Nathanson [88]: Prove that there does not exist a polynomial identity of the form:

$$(x_1^2+x_2^2+x_3^2)(y_1^2+y_2^2+y_3^2) = z_1^2+z_2^2+z_3^2$$

where z_1, z_2, z_3 are polynomials in $x_1, x_2, y_1, y_2, x_3, y_3$ with integral coefficients:

Proof: Setting $x_3 = y_3 = 0$ we obtain:

$$(x_1^2+x_2^2)(y_1^2+y_2^2) = z_1'^2+z_2'^2+z_3'^2$$

(with $x_3 = y_3 = 0$)

$$\Leftrightarrow \quad x_1^2y_1^2 + x_2^2y_1^2 + x_1^2y_2^2 + x_2^2y_2^2$$
$$= z_1'^2 + z_2'^2 + z_3'^2$$

where z_1', z_2', z_3' are polynomials in x_1, x_2, y_1, y_2 only:

Any general polynomial in x_1, x_2, y_1, y_2 is of the form:

$$ax_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2)$$

$$a, b, c, d \in \mathbb{Z}.$$

So let us assume that:

$$x_1^2y_1^2 + x_2^2y_1^2 + x_1^2y_2^2 + x_2^2y_2^2$$

$$= (ax_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2)^2$$

$$+ (ex_1y_1 + fx_1y_2 + gx_2y_1 + hx_2y_2)^2$$

$$+ (ix_1y_1 + jx_1y_2 + kx_2y_1 + lx_2y_2)^2$$

comparing coefficients of x_1y_1, x_1y_2, x_2y_1 and x_2y_2 on both sides give us:

$$\begin{cases} a^2 + e^2 + i^2 = 1 & (1) \\ b^2 + f^2 + j^2 = 1 & (2) \\ c^2 + g^2 + k^2 = 1 & (3) \\ d^2 + h^2 + l^2 = 1 & (4) \end{cases}$$

Next coefficients of cross terms like $x_1^2y_1y_2$, $x_2^2y_1y_2$ etc should be zero on the RHS because no such terms appear on the LHS. ~~This~~

This gives us the equations:

$$\begin{cases} ab + ef + ij = 0 & \textcircled{1}' \\ ac + eg + ik = 0 & \textcircled{2}' \\ ad + eh + il = 0 & \textcircled{3}' \\ bc + fg + jk = 0 & \textcircled{4}' \\ bd + fh + jl = 0 & \textcircled{5}' \\ cd + gh + kl = 0 & \textcircled{6}' \end{cases}$$

$\textcircled{1} \Rightarrow a^2 + e^2 + i^2 = 1$

where $a, e, i \in \mathbb{Z}$.

w.l.o.g assume $a = \pm 1$, $e = i = 0$.

With $e = i = 0$ and $a = \pm 1$, $\textcircled{1}'$ implies

$b = 0$.

Now $\textcircled{2} \Rightarrow f^2 + j^2 = 1$.

w.l.o.g assume $f = \pm 1$ ~~and~~ $j = 0$.

$\textcircled{4}' \Rightarrow g = 0$

$\textcircled{5}' \Rightarrow h = 0$.

③ $\Leftrightarrow c^2+g^2+k^2=1$

$\Rightarrow c^2+k^2=1$ ($\because g=0$)

~~w.l.o.g assume~~ But,

$ac+eg+ik=0$

$\Leftrightarrow (\pm1)c+0\cdot0+0\cdot k=0$

$\Rightarrow c=0$.

so ③ $\Rightarrow c^2+k^2=1$

$\Leftrightarrow k=\pm1$.

④' $\Rightarrow j=0$

⑥' $\Rightarrow l=0$

③' $\Rightarrow ad+0+0=0$

$\Rightarrow d=0$

So $d^2+h^2+l^2=0^2+0^2+0^2$

$=0$

But ④ $\Rightarrow d^2+h^2+l^2=1$

A contradiction.