{y\in[N]\setminus S} |S,y\rangle$$

$$= \frac{1}{\sqrt{N-r}} \sum_{\substack{S\in S_r \\ |S\cap\{i_1,i_2\}|=0}} \; \sum_{\substack{y\in[N]\setminus S \\ y\notin\{i_1,i_2\}}} |S,y\rangle$$

$$+ \frac{1}{\sqrt{N-r}} \sum_{\substack{S\in S_r \\ |S\cap\{i_1,i_2\}|=0}} \; \sum_{\substack{y\in[N]\setminus S \\ y\in\{y_1,y_2\}}} |S,y\rangle$$

$$= \frac{1}{\sqrt{N-r}}\left(\sqrt{|\eta_1|} \cdot \sum_{(S,y)\in\eta_1} |S,y\rangle + \sqrt{|\eta_2|}\sum_{(S,y)\in\eta_2} |S,y\rangle\right)$$

$$= \frac{1}{\sqrt{N-r}}\left(\sqrt{|\eta_1|}\cdot|\eta_1\rangle + \sqrt{|\eta_2|}\cdot|\eta_2\rangle\right)$$

as desired.

Similarly we can get ~~for~~ answers for $|S \cap \{i_1, i_2\}| = 1$ and $|S \cap \{i_1, i_2\}| = 2$.

The answers will be naturally,

$$\sum_{\substack{S \in S_r \\ |S \cap \{i_1, i_2\}| = 1}} |\alpha_S\rangle$$

$$= \frac{1}{\sqrt{N-r}} \left(\sqrt{|\eta_3|} \cdot |\eta_3\rangle + \sqrt{|\eta_4|} \cdot |\eta_4\rangle \right)$$

and,

$$\sum_{|S \cap \{i_1, i_2\}| = 2} |\alpha_S\rangle = \frac{1}{\sqrt{N-r}} \left(\sqrt{|\eta_0|} \cdot |\eta_0\rangle \right.$$

Next we move to the 3rd part of the problem

$$U_\alpha = 2 \sum_{S \in S_r} |\alpha_S\rangle\langle\alpha_S| - I$$

$$\Rightarrow U_\alpha |\eta_\ell\rangle = 2 \sum_{S \in S_r} \langle\alpha_S|\eta_\ell\rangle |\alpha_S\rangle - |\eta_\ell\rangle$$

$$= \cancel{2 \sum_{S \in S_r}}$$

$$= \frac{2c_e}{\sqrt{|\eta_e|}\cdot\sqrt{N-r}}\left[\left(\sum_{s\in S_r}|\alpha_s\rangle\right) - |\eta_e\rangle\right.$$

$$= \frac{2c_e}{\sqrt{|\eta_e|}\sqrt{N-r}} \times \left[\sum_{\substack{s\in S_r \\ |s\cap\{i,i_2\}|=0}} |\alpha_s\rangle + \sum_{\substack{s\in S_r \\ |s\cap\{i,i_2\}|=1}} |\alpha_s\rangle + \sum_{\substack{s\in S_r \\ |s\cap\{i,i_2\}|=2}} |\alpha_s\right] - |\eta_e\rangle$$

Now we know the expressions for each of the 3 ~~term~~ sums insid the square bracket ~~So and to~~ by the second part of the problem. we substitute those expressions and consider

$$U_\alpha|\eta_0\rangle = \frac{2c_0}{\sqrt{|\eta_0|}\cdot\sqrt{N-r}} \times \frac{1}{\sqrt{N-r}}$$

$$\left[\sqrt{|\eta_1|}\,|\eta_1\rangle + \sqrt{|\eta_2|}\cdot|\eta_2\rangle + \sqrt{|\eta_3|}\cdot|\eta_3\rangle + \sqrt{|\eta_4|}\,|\eta_4\rangle + \sqrt{|\eta_0|}\cdot|\eta_0\rangle\right] - |\eta_0\rangle$$

The first 4 terms inside the square brackets evaluate to 0 because their sums run over the ~~range~~ range, $(s, y) \notin \eta_0$.

Hence,

$$U_\alpha|\eta_0\rangle = \left(\frac{2c_0}{(N-r)} - 1\right) = 1$$

$$(c_0 = 1 \ \because \ (s, y) \in \eta_0).$$

Thus,

$$U_\alpha|\eta_0\rangle = 1\cdot|\eta_0\rangle + 0|\eta_1\rangle + 0\cdot|\eta_2\rangle + 0\cdot|\eta_3\rangle + 0\cdot|\eta_4\rangle.$$

$\Rightarrow$ the first row of the 5-dimensional matrix is $(1, 0, 0, 0, 0)$.

Next

$$U_\alpha|\eta_1\rangle = \frac{2c_1}{(N-r)\sqrt{|\eta_1|}} \times$$

$$\left[\sqrt{|\eta_1|}\,|\eta_1\rangle + \sqrt{|\eta_2|}\cdot|\eta_2\rangle + \sqrt{|\eta_3|}\,|\eta_3\rangle + \sqrt{|\eta_4|}\,|\eta_4\rangle + \sqrt{|\eta_0|}\cdot|\eta_0\rangle\right] - |\eta_1\rangle$$

only the terms corresponding to $|\eta_1\rangle$ and $|\eta_2\rangle$ are non-zero. Rest all are zeros

$\left(\because \cancel{(s,y) \in \eta_1}\right)$.

$\left(\because |S \cap \{i_1, i_2\}| = 1\right)$.

$$\Rightarrow U_\alpha|\eta_1\rangle = \left(\frac{2c_1}{N-r} - 1\right)|\eta_1\rangle + \frac{2c_1}{N-r} \cdot \sqrt{\frac{|\eta_2|}{|\eta_1|}} \cdot |\eta_2\rangle$$

Let $N - r = a$. Then, $c_1 = N - r - 2 = a - 2$

$$\frac{2c_1}{N-r} - 1 = \frac{2(a-2)}{a} - 1 = \frac{a-4}{a}.$$

and, $$\frac{2c_1}{N-r}\sqrt{\frac{|\eta_2|}{|\eta_1|}} = \frac{2(a-2)}{a}\sqrt{\frac{2\binom{N-r}{r}}{\binom{N-2}{r}(N-r-2)}}$$

$$= \frac{2(a-2)}{a}\sqrt{\frac{2}{a-2}} = \frac{2\sqrt{2}\sqrt{a-2}}{a}$$

$\Rightarrow$ second row of the 5-dimensional matrix is

$$\left(0, \frac{a-4}{a}, \frac{2\sqrt{2}\sqrt{a-2}}{a}, 0, 0\right)$$

$\cancel{U_\alpha|\eta_2\rangle = \left(\frac{2C_1}{N-r} - 1\right)|\eta_2\rangle +}$

By similar reasoning we can find the rest of the rows also.
(skipping the tedious calculations to ~~make the~~ keep the document short)
Finally we get the 5-dimensional matrix

$$U_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{a-4}{a} & \frac{2\sqrt{2}\sqrt{a-2}}{a} & 0 & 0 \\ 0 & \frac{2\sqrt{2}\sqrt{a-2}}{a} & \frac{4-a}{a} & 0 & 0 \\ 0 & 0 & 0 & \frac{a-2}{a} & \frac{2\sqrt{a-1}}{a} \\ 0 & 0 & 0 & \frac{2\sqrt{a-1}}{a} & \frac{2-a}{a} \end{bmatrix}$$

[The matrix matches with the answer given in the book and is correct.
This completes the solution to the problem.