Catalan, Schröder, Baxter, and other sequences

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Outline

Introduction I: Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$(C_n)_{n\geq 1} : 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, \ldots$$

- OEIS: A000108
- Count dozens (hundreds?) of different combinatorial objects, including:
  - binary trees
  - Dyck paths
  - sets of balanced parentheses
  - polygon triangulations
  - 123-avoiding permutations
  - paralellogram polyominoes
  - ...
- Satisfy recurrence relations:

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} = \frac{2(2n+1)}{n+2} C_n$$

- Algebraic generating function:

$$C(z) = \sum_n C_n z^n = \frac{1 - \sqrt{1-4z}}{2z}$$

Introduction II: Baxter numbers

$$B_n = \frac{2}{n(n+1)^2} \sum_{k=1}^{n} \binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1}$$

$(B_n)_{n \geq 1}$: $1, 2, 6, 22, 92, 422, 2074, 10754, 58202, 326240, 1882960, 11140560, \ldots$

- OEIS: A001181
- Introduced by G. Baxter in 1964: Let $f, g : [0,1] \to [0,1]$ be continuous functions such that $(f \circ g)(x) = (g \circ f)(x) = h(x)$. Say x^* is a fixed point of h. Then

$$h(f(x)) = (f \circ g)(f(x)) = f((g \circ f)(x)) = f(x),$$

  i.e. $f(x)$ is also a fixed point of h. If the set of fixed points $\mathcal{X}$ is finite $\{x_1, \ldots, x_N\}$, then f induces a permutation π on $\mathcal{X}$, by $\pi(i) = j \iff f(x_i) = x_j$.
- Turns out $N = 2n - 1$ for some n, and Baxter permutations on $[2n-1]$ are restricted:
  - $\pi(i) = j$ only if i and j have same parity
  - if $\pi(x) = i$ and $\pi(y) = i + 1$, and z is between x and y, then $\pi(z) \geq i$ if i is even and $\pi(z) \geq i + 1$ if i is odd

Introduction II: Baxter numbers ct'd

- Baxter permutation π on $[2n-1]$ is actually determined by its action on the odd numbers. Define the reduced Baxter permutation $\hat{\pi}$ on $[n]$ by

$$\hat{\pi}(k) = \tfrac{1}{2}(1 + \pi(2k+1))$$

- Then reduced Baxter permutations on $[n]$ are those avoiding the vincular patterns $2\underbracket{41}3$ and $3\underbracket{14}2$.
- Linear three-term recurrence:

$$B_{n+1} = \frac{7n^2 + 21n + 12}{(n+3)(n+4)} B_n + \frac{8n(n-1)}{(n+3)(n+4)} B_{n-1}$$

- D-finite generating function

Introduction III: (large) Schröder numbers

$$S_n = \frac{1}{n}\sum_{k=0}^{n} 2^k \binom{n}{k}\binom{n}{k-1}$$

$(S_n)_{n\geq 1}$: $1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, 1037718, 5293446, \ldots$

- OEIS: A006318
- S_n is the number of separable permutations of length n
- Equivalently, the number of permutations avoiding patterns 2413 and 3142
- Number of paths from $(0,0)$ to $(2n-2,0)$ with steps $(1,1),(1,-1)$ and $(2,0)$ which stay above the line $y=0$.
- Number of unary-binary trees with n internal nodes and unary root
- ...
- Linear three-term recurrence:

$$S_{n+1} = \frac{3(2n+1)}{n+2}S_n - \frac{n-1}{n+2}S_{n-1}$$

- Algebraic generating function:

$$S(z) = \sum_n S_n z^n = \frac{1-z-\sqrt{1-6z+z^2}}{2}$$

Introduction IV

For all $n \geq 1$,

$$C_n \leq S_n \leq B_n$$

The goal is to understand this combinatorially, particularly in the way that the different combinatorial objects grow.

Succession rules

We will grow objects of size $n+1$ from those of size n.

For Dyck paths (counted by Catalan): from a path of length $2n$, grow a path of length $2n+2$ by inserting a peak somewhere in the last descent (or immediately before/after).

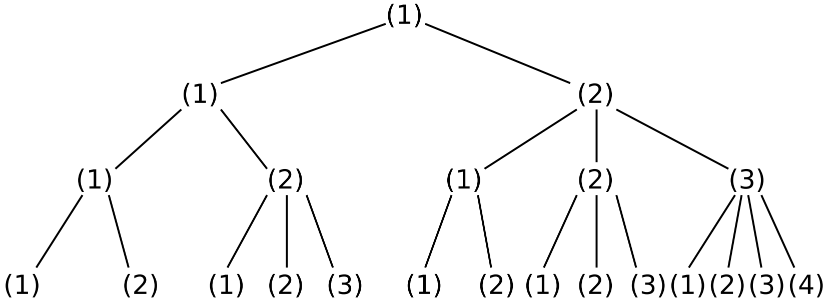
A succession rule describes a process like this by assigning a label to each object.

- Label a Dyck path of length $2n$ according to the length of the last descent. Possible labels are $\ell = 1, 2, \ldots, n$.
- The rule is

$$(\ell) \rightsquigarrow (1), (2), \ldots, (\ell), (\ell+1) \qquad \text{(Cat)}$$

with initial label (1).

Viewed as a generating tree:



Succession rule for Baxter permutations

Baxter permutations can also be grown recursively [Bousquet-Mélou 2003]: In a Baxter permutation σ on $[n]$, can insert a new element $n+1$ immediately to the left of a left-to-right maximum of σ, or immediately to the right of a right-to-left maximum of σ. (Impossible to do both at same time.) These are the active sites.

- Label a Baxter permutation (h,k), where h is number of left-to-right maxima and k is number of right-to-left maxima.
- Then the rule is

$$(h,k) \rightsquigarrow \begin{cases} (1,k+1),(2,k+1),\dots,(h,k+1) \\ (h+1,1),(h+1,2),\dots,(h+1,k) \end{cases} \tag{Bax}$$

  with initial label $(1,1)$.
- Note that the sub-rule

$$(h,k) \rightsquigarrow (1,k+1),(2,k+1),\dots,(h,k+1),(h+1,k)$$

  is isomorphic to (Cat).

Succession rule for Schröder numbers

Separable permutations are those avoiding patterns 2413 and 3142. For a separable permutation σ on $[n]$, let t be the number of active sites, ie. positions where we can insert $n+1$ and get a separable permutation on $[n+1]$.

It is known [West 1995] that separable permutations follow the succession rule

$$(t) \rightsquigarrow (3),(4),\ldots,(t),(t+1),(t+1) \qquad \text{(Schr)}$$

with initial label (2).

Theorem

The succession rule

$$(h,k) \rightsquigarrow \begin{cases} (1,k+1),(2,k+1),\ldots,(h,k+1) \\ (2,1),(2,2),\ldots,(2,k-1),(h+1,k) \end{cases} \qquad \text{(NewSchr)}$$

with initial label $(1,1)$ *produces the enumeration sequence* $\{S_n\}$.

Proof.

Replace (h,k) with $(h+k)$ to get (Schr). □

(Cat) is once again isomorphic to a sub-rule:

$$(h, k) \rightsquigarrow (1, k+1), (2, k+1), \dots, (h, k+1), (h+1, k)$$

Theorem

(NewSchr) *is isomorphic to a sub-rule of* (Bax).

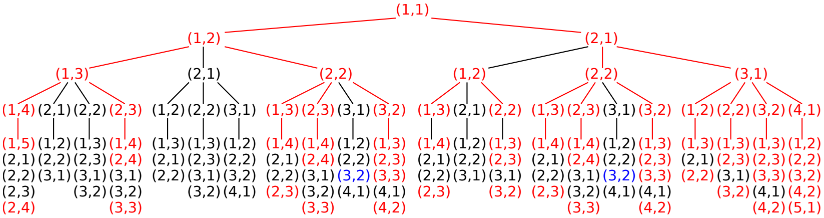
Proof.

If (h, k) is a label in (NewSchr) and (h', k) is a label in (Bax) with $h' \geq h$, then there is an injective map from the successors of (h, k) to the successors of (h', k) such that the image of (i, j) is (i', j) for some $i' \geq i$.

$$\begin{array}{ccccccccc} (h,k) \rightsquigarrow & (1,k+1), & (2,k+1), & \dots, & (h,k+1), & (2,1), & \dots, & (2,k-1), & (h+1,k) \\ & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow \\ (h',k) \rightsquigarrow & (1,k+1), & (2,k+1), & \dots, & (h',k+1), & (h'+1,1), & \dots, & (h'+1,k-1), & (h'+1,k) \end{array}$$

Then start from the initial label $(1, 1)$ and apply induction. $\square$

Viewed as a generating tree:



- red is the Catalan subtree
- blue are the objects which are Baxter but not Schröder

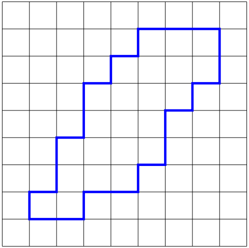
Polyomino slicings

This nesting structure

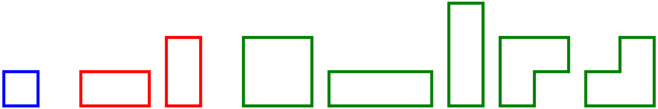
$$\text{Catalan} \subseteq \text{Schröder} \subseteq \text{Baxter}$$

can be viewed through various combinatorial objects.

A parallelogram polyomino (PP) of perimeter $2n$ is the union of two directed paths of length n, stepping $\uparrow$ and $\rightarrow$, starting and ending at the same points but otherwise not intersecting.

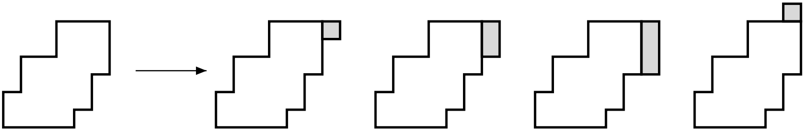


The number of PPs of size n (perimeter $2n+2$) is C_n:

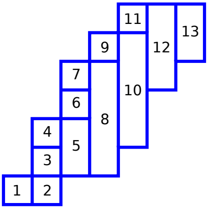


To see this using (Cat):

- in a PP of size n, assign label ℓ as the height of the rightmost column
- to grow a new PP of size $n+1$:
  - append a new column to the right of height $1, 2, \ldots, \ell$, or
  - attach a single cell above the rightmost column



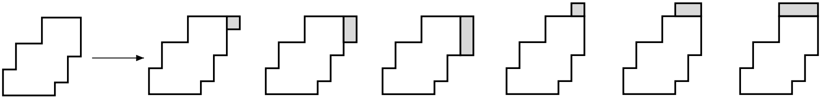
We can uniquely slice any PP into pieces according to the way it was grown using (Cat):



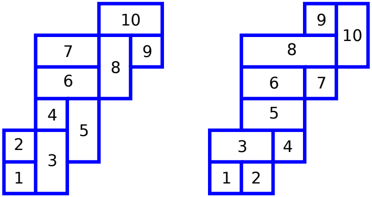
This slicing is asymmetric: can add a new column of height $1, 2, \ldots, \ell$, but only a new row of width 1.

(Bax) can be viewed as the symmetrical version:

- in a PP of size n, assign label (h, k) according to the width of the top row and height of the rightmost column
- to grow a new PP of size $n + 1$:
  - append a new row to the top of width $1, 2, \ldots, h$, or
  - append a new column to the right of height $1, 2, \ldots, k$



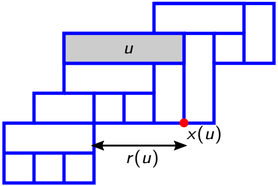
Now a PP can be grown in multiple ways, so B_n counts the number of Baxter slicings.



For (NewSchr), things are a bit more complicated.

In a Baxter slicing S of a PP, let u be a horizontal block:

- let $w(u)$ be the width of u
- let $x(u)$ be the lowest point below the right edge of u
- let $r(u)$ be the number of horizontal edges immediately to the left of $x(u)$



Then S is a Schröder slicing if for every horizontal block u,

$$w(u) \leq r(u) + 1.$$

Theorem

The number of Schröder slicings of size n is S_n.

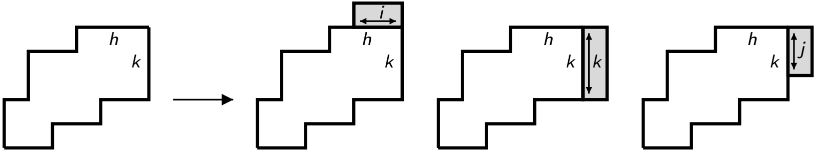
Proof.

Recall (NewSchr):

$$(h,k) \rightsquigarrow \begin{cases} (1,k+1),(2,k+1),\ldots,(h,k+1) \\ (2,1),(2,2),\ldots,(2,k-1),(h+1,k) \end{cases}$$

Let a Schröder slicing S have label (h,k) according to the maximum width (resp. height) of a new row (resp. column) which can be added.

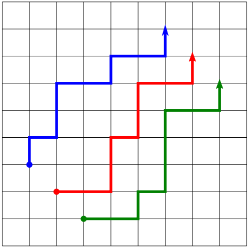
- If we add a new row of width i, $1 \le i \le h$, the resulting slicing will have label $(i, k+1)$.
- If we add a new column of height j, $1 \le j < k$, the resulting slicing will have label $(2, j)$.
- If we add a new column of height k the resulting slicing will have label $(h+1, k)$. □



Non-intersecting lattice paths

A triple of non-intersecting lattice paths (3NILP) is a set of three directed paths (u, m, d) with

- u starting at $(0, 2)$
- m starting at $(1, 1)$
- d starting at $(2, 0)$
- all three taking k east steps and $n - k$ north steps for some $0 \leq k \leq n$
- no paths sharing a vertex



Theorem (Viennot 1984?)

The number of 3NILPs of length n is B_{n+1}

A new bijective proof:

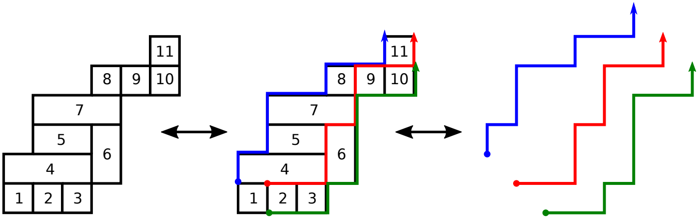
Proof.

Given a Baxter slicing S of size n:

- Let u' be the upper boundary of S, except for the first and last edges
- Let d' be the lower boundary of S, except for the first and last edges
- Let m be the path obtained by starting at $(1, 1)$ and following the upper/left boundaries of vertical blocks and lower/right boundaries of horizontal block

Shift u' up by 1 to get u, and d' right by 1 to get d. Then (u, m, d) is a 3NILP.

Reverse procedure is easily defined. □



The 3NILPs corresponding to Schröder slicings can also be described (characterisation is fairly complicated).

Pattern-avoiding permutations revisited

Already know that S_n counts separable permutations (avoiding 2413 and 3142) of length n, and these can be grown according to (Schr). But don't know how to grow these according to (NewSchr)!

However, viewing (NewSchr) as a sub-rule of (Bax), we observe a different set of permutations which grow according to (NewSchr).

First, a new definition: a subsequence $\sigma_i\sigma_j\sigma_k\sigma_\ell\sigma_m$ in permutation σ is an occurrence of the pattern 41323^+ (resp. 42313^+) if

- $\sigma_i\sigma_j\sigma_k\sigma_\ell\sigma_m$ is an occurrence of the pattern 51324 (resp. 52314), and
- $\sigma_m = \sigma_k + 1$

Define a Schröder* permutation to be one which avoids $2\underline{41}3$, $3\underline{14}2$, 41323^+ and 42313^+.

Theorem

The number of Schröder* permutations of length n is S_n.

Proof (idea):

Schröder* permutations are a subset of Baxter permutations (avoiding $2\underline{41}3$ and $3\underline{14}2$). So their active sites are a subset of the active sites of Baxter permutations. Can show that these are:

- sites immediately to the right of right-to-left maxima; and
- sites immediately to the left of left-to-right maxima σ_i, when $\sigma_{i+1}\dots\sigma_n$ contains no subsequence $\sigma_a\sigma_b\sigma_c$ with $\sigma_a > \sigma_i$, $\sigma_b < \sigma_a$ and $\sigma_c = \sigma_a + 1$.

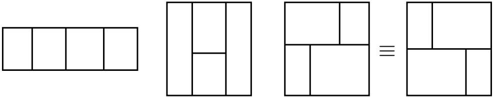
Let k count the former and h count the latter. Then by labelling a Schröder* permutation (h, k), the growth exactly follows (NewSchr). □

Floorplans

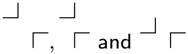
A mosaic floorplan (MFP) [Hong et al 2000] is a partition of a rectangle into smaller rectangles using segments which may meet but not cross, ie. intersections of type

$$\perp, \top, \vdash, \text{ or } \dashv \text{ but not } +.$$

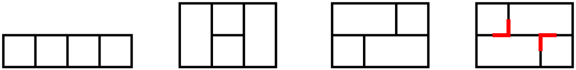
These are defined up to the sliding of segments (equivalence classes). The size of an MFP is the number of parts in the partition.



[Aval et al 2016] A packed floorplan (PFP) of dimension (d, ℓ) is a partition of a $d \times \ell$ rectangle into $d + \ell - 1$ rectangles of integer lengths, such that the patterns



are avoided. The size of a PFP is $d + \ell - 1$.



Every equivalence class of MFPs corresponds to a unique PFP.

Theorem (Yao et al 2003, Aval et al 2016)

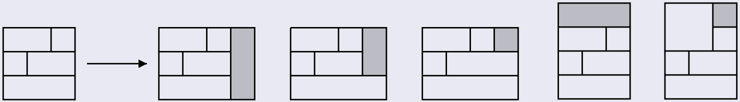
The number of PFPs of size n is B_n.

A new proof:

Proof (idea):

Label a PFP P of size n by (h, k), where h is the number of rectangles which meet the top boundary, and k is the number which meet the right boundary. Then construct a PFP P' of size $n+1$ by

- For $i = 1, \ldots, h$, insert new rectangle at top of P, aligned with the i rightmost rectangles at top of P. Extend the remaining $h - i$ rectangles at top of P up.
- For $j = 1, \ldots, k$, insert new rectangle at right of P, aligned with the j uppermost rectangles at right of P. Extend the remaining $k - j$ rectangles at right of P up.



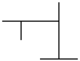
This growth follows (Bax).

Straightforward to show that P' is also a PFP, and this operation is easily reversible. □

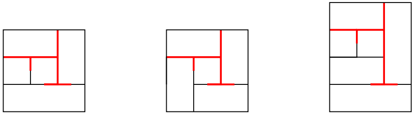
How do (Cat) and (NewSchr) fit into this?

A Catalan PFP is one whose internal segments avoid the configuration

A Schröder PFP is one whose internal segments avoid the configuration



eg. The following are not Schröder PFPs



Theorem

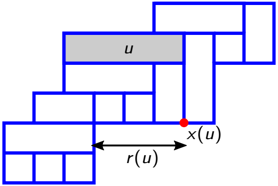
The number of Catalan PFPs of size n is C_n, and the number of Schröder PFPs of size n is S_n.

Proof follows by restricting the growth of PFPs according to the rules.

Generalisations of Schröder slicings

Recall that Schröder slicings were defined by restricting the widths of horizontal blocks:

$$w(u) \leq r(u) + 1$$



This can be "generalised" in two ways:

- An *m*-skinny slicing is one for which every horizontal block u satisfies

$$w(u) \leq r(u) + m.$$

- An *m*-row-restricted slicing is one for which every horizontal block u satisfies

$$w(u) \leq m.$$

Succession rules can be given for both:

$$(h,k) \rightsquigarrow \begin{cases} (1,k+1),(2,k+1),\dots,(h,k+1), & \\ (h+1,1),\dots,(h+1,k-1),(h+1,k), & \text{if } h < m, \\ (m+1,1),\dots,(m+1,k-1),(h+1,k). & \text{if } h \geq m. \end{cases} \qquad (m\text{-Sk})$$

$$(h,k) \rightsquigarrow \begin{cases} (1,k+1),(2,k+1),\dots,(h,k+1), & \\ (h+1,1),(h+1,2),\dots,(h+1,k), & \text{if } h < m \\ (m,1),(m,2),\dots,(m,k). & \text{if } h = m \end{cases} \qquad (m\text{-RR})$$

These rules can be encoded as linear systems of functional equations satisfied by the generating functions of m-skinny and m-row-restricted.

For $m \leq 5$, we have solved these using the kernel method. Believe that this can always be done:

Conjecture

For all $m \geq 0$, the generating functions for m-skinny and m-row-restricted slicings are algebraic.

Theorem

The number of 0-skinny slicings of size n is the same as the number of 2-row-restricted slicings of size n, for all n. The generating function $H(z)$ satisfies

$$H(z) = \frac{z(H(z)+1)}{1 - z(H(z)+1)^2}.$$

Can prove this algebraically using the kernel method. We do not have a bijective proof!

Note that for $m \geq 1$, it is not the case that the number of m-skinny slicings is the same as the number of $(m+2)$-row-restricted slicings.

References

NRB, Mathilde Bouvel, Veronica Guerrini and Simone Rinaldi, *Slicings of parallelogram polyominoes: Catalan, Schröder, Baxter, and other sequences*. arXiv:1511.04864.

Thank you!