Book: Quantum Computation & Quantum Information.

Authors: Chuang & Nielsen

Problem 5.1: Give a direct proof that the linear transformation defined by Equation (5.2) is unitary.

Proof: The equation (5.2) is:

$$|j\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i jk/N} |k\rangle$$

let $e^{2\pi i/N} = \omega$.

Then the corresponding linear transformation U is given by:

$$\begin{array}{cc} U = \frac{1}{\sqrt{N}} \begin{pmatrix} & & \\ & \omega^{kj} & \\ & & \end{pmatrix} & \begin{array}{l} \text{k-th} \\ \text{row} \end{array} \\ \text{j-th column.} & \end{array}$$

To show that U is unitary we need to show that

$$UU^{\dagger} = I \quad (\dagger \text{ denotes adjoint})$$

Note that U is a symmetric matrix.

$\Rightarrow \quad U^{T} = U$

$\Rightarrow \quad (U^{T})^{*} = U^{*}$

(U^{T} = transpose of T and $*$ denotes complex conjugation).

$\Rightarrow \quad U^{\dagger} = U^{*}$

$$\Rightarrow \quad U^{\dagger} = \frac{1}{\sqrt{N}} \begin{pmatrix} & & \\ & \omega^{-jk} & \\ & & \end{pmatrix}$$

k-th row

j-th column.

$\Rightarrow$ the k-th row & j-th column element of $UU^{\dagger}$ is given by the ~~product~~ ~~of~~ inner product of the k-th row of U & the j-th column of $U^{\dagger}$.

$$\cancel{\Rightarrow (p-q)(k-j) = 0 \pmod{N}}.$$

that is:

$$(UU^{+})_{kj} = \frac{1}{N}\left(1\ \omega^{k} \ldots + \omega^{(N-1)k}\right)\begin{pmatrix} 1 \\ \omega^{j} \\ \vdots \\ \omega^{(N-1)j} \end{pmatrix}$$

$$= \frac{1}{N}\left[1 + \omega^{(k-j)} + \cdots + \omega^{(N-1)(k-j)}\right]$$

case i: $k = j$. Then:

$$(UU^{+})_{kk} = (UU^{+})_{jj} = \frac{1}{N}\left(1 + \omega^{0} + \cdots + \omega^{0}\right)$$

$$= \frac{1}{N} \times N = 1$$

case ii: $k \neq j$. Then $\omega^{k-j} \neq 1$

$$(\because\ 0 \leq k-j \leq N-1)$$

$$\Rightarrow (UU^{+})_{kj} = \frac{1 - \omega^{(k-j)N}}{1 - \omega^{(k-j)}}.$$

$$= \frac{1 - (\omega^{N})^{k-j}}{1 - \omega^{k-j}} = \frac{1 - 1^{k-j}}{1 - \omega^{k-j}} = 0$$

$\Rightarrow \quad UU^{\dagger} = I$

$\Rightarrow \quad U$ is unitary matrix.

This completes the proof.

Problem 2: Verification of Equation (5.12)

Proof: we will derive equation (5.13) using the equations (5.11) & (5.12).

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i/2k} \end{bmatrix} \quad \text{——} (5.11)$$

The initial state is

$$|j_1 j_2 \cdots j_n\rangle.$$

Applying the Hadamard gate to the first bit produces the state:

$$\frac{1}{2^{1/2}}\left(|0\rangle + e^{2\pi i\, 0.j_1}|1\rangle\right)|j_2 \cdots j_n\rangle \quad \text{——} (5.12)$$

where, $0.j_l j_{l+1} \cdots j_m$ represents the binary fraction $\frac{j_l}{2} + \frac{j_{l+1}}{2^2} + \cdots + \frac{j_m}{2^{m-l+1}}$

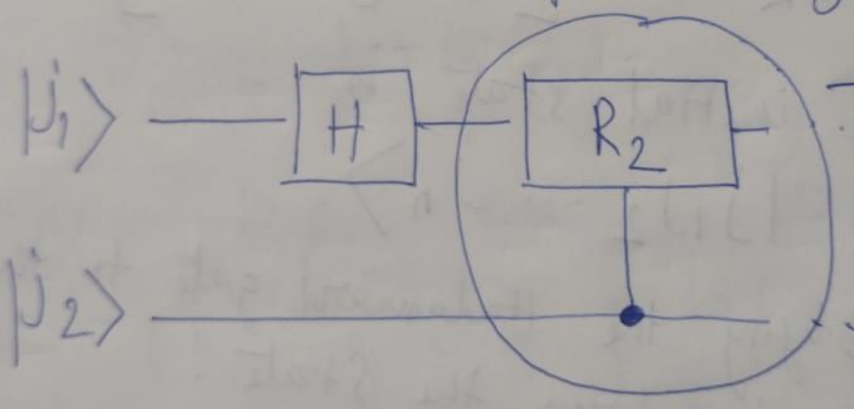
(5.12) is true because

$$e^{2\pi i\, 0.j_1} = \begin{cases} -1 & \text{when } j_1 = 1 \\ +1 & \text{when } j_1 = 0 \end{cases}$$

we need to show that applying the controlled-R_2 gate produces the state

$$\frac{1}{2^{1/2}}\left(|0\rangle + e^{2\pi i\, 0.j_1 j_2}|1\rangle\right)|j_2 \cdots j_n\rangle \qquad (5.13).$$

The relevant part of the circuit for the Quantum Fourier Transformer is given below:



→ upside-down controlled R_2.

$$R_2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i/4} \end{bmatrix} \quad \text{from eqn (5.11).}$$

$$\Rightarrow \text{controlled-}R_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi i/4} \end{bmatrix}$$

But as shown in the diagram we are dealing with an upside-down-controlled R_2 (= G let's say).

then $G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi i/4} \end{bmatrix}$

$$\Rightarrow G|0\rangle|j_2\rangle = \frac{1}{2^{1/2}}\begin{cases} |0\rangle|j_2\rangle & \text{if } j_2 = 0 \\ |0\rangle(|j_2\rangle) & \text{if } j_2 = 1 \end{cases}$$

and,

$$G|1\rangle|j_2\rangle = \frac{1}{2^{1/2}}\begin{cases} |1\rangle|j_2\rangle & \text{if } j_2 = 0 \\ |1\rangle(R_2|j_2\rangle) & \text{if } j_2 = 1 \end{cases}$$

So when G acts on the expression in (5.12) the result is:

$$= \frac{1}{2^{1/2}}\left[G|0\rangle|j_2\rangle + e^{2\pi i\, 0.j_1} G|1\rangle|j_2\rangle\right]$$

$$= \frac{1}{2^{1/2}}\begin{cases} |0\rangle|j_2\rangle + e^{2\pi i\, 0.j_1}|1\rangle|j_2\rangle & \text{if } j_2 = 0 \\ |0\rangle(|j_2\rangle) + e^{2\pi i\, 0.j_1} \times e^{2\pi i/4}|1\rangle|j_2\rangle & \text{if } j_2 = 1 \end{cases} \qquad (1)$$

Now the expression in (5.13) is:

$$\frac{1}{2^{1/2}}\left(|0\rangle + e^{2\pi i\, 0.j_1 j_2}|1\rangle\right)|j_2\rangle$$

$$= \frac{1}{2^{1/2}}|0\rangle|j_2\rangle + e^{2\pi i\, 0.j_1} \times e^{2\pi i\, 0.0 j_2}|1\rangle|j_2\rangle$$

$$= \frac{1}{2^{1/2}}\begin{cases} |0\rangle|j_2\rangle + e^{2\pi i\, 0.j_1}|1\rangle|j_2\rangle & \text{if } j_2 = 0 \\ |0\rangle|j_2\rangle + e^{2\pi i\, 0.j_1} \times e^{2\pi i/4}|1\rangle|j_2\rangle & \text{if } j_2 = 1 \end{cases} \qquad (2)$$

(because $e^{2\pi i\, 0.01} = e^{2\pi i \times \frac{1}{4}}$

$= e^{2\pi i/4}$) using the definition of a decimal notation as given earlier)

The right hand sides of both the equations ① & ② are the same.

This completes the proof.

Book: Explorations in Quantum Computing.

Author: Colin . P. Williams

Problem 6.7: Prove the convolution property of the n-qubit Quantum Fourier transform. Suppose we have two n-qubit quantum states:

$$|\psi\rangle = \sum_{k=0}^{N-1} c_k |k\rangle$$

$$|\varphi\rangle = \sum_{k=0}^{N-1} d_k |k\rangle$$

(where $N = 2^n$)

Define the convolution of $|\psi\rangle$ & $|\varphi\rangle$ as

convolution $(|\psi\rangle, |\varphi\rangle)$

$$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} c_l d_{j-l} |j\rangle$$

where $d_{j-l} = d_{N+j-l}$ if $j-l < 0$.

Your task is to show that the QFT of the convolution is related to the QFT of the states themselves. To see this write the QFTs of $|\psi\rangle$ and $|\varphi\rangle$ as:

$$\left.\begin{aligned} QFT|\psi\rangle &= \sum_{k=0}^{N-1} \alpha_k |k\rangle \\ QFT|\varphi\rangle &= \sum_{k=0}^{N-1} \beta_k |k\rangle \end{aligned}\right\} \quad (6.28)$$

and based on these definitions prove that:

QFT (convolution $(|\psi\rangle, |\varphi\rangle)$)

$$= \sum_{j=0}^{N-1} \alpha_j \beta_j |j\rangle \quad (6.29)$$

<u>Proof</u>: First we obtain the α_k's and β_k's for the representation given in (6.28).

We start with $|\psi\rangle = \sum_{k=0}^{N-1} c_k|k\rangle$

Then,

$$QFT|\psi\rangle = \sum_{k=0}^{N-1} c_k \, QFT|k\rangle$$

(by the linearity of the QFT)

$$= \sum_{k=0}^{N-1} c_k \cdot \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{2\pi i jk}|j\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \left(\sum_{k=0}^{N-1} c_k \cdot e^{2\pi i jk} \right) |j\rangle$$

(by interchanging the sums over j & k)

$$\Rightarrow \alpha_k = \sum_{j=0}^{N-1} c_j e^{2\pi i jk}$$

Similarly,

$$\beta_k = \sum_{j=0}^{N-1} d_j e^{2\pi i jk}$$

$\Rightarrow$ α_k's & β_k's for (6.28) are found.

Now,

QFT (convolution $(|\psi\rangle, |\varphi\rangle)$).

$$= QFT \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} c_l \cdot d_{j-l} |j\rangle$$

$$= QFT \frac{1}{\sqrt{N}} \left[\sum_{j=0}^{N-1} \left(\sum_{l=0}^{N-1} c_l \cdot d_{j-l} \right) |j\rangle \right]$$

$$= QFT \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e_j' |j\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \alpha_k' |k\rangle$$

where, $$\alpha_k' = \sum_{j=0}^{N-1} c_j' \cdot e^{2\pi i j k}$$

$$= \sum_{j=0}^{N-1} \left(\sum_{l=0}^{N-1} c_l d_{j-l} \right) e^{2\pi i j k}$$

(1)

On the other hand, the RHS of (6.29) is:

$$= \left(\sum_{p=0}^{N-1} c_p \cdot e^{2\pi i k p} \right) \left(\sum_{q=0}^{N-1} d_q \cdot e^{2\pi i k q} \right)$$

So ~~first term~~ is :

$$= \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} c_p d_q e^{2\pi i k(p+q)}$$

$$= \sum\sum_{p+q \le N-1} c_p d_q e^{2\pi i k(p+q)}$$

$$+ \sum\sum_{p+q \ge N} c_p d_q e^{2\pi i k(p+q)}$$

For the first summation, make the change of variable $p+q = m \Rightarrow q = m - p$

$$q \geq 0 \Rightarrow p \le m$$

So first term becomes

$$= \sum_{p=0}^{m} \sum_{m=0}^{N-1} c_p d_{m-p} e^{2\pi i m k}$$

$$= \sum_{m=0}^{N-1} \sum_{p=0}^{m} c_p d_{m-p} e^{2\pi i m k}$$

$$= \sum_{j=0}^{N-1} \sum_{l=0}^{j} c_l d_{j-l} e^{2\pi i j k}$$

For the second summation, make the change of variable:

$$m = p + q - N$$

$$\Rightarrow q = m - p + N$$

when $q = 0 \Rightarrow m = N - p$

& $q = N-1 \Rightarrow m = p+1$.

So second term becomes

$$= \sum_{p=\cancel{m}m+1}^{N-m} \sum_{m=0}^{N-1} c_p d_q \, e^{2\pi i k(p+q-N)}$$

$$= \sum_{m=0}^{N-1} \sum_{p=m+1}^{N-m} c_p d_q \cdot e^{2\pi i jk} \quad \left(\because e^{2\pi i N} = 1\right)$$

$$= \sum_{j=0}^{N-1} \sum_{l=m+1}^{N-m} c_l d_{j-l} \cdot e^{2\pi i jk}$$

So the sum of the two terms is

$$= \sum_{j=0}^{N-1} \sum_{l=0}^{j} c_l d_{j-l} e^{2\pi i jk} + \sum_{j=0}^{N-1} \sum_{l=j+1}^{N-j} c_l d_j e^{2\pi i jk}$$

$$= \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} c_l d_{j-l} e^{2\pi i jk} \quad \text{———} \quad (2)$$

RHS of (1) & (2) are same which completes the proof!

Book: Graph Theory - Undergraduate Mathematics

Authors: Khee Mong Koh et. al.

Publisher: World Scientific.

Problem 1: Let G be a bipartite graph with bipartition (X, Y). Let A, B be subsets of X. Show that

(a) $N(A \cup B) = N(A) \cup N(B)$

(b) $N(A \cap B) \subseteq N(A) \cap N(B)$

(c) $|N(A \cup B)| + |N(A \cap B)| \leq |N(A)| + |N(B)|$.

Solution: (a) Let $y \in N(A \cup B)$

$\Rightarrow \exists\ x \in A \cup B$ s.t $xy \in E(G)$

$\Rightarrow$ if $x \in A$ then $xy \in N(A)$

$\Rightarrow$ if $x \in B$ then $xy \in N(B)$

& if $xy \in N(A) \cup N(B)$

So $x \in A \cup B \Rightarrow xy \in N(A) \cup N(B)$.

$\Rightarrow N(A \cup B) \subseteq$

Next let $y \in N(A) \cup N(B)$.
then $y \in N(A)$ or $y \in N(B)$ or both.

$\Rightarrow$ if $y \in N(A)$, $\exists\ x \in A$ s.t. $xy \in E$

& if $y \in N(B)$, $\exists\ x \in B$ s.t. $xy \in E(G)$

$\Rightarrow$ if $y \in N(A) \cup N(B)$, $\exists\ x \in A \cup B$ such that $xy \in E(G) \Rightarrow y \in N(A \cup B)$

$\Rightarrow$ $N(A \cup B) \supseteq N(A) \cup N(B)$

$\Rightarrow$ $N(A \cup B) = N(A) \cup N(B)$.
(both inclusions proved).

(b) $A \cap B \subseteq A \Rightarrow N(A \cap B) \subseteq N(A)$

$A \cap B \subseteq B \Rightarrow N(A \cap B) \subseteq N(B)$

$\Rightarrow$ $N(A \cap B) \subseteq N(A) \cap N(B)$
($\because$ of inclusion in both $N(A)$ & $N(B)$)

(c) $|N(A \cup B)| \leq |N(A) \cup N(B)|$

$= |N(A)| + |N(B)| - |N(A) \cap N(B)|$

$\leq |N(A)| + |N(B)| - |N(A \cap B)|$
(by the result in (b))

$\Rightarrow$ $|N(A \cup B)| + |N(A \cap B)| \leq |N(A)| + |N(B)|$

Problem 2: Let G be a bipartite graph with bipartition (X, Y). Prove that G contains a complete matching from X to Y if and only if

$$|X \setminus N(T)| \leq |Y \setminus T| \quad \forall\, T \subseteq Y.$$

Solution. Suppose the condition holds i.e.,

$$|X \setminus N(T)| \leq |Y \setminus T| \quad \forall\, T \subseteq Y.$$

Now let $S \subseteq X$ be any subset of X.
Taking $T = N(S)^c$. we get

$$|X \setminus N(N(S)^c)| \leq |Y| - |N(S)^c|$$

or $$|X| - |N(N(S)^c)| \leq |N(S)|$$

It suffices to prove that

$$|S| \leq |X| - |N(N(S)^c)|$$

$$\iff |S| + |N(N(S)^c)| \leq |X|. \quad (*)$$

claim: $N(N(S)) \cap N(N(S)^c) = \phi$.

If ~~not~~, then $\exists\, y_1 \in N(S)$ and $\exists\, y_2 \in N(S)^c$ such that $x y_2 \in E(G)$
$x y_1 \in E(G)$ & $x y_2$

~~But then Since~~

Now let M be the complete matching of G.. Then since xy_1 & xy_2 have a common vertex, so,

xy_1 & xy_2 cannot be in the matching M. (by definition of a matching)

$\Rightarrow$ $|M| < |X|$

$\Rightarrow$ M is not a complete matching.

$\Rightarrow$ a contradiction.

$\Rightarrow$ there cannot exist an x such that $x \in N(N(S)) \cap N(N(S)^c)$

$\Rightarrow$ $N(N(S)) \cap N(N(S)^c) = \phi$

as claimed.

But then this implies that

$|N(N(S) \cup N(S)^c)|$

$= |N(N(S))| + |N(N(S)^c)|$

Now $N(S) \cup N(S)^c \subseteq Y$

$\Rightarrow$ $N(N(S) \cup N(S)^c) \subseteq X$.

$$|N(N(S) \cup N(S)^c)| \leq |X|.$$

$\Rightarrow$

$$|N(N(S))| + |N(N(S)^c)| \leq |X|$$

$\Rightarrow$

Now it is trivial to see that

$$|S| \leq |N(N(S))|. \quad \forall S \subseteq X.$$

$$|S| + |N(N(S)^c)| \leq |X|$$

$\Rightarrow$

Thus we proved the sufficient condition ⊛.

$$|S| \leq |X| - |N(N(S)^c)| \leq |N(S)|.$$

$\Rightarrow$

$$|S| \leq |N(S)| \quad \forall S \subseteq X$$

$\Rightarrow$

So the condition of Hall's theorem is met

$\Rightarrow$ G has a complete matching from X to Y.

Next, let's prove the "only if" part.

Assume G has a complete matching M.

Then by Hall's theorem,

$$|S| \leq |N(S)| \quad \forall S \subseteq X.$$

So now let $T \subseteq Y$.

Let $S = N(T)^c$. Then we get

$$|N(T)^c| \leq |N(N(T)^c)|$$

$$\Rightarrow |X| - |N(T)| \leq |N(N(T)^c)| \quad \text{— ①}$$

But $N(N(T)) \cup N(N(T)^c) \subseteq Y$

and $N(N(T)) \cap N(N(T)^c) = \phi$.

(as proved in the "if" part)

$$\Rightarrow |N(N(T)) \cup N(T)^c)|$$
$$= |N(N(T))| + |N(N(T)^c)|$$

$$\Rightarrow |N(N(T))| + |N(N(T)^c)| \leq |Y|. \quad \text{— ②}$$

Adding inequalities ① & ② we get after cancellations of some terms:

$$|X| - |N(T)| \leq |Y| - |N(N(T))| \quad \text{— ③}$$

But $|T| \leq |N(N(T))|$ — ④

Adding ③ & ④ we get,

$$|X| - |N(T)| \leq |Y| - |T|$$

In other words:

$$|X \setminus N(T)| \leq |Y \setminus T| \quad \forall\, T \subseteq Y$$

This proves the "only if" part.
And thus the solution to the problem is completed.

Book: Modern Graph Theory.

Author: Bela Bollobas

Problem: Let $G = G_2(m,n)$ be a bipartite graph with vertex classes V_1 & V_2 containing a complete matching from V_1 to V_2.

Prove that there is a vertex $x \in V_1$ such that for every edge xy there is a matching from V_1 to V_2 that contains the edge xy.

Show that matchings of size at least 2 exist.

Solution : Let M be the complete matching from V_1 to V_2.

Assume that what we want to prove is not true. So the converse is true. That is, for each $x \in V_1$, $\exists\, y \in V_2$ such that there is no matching from V_1 to V_2 that contains the edge xy.

Let $V_1 = \{x_1, x_2, \dots, x_m\}$

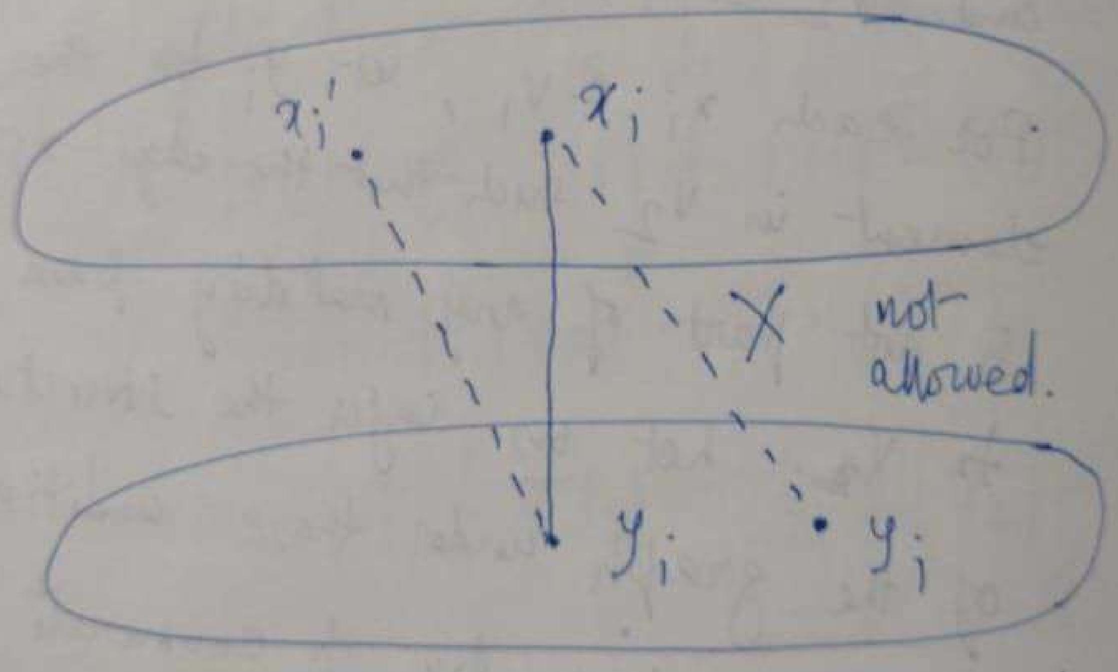
and $V_2 = \{y_1, y_2, \dots, y_n\}$

For each $x_i \in V_1$, let y_i be the element in V_2 such that the edge $x_i y_i$ is not part of any matching from V_1 to V_2. Let us infer the structure of the graph under these conditions.

That is let us try to enumerate all possible structures the graph can have under these circumstances.

Since there is no matching from V_1 to V_2 which contains the edge x_iy_i, it follows that each edge x_iy_i must fall into one of the following categories.

<u>Scenario 1</u>: $\exists\, x_i' \in V_1$ such that $x_i'y_i \in E(G)$ and $\nexists\, y_i' \in V_2$ such that $x_iy_i' \in E(G)$. The situation is shown below:



In this case, let $S = \{x_i, x_i'\}$ then we have,

$$|N(S)| = |\{y_i\}| = 1 \lneq 2 = |S|$$

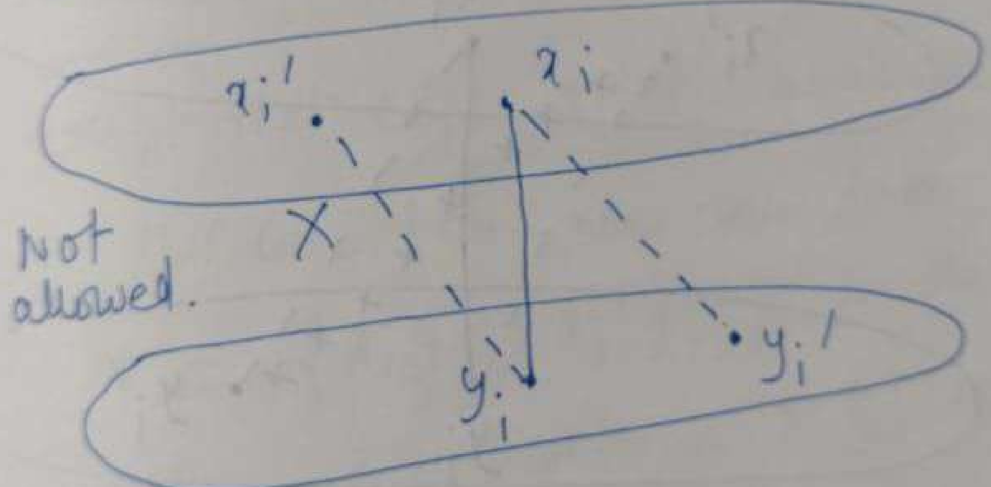
So Hall's condition is violated.

$\Rightarrow$ G does not have a complete matching from V_1 to V_2.

This is a contradiction

$\Rightarrow$ Scenario 1 does not arise.

Scenario 2: $\exists\, y_i' \in V_2$ such that $x_i y_i' \in E(G)$ and $\nexists\, x_i' \in V_1$ such that $x_i' y_i \in E(G)$. The situation is shown below:



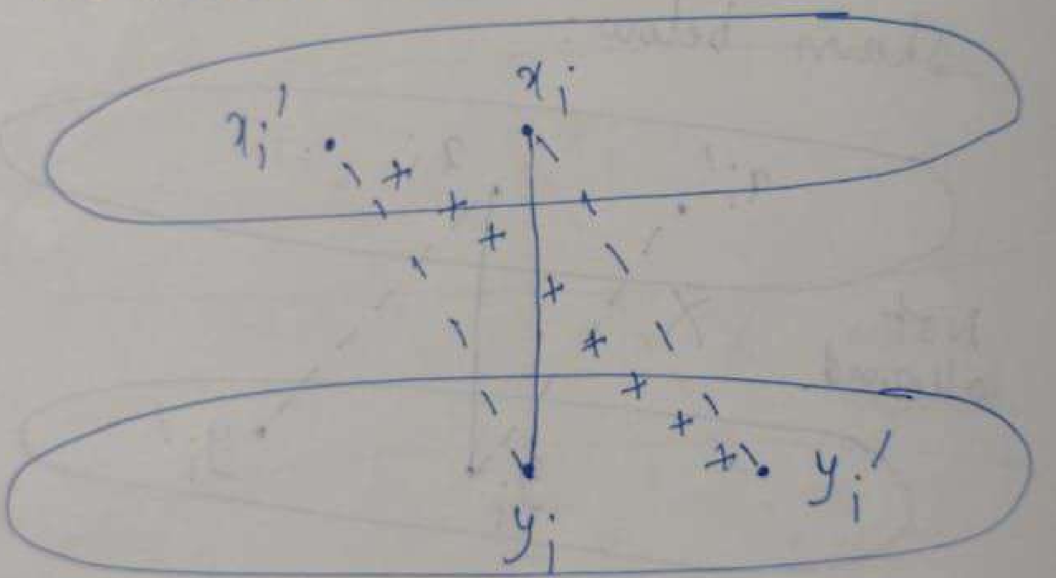
This is a valid scenario.

If $\exists$ at least 2 such x_i's with this property, then $M' = \{x_i y_i, x_{i+1} y_{i+1}\}$ is a matching of size at least 2 which contains the edge $x_i y_i$.

This contradicts the assumption that there is no such matching that we made at the beginning. So this scenario doesn't really arise and is ruled out.

Scenario 3 : $\exists\ x_i' \in V_1$ such that $x_i' y_i \in E(G)$ & $\exists\ y_i' \in V_2$ such that $x_i y_i' \in E(G)$ but $x_i' y_i \notin E(G)$.

The scenario is shown below:



Edge xxx is not allowed..

In this case too defining $S = \{x_i, x_i'\}$

$\Rightarrow N(S) = \{y_i\}$. So Hall's condition is violated.

$\Rightarrow$ this scenario is impossible.

~~scenario 4: this is same as scenario 3 except that edge~~ ~~$x_i' y_i'$~~ ~~is allowed.~~

In this scenario 3, we can take the matching as $M' = \{x_i y_i, x_i' y_i'\}$ contradicting the assumption that there is no matching of size at least 2 which contains $x_i y_i$.

Scenario 4: this is same as scenario 3 except that edge $x_i' y_i'$ is allowed.

In this case too, we can take $M' = \{x_i y_i, x_i' y_i'\}$ as the matching which contains $x_i y_i$ thus obtaining a contradiction.

Thus all possible scenarios coming out of the assumption that converse is true lead to impossible situation. So the converse is false. This completes the proof.

Book - Graph Theory - Undergraduate Mathematics

Authors - Khee Meng Koh et. al.

Publisher - World Scientific.

Problem: Let G be a bipartite graph with bipartition (X, Y). Assume that there exists a positive integer k such that

$$d(y) \leq k \leq d(x) \quad \text{— } (0)$$

for each vertex y in Y and each vertex x in X. Let $S \subseteq X$ and denote by E_1 the set of edges in G incident with some vertex in S, and by E_2 the set of edges in G incident with some vertex in $N(S)$.

(a) Show that $k|S| \leq |E_1| \leq |E_2| \leq k|N(S)|$

(b) Show that G has a complete matching from X to Y.

(c) Deduce from (b) that every k-regular bipartite graph with $k \geq 1$ has a perfect matching.

Proof: (a) $d(x) \geq k \ \forall \ x \in X$.

and $S \subseteq X$ together imply

$$d(x) \geq k \ \forall \ x \in S.$$

Summing over all $x \in S$ we get

$$\sum_{x \in S} d(x) \geq k|S|$$

But $\sum_{x \in S} d(x) =$ # edges incident with some vertex in S

$$= |E_1|.$$

Thus $|E_1| \geq k|S|$ —— ①

Similarly $d(y) \leq k \ \forall \ y \in Y$

and $N(S) \subseteq Y$ together imply.

$$d(y) \leq k \ \forall \ y \in N(S).$$

Summing over all $y \in N(S)$ we get

$$\sum_{y \in N(S)} d(y) \leq k|N(S)|.$$

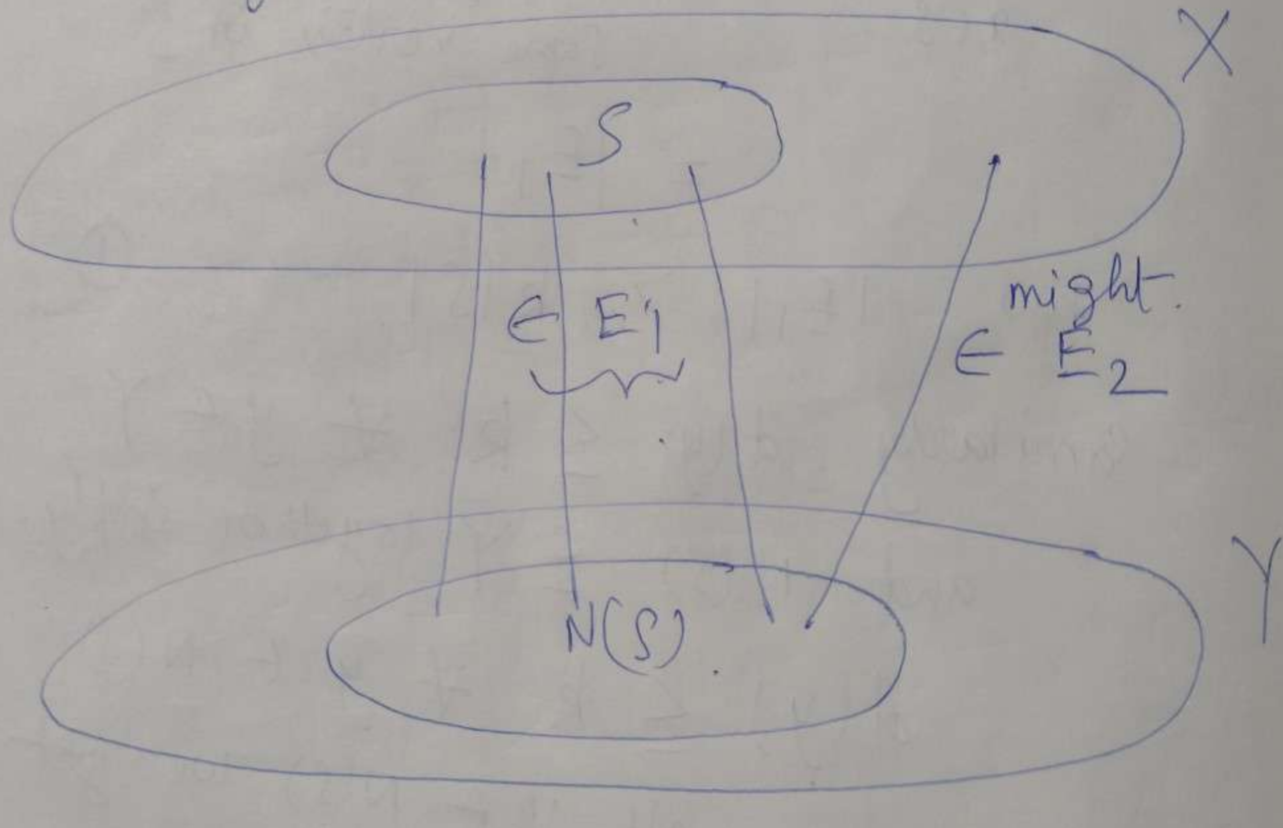
But $\sum_{y \in N(S)} d(y) =$ # edges incident with some vertex in $N(S)$

$$= |E_2|.$$

Thus $|E_2| \leq k|N(S)|$ — (2)

So to prove (a) it suffices to show that $|E_1| \leq |E_2|$.

By definition every edge in E_1 belongs to E_2. But every edge in E_2 may not belong to E_1. See diagram below:



Therefore $E_1 \subseteq E_2$ always.

$\Rightarrow |E_1| \leq |E_2|$.

Therefore.

$$k|S| \leq |E_1| \leq |E_2| \leq k|N(S)|$$

(b). Since,

$$k|S| \leq |E_1| \leq |E_2| \leq k|N(S)|$$

holds for all subsets $S \subseteq X$

$$\Rightarrow \cancel{k}|S| \leq \cancel{k}|N(S)| \quad \forall S \subseteq X$$

$$\Rightarrow |N(S)| \geq |S| \quad \forall S \subseteq X$$

So Hall's condition is true.

Therefore Hall's theorem implies that G has a complete matching.

(c) Now let G be k-regular bipartite.

Then,

$$k = d(y) \leq k \leq d(x) = k$$

$\forall x \in X$ & $y \in Y$

$\Rightarrow$ condition ~~of~~ (a) of the problem is satisfied. So from (b), it follows that G has a complete matching M.

i.e., a matching M such that $|X| = |M|$

But for a k-regular bipartite, $|X| = |Y|$

$$\Rightarrow |X| = |M| = |Y|$$

$\Rightarrow$ M is a perfect matching.