Linear Algebra Notes

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November 1, 2023

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1 Introduction

Throughout these notes we let $\mathbb{F}$ denote a field, either $\mathbb{F} = \mathbb{R}$ (the field of real numbers) or $\mathbb{F} = \mathcal{C}$ (the field of complex numbers). However the most of the results of linear algebra generalizes to arbitrary fields.

I have tried to keep definitions to a minimum

2 Linear Systems

2.1 Augmented matricies

3 Vector Spaces

4 Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors are in my opinion the crown jewel's of linear algebra. They tie together every concept introduced so far.

Definition 4.1. Let $A \in \mathbb{F}^{n \times n}$, suppose that there exists $\lambda \in \mathbb{F}$ and a non-zero $\mathbf{v} \in \mathbb{F}^n$ then λ is called an *eigenvalue* and $\mathbf{v}$ and *eigenvector* of A if and only if:

$$A\mathbf{v} = \lambda \mathbf{v} \tag{1}$$

Remark 4.2. It is important to note that $\mathbf{v} = \mathbf{0}$ is never an eigenvector, for technical reasons which will become apparent later. However it does obey Equation (1).

Example 4.3. Show that $v = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ is an eigenvector of $A \mp$ and find the corresponding eigenvalue λ.

Solution: □

4.1 Computing Eigenvalues and Eigenvectors

Let $A \in \mathbb{F}^{n \times n}$, suppose we want to find an Suppose $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, that is $A\mathbf{v} = \lambda \mathbf{v}$. Then:

$$\mathbf{0} = A\mathbf{v} - \lambda \mathbf{v} = A\mathbf{v} - \lambda I \mathbf{v} = (A - \lambda I)\mathbf{v}$$

If the matrix $A - \lambda I$ is invertible, then:

$$(A - \lambda I)^{-1}\mathbf{0} = \mathbf{v}$$

Hence we must have $\mathbf{v} = 0$ by Lemma **??**. However for $\gtrapprox$ to be an eigenvector we must have $\gtrapprox \neq \mathbf{0}$. Hence $(A - \lambda I)$ cannot be invertible and hence:

$$\det(A - \lambda I) = 0$$

This leads us to the following theorem:

Theorem 4.4. *Let $A \in \mathbb{F}^{n \times n}$ be a matrix, then the charateristic polynomial of A is:*

$$f(\lambda) = \det(A - \lambda I)$$

Furthermore $\lambda' \in \mathbb{F}$ is an eigenvalue of A if and only if $f(\lambda') = 0$.

After we have found an eigenvalue λ' using Theorem 4.4 we may obtain an eigenvector by solving the system:

$$(A - \lambda' I)\gtrapprox = \nvdash$$

We will get a parametric solution. The basis of the solution space is a basis of the eigenspace associated with λ'.

Example 4.5. Consider the matrix $A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix}$ find the eigenvalues of A and their associated eigenspaces.

Solution: □

5 Factorizations