Complex Analysis

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Contents

Lecture 20: exam self study

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Exercise 0.1. *Let $f : U \to \mathbb{C}$ be holomorphic on $U \subseteq \mathbb{C}$, with $z_0 \in U$. A powerseries for f with radius of convergence R is given by $f(z) = \sum_{k=0}^{\infty} a_n(z-z_0)^n$. Show:*

i) $g : \bar{U} \to \mathbb{C}, z \mapsto f(\bar{z})$ is holomorphic

ii) Give a power series expansion for g in $\bar{z_0} \in \bar{U}$, and its radius of convergence

Proof. i) Since g is holomorphic it satisfies the cauchy reimann equations, so

$$\frac{\partial}{\partial x}u(x,y) = \frac{\partial}{\partial y}v(x,y)$$
$$\frac{\partial}{\partial y}u(x,y) = -\frac{\partial}{\partial x}v(x,y)$$

Since $g : x + iy \mapsto \overline{f(x-iy)} = u(x,-y) - iv(x,-y) = \hat{u}(x,y) + \hat{v}(x,y)$. g also satisfies these equations.

ii) Suppose $\overline{z} \in U$ then

$$g(z) = \overline{\sum_{k=1}^{\infty} a_k(\overline{z} - z_0)^n} = \sum_{k=1}^{\infty} \overline{a_k}\,\overline{(\overline{z} - z_0)^n} = \sum_{k=1}^{\infty} \overline{a_k}(z - \overline{z_0})^n \tag{1}$$

From this it also follows that the radius of convergence is the same

■

Exercise 0.2. *Compute the curve intergral*

$$\int_{\partial B(0,1/2)} \frac{\exp 1 - z}{z^3(1-z)}dz = \int_{\partial B(0,1/2)} f(z)dz \tag{2}$$

Proof. We have singular points on the outside of the ball at 0 with radi 1/2. Thus we know that f is holomorphic on $B(0,1/2)$, since its the quotient of two holomorphic functions. This implies that f has a primitive and since $\partial B(0,1/2)$ is a closed circuit $\int_{\partial B(0,1/2)} f(z)dz = 0$. ■

Exercise 0.3. *Let $f, gU \to \mathbb{C}$ be holomorphic functions on the domain $U \subseteq \mathbb{C}$, and let $z_0 \in U$, be a root of order n of f and m og g, let $h(z) = \frac{f(z)}{g(z)}$. Show that* *i) If $n \geq m$. Then h has a removable singularity in z_0 and* $\lim_{z \to z_0} h(z) = \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)}$.

ii) $n > m$. Then h has a pole of order $m - n$ in z_0.

Proof. i) Assumme $n \geq m$, then we apply l'hopital **Maybe dont use l'hopital, instead use the definition of the derivative of f and g** m times to optain

$$\lim_{z \to z_0} h(z) = \lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f^{(m)}(z)}{g^{(m)}(z)}$$

but z_0 is not a pole of $f^{(m)}$ and $g^{(m)}$, since these are holomorphic, they are also continus, and thus $\lim_{z \to z_0} \frac{f^{(m)}(z)}{g^{(m)}(z)} = \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)} = c$. Now let $\varepsilon > 0$, then by the limit there exists $\delta > 0$ st. $||z - z_0|| \implies ||h(z) - c|| < \varepsilon$, thus $||h(z)|| < ||c|| + \varepsilon$, and the function h is bounded on the ball $B(z_0, \delta)$, and the pole is removable. by theorem 7.6

ii) Now assume $n < m$, then $f(z) = (z - z_0)^m f_1(z)$ og $g(z) = (z - z_0)^n g_1(z)$, hvor $f_1(z_0), g_1(z_0) \neq 0$, $g_1, f_1 \in H(G)$. then $\frac{f(z)}{g(z)} = (z - z_0)^{m-n} \frac{f_1(z)}{g_1(z)}$, and from this we get $\lim_{z \to z_0} (z - z_0)^{n-m} \frac{f(z)}{g(z)} = \frac{f_1(z_0)}{g_1(z_0)} \neq 0$

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