Consider the following Block Diagonal matrix :

$$M = \left[\begin{array}{ccc|cc} 1 & 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{array}\right]$$

Python calculations show that for M, the SVD decomposition is as follows:

$$U = \begin{bmatrix} -0.54 & 0 & 0 & 0.83 \\ -0.83 & 0 & 0 & -0.54 \\ 0 & 0.7 & -0.7 & 0 \\ 0 & 0.7 & 0.7 & 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 4.45 & 0 & 0 & 0 & * \\ 0 & 3 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0.38 & 0 \end{bmatrix}$$

and,

$$V^T = \begin{bmatrix} -4.5 & -0.3 & -0.09 & 0 & 0 \\ 0 & 0 & 0 & 0.7 & 0.7 \\ 0 & 0 & 0 & 0.7 & -0.7 \\ -0.64 & 7.5 & 1.08 & 0 & 0 \\ -0.577 & 0.57 & -0.57 & 0 & 0 \end{bmatrix}$$

what transformations will make U and VT block diagonal?

→ If we swap the 2nd & 4th columns of U, then U will become a block diagonal matrix. Suppose this transformation is brought about by right-multiplying U by a matrix P_1.

→ Similarly VT can be made block diagonal by 2 operations:

1) swapping 2nd & 4th rows.
2) swapping 3rd & 5th rows.

Suppose this transformation can be caused by left-multiplying V^T by a matrix P_2.

Then M can be written as:

$$M = U \cdot P_1 \cdot P_1^{-1} \Sigma \cdot P_2^{-1} P_2 \cdot V_2^T$$

$$= (UP_1)(P_1^{-1} \Sigma P_2^{-1})(P_2 V_2^T)$$

Now note that UP_1 & $P_2 V_2^T$ are both in block-diagonal form.

Also P_1 & P_2 are permutation matrices.

$$M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Again Python calculations show that

$$U_A = \begin{bmatrix} -0.54 & 0.83 \\ -0.83 & -0.54 \end{bmatrix}$$

$$\Sigma_A = \begin{bmatrix} 4.45 & 0 \\ 0 & 0.38 \end{bmatrix}$$

$$V_A^T = \begin{bmatrix} -0.49 & -0.31 & -0.809 \\ -0.64 & 0.75 & 0.108 \\ -0.57 & -0.57 & 0.57 \end{bmatrix}$$

$$U_B = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}$$

$$\Sigma_B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V_B^T = \begin{bmatrix} -0.7 & -0.7 \\ 0.7 & -0.7 \end{bmatrix}$$

comparison of the matrices

Σ and $\begin{bmatrix} \Sigma_A & 0 \\ 0 & \Sigma_B \end{bmatrix}$

shows that the singular values are swapped.

Hence if we are given only the singular value decomposition (U, Σ, V^T) of M alone, then, $P_1^{-1} \Sigma P_2^{-1}$ will ~~give~~ ~~us the~~ be a diagonal matrix of singular values in which the singular values of A & B will be in the right order.

The only question which remains now is how do we obtain the permutation matrices P_1 and P_2?

As noted earlier U requires a column permutation and V_2 transpose requires a row permutation.

It suffices to show how to construct a permutation matrix P to shuffles the rows of a shuffled-block-digonal matrix to bring it into its block diagonal form. Because for column permutation, ~~the~~ we just look at the transpose of the matrix and apply the aforementioned technique of row permutation and the transpose back.

So I will address only the problem of row permutation matrix here.

Consider a matrix A which we already know is a block diagonal matrix BUT whose rows have been shuffled. How to construct P such that $P \cdot A$ is in block diagonal form?

Let us look at the first column of A. Let A ~~be~~ have n rows. Now consider ~~the~~ the 2 subsets Z & NZ of the set $S = \{1, 2, \ldots, n\}$ where

$$Z = (z_1, z_2, \ldots, z_{n_1})$$

$\boxed{n_1 + n_2 = n}$

and $NZ = (nz_1, nz_2, \ldots, nz_{n_2})$

are ordered subsets of S. satisfying the conditions:

z_i = # i-th row whose first entry is a 0.

nz_i = # i-th row whose first entry is a non-zero.

and

$$z_1 < z_2 < \ldots < z_{n_1}$$

$$Nz_1 < nz_2 < \ldots < nz_{n_2}.$$

Now we construct our P as follows.

In the first row of P, put 0's in all positions except the nz_1'th column. (put a 1 here).

Similarly, for $i = 1, 2, \ldots, n_2$, in the i-th row of P (from the top) put 0's in all columns except the nz_i'th column where we put a 1

Next we populate the remaining rows of P with the rows originally numbered as $z_1, z_2, \ldots, z_{n_1}$. That is:

For $j = 1, 2, \ldots, n_1$,

~~in the i-th row~~ in the $(n_2 + i)$-th row of P (from the top) put 0's in all columns except the z_i'th column where we put a 1.

<u>Example</u>: In our case,

$$V^T = \begin{bmatrix} -4.5 & -0.3 & -0.09 & 0 & 0 \\ 0 & 0 & 0 & 0.7 & 0.7 \\ 0 & 0 & 0 & 0.7 & -0.7 \\ -0.84 & 0.75 & 1.08 & 0 & 0 \\ -0.57 & 0.57 & -0.57 & 0 & 0 \end{bmatrix}$$

Looking at the 1st column, we obtain

$$z = (z_1, z_2) = (2, 3)$$

$$nz = (nz_1, nz_2, nz_3) = (1, 4, 5).$$

So our constructed permutation matrix is:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

It can be easily verified that $P.V^T$ is indeed in the block-diagonal form as was required.