Exam in probability theory

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Chapter 1

1: Basics of probability

> **Definition 1.1.** Let B be an event st. $P(B) > 0$, then for $A \subseteq S$, we definie **conditional** probability of A given B as
>
> $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Theorem 1.2 (Law of total probability and Bayes formula)**.** *Let $B_1, B_2, \ldots,$ be sequence of events such that.*

i) $P(B_k) > 0\ \forall k \in \mathbb{N}$

ii) $B_1, B_2, \ldots$ ***partion*** *the* ***sample space*** S.

Then for any $A \subseteq S$ we have

i) Law of total probability:

$$P(A) = \sum_{k=1}^{\infty} P(A|B_k)P(b_k)$$

ii) Assuming $P(A) > 0$, then $\forall n \in \mathbb{N}$ we have

$$P(B_n|A) = \frac{P(A|B_n)P(B_n)}{\sum_{k=1}^{\infty} P(A|B_k)P(B_k)}$$

Proof. We have $A = A \cap S = A \cap (\bigcup_{k=1}^{\infty} B_k) = \bigcup_{k=1}^{\infty} A \cap B_k$. By the **distributive law of infinite unions**, however $(A \cap B_i) \cap (A \cap B_j) = \emptyset$, **pairwise disjoint**. So

$$P(A) \overset{(a)}{=} \sum_{k=1}^{\infty} P(A \cap B_k) \overset{(b)}{=} sum_{k=1}^{\infty} P(A|B_k)P(B_k)$$

Where (a) follows from the **3. Axiom of probability** and (b) from the **definition of the conditional probability**. This proves the law of total probability.
Now assume $P(A) > 0$ we have

$$P(A|B_k)P(B_k) = P(A \cap B_k) = P(B_k|A)P(A)$$

from the definition of conditional probability. This implies

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{P(A)} \overset{(c)}{=} \frac{P(A|B_k)P(B_k)}{\sum_{k=1}^{\infty} P(A|B_k)P(B_k)}$$

where (c) follows from law of total probability. ■

Chapter 2

Discrete random variables

Definition 2.1. Let X, be an **discrete stocastic random variable** with the **discrte** range $\{x_1, X_2, \ldots\}$ and define the function as

$$p(x_k) = P(X = x_k)$$

then, $p(x_k)$ is definied as the **probability mass function**.

Definition 2.2. Let X have pmf

$$p(k) = \lambda(1-\lambda)^{k-1}, k \in \mathbb{N}.$$

then X is said to follow the **geometric distrobution** with parameter $\lambda > 0$ and we denote $X \sim \mathrm{geom}(\lambda)$.

Theorem 2.3. *Let $\lambda > 0$ and $X \sim geom(\lambda)$, then*

$$E[X] = \frac{1}{\lambda}, \quad var[X] = \frac{1-p}{p^2}.$$

Proof. From proposition 2.9 we have

$$E[X] = \sum_{k=0}^{\infty} P(X > k) = \sum_{k=0}^{\infty} (1-p)^k = \frac{1}{p}$$

where we have that $P(X > k) = (1-p)^k$, since we need to have atleast k **indepependent failures**. Now we plan to use the fact that $var[X] = E[X^2] -$

$E[X]^2$ and $E[X^2] = E[X] + E[X(X-1)]$ (these are both known results) To get

$$
\begin{aligned}
E[X^2] &= E[X] + E[X(X-1)] \\
&= \frac{1}{\lambda} + \sum_{k=0}^{\infty} k(k-1)\lambda(1-\lambda)^{k-1} \\
&= \frac{1}{\lambda} + \lambda(1-\lambda) \\
&= \frac{1}{\lambda} + \lambda(1-\lambda)\sum_{k=0}^{\infty} k(k-1)(1-\lambda)^{k-2} \\
&= \frac{1}{\lambda} + \lambda(1-\lambda)\frac{2}{(1-(1-\lambda))^3} \\
&= \frac{2-\lambda}{\lambda^2}
\end{aligned}
$$

This now gives us

$$
Var[X] = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}
$$

which finishes the proof. ■

Remark 1. Note that there is a way easier proof using the theory of **generating functions**